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Normal family of meromorphic functions concerning fixed-points

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Abstract Let $A > 1$ be a constant and $\mathcal{F}$ be a family of meromorphic functions defined in a domain D. For each $f \in \mathcal{F}$, f has only zeros of multiplicity at least 3 and satisfies the following conditions: (1) $|f'''(z)| \leq A|z|$ when $f(z) = 0$; (2) $f'''(z) \neq z$; (3) all poles of f are multiple. In this paper, we characterize the non-normal sequences of $\mathcal{F}$.

Keywords Normal family · Meromorphic function · Fixed-point

Mathematics Subject Classification 30D45

1 Introduction and main results

Let $D \subseteq \mathbb{C}$ be a domain, and $\mathcal{F}$ be a family of meromorphic functions defined on D. $\mathcal{F}$ is said to be normal on D, in the sense of Montel, if for each sequence $\{f_n\} \subset \mathcal{F}$ there exists a subsequence $\{f_{n_k}\}$, such that $\{f_{n_k}\}$ converges spherically locally uniformly on D, to a meromorphic function or ∞ (see [4,9,13]).

The following well-known normality criterion was conjectured by Hayman [4], and proved by Gu [3].

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Theorem A *Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D, and k be a positive integer. If for every function $f \in \mathcal{F}$, $f \neq 0$ and $f^{(k)} \neq 1$ in D, then $\mathcal{F}$ is normal in D.*

This result has undergone various extensions and improvements. In [6] (cf. [8,11]), Pang–Yang–Zalcman obtained.

Theorem B *Let k be a positive integer. Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D, all of whose zeros have multiplicity at least $k+2$ and whose poles are multiple. Let $h(z)(\not\equiv 0)$ be a holomorphic functions on D. If for each $f \in \mathcal{F}$, $f^{(k)}(z) \neq h(z)$, then $\mathcal{F}$ is normal in D.*

When $k = 1$, an example [8, Example 1] shows that the condition on the multiplicity of zeros of functions in $\mathcal{F}$ cannot be weakened. When $k \geq 2$, Zhang–Pang–Zalcman [14] proved that the multiplicity of zeros of functions in $\mathcal{F}$ can be reduced from $k+2$ to $k+1$ in Theorem C.

Theorem C *Let $k \geq 2$ be a positive integer. Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D, all of whose zeros have multiplicity at least $k+1$ and whose poles are multiple. Let $h(z)(\not\equiv 0)$ be a holomorphic functions on D. If for each $f \in \mathcal{F}$, $f^{(k)}(z) \neq h(z)$, then $\mathcal{F}$ is normal in D.*

In [12], Xu reduced the multiplicity of the zeros of functions in $\mathcal{F}$ to k for the case $h(z) = z$, but restricting the values $f^{(k)}$ can take at the zeros of f, as follows.

Theorem D *Let $k \geq 4$ be a positive integer, $A > 1$ be a constant. Let $\mathcal{F}$ be a family of meromorphic functions in a domain D. If, for every function $f \in \mathcal{F}$, f has only zeros of multiplicity at least k and satisfies the following conditions:*

(a) $f(z) = 0 \Rightarrow |f^{(k)}(z)| \leq A|z|$.
(b) $f^{(k)}(z) \neq z$.
(c) *All poles of f are multiple .*

Then $\mathcal{F}$ is normal in D.

Theorem E *Let $A > 1$ be a constant. Let $\mathcal{F}$ be a family of meromorphic functions in a domain D. If, for every function $f \in \mathcal{F}$, f has only zeros of multiplicity at least 3 and satisfies the following conditions:*

(a) $f(z) = 0 \Rightarrow |f'''(z)| \leq A|z|$.
(b) $f'''(z) \neq z$.
(c) *All poles of f have multiplicity at least* 3.

Then $\mathcal{F}$ is normal in D.

Also in [12], Xu gave the following example to show that the condition (c) in Theorem E is necessary and the number 3 is best possible.

Example 1 (See [12]) *Let $\Delta = \{z : |z| < 1\}$, and let*

$$\mathcal{F} = \left\{ f_n(z) = \frac{(z - 1/n)^3 (z + 1/n)^3}{24 z^2} \right\}.$$

Clearly,

$$f_n'''(z) = z + \frac{1}{n^6 z^5} \neq z.$$

For each n, f_n *has two zeros* $z_1 = 1/n$ *and* $z_2 = -1/n$ *of multiplicity 3. It's easy to see that*

$$f_n'''\left(\frac{1}{n}\right) = \frac{2}{n}, \ f_n'''\left(-\frac{1}{n}\right) = -\frac{2}{n},$$

and $|f_n'''(z_i)| \leq 2|z_i| (i = 1, 2)$, *then* $f_n(z) = 0 \Rightarrow |f_n'''(z)| \leq 2|z|$. *However* $\mathcal{F}$ *is not normal at* 0 *since* $f_n(1/n) = 0$ *and* $f_n(0) = \infty$.

In this paper, inspired by the idea in [1,2], we prove the following result, which shows that the counterexample above is unique in some sense.

Theorem 1 *Let* $A > 1$ *be a constant and* $\mathcal{F}$ *be a family of meromorphic functions defined in* D, *all of whose zeros have multiplicity at least* 3 *and whose poles all are multiple, such that for each* $f \in \mathcal{F}$, $f(z) = 0 \Rightarrow |f'''(z)| \leq A|z|$, *and* $f'''(z) \neq z$. *If* $\mathcal{F}$ *is not normal at* $z_0 \in D$, *then* $z_0 = 0$ *and there exist* $r > 0$ *and* $\{f_n\} \subset \mathcal{F}$ *such that*

$$f_n(z) = \frac{\left(z - \xi_n^1\right)^3 \left(z - \xi_n^2\right)^3}{(z - \eta_n)^2} \hat{f}_n(z)$$

on $\Delta_r = \{z : |z| < r\}$, *where* $\xi_n^i/\rho_n \to c_i (i = 1, 2)$ *and* $\eta_n/\rho_n \to (c_1+c_2)/2$ *for some sequence of positive numbers* $\rho_n \to 0$ *and two distinct constants* c_1 *and* c_2. *Moreover,* $\hat{f}_n(z)$ *is holomorphic and non-vanishing on* Δ_r *such that* $\hat{f}_n(z) \to \hat{f}(z) \equiv 1/24$ *locally uniformly on* Δ_r.

In this paper, we denote $\Delta_r = \{z : |z| < r\}$ and $\Delta_r' = \{z : 0 < |z| < r\}$, and the number r may be different in different place. When $r = 1$, we drop the subscript.

2 Lemmas

To prove our results, we need the following lemmas.

Lemma 1 ([7, Lemma 2]) *Let* k *be a positive integer and let* $\mathcal{F}$ *be a family of meromorphic functions in a domain* D, *all of whose zeros have multiplicity at least* k, *and suppose that there exists* $A \geq 1$ *such that* $|f^{(k)}(z)| \leq A$ *whenever* $f(z) = 0$, $f \in \mathcal{F}$. *If* $\mathcal{F}$ *is not normal at* $z_0 \in D$, *then for each* α, $0 \leq \alpha \leq k$, *there exist a sequence of complex numbers* $z_n \in D$, $z_n \to z_0$, *a sequence of positive numbers* $\rho_n \to 0$, *and a sequence of functions* $f_n \in \mathcal{F}$ *such that*

$$g_n(\zeta) = \frac{f_n(z_n + \rho_n \zeta)}{\rho_n^{\alpha}} \to g(\zeta)$$

locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on $\mathbb{C}$*, all of whose zeros have multiplicity at least k, such that* $g^{\#}(\zeta) \le g^{\#}(0) = kA + 1$. *Moreover,* $g(\zeta)$ *has order at most 2.*

Lemma 2 ([12, Lemma 6]) *Let* f *be a transcendental meromorphic function of finite order* ρ*, and let* $k(\ge 2)$ *be a positive integer. If* f *has only zeros of multiplicity at least* k*, and there exists* $A > 1$ *such that* $f(z) = 0 \Rightarrow |f^{(k)}(z)| \le A|z|$*, then* $f^{(k)}$ *has infinitely many fix-points.*

Lemma 3 ([11, Lemma 8]) *Let* f *be a non-polynomial rational function and* k *be a positive integer. If* $f^{(k)}(z) \ne 1$*, then*

$$f(z) = \frac{1}{k!}z^k + a_{k-1}z^{k-1} + \cdots + a_0 + \frac{a}{(z-b)^m},$$

where $a_{k-1}, \ldots, a_0, a(\ne 0), b$ *are constants and* m *is a positive integer.*

Lemma 4 ([12, Lemma 10]) *Let* $k \ge 3$ *be a positive integer,* $A > 1$ *be a constant. Let* $\mathcal{F}$ *be a family of meromorphic functions in a domain* D*. Suppose that, for every* $f \in \mathcal{F}$*,* f *has only zeros of multiplicity at least* k*, and satisfies the following conditions:*

(a) $f(z) = 0 \Rightarrow |f^{(k)}(z)| \le A|z|$.
(b) $f^{(k)}(z) \ne z$.
(c) *all poles of* f *are multiple.*

Then $\mathcal{F}$ *is normal in* $D\backslash\{0\}$.

Lemma 5 *Let* $\mathcal{F}$ *be a family of functions meromorphic on* Δ_r*,* $b \in \overline{\mathbb{C}}$ *to be a constant which satisfies* $f(z) \ne b$ *on* Δ_r *for each* $f \in \mathcal{F}$*. If* $\mathcal{F}$ *is normal on* Δ_r'*, but not normal on* Δ_r*, then there exists a subsequence* $\{f_n\} \subset \mathcal{F}$ *such that* $f_n(z) \overset{\chi}{\Rightarrow} b$ *on* Δ_r'*.*

Proof Without loss of generality, we assume that $b = 0$. Since $\mathcal{F}$ is normal on Δ_r', then there exists a subsequence $\{f_n\} \subset \mathcal{F}$ such that $f_n(z) \to f(z)$ spherically locally uniformly on Δ_r'. Set $g_n(z) = 1/f_n(z)$. Thus $g_n(z) \to g(z) = 1/f(z)$ on Δ_r'. Noting that $f_n(z) \ne 0$, it follows that $f(z) \ne 0$ or $f(z) \equiv 0$ by Hurwitz's theorem and $g_n(z)$ is holomorphic on Δ_r. If $f(z) \ne 0$, then the maximum modulus principle implies that $g_n(z) \to g(z)$ on Δ_r. Hence $f_n(z) \to$ on Δ_r, a contradiction. So, $f(z) \equiv 0$. This finishes the proof of Lemma 5. □

Lemma 6 *Let* f *be a rational function, all of whose zeros are of multiplicity at least 3. If* $f'''(z) \ne z$*, then one of the following three cases must occur:*

(i)

$$f(z) = \frac{(z+c)^4}{24}; \tag{2.1}$$

(ii)

$$f(z) = \frac{(z-c_1)^5}{24(z-b)}; \tag{2.2}$$

(iii)

$$f(z) = \frac{(z-c_1)^3(z-c_2)^3}{24[z-(c_1+c_2)/2]^2}, \tag{2.3}$$

where c is nonzero constant, $b(\neq c_1)$ is a constant and c_1, c_2 are two distinct constants.

Proof First, suppose that f is a polynomial. Since $f'''(z) \neq z$, then $f'''(z) = z + c$, where $c(\neq 0)$ is a constant. Thus,

$$f(z) = \frac{1}{24}z^4 + \frac{c}{6}z^3 + a_1z^2 + a_2z + a_3$$

where a_1, a_2 and a_3 are three constants. Noting that f has only zeros of multiplicity at least 3, it follows that f has only one zero of multiplicity 4. Thus, f has the form (2.1).

Then, suppose that f is a non-polynomial rational function. Set

$$g(z) = f(z) - \frac{1}{24}z^4 + \frac{1}{6}z^3.$$

Then $g'''(z) \neq 1$, so by Lemma 3

$$g(z) = \frac{1}{6}z^3 + a_2z^2 + a_1z + a_0 + \frac{a}{(z-b)^m},$$

where $a_2, a_1, a_0, a(\neq 0), b$ are constants and m is a positive integer. Thus

$$f(z) = p_4(z) + \frac{a}{(z-b)^m} = \frac{p_4(z)(z-b)^m + a}{(z-b)^m}, \tag{2.4}$$

where

$$p_4(z) = \frac{1}{24}z^4 + a_2z^2 + a_1z + a_0.$$

Let $c_1, c_2, \ldots, c_q$ be q distinct zeros of $p_4(z)(z-b)^m + a$, with multiplicity $n_1, n_2, \ldots, n_q$. Clearly, $n_i \geq 3$, $c_i \neq b$, and c_i is a zero of $[p_4(z)(z-b)^m + a]'$ with multiplicity $n_i - 1 \geq 2 (1 \leq i \leq q)$. Since

$$\left[p_4(z)(z-b)^m + a\right]' = (z-b)^{m-1}\left[p_4'(z)(z-b) + mp_4(z)\right], \tag{2.5}$$

then c_i must be a zero of $p_4'(z)(z-b) + mp_4(z)$ with multiplicity $n_i - 1 (\geq 2)$. Comparing the degree on both sides of (2.5), it follows that $\deg[p_4'(z)(z-b) + mp_4(z)] = 4$. Now we divide two cases:

(a) $p_4'(z)(z-b) + mp_4(z)$ has only one zero c_1 with multiplicity 4;
(b) $p_4'(z)(z-b) + mp_4(z)$ has two distinct zeros c_1 and c_2 with multiplicity 2.

For case (a), it follows that $m = 1$ and

$$p_4(z)(z-b) + a = \frac{1}{24}(z-c_1)^5.$$

Thus, by (2.4), f has the form (2.2).

For case (b), it's easy to see that $m = 2$ and

$$\begin{aligned} p_4'(z)(z-b) + 2p_4(z) &= \frac{1}{4}(z-c_1)^2(z-c_2)^2, \\ p_4(z)(z-b)^2 + a &= \frac{1}{24}(z-c_1)^3(z-c_2)^3. \end{aligned}$$

These, together with (2.5) give

$$z - b = \frac{1}{2}(z - c_1 + z - c_2).$$

Thus, $b = (c_1 + c_2)/2$. Hence, by (2.4), f has the form (2.3).

This completes the proof of Lemma 6. □

3 Proof of Theorem 1

Since $\mathcal{F}$ is not normal at z_0, by Lemma 4, $z_0 = 0$. Without loss of generality, we assume that $\mathcal{F}$ is normal on Δ' but not normal at the origin.

Consider the family

$$\mathcal{G} = \left\{ g(z) = \frac{f(z)}{z} : f \in \mathcal{F} \right\}.$$

It's easy to know that $f(0) \neq 0$ for every $f \in \mathcal{F}$. Thus, for each $g \in \mathcal{G}$, $g(0) = \infty$. Furthermore, all zeros of $g(z)$ have multiplicity at least 3. On the other hand, by simple calculation, we have

$$g'''(z) = \frac{f'''(z)}{z} - \frac{3g''(z)}{z}. \tag{3.1}$$

Since $f(z) = 0 \Rightarrow |f'''(z)| \le A|z|$, it follows that $g(z) = 0 \Rightarrow |g'''(z)| \le A$.

Clearly, $\mathcal{G}$ is normal on Δ'. We claim that $\mathcal{G}$ is not normal at $z = 0$. Indeed, if $\mathcal{G}$ is normal at $z = 0$, then $\mathcal{G}$ is normal on the whole disk Δ and hence equicontinuous on Δ with respect to the spherical distance. Noting that $g(0) = \infty$ for each $g \in \mathcal{G}$, so there exists $r > 0$ such that for every $g \in \mathcal{G}$ and $|g(z)| \ge 1$ for every $z \in \Delta_r$. Then $f(z) \neq 0$ on Δ_r for all $f \in \mathcal{F}$. Since $\mathcal{F}$ is normal on Δ' but not normal on Δ, there exists a sequence $\{f_n\} \subset \mathcal{F}$ such that $f_n \to 0$ on Δ_r' according to Lemma 5. So does $\{g_n\} \subset \mathcal{G}$, where $g_n(z) = f_n(z)/z$. However $|g_n(z)| \ge 1$ for $z \in \Delta_r$, a contradiction.

Then, by Lemma 1, there exist functions $g_n \in \mathcal{G}$, points $z_n \to 0$ and positive numbers $\rho_n \to 0$ such that

$$G_n(\zeta) = \frac{g_n(z_n + \rho_n \zeta)}{\rho_n^3} \to G(\zeta), \tag{3.2}$$

converges spherically uniformly on compact subsets of $\mathbb{C}$, where G is a non-constant meromorphic function on $\mathbb{C}$ and of finite order, all zeros of G have multiplicity at least 3, and $G^{\#}(\zeta) \leq G^{\#}(0) = 3A + 1$ for all $\zeta \in \mathbb{C}$.

By [12, pp. 480–482], we can assume that $z_n/\rho_n \to \alpha$, a finite complex number. Then

$$\frac{g_n(\rho_n \zeta)}{\rho_n^3} = G_n(\zeta - z_n/\rho_n) \overset{\chi}{\to} G(\zeta - \alpha) = \widetilde{G}(\zeta)$$

on $\mathbb{C}$. Clearly, all zeros of $\widetilde{G}$ have multiplicity at least 3, and all poles of $\widetilde{G}$ are multiple, except possibly the pole at 0.

Set

$$H_n(\zeta) = \frac{f_n(\rho_n \zeta)}{\rho_n^4}. \tag{3.3}$$

Then

$$H_n(\zeta) = \frac{f_n(\rho_n \zeta)}{\rho_n^4} = \zeta \frac{g_n(\rho_n \zeta)}{\rho_n^3} \to \zeta \widetilde{G}(\zeta) = H(\zeta) \tag{3.4}$$

spherically uniformly on compact subsets of $\mathbb{C}$, and

$$H_n'''(\zeta) = \frac{f_n'''(\rho_n \zeta)}{\rho_n} \to H'''(\zeta) \tag{3.5}$$

locally uniformly on $\mathbb{C}\backslash H^{-1}(\infty)$. Obviously, all zeros of H have multiplicity at least 3, and all poles of H are multiple. Since $\widetilde{G}(0) = \infty$, $H(0) \neq 0$.

Claim (I) $H(\zeta) = 0 \Rightarrow |H'''(\zeta)| \leq A|\zeta|$; (II) $H'''(\zeta) \neq \zeta$.

If $H(\zeta_0) = 0$, by Hurwitz's theorem and (3.4), there exist $\zeta_n \to \zeta_0$ such that $f_n(\rho_n \zeta_n) = 0$ for for n sufficiently large. By the assumption, $|f_n'''(\rho_n \zeta_n)| \leq A|\rho_n \zeta_n|$. Then, it follows from (3.5) that $|H'''(\zeta_0)| \leq A|\zeta_0|$. Claim (I) is proved.

Suppose that there exists ζ_0 such that $H'''(\zeta_0) = \zeta_0$. By (3.5),

$$0 \neq \frac{f_n'''(\rho_n \zeta) - \rho_n \zeta}{\rho_n} = H_n'''(\zeta) - \zeta \to H'''(\zeta) - \zeta,$$

uniformly on compact subsets of $\mathbb{C}\backslash H^{-1}(\infty)$. Hurwitz's theorem implies that $H'''(\zeta) \equiv \zeta$ on $\mathbb{C}\backslash H^{-1}(\infty)$, and then on $\mathbb{C}$. It follows that H is a polynomial of degree 4. Since all zeros of H have multiplicity at least 3, we know that H has a single zero ζ_1 with multiplicity 4, so that $H'''(\zeta_1) = 0$, and hence $\zeta_1 = 0$ since $H'''(\zeta) \equiv \zeta$. But $H(0) \neq 0$, we arrive at a contradiction. This proves claim (II).

Then, by Lemma 2, H must be a rational function. Since all poles of H are multiple, it derives from Lemma 6 that $H(\zeta) = (\zeta + b)/24$ or

$$H(\zeta) = \frac{(\zeta - c_1)^3(\zeta - c_2)^3}{24[\ zeta - (c_1 + c_2)/2]^2},$$

where b is a constant, c_1 and c_2 are two distinct constants. But, $H(\zeta) = (\zeta + b)/24$ is impossible(for details, see [12, pp. 483–485]). By (3.3) and (3.4), it follows that

$$\frac{f_n(\rho_n\zeta)}{\rho_n^4} \to \frac{(\zeta - c_1)^3(\zeta - c_2)^3}{24[\zeta - (c_1 + c_2)/2]^2}. \tag{3.6}$$

Noting that all zeros of f_n have multiplicity at least 3, there exist $\zeta_n^1 \to c_1, \zeta_n^2 \to c_2$ and $\zeta_n^3 \to (c_1 + c_2)/2$ such that $\xi_n^1 = \rho_n\zeta_n^1$ and $\xi_n^2 = \rho_n\zeta_n^2$ are zeros of f_n with exact multiplicity 3, and $\eta_n = \rho_n\zeta_n^3$ is the pole of f_n with exact multiplicity 2.

Now write

$$f_n(z) = \frac{\left(z - \xi_n^1\right)^3 \left(z - \xi_n^2\right)^3}{(z - \eta_n)^2} \hat{f}_n(z) \tag{3.7}$$

Then by (3.6) and (3.7), it follows that

$$\hat{f}_n(\rho_n\zeta) \to \frac{1}{24} \tag{3.8}$$

on $\zeta \in \mathbb{C}$.

Next, we complete our proof in three steps.

Step 1. Claim that *there exists a* $r > 0$ *such that* $\hat{f}_n(z) \neq 0$ *on* Δ_r.

Suppose not, taking a sequence and renumbering if necessary, $\hat{f}_n$ has zeros tending to 0. Assume $\hat{z}_n \to 0$ is the zero of $\hat{f}_n$ with the smallest modulus. Then by (3.8), it's easy to know that $\hat{z}_n/\rho_n \to \infty$.

Set

$$\widehat{f_n^*}(z) = \hat{f}_n(\hat{z}_n z). \tag{3.9}$$

Thus, $\widehat{f_n^*}(z)$ is well-defined on $\mathbb{C}$ and non-vanishing on Δ. Moreover, $\widehat{f_n^*}(1) = 0$.

Now let

$$M_n(z) = \frac{\left(z - \xi_n^1/\hat{z}_n\right)^3 \left(z - \xi_n^2/\hat{z}_n\right)^3}{\left(z - \eta_n/\hat{z}_n\right)^2} \widehat{f_n^*}(z). \tag{3.10}$$

According to (3.7), (3.9) and (3.10), it follows that

$$M_n(z) = \frac{\left(z\hat{z}_n - \xi_n^1\right)^3 \left(z\hat{z}_n - \xi_n^2\right)^3}{\left(z\hat{z}_n - \eta_n\right)^2} \frac{\hat{f}_n\left(\hat{z}_n z\right)}{\left(\hat{z}_n\right)^4} = \frac{f_n\left(\hat{z}_n z\right)}{\left(\hat{z}_n\right)^4}.$$

Obviously, all zeros of $M_n(z)$ have multiplicity at least 3 and all poles of $M_n(z)$ have multiplicity at least 2. Since $f_n(z) = 0 \Rightarrow |f_n'''(z)| \leq A|z|$, it follows that $M_n(z) = 0 \Rightarrow |M_n'''(z)| \leq A|z|$. Now that $f_n'''(z) \neq z$, it derives that

$$M_n'''(z) - z = \frac{\left(f_n'''\hat{z}_n z\right) - \hat{z}_n z}{\hat{z}_n} \neq 0. \tag{3.11}$$

Hence, by Lemma 4, $\{M_n(z)\}$ is normal on $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$.

Noting that

$$\frac{\xi_n^1}{\hat{z}_n} = \frac{\xi_n^1}{\rho_n}\frac{\rho_n}{\hat{z}_n} \to 0,$$
$$\frac{\xi_n^2}{\hat{z}_n} = \frac{\xi_n^2}{\rho_n}\frac{\rho_n}{\hat{z}_n} \to 0,$$
$$\text{and} \quad \frac{\eta_n}{\hat{z}_n} = \frac{\eta_n}{\rho_n}\frac{\rho_n}{\hat{z}_n} \to 0,$$

we deduce from (3.10) that $\{\widehat{f_n^*}\}$ is also normal on $\mathbb{C}^*$. Thus by taking a subsequence, we assume that $\widehat{f_n^*} \to \widehat{f^*}$ spherically locally uniformly on $\mathbb{C}^*$. Clearly, $\widehat{f^*}(z)$ has a zero at 1 with multiplicity at least 3 since $\widehat{f_n^*}(1) = 0$.

Set

$$L_n(z) = M_n'''(z) - z. \tag{3.12}$$

Then $L_n \neq 0$ from (3.11).

Now we prove that $\hat{f}^*(z) \not\equiv 0$. Otherwise $\hat{f}_n^*(z) \to 0$, thus $L_n(z) \to -z$ and $L_n'(z) \to -1$ locally uniformly on $\mathbb{C}^*$. By the argument principle, it derives that

$$\left| n(1, L_n) - n\left(1, \frac{1}{L_n}\right) \right| = \frac{1}{2\pi}\left| \int_{|z|=1} \frac{L_n'}{L_n} dz \right| \to \frac{1}{2\pi}\left| \int_{|z|=1} \frac{1}{z} dz \right| = 1, \tag{3.13}$$

where $n(r, f)$ denotes the number of poles of f in Δ_r, counting multiplicity. It follows that $n(1, L_n) = 1$. On the other hand, the poles of $L_n(z) = M_n'''(z) - z$ have multiplicity at least 4. A contradiction.

Then $\hat{f}_n^* \to \hat{f}^* \not\equiv 0$ spherically locally uniformly on $\mathbb{C}^*$. Since $\hat{f}_n^*$ is non-vanishing on Δ, then $\hat{f}_n^* \to \hat{f}^*$ on Δ by Lemma 5. Hence, $\hat{f}_n^* \to \hat{f}^*$ on $\mathbb{C}$.

By (3.10) and (3.12), we see that

$$L_n(z) \to L(z) = \left(z^4 \widehat{f}^*(z)\right)''' - z$$

on $\mathbb{C}^* \backslash (\widehat{f}^*)^{-1}(\infty)$. Obviously, $\{L_n(z)\}$ is normal on Δ_r. If not, Lemma 5 derives that $L(z) = (z^4 \widehat{f}^*(z))''' - z \equiv 0$ since $L_n \neq 0$ on $\mathbb{C}$. Thus,

$$\hat{f}^*(z) = \frac{z^4 + a_1 z^2 + a_2 z + a_3}{24 z^4},$$

where a_1, a_2 and a_3 be three constants. Now that the zeros of $\hat{f}^*(z)$ have multiplicity at least 3 and $\hat{f}^*(1) = 0$, then

$$\hat{f}^*(z) = \frac{(z-1)^4}{24 z^4},$$

which is impossible since $z^4 + a_1 z^2 + a_2 z + a_3 \neq (z-1)^4$. So $L_n(z) \to L(z)$ on $\mathbb{C}$.

Since $L_n(z) \neq 0$, Hurwitz's theorem implies that either $L(z) \equiv 0$ or $L(z) \neq 0$. $\hat{f}^*(1) = 0$ follows that $L(z) \neq 0$. On the other hand, $\hat{f}_n^*(0) = \hat{f}_n(0) \to \hat{f}^*(0) = 1/24$, it follows that $L(0) = 0$, a contradiction. The claim is completed.

Step 2. Show that *there exists a* $r > 0$ *such that* $\hat{f}_n(z)$ *is holomorphic on* Δ_r.

Since $\{f_n\}$ and hence $\{\hat{f}_n\}$ is normal on Δ', taking a subsequence and renumbering, we have $\hat{f}_n \to \hat{f}$ spherically locally uniformly on Δ'.

It's easy to see that $\hat{f}(z) \not\equiv 0$ on Δ'. Otherwise, we have $f_n'''(z) \to 0$ and $f_n^{(4)}(z) \to 0$ locally uniformly on Δ'. Then the argument principle yields that

$$\left| n\left(\frac{1}{2}, f_n''' - z\right) - n\left(\frac{1}{2}, \frac{1}{f_n''' - z}\right) \right| = \frac{1}{2\pi} \left| \int_{|z|=\frac{1}{2}} \frac{f_n^{(4)} - 1}{f_n''' - z} dz \right| \to \frac{1}{2\pi} \left| \int_{|z|=\frac{1}{2}} \frac{1}{z} dz \right|$$
$$= 1.$$

Now that $f_n'''(z) \neq z$, it follows that $n(\frac{1}{2}, f_n''') = n(\frac{1}{2}, f_n''' - z) = 1$, which is impossible. Thus, $\hat{f}_n \to \hat{f} \not\equiv 0$.

Recalling that $\hat{f}_n(z) \neq 0$, and by Lemma 5, it gives that $\hat{f}_n \to \hat{f}$ spherically locally uniformly on Δ. Since $\hat{f}_n(0) \to 1/24$, then $\hat{f}(0) = 1/24$. Thus, there exists a positive number r such that $\hat{f}$ is holomorphic on Δ_r. Hence $\hat{f}_n$ is holomorphic on Δ_r.

Step 3. Prove that *there exists a* $r > 0$ *such that* $\hat{f}_n(z) \to \hat{f}(z) \equiv 1/24$ *on* Δ_r.

By (3.7), we get $f_n(z) \to z^4 \hat{f}(z)$ on Δ'. Thus

$$f_n'''(z) - z \to \left[z^4 \hat{f}(z)\right]''' - z, \tag{3.14}$$

on $\Delta' \backslash \hat{f}^{-1}(\infty)$.

Hence there exists $r > 0$ such that $f_n'''(z) - z \to [z^4 \hat{f}(z)]''' - z$ on Δ_r'.

If $\{f_n'''(z) - z\}$ is not normal on Δ_r, combining $f_n'''(z) \neq z$ with Lemma 5, it follows that $[z^4 \hat{f}(z)]''' - z \equiv 0$ on Δ_r'. Hence

$$z^4 \hat{f}(z) = \frac{1}{24} z^4 + a_1 z^2 + a_2 z + a_3$$

on Δ'_r. Recalling that $\hat{f}_n \to \hat{f}$ on Δ and $\hat{f}(0) = 1/24$, so $\hat{f}_n(z) \to \hat{f}(z) \equiv 1/24$ on Δ_r.

If $\{f_n'''(z) - z\}$ is normal on Δ_r, then either $[z^4 \hat{f}(z)]''' - z \equiv 0$ or $[z^4 \hat{f}(z)]''' - z \neq 0$ according to $f_n'''(z) \neq z$. Noting the fact that $[(z^4 \hat{f}(z))''' - z]|_{z=0} = 0$, it derives that $[z^4 \hat{f}(z)]''' - z \equiv 0$. Similarly, it follows that $\hat{f}_n(z) \to \hat{f}(z) \equiv 1/24$ on Δ_r.

The proof of Theorem 1 is finished. □

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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