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Inverse problems for Dirac operator with the potential known on an interior subinterval

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Abstract The inverse spectral problems for Dirac operator with the potential known on an interior subinterval are considered. We prove that the potential on the entire interval and boundary conditions are uniquely determined in terms of the potential on an interior subinterval including midpoint, the known partial eigenvalues and partial interior spectral data.

Keywords Dirac operator · Inverse spectral problem · Interior spectral data

Mathematics Subject Classification Primary 34L05 · 34A55; Secondary 34L40

1 Introduction

Consider the inverse spectral problems for the Dirac operator, denoted by $L := L(Q(x); \alpha, \beta)$, of the form

$$ly := By\prime - Q(x)y = \lambda y, \quad 0 < x < 1 \tag{1.1}$$

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with

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Q(x) = \begin{pmatrix} p(x) & 0 \\ 0 & q(x) \end{pmatrix}, \quad y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$$

subject to the boundary conditions

$$\begin{cases} U(y) := y_1(0)\cos\alpha + y_2(0)\sin\alpha = 0, \\ V(y) := y_1(1)\cos\beta + y_2(1)\sin\beta = 0, \end{cases} \quad 0 \le \alpha, \beta < \pi. \tag{1.2}$$

Here λ is a spectral parameter, $p(x), q(x) \in C[0, 1]$ and are real-valued functions. It is well known [9] that the operator L is self-adjoint and has a discrete spectrum consisting of simple and real eigenvalues denoted by $\sigma(L) = \{\lambda_n\}_{n\in\mathbb{Z}}$.

The Dirac operator is the relativistic Schrödinger operator in quantum physics. The basic and comprehensive results about Dirac operators were given in [9]. Furthermore, spectral problems for Sturm–Liouville or Dirac operators were extensively studied in various publications, see e.g. [3,4,10,11,17].

Research of inverse problems for Dirac operator follows investigations of closely related inverse problems for Sturm–Liouville operator. Arutyunyan [1] obtained an analog of Marchenko theorem [12]: one full spectrum and the corresponding normalising coefficients uniquely determined the potential $Q(x)$. Malamud [13] proved an analog of Borg theorem [2]: two spectra (defined by different boundary conditions at one end and identical conditions repeated at the other end) uniquely determined the potential $Q(x)$. He also proved an analog of the theorem of Hochstadt and Lieberman [7]: one spectrum and a potential on the interval $[0, 1/2]$ uniquely determined the potential $Q(x)$ on the whole interval $[0, 1]$. Horváth [8] obtained an analog of the theorem of Gesztesy and Simon [5]: certain part of the spectrum and a potential on an interval $[0, a_2]$ for any $a_2 > 1/2$ completely determine the potential $Q(x)$ uniquely on the interval $[0, 1]$.

The main aim of this paper is to investigate in detail the uniqueness problem for Dirac operator with the potential $Q(x)$ known on an interior subinterval $[a_1, a_2] \subset [0, 1]$ with $1/2 \in [a_1, a_2]$, and solve it by virtue of the known eigenvalues and some information on the eigenfunctions at the point a_1. The later is called interior spectral data, which together with the associated eigenvalues has been used to recover the potentials uniquely for the Dirac operators and the Sturm–Liouville problems, etc. (see [6,14,15,18] and references therein). The technique which we used to obtain this result is based on the method discussed in Horváth [8].

2 Statement of results

In this section, we will provide a new result, analogous to the theorem of Horváth [8], on the unique determination problem of the potential $Q(x)$ and α, β under the circumstance where only partial information of the potential $Q(x)$ (on an interior subinterval $[a_1, a_2] \subset [0, 1]$), of the eigenvalues $\{\lambda_n\}_{n\in\mathbb{Z}}$, and of the interior spectral data $\{\varphi_2(a_1, \lambda_n)/\varphi_1(a_1, \lambda_n)\}_{n\in\mathbb{Z}}$ is available. The function $\varphi(x, \lambda) = (\varphi_1(x, \lambda), \varphi_2(x, \lambda))^T$ is the solution of Eq. (1.1) with the initial conditions $\varphi_1(0, \lambda) =$

$\sin\alpha$ and $\varphi_2(0,\lambda) = -\cos\alpha$. Let us mention here that we allow the interior spectral data be infinite, that is, $\varphi_1(a_1,\lambda_n) = 0$.

Without loss of generality, we always assume that $a_1 \le 1 - a_2$. Otherwise, the similar approach of this paper can be used to derive similar result. Given a sequence $\Lambda := \{x_{n_k}\}_{k\in\mathbb{Z}}$ of arbitrary real numbers, suppose the set $S := \{n_k : x_{n_k} \in \Lambda\}$ is almost symmetric with respect to the origin, which means that if $n_k \in S$ then $-n_k \in S$ with finitely many exceptions. Define

$$N_\Lambda(t) = \begin{cases} \sum\limits_{0\le x_{n_k}\le t} 1 & \text{if } t > 0, \\ -\sum\limits_{t<x_{n_k}<0} 1 & \text{if } t < 0. \end{cases} \tag{2.1}$$

We give our uniqueness results for the known potential on interior subinterval $[a_1, a_2]$ through the following two cases: $a_1 < 1 - a_2$ and $a_1 = 1 - a_2$.

Theorem 2.1 *Let* $1/2 \in [a_1, a_2] \subset [0, 1]$ *with* $a_1 < 1 - a_2$. *Let* $\sigma_2 \subset \sigma := \sigma(L)$ *and* $\sigma_1 \subset \sigma_2$, *where the sets* $S_j = \{n : \lambda_n \in \sigma_j\}$ *are almost symmetric with respect to the origin for* $j = 1, 2$. *Suppose that the limits*

$$\lim_{|t|\to\infty} \frac{N_{\sigma_j}(t)}{t} = \gamma_j \tag{2.2}$$

exist for $j = 1, 2$ *and there are the constants* $t_0 > 0$ *and* $\mu_j \in \mathbb{R}$ *for* $j = 1, 2$ *such that*

$$N_{\sigma_1}(t) \begin{cases} \ge 2a_1 N_\sigma(t) + \mu_1 - a_1 & \text{if } t \ge t_0, \\ \le 2a_1 N_\sigma(t) + \mu_1 - a_1 & \text{if } t \le -t_0, \end{cases} \tag{2.3}$$

and

$$N_{\sigma_2}(t) \begin{cases} \ge 2(1-a_2) N_\sigma(t) + \mu_2 - (1-a_2) & \text{if } t \ge t_0, \\ \le 2(1-a_2) N_\sigma(t) + \mu_2 - (1-a_2) & \text{if } t \le -t_0, \end{cases} \tag{2.4}$$

where $N_\sigma(t)$ *and* $N_{\sigma_j}(t)$ *for* $j = 1, 2$ *are defined as* (2.1) *by replacing* Λ *with* σ *and* σ_j *respectively.*

Then $Q(x)$ *on* $[a_1, a_2]$, $\{\varphi_2(a_1,\lambda_n)/\varphi_1(a_1,\lambda_n)\}_{\lambda_n\in\sigma_1}$ *and* $\{\lambda_n\}_{\lambda_n\in\sigma_2}$ *uniquely determine* α, β *and* $Q(x)$ *on* $[0, 1]$.

Remark 2.2 The obtained result here is a natural generalization of the result of Horváth [8] where the case $a_1 = 0$ was treated.

For the case $a_1 = 1 - a_2$, we have the following theorem.

Theorem 2.3 *Let* $a_1 = 1 - a_2$ *with* $1/2 \in (a_1, a_2)$. *Let* $\sigma_1 \subset \sigma := \sigma(L)$, *where the set* $S_1 = \{n : \lambda_n \in \sigma_1\}$ *is almost symmetric with respect to the origin. Assume that the limit*

$$\lim_{|t|\to\infty} \frac{N_{\sigma_1}(t)}{t} = \gamma \tag{2.5}$$

exists and there are the constants $t_0 > 0$ *and* $\mu \in \mathbb{R}$ *such that*

$$N_{\sigma_1}(t) \begin{cases} \ge 2a_1 N_\sigma(t) + \mu - a_1 & \text{if } t \ge t_0, \\ \le 2a_1 N_\sigma(t) + \mu - a_1 & \text{if } t \le -t_0, \end{cases} \tag{2.6}$$

where $N_\sigma(t)$ and $N_{\sigma_1}(t)$ are defined as (2.1) by replacing Λ with σ and σ_1 respectively.

Then $Q(x)$ on $[a_1, a_2]$, $\{\lambda_n\}_{\lambda_n \in \sigma_1}$ and $\{\varphi_2(a_1, \lambda_n)/\varphi_1(a_1, \lambda_n)\}_{\lambda_n \in \sigma_1}$ uniquely determine α, β and $Q(x)$ on $[0, 1]$.

As a special case of Theorem 2.3, we have the following corollary.

Corollary 2.4 (See Theorem 2.1 in [15]) *Let $a_1 = 1/2 = a_2$. Then $\{\lambda_n\}_{n\in\mathbb{Z}}$ and $\{\varphi_2(1/2, \lambda_n)/\varphi_1(1/2, \lambda_n)\}_{n\in\mathbb{Z}}$ uniquely determine α, β and $Q(x)$ on $[0, 1]$.*

3 Proofs

We begin by recalling some classical results, which will be needed later. Let $\varphi(x, \lambda) = (\varphi_1(x, \lambda), \varphi_2(x, \lambda))^T$ be the solution of Eq. (1.1) under the initial conditions

$$\varphi_1(0, \lambda) = \sin\alpha, \quad \varphi_2(0, \lambda) = -\cos\alpha.$$

It is clear that for each fixed $x \in [0, 1]$, these solutions are entire in λ. Let $\tau = \mathrm{Im}\lambda$. Then the following representations hold uniformly in x as $|\lambda| \to \infty$ (see [9]: page 208, (5.11) and (5.12):

$$\varphi_1(x, \lambda) \;=\; \sin(\lambda x + \alpha + \eta(x)) + O\left(\frac{e^{|\tau|x}}{|\lambda|}\right), \tag{3.1}$$

$$\varphi_2(x, \lambda) = -\cos(\lambda x + \alpha + \eta(x)) + O\left(\frac{e^{|\tau|x}}{|\lambda|}\right). \tag{3.2}$$

Here

$$\eta(x) = \frac{1}{2}\int_0^x (p(t) + q(t))\,\mathrm{d}t. \tag{3.3}$$

It is well known [9] that the eigenvalues $\{\lambda_n\}_{n\in\mathbb{Z}}$ of the operator L are precisely the zeros of its characteristic function

$$g_\sigma(\lambda) = \varphi_1(1, \lambda)\cos\beta + \varphi_2(1, \lambda)\sin\beta, \tag{3.4}$$

and satisfy the classical asymptotic form

$$\lambda_n = n\pi + c_0 + O\left(\frac{1}{n}\right) \tag{3.5}$$

as $|n| \to \infty$, where

$$c_0 = \beta - \alpha - \eta(1).$$

Moreover, it follows from (3.1) and (3.2) that for sufficiently large $|\lambda|$

$$|g_\sigma(\lambda)| \geq C_\delta \exp(|\mathrm{Im}\lambda|),\;\; \lambda \in G_\delta = \{\lambda : |\lambda - n\pi - c_0| \geq \delta, n \in \mathbb{Z}\}. \tag{3.6}$$

Throughout of this paper we always assume that 0 is not an eigenvalue of the operator L defined by (1.1)–(1.2), otherwise we can make a shift. For our purpose of this paper, together with the operator L, we consider another operator $\tilde{L}$ of the same form but with different coefficients $\tilde{Q}(x), \tilde{\alpha}, \tilde{\beta}$. We agree that, everywhere below if a certain symbol δ denotes an object related to L, then $\tilde{\delta}$ will denote an analogous object related to $\tilde{L}$.

Next we give the proof of Theorem 2.1

Proof of Theorem 2.1 Let us consider another Dirac operator $\tilde{L}$ of the same form (1.1)–(1.2) but with different coefficients $(\tilde{Q}(x), \tilde{\alpha}, \tilde{\beta})$. Then both operators satisfy $\tilde{Q}(x) = Q(x)$ for $x \in [a_1, a_2]$, and have common eigenvalues $\{\lambda_n\}_{\lambda_n \in \sigma_2}$ and common interior spectral data $\{\varphi_2(a_1, \lambda_n)/\varphi_1(a_1, \lambda_n)\}_{\lambda_n \in \sigma_1}$. Under the hypothesis of Theorem 2.1, we will prove $L = \tilde{L}$ through the following two steps.

(1) We first show that $\alpha = \tilde{\alpha}$, $\tilde{Q}(x) = Q(x)$ on $[0, a_1]$. To this end, let us define function $g_{\sigma_1}(\lambda)$ by

$$g_{\sigma_1}(\lambda) = \text{p.v.} \prod_{\lambda_n \in \sigma_1} \left(1 - \frac{\lambda}{\lambda_n}\right), \tag{3.7}$$

it is known [8] that this product converges locally uniformly and defines an entire function with zeros $\{\lambda_n | \lambda_n \in \sigma_1\}$. Consider the function

$$F(\lambda) = \frac{\langle \varphi, \tilde{\varphi} \rangle (a_1, \lambda)}{g_{\sigma_1}(\lambda)}, \tag{3.8}$$

where $\langle \varphi, \tilde{\varphi} \rangle (a_1, \lambda) = (\varphi_1 \tilde{\varphi}_2 - \varphi_2 \tilde{\varphi}_1)(a_1, \lambda)$. Under the hypothesis of Theorem 2.1 we have for $\lambda_n \in \sigma_1$ that

$$\frac{\varphi_2(a_1, \lambda_n)}{\varphi_1(a_1, \lambda_n)} = \frac{\tilde{\varphi}_2(a_1, \lambda_n)}{\tilde{\varphi}_1(a_1, \lambda_n)}.$$

In particular, if both sides in the above equation are infinite, then $\varphi_1(a_1, \lambda_n) = \tilde{\varphi}_1(a_1, \lambda_n) = 0$. In all, we infer that $\langle \varphi, \tilde{\varphi} \rangle (a_1, \lambda)$ vanishes at each point where $g_{\sigma_1}(\lambda)$ vanishes, hence $F(\lambda)$ is an entire function since $g_{\sigma_1}(\lambda)$ necessarily has simple zeros.

We estimate the numerator of $F(\lambda)$ using (3.1) and (3.2) that

$$\begin{aligned} \langle \varphi, \tilde{\varphi} \rangle (a_1, \lambda) = &\ \sin(\lambda a_1 + \tilde{\alpha} + \tilde{\eta}(a_1)) \cos(\lambda a_1 + \alpha + \eta(a_1)) \\ &- \sin(\lambda a_1 + \alpha + \eta(a_1)) \cos(\lambda a_1 + \tilde{\alpha} + \tilde{\eta}(a_1)) + O\left(\frac{e^{2a_1|\text{Im}\lambda|}}{|\lambda|}\right) \\ = &\ \sin x_0 + O\left(\frac{e^{2a_1|\text{Im}\lambda|}}{|\lambda|}\right), \end{aligned}$$

where $x_0 = \tilde{\alpha} - \alpha + \tilde{\eta}(a_1) - \eta(a_1)$. We know that this function has infinitely many real zeros and the zeros are not bounded. This is compatible with the above estimate only when $\sin x_0 = 0$ and then

$$\langle \varphi, \tilde{\varphi} \rangle (a_1, \lambda) = O\left(\frac{e^{2a_1|\text{Im}\lambda|}}{|\lambda|}\right). \tag{3.9}$$

We estimate the denominator of $F(\lambda)$ in virtue of Lemma 2.6 in [8]. If $|\lambda-\lambda_n|\geq\delta$ and $|\lambda-n\pi-c_0|\geq\delta$ for all $\lambda_n\in\sigma_1$, then from (3.5) we have

$$|g_{\sigma_1}(\lambda)|\asymp|\hat{g}_{\sigma_1}(\lambda)| \tag{3.10}$$

where the notation $\asymp$ means that both $|g_{\sigma_1}(\lambda)/\hat{g}_{\sigma_1}(\lambda)|$ and $|\hat{g}_{\sigma_1}(\lambda)/g_{\sigma_1}(\lambda)|$ are bounded, and

$$\hat{g}_{\sigma_1}(\lambda)=\text{p.v.}\prod_{\{n\in\mathbb{Z}|\lambda_n\in\sigma_1\}}\left(1-\frac{\lambda}{n\pi+c_0}\right).$$

We use

$$1-\frac{\lambda+c_0}{n\pi+c_0}=\left(1-\frac{\lambda}{n\pi}\right)\left(1-\frac{c_0}{n\pi+c_0}\right)=:c_n\left(1-\frac{\lambda}{n\pi}\right)$$

to obtain

$$\hat{g}_{\sigma_1}(\lambda+c_0)=c\bar{g}_{\sigma_1}(\lambda), \tag{3.11}$$

where

$$c=\text{p.v.}\prod_{\{n\in\mathbb{Z}|\lambda_n\in\sigma_1\}}\left(1-\frac{c_0}{n\pi+c_0}\right)\text{ and }\bar{g}_{\sigma_1}(\lambda)=\text{p.v.}\prod_{\{n\in\mathbb{Z}|\lambda_n\in\sigma_1\}}\left(1-\frac{\lambda}{n\pi}\right).$$

Arrange the values $\{n\in\mathbb{Z}|\lambda_n\in\sigma_1\}$ in an increasing sequence $\{z_k\}$. Since $N_{\sigma_1}(z_k)=k/\pi+O(1)$, we have from (2.2) that

$$\frac{k}{z_k\pi}=\frac{N_{\sigma_1}(z_k)}{z_k}+o(1)\to\gamma_1$$

as $|k|\to\infty$. Now the almost symmetric property of S_1 implies a lower estimate by Lemma 2.8 in [8]: for every $\varepsilon>0$ there exists a $c>0$ such that if $|\lambda-n\pi|\geq\delta$ for all $\{n\in\mathbb{Z}|\lambda_n\in\sigma_1\}$ that

$$|\bar{g}_{\sigma_1}(\lambda)|\geq c\exp(\pi\gamma_1|\text{Im}\lambda|-\varepsilon|\lambda|).$$

By the above considerations, one infers that

$$|g_{\sigma_1}(\lambda)|\asymp|\hat{g}_{\sigma_1}(\lambda)|\asymp|\bar{g}_{\sigma_1}(\lambda-c_0)|\geq c\exp(\pi\gamma_1|\text{Im}\lambda|-2\varepsilon|\lambda|) \tag{3.12}$$

for $|\lambda|$ large enough. If both $|\lambda-\lambda_n|\geq\delta$ and $|\lambda-n\pi-c_0|\geq\delta$ hold for $\lambda_n\in\sigma_1$, then it follows from (3.12) that the whole denominator of $F(\lambda)$ has a lower estimate

$$|g_{\sigma_1}(\lambda)|\geq c\exp(2a_1|\text{Im}\lambda|-2\varepsilon|\lambda|). \tag{3.13}$$

Combined with (3.9), there exists a positive number C such that if $|\lambda-\lambda_n|\geq\delta$ and $|\lambda-n\pi-c_0|\geq\delta$ hold for $\lambda_n\in\sigma_1$, then

$$|F(\lambda)|=\left|\frac{\langle\varphi,\tilde{\varphi}\rangle(a_1,\lambda)}{g_{\sigma_1}(\lambda)}\right|\leq C\exp(3\varepsilon|\lambda|) \tag{3.14}$$

for $|\lambda|$ large enough. Consequently, the maximum modulus principle [16] yields that $|F(\lambda)|\leq C\exp(3\varepsilon|\lambda|)$ for all $\lambda\in\mathbb{C}$, which implies that if $F(iy)\to 0$ as y (real) $\to\infty$, then by virtue of [8, Lemma 2.9] one can derive that $F\equiv 0$.

Since we always assume 0 is not an eigenvalue of the operator L, without loss of generality, we assume that $N_\sigma(t)=0$ for $-1\leq t\leq 1$, which implies that $N_{\sigma_1}(t)=0$ also holds in the same interval. It follows from [8, Lemma 2.5] that

$$\begin{aligned}\ln|g_{\sigma_1}(iy)| &= \text{p.v.}\int_{-\infty}^{\infty}\frac{N_{\sigma_1}(t)}{t}\frac{y^2}{y^2+t^2}\mathrm{d}t\\ &=\int_{-\infty}^{-1}\frac{N_{\sigma_1}(t)}{t}\frac{y^2}{y^2+t^2}\mathrm{d}t+\int_{1}^{\infty}\frac{N_{\sigma_1}(t)}{t}\frac{y^2}{y^2+t^2}\mathrm{d}t.\end{aligned} \tag{3.15}$$

By (2.3), it is easy to infer that there exists a constant C_0 satisfying

$$N_{\sigma_1}(t)\begin{cases}\geq 2a_1N_\sigma(t)+C_0 & \text{if } 1<t\leq t_0,\\ \leq 2a_1N_\sigma(t)+C_0 & \text{if } -t_0\leq t<-1.\end{cases} \tag{3.16}$$

Substituting the above inequality and (2.3) into (3.15), one yields that

$$\begin{aligned}\ln|g_{\sigma_1}(iy)| \geq &\int_{-\infty}^{-t_0}\frac{(2a_1N_\sigma(t)+\mu_1-a_1)y^2}{ty^2+t^3}\mathrm{d}t+\int_{t_0}^{\infty}\frac{(2a_1N_\sigma(t)+\mu_1-a_1)y^2}{ty^2+t^3}\mathrm{d}t+O(1)\\ =&\ 2a_1\int_{-\infty}^{-1}\frac{N_\sigma(t)}{t}\frac{y^2}{y^2+t^2}\mathrm{d}t+2a_1\int_1^{\infty}\frac{N_\sigma(t)}{t}\frac{y^2}{y^2+t^2}\mathrm{d}t\\ &+(\mu_1-a_1)\int_{-\infty}^{-1}\frac{y^2}{t^3+ty^2}\mathrm{d}t+(\mu_1-a_1)\int_1^{\infty}\frac{y^2}{t^3+y^2}\mathrm{d}t+O(1).\end{aligned} \tag{3.17}$$

Here we have used the following formula:

$$\int_1^{t_0}\frac{y^2}{t^3+ty^2}\mathrm{d}t=-\frac{1}{2}\ln\left(1+\frac{y^2}{t^2}\right)\Bigg|_1^{t_0}=O(1) \tag{3.18}$$

as y (real) $\to\infty$. It follows from (3.6) that In$|g_\sigma(iy)|=|y|+O(1)$, as $y\to\infty$ (y real), analogous to (3.15), we can infer that

$$\int_{-\infty}^{-1}\frac{N_\sigma(t)}{t}\frac{y^2}{t^2+y^2}\mathrm{d}t+\int_1^{\infty}\frac{N_\sigma(t)}{t}\frac{y^2}{t^2+y^2}\mathrm{d}t=|y|+O(1). \tag{3.19}$$

Moreover, in virtue of (3.18) we have

$$\int_1^{\infty}\frac{y^2}{t^3+ty^2}\mathrm{d}t=-\frac{1}{2}\ln\left(1+\frac{y^2}{t^2}\right)\Bigg|_{t=1}^{\infty}=\ln|y|+O(1), \tag{3.20}$$

and analogously

$$\int_{-\infty}^{-1} \frac{y^2}{t^3 + ty^2} \mathrm{d}t = -\ln|y| + O(1). \tag{3.21}$$

Thus, by virtue of (3.17)–(3.21), we have

$$|g_{\sigma_1}(iy)| \geq C \exp(2a_1|y|). \tag{3.22}$$

Using (3.9), we obtain $|\langle\varphi, \tilde{\varphi}\rangle(a_1, iy)| \leq C(\exp(2a_1|y|)/|y|)$, which together with (3.8) and (3.22) yields that for $|y|$ sufficiently large

$$\begin{aligned} |F(iy)| &\leq C \frac{\exp(2a_1|y|)}{|y| \exp(2a_1|y|)} \\ &= O(|y|^{-1}). \end{aligned} \tag{3.23}$$

This implies that $|F(iy)| \to 0$ as $y \to \infty$ (y real). This together with (3.14) implies that $F \equiv 0$. Therefore, we obtain $\langle\varphi, \tilde{\varphi}\rangle(a_1, \lambda) = 0$ and

$$m(a_1, \lambda) := \frac{\varphi_2(a_1, \lambda)}{\varphi_1(a_1, \lambda)} = \frac{\tilde{\varphi}_2(a_1, \lambda)}{\tilde{\varphi}_1(a_1, \lambda)} =: \tilde{m}(a_1, \lambda) \tag{3.24}$$

for all $\lambda \in \mathbb{C}$. According to the uniqueness theorem [8, Theorem 1.3], we get $\alpha = \tilde{\alpha}$, $\tilde{Q}(x) = Q(x)$ on $[0, a_1]$.

(2) We next show that $\beta = \tilde{\beta}$, $\tilde{Q}(x) = Q(x)$ on $[a_2, 1]$. Notice that here we have known $\alpha = \tilde{\alpha}$, $\tilde{Q}(x) = Q(x)$ on $[0, a_2]$. In this situation, the uniqueness of determining $Q(x)$ and β needs to be in virtue of the set σ_2 of common eigenvalues. This can be followed by Step (1) and Theorem 1.9 in [8], and the proof of this theorem is completed. □

Proof of Theorem 2.3 The proof of this theorem is analogous to that of Theorem 2.1 and therefore is omitted. □

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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