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An inverse spectral problem for Sturm–Liouville operators with a large constant delay

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Abstract We consider the Sturm–Liouville differential equation with a constant delay, which is not less than the half length of the interval. An inverse spectral problem is studied of recovering the potential from subspectra of two boundary value problems with one common boundary condition. The conditions on arbitrary subspectra are obtained that are necessary and sufficient for the unique determination of the potential by specifying these subspectra, and a constructive procedure for solving the inverse problem is provided along with necessary and sufficient conditions of its solvability.

Keywords Differential operators · Deviating argument · Constant delay · Inverse spectral problems

Mathematics Subject Classification 34A55 · 34K29 · 45J05

1 Introduction

Inverse problems of spectral analysis consist in recovering operators from their spectral characteristics. The problems of this type often appear in mathematics, physics, mechanics, geophysics, electronics, meteorology, etc. The greatest success in the inverse problem theory has been achieved for the classical Sturm–Liouville operator (see [1–4] and the references therein) and afterwards for higher order differential operators [5–7]). In particular, it is known that the spectra of two boundary value problems

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for one and the same classical Sturm–Liouville equation with one common boundary condition uniquely determine the potential along with the coefficients of the boundary conditions (see [1,4]). In the present paper we study an inverse problem for the Sturm–Liouville operator with a constant delay.

Consider the boundary value problems $L_j = L_j(q),\ j = 0, 1$, of the form

$$-y''(x) + q(x)y(x-a) = \lambda y(x), \quad 0 < x < \pi, \tag{1}$$

$$y(0) = y^{(j)}(\pi) = 0, \tag{2}$$

where $q(x)$ is a complex-valued function, $q(x) \in L_2(0, \pi)$ and $q(x) = 0$ a.e. on $(0, a)$. It is known that the problem $L_j,\ j = 0, 1$, has infinitely many complex eigenvalues $\lambda_{n,j}$, $n \geq 1$, of the form

$$\lambda_{n,j} = \Big(n - \frac{j}{2} + O\Big(\frac{1}{n}\Big)\Big)^2. \tag{3}$$

More precise asymptotics is given in formula (11) below.

The interest in differential equations with delay has started intensively growing in twentieth century stimulated by the appearance of various applications in natural sciences and engineering, including the theory of automatic control, the theory of self-oscillating systems, long-term forecasting in the economy, biophysics, etc (see, e.g., [8–13] and the references therein). The presence of a delay in the mathematical model frequently causes phenomena that essentially influence the entire process. Technological and constructive improvements require taking into account such phenomena even in the classical areas of engineering. That is why admitting a delay in the mathematical model makes an essential advance as compared with mathematical models of ideal processes based on the assumption that the process has a local nature. In many situations even the presence of a constant delay $a > 0$ in the mathematical model describes real processes much more adequately than after its simplification by assuming $a = 0$. In particular, it is vital for transmission of acoustic signals as well as in modelling a hydraulic shock or other wave processes. For example, a second order differential equation with delay appears in modelling the combustion process in liquid-propellant rocket engines [13]. The analysis of such processes is based on the concept that there occurs a combustion time delay from the moment of propellant injection to the moment of propellant combustion. In this case the function $q(x)$ characterizes the influence of all parameters responsible for this delay, such as: cross-sectional area of propellant feed line, rocket nozzle throat area, etc.

There exists a number of results revealing spectral properties of differential operators with delay (see, e.g., [10] and the references therein). At the same time, concerning the inverse spectral theory, its classical methods do not work for such operators as well as for other classes of nonlocal operators, and therefore there are only few separate results in this direction, which do not form a comprehensive picture. However, some aspects of inverse problems for differential operators with a constant delay were studied in [14–17]. For example, in [15] it was proven that if the spectra of the problems $L_j(q),\ j = 0, 1$, coincide with the spectra of $L_j(0),\ j = 0, 1$, respectively, then $q(x) = 0$ a.e. on $(0, \pi)$. In [16] the reconstruction of $q(x)$ was studied from the two spectra if $a \in (\pi/2, \pi)$. We note that inverse problems for operators with an integral delay were studied in [16–24] and other works.

As will be seen below, unlike the classical case ($a = 0$) two spectra $\{\lambda_{n,j}\}_{n\ge1},\ j = 0, 1,$ carry an excessive information about the potential $q(x)$. For example, when $a = \pi/2$ it is sufficient to specify the eigenvalues only with, e.g., even indices. In this paper we study the inverse problem of recovering the potential $q(x)$ from given subsequences of the spectra $\{\lambda_{n,j}\}_{n\ge1},\ j = 0, 1$. Assuming in the sequel that $a \in [\pi/2, \pi)$, we note that the case $a \in (0, \pi/2)$ is essentially more difficult and requires a separate investigation. We obtain conditions on the increasing natural sequence $\{n_k\}_{k\ge1}$ that are necessary and sufficient for the unique determination of the potential $q(x)$ by specifying the subspectra $\{\lambda_{n_k,0}\}_{k\ge1}$ and $\{\lambda_{n_k,1}\}_{k\ge1}$. Moreover, we obtain also sufficient conditions for the solvability of the inverse problem, i.e. requirements on two arbitrary sequences of complex numbers to be subspectra for certain problems $L_j(q),\ j = 0, 1,$ of the form (1), (2) with a common potential $q(x)$. The related proof is constructive and generates an algorithm for solving the inverse problem. In some cases this result allows one to obtain *necessary and sufficient* conditions for the solvability of the inverse problem in terms of asymptotics. In particular, we prove that a proper asymptotic behavior is a necessary and sufficient condition for two arbitrary sequences of complex numbers $\{\lambda_{2k,0}\}_{k\ge1}$ and $\{\lambda_{2k,1}\}_{k\ge1}$ to be the even subspectra in the case $a = \pi/2$. The term "even" here indicates that each such subspectrum consists of eigenvalues with even indices. Since the eigenvalues are indexed according to their asymptotic behavior at infinity (3), in any spectrum there can be chosen infinitely many such even subspectra pairwise differing in an at most finite number of elements. In view of this, there is no restriction which namely even subspectrum to use. Analogous results also hold for other types of subspectra $\{\lambda_{n_k,0}\}_{k\ge1}$ and $\{\lambda_{n_k,1}\}_{k\ge1}$ that are not asymptotically even but for which the corresponding functional systems $\{1\} \cup \{\cos n_k x\}_{k\ge1}$ and $\{\sin(n_k - 1/2)x\}_{k\ge1}$ are Riesz bases in $L_2(0, \pi - a)$.

The paper is organized as follows. In the next section we prove some auxiliary assertions and provide the asymptotics of the spectra $\{\lambda_{n,j}\}_{n\ge1},\ j = 0, 1$. In Sect. 3 we study recovering the potential $q(x)$ from subspectra.

2 Preliminary information

Let $S(x, \lambda)$ be a solution of equation (1) satisfying the initial conditions $S(0, \lambda) = 0$, $S'(0, \lambda) = 1$. The eigenvalues of L_j coincide with the zeros of its characteristic function

$$\Delta_j(\lambda) := S^{(j)}(\pi, \lambda). \tag{4}$$

The following representations hold (see [15]):

$$S(x, \lambda) = \frac{\sin\rho x}{\rho} - \frac{\cos\rho(x-a)}{2\lambda}\int_a^x q(t)\,dt + \frac{1}{2\lambda}\int_a^x q(t)\cos\rho(x-2t+a)\,dt, \tag{5}$$

$$S'(x,\lambda) = \cos\rho x + \frac{\sin\rho(x-a)}{2\rho}\int_a^x q(t)\,dt - \frac{1}{2\rho}\int_a^x q(t)\sin\rho(x-2t+a)\,dt, \tag{6}$$

where $\rho^2 = \lambda$. Substituting (5), (6) into (4) we get

$$\Delta_0(\lambda) = \frac{\sin\rho\pi}{\rho} - \omega\frac{\cos\rho(\pi-a)}{\lambda} + \frac{1}{\lambda}\int_0^{\pi-a} w_0(x)\cos\rho x\,dx, \tag{7}$$

$$\Delta_1(\lambda) = \cos\rho\pi + \frac{\omega\sin\rho(\pi-a)}{\rho} + \frac{1}{\rho}\int_0^{\pi-a} w_1(x)\sin\rho x\,dx, \tag{8}$$

where

$$w_0(x) = v(-x) + v(x), \quad w_1(x) = v(-x) - v(x), \tag{9}$$

$$v(x) = \frac{1}{4}q\Big(\frac{\pi+a-x}{2}\Big), \quad \omega = \frac{1}{2}\int_a^\pi q(x)\,dx = \int_0^{\pi-a} w_0(x)\,dx. \tag{10}$$

By the standard approach (see, e.g., [4]) involving Rouché's theorem and by using representations (7), (8) one can prove the following theorem.

Theorem 1 *The spectrum* $\{\lambda_{n,j}\}_{n\ge1}$ *of the problem* $L_j,\ j=0,1,$ *has the form*

$$\lambda_{n,j} = \rho_{n,j}^2, \quad \rho_{n,j} = n - \frac{j}{2} + \frac{\omega\cos(n-j/2)a}{\pi n} + \frac{\kappa_n}{n}, \quad \{\kappa_n\}_{n\ge1}\in l_2. \tag{11}$$

In (11) and everywhere below we assume that one and the same symbol $\{\kappa_n\}$ denotes different sequences in l_2.

Let $\{n_k\}_{k\ge1}$ be an increasing sequence of natural numbers and $n_0 := 0$. Consider the following inverse problem.

Inverse Problem 1 Given the subspectra $\{\lambda_{n_k,j}\}_{k\ge1},\ j=0,1,$ find $q(x)$.

We begin the next section with providing conditions on the sequence $\{n_k\}_{k\ge1}$ that are necessary and sufficient for the unique determination of $q(x)$ by the subspectra. However, the value ω is always determined uniquely by their specification. Indeed, according to Theorem 1 we have

$$\lambda_{n_k,j} = \rho_{n_k,j}^2, \quad \rho_{n_k,j} = n_k - \frac{j}{2} + \frac{\omega\cos(n_k-j/2)a}{\pi n_k} + \frac{\kappa_k}{n_k}. \tag{12}$$

If $\cos n_k a \to 0$ as $k\to\infty$, then $\sin^2 n_k a \to 1$ and

$$\sin n_k a\cos\Big(n_k-\frac{1}{2}\Big)a = \sin n_k a\cos n_k a\cos\frac{a}{2} + \sin^2 n_k a\sin\frac{a}{2} \to \sin\frac{a}{2} \ne 0, \quad k\to\infty.$$

Hence

$$\omega = \lim_{k\to\infty} \pi n_k\Big(\rho_{n_k,1} - n_k + \frac{1}{2}\Big)\sin n_k a \sin^{-1}\frac{a}{2}, \quad \lim_{k\to\infty}\cos n_k a = 0. \tag{13}$$

Otherwise, for finding ω one can use, for example, the following formula

$$\omega = \lim_{\nu\to\infty} \pi n_{k_\nu}(\rho_{n_{k_\nu},0} - n_{k_\nu})\Big(\lim_{\nu\to\infty}\cos n_{k_\nu} a\Big)^{-1}, \quad \lim_{\nu\to\infty}\cos n_{k_\nu} a \neq 0. \tag{14}$$

In particular, if $a = \pi/2$ and $n_k = 2k$, then

$$\omega = \lim_{\nu\to\infty} 4\pi\nu(\rho_{4\nu,0} - 4\nu).$$

Before proceeding with the inverse problem, we introduce and study special functional systems. Consider the sequences

$$\Lambda_0 := \{\mu_{k,0}\}_{k\geq 0}, \quad \Lambda_1 := \{\mu_{k,1}\}_{k\geq 1},$$

where $\mu_{0,0} = 0,\ \mu_{k,j} := \lambda_{n_k,j},\ k \geq 1,\ j = 0, 1$. Denote by $m_{k,j}$ the multiplicity of the value $\mu_{k,j}$ in the sequence Λ_j. Note that in $\{\lambda_{n_k,0}\}_{k\geq 1}$ the multiplicity of 0 equals $m_{0,0} - 1$, while the multiplicity of $\lambda_{n_k,0} \neq 0$ is $m_{k,0}$. Without loss of generality one can assume that multiple elements in Λ_j are neighboring, i.e.

$$\mu_{k,j} = \mu_{k+1,j} = \ldots = \mu_{k+m_{k,j}-1,j}.$$

Put $S_j := \{j\}\cup\{k : \mu_{k,j} \neq \mu_{k-1,j},\ k \geq j+1\}$ and consider two systems of functions

$$\{c_n(x)\}_{n\geq 0}, \quad \{s_n(x)\}_{n\geq 1}, \tag{15}$$

where

$$c_{k+\nu}(x) = \frac{d^\nu}{d\lambda^\nu}\cos\rho x\Big|_{\lambda=\mu_{k,0}}, \quad k \in S_0,\ \nu = \overline{0, m_{k,0}-1},$$
$$s_{k+\nu}(x) = n_k\frac{d^\nu}{d\lambda^\nu}\frac{\sin\rho x}{\rho}\Big|_{\lambda=\mu_{k,1}}, \quad k \in S_1,\ \nu = \overline{0, m_{k,1}-1}.$$

In particular, we have $c_0(x) = 1$.

We remind that the functional systems in (15) are dependent on the initial choice of the increasing natural sequence $\{n_k\}_{k\geq 1}$. The following lemma gives conditions on $\{n_k\}_{k\geq 1}$ that are necessary and sufficient for systems in (15) to be complete in $L_2(0, b)$ (to be a Riesz basis in $L_2(0, b)$).

Lemma 1 (i) *The system* $\{c_n(x)\}_{n\geq 0}$ *is complete (is a Riesz basis) in* $L_2(0, b)$ *if and only if so is the system* $\{\cos n_k x\}_{k\geq 0}$.

(ii) *The system* $\{s_n(x)\}_{n\geq 1}$ *is complete (is a Riesz basis) in* $L_2(0, b)$ *if and only if so is the system* $\{\sin(n_k - 1/2)x\}_{k\geq 1}$.

Proof First let us prove that the systems $\{c_n(x)\}_{n\geq 0}$ and $\{\cos n_k x\}_{k\geq 0}$ can be complete in $L_2(0,b)$ only simultaneously. Let $\{\cos n_k x\}_{k\geq 0}$ be not complete in $L_2(0,b)$. Then there exists a nonzero entire function $u(\lambda)$ of the form

$$u(\lambda) = \int_0^b f(x)\cos\rho x\,dx, \quad f(x)\in L_2(0,b),$$

whose zeros include the sequence $\{n_k^2\}_{k\geq 0}$. Consider the meromorphic function

$$F(\lambda) := \prod_{k=1}^{\infty} \frac{\mu_{k,0}-\lambda}{n_k^2-\lambda}.$$

Let us show that the function $v(\lambda) := F(\lambda)u(\lambda)$ also has the form

$$v(\lambda) = \int_0^b g(x)\cos\rho x\,dx, \quad g(x)\in L_2(0,b). \tag{16}$$

As in the proof of Lemma 3.3 in [19] one can show that $|F(\lambda)| < C_\delta$ in $G_{\delta,1} := \{\lambda = \rho^2 : |\rho \pm n_k| \geq \delta,\ k\in\mathbb{N}\}$ for each fixed $\delta > 0$. Obviously, the function $v(\lambda)$, after removing the singularities, is entire in λ and, by virtue of the maximum modulus principle, we have $|v(\lambda)| \leq C_\delta |u(\lambda)|$ on $\mathbb{C}$. Hence, $\{v((\pi n/b)^2)\}_{n\geq 0} \in l_2$ and one can construct a function $g(x)\in L_2(0,b)$ such that

$$v\Big(\Big(\frac{\pi n}{b}\Big)^2\Big) = \int_0^b g(x)\cos\frac{\pi n x}{b}\,dx, \quad n+1\in\mathbb{N}.$$

Consider the function

$$R(\lambda) := \Big(v(\lambda) - \int_0^b g(x)\cos\rho x\,dx\Big)\rho^{-1}\sin^{-1}\rho b,$$

which, by definition of $g(x)$, after removing the singularities, is entire in λ. Further, it is clear that $R(\lambda) = o(\rho^{-1})$ as $\lambda\to\infty$ in $G_{\delta,2} := \{\lambda = \rho^2 : |\rho - \pi n/b| \geq \delta,\ n\in\mathbb{Z}\}$ for each fixed $\delta > 0$. Again using the maximum modulus principle along with Liouville's theorem we get $R(\lambda)\equiv 0$, and hence the representation (16) holds. Since the function $v(\lambda)$ is not identically zero and its zeros include the elements of the sequence Λ_0 with account of their multiplicities, the functional system $\{c_n(x)\}_{n\geq 0}$ is not complete in $L_2(0,b)$. In the inverse way, assuming the incompleteness of $\{c_n(x)\}_{n\geq 0}$ one can analogously show that the system $\{\cos n_k x\}_{k\geq 0}$ is not complete too.

Let us show the simultaneous Riesz basisness of these systems. According to Proposition 1.8.5 in [4], a system of functions forms a Riesz basis if and only if it is complete and quadratically close to a Riesz basis. Thus, it is sufficient to prove the quadratical closeness of $\{c_n(x)\}_{n\geq 0}$ and $\{\cos n_k x\}_{k\geq 0}$ in $L_2(0,b)$, which is equivalent to the inequality

$$\sum_{k=0}^{\infty}\int_0^b |c_k(x)-\cos n_k x|^2\,dx<\infty. \tag{17}$$

Indeed, (17) follows from (11) and the estimate for large k (and as $k\to\infty$)

$$c_k(x)-\cos n_k x=\cos\rho_{n_k,0}x-\cos n_k x$$
$$=2\sin\frac{(\rho_{n_k,0}+n_k)x}{2}\sin\frac{(n_k-\rho_{n_k,0})x}{2}=O\Big(\frac{1}{k}\Big).$$

Thus, (i) is proven. The second assertion (ii) can be proven analogously. □

3 Solution of the inverse problem

The following theorem gives conditions on the sequence $\{n_k\}_{k\ge1}$ that are necessary and sufficient for the unique determination of the potential $q(x)$ by specifying the subspectra $\{\lambda_{n_k,0}\}_{k\ge1}$ and $\{\lambda_{n_k,1}\}_{k\ge1}$.

Theorem 2 *Specification of the subspectra $\{\lambda_{n_k,0}\}_{k\ge1}$ and $\{\lambda_{n_k,1}\}_{k\ge1}$ of the problems $L_0(q)$ and $L_1(q)$ uniquely determines the potential $q(x)$ if and only if the functional systems $\{\cos n_k x\}_{k\ge0}$ and $\{\sin(n_k-1/2)x\}_{k\ge1}$ are complete in $L_2(0,\pi-a)$.*

Proof We start with the necessity. For $j=0,1$ differentiating $\nu=\overline{0,m_{k,j}-1}$ times the functions $\lambda^{1-j}\Delta_j(\lambda)$ using (7) and (8), where $k\in S_j$, and afterwards substituting $\lambda=\lambda_{n_k,j}$ into the obtained derivatives, we arrive at

$$\beta_{k+\nu,0}:=\frac{d^\nu}{d\lambda^\nu}\Big(\omega\cos\rho(\pi-a)-\rho\sin\rho\pi\Big)\Big|_{\lambda=\lambda_{n_k,0}}$$
$$=\int_0^{\pi-a} w_0(t)c_{k+\nu}(t)\,dt,\quad k\in S_0,\ \nu=\overline{0,m_{k,0}-1}, \tag{18}$$

where $\lambda_{n_0,0}=0$, and

$$\beta_{k+\nu,1}:=-n_k\frac{d^\nu}{d\lambda^\nu}\Big(\cos\rho\pi+\frac{\omega\sin\rho(\pi-a)}{\rho}\Big)\Big|_{\lambda=\lambda_{n_k,1}}$$
$$=\int_0^{\pi-a} w_1(t)s_{k+\nu}(t)\,dt,\quad k\in S_1,\ \nu=\overline{0,m_{k,1}-1}. \tag{19}$$

Since by the assumption the specification of the subspectra uniquely determines $q(x)$ and according to (9) and (10) the specification of $q(x)$, in turn, uniquely determines the functions $w_0(x)$ and $w_1(x)$, we observe that the specification of the subspectra uniquely determines $w_0(x)$ and $w_1(x)$. In other words, $w_0(x)$ and $w_1(x)$ are unique functions in $L_2(0,\pi-a)$ obeying (18) and (19), respectively, which is equivalent to the completeness in $L_2(0,\pi-a)$ of each functional system in (15), and by Lemma 1, the necessity is proven.

Let us prove the sufficiency. By virtue of Lemma 1, each functional system in (15) is complete in $L_2(0, \pi - a)$, then according to (18) and (19) the functions $w_0(x)$ and $w_1(x)$ are uniquely determined by the specification of the subspectra. Further, according to (9) and (10), the function $q(x)$ on (a, π) is uniquely determined by the formula

$$q(x) = 2\begin{cases} w_0(\pi + a - 2x) - w_1(\pi + a - 2x), & a < x < \dfrac{\pi + a}{2}, \\ w_0(2x - \pi - a) + w_1(2x - \pi - a), & \dfrac{\pi + a}{2} < x < \pi, \end{cases}$$

which finishes the proof. □

The following theorem gives sufficient conditions for the solvability of Inverse Problem 1.

Theorem 3 *For arbitrary sequences of complex numbers* $\{\mu_{k,j}\}_{k\ge 1},\ j = 0, 1,$ *to be subspectra of boundary value problems* $L_j(q),\ j = 0, 1,$ *respectively, it is sufficient that the following two conditions are fulfilled:*

(i) *These sequences* $\{\mu_{k,j}\}_{k\ge 1},\ j = 0, 1,$ *have the asymptotics*

$$\mu_{k,j} = \Big(n_k - \frac{j}{2} + \frac{\omega\cos(n_k - j/2)a}{\pi n_k} + \frac{\kappa_k}{n_k}\Big)^2, \quad \{\kappa_k\}_{k\ge 1} \in l_2, \tag{20}$$

with a certain increasing sequence of natural numbers $\{n_k\}_{k\ge 1}$;

(ii) *Each of the systems* $\{\cos n_k x\}_{k\ge 0}$ *and* $\{\sin(n_k - 1/2)x\}_{k\ge 1}$ *is a Riesz basis in* $L_2(0, \pi - a)$.

Remark 1 According to Theorem 2, under the condition (ii) of Theorem 3, the solution of Inverse Problem 1 is unique.

Proof of Theorem 3. Put $\lambda_{n_k,j} := \mu_{k,j},\ k \ge 1,\ j = 0, 1$. Then (20) implies (12). Find ω by formula (13) or (14). Construct the sequences $\{\beta_{n,j}\}_{n\ge j},\ j = 0, 1,$ by formulae

$$\begin{aligned} \beta_{k+\nu,0} &:= \frac{d^\nu}{d\lambda^\nu}\Big(\omega\cos\rho(\pi - a) - \rho\sin\rho\pi\Big)\Big|_{\lambda=\lambda_{n_k,0}}, \quad k \in S_0,\ \nu = \overline{0, m_{k,0} - 1}, \\ \beta_{k+\nu,1} &:= -n_k\frac{d^\nu}{d\lambda^\nu}\Big(\cos\rho\pi + \frac{\omega\sin\rho(\pi - a)}{\rho}\Big)\Big|_{\lambda=\lambda_{n_k,1}}, \ k \in S_1,\ \nu = \overline{0, m_{k,1} - 1}, \end{aligned} \tag{21}$$

where $\lambda_{0,n_0} = 0$. Let us show that $\{\beta_{n,j}\}_{n\ge j} \in l_2,\ j = 0, 1,$ i.e. $\beta_{n,j} = \kappa_n$. For briefness we denote

$$\varepsilon_{k,j} := \rho_{n_k,j} - n_k + \frac{j}{2}, \quad k \ge 1, \quad j = 0, 1.$$

Then, according to (12) and (21) for large k we have

$$\beta_{k,0} = \omega\cos n_k(\pi-a) - n_k\sin\varepsilon_{k,0}\pi\cos n_k\pi + O\Big(\frac{1}{n_k}\Big)$$
$$= \omega(-1)^{n_k}\cos n_k a - n_k\pi\varepsilon_{k,0}(-1)^{n_k} + O\Big(\frac{1}{k}\Big) = \kappa_k,$$
$$\beta_{k,1} = -n_k\Big((-1)^{n_k}\sin\varepsilon_{k,1}\pi + \frac{\omega}{n_k}\sin\Big(n_k-\frac{1}{2}\Big)(\pi-a)\Big) + O\Big(\frac{1}{n_k}\Big)$$
$$= -n_k(-1)^{n_k}\Big(\varepsilon_{k,1}\pi - \frac{\omega}{n_k}\cos\Big(n_k-\frac{1}{2}\Big)a\Big) + O\Big(\frac{1}{k}\Big) = \kappa_k.$$

According to Lemma 1, the functional systems in (15) are Riesz bases in $L_2(0,\pi-a)$. Find the functions $W_0(x),\ W_1(x)\in L_2(0,\pi-a)$ by the formulae

$$W_0(x) = \sum_{n=0}^{\infty}\beta_{n,0}c_n^*(x),\quad W_1(x) = \sum_{n=1}^{\infty}\beta_{n,1}s_n^*(x), \tag{22}$$

where $\{c_n^*(x)\}_{n\ge 0}$ and $\{s_n^*(x)\}_{n\ge 1}$ are Riesz bases that are biorthogonal to the bases $\{c_n(x)\}_{n\ge 0}$ and $\{s_n(x)\}_{n\ge 1}$, respectively. Construct the function $q(x)$ by the formula

$$q(x) = 2\begin{cases} 0, & 0<x<a,\\ W_0(\pi+a-2x) - W_1(\pi+a-2x), & a<x<\dfrac{\pi+a}{2},\\ W_0(2x-\pi-a) + W_1(2x-\pi-a), & \dfrac{\pi+a}{2}<x<\pi. \end{cases} \tag{23}$$

It remains to show that the sequences $\{\lambda_{n_k,0}\}_{k\ge 1}$ and $\{\lambda_{n_k,1}\}_{k\ge 1}$ are subspectra of the boundary value problems $L_0(q)$ and $L_1(q)$, respectively. In other words, one needs to prove that these sequences with account of multiplicities of their elements are subsequences of zeros of the corresponding characteristic functions, i.e.

$$\Delta_0^{(\nu)}(0) = 0,\ \nu = \overline{0, m_{0,0}-2},\quad \Delta_0^{(\nu)}(\lambda_{n_k,0}) = 0,\ k\in S_0\backslash\{0\},$$
$$\nu = \overline{0, m_{k,0}-1},$$
$$\Delta_1^{(\nu)}(\lambda_{n_k,1}) = 0,\quad k\in S_1,\ \nu = \overline{0, m_{k,1}-1}, \tag{24}$$

where $m_{k,j}$ and S_j are determined in Sect. 2.

Indeed, the functions $\Delta_0(\lambda)$ and $\Delta_1(\lambda)$ have the forms (7) and (8) with $w_0(x)$, $w_1(x)$ determined in (9), (10). Substituting (23) into (9) and (10) we get $w_j(x) = W_j(x),\ j=0,1$, a.e. on $(0,\pi-a)$, and hence

$$\Delta_0(\lambda) = \frac{\sin\rho\pi}{\rho} - \omega\frac{\cos\rho(\pi-a)}{\lambda} + \frac{1}{\lambda}\int_0^{\pi-a} W_0(x)\cos\rho x\,dx,$$
$$\Delta_1(\lambda) = \cos\rho\pi + \frac{\omega\sin\rho(\pi-a)}{\rho} + \frac{1}{\rho}\int_0^{\pi-a} W_1(x)\sin\rho x\,dx. \tag{25}$$

For $j = 0, 1$ and $k \in S_j$ differentiating $\nu = \overline{0, m_{k,j} - 1}$ times the functions $\lambda^{1-j}\Delta_j(\lambda)$ using (25) and afterwards substituting $\lambda = \lambda_{n_k,j}$ into the obtained derivatives, we arrive at

$$\begin{aligned}
&\lambda_{n_k,0}\Delta_0^{(\nu)}(\lambda_{n_k,0}) + \nu\Delta_0^{(\nu-1)}(\lambda_{n_k,0}) \\
&= -\beta_{k+\nu,0} + \int_0^{\pi-a} W_0(x)c_{k+\nu}(x)\,dx, \quad k \in S_0, \quad \nu = \overline{0, m_{k,0} - 1}, \\
&\lambda\Delta_1^{(\nu)}(\lambda_{n_k,1}) = -\beta_{k+\nu,1} + \int_0^{\pi-a} W_0(x)s_{k+\nu}(x)\,dx, \quad k \in S_1, \quad \nu = \overline{0, m_{k,1} - 1}.
\end{aligned} \tag{26}$$

Further, by (22) we have

$$\beta_{n,0} = \int_0^{\pi-a} W_0(t)c_n(t)\,dt, \quad n \geq 0, \quad \beta_{n,1} = \int_0^{\pi-a} W_1(t)s_n(t)\,dt, \quad n \geq 1,$$

which along with (26) give (24). □

The proof of Theorem 3 is constructive and gives the following algorithm for solving Inverse Problem 1.

Algorithm 1 *Let the subspectra* $\{\lambda_{n_k,j}\}_{k\geq 1}$, $j = 0, 1$, *be given. Then:*

(i) *Find* ω *by formula* (13) *or* (14)*;*
(ii) *Construct the sequences* $\{\beta_{n,j}\}_{n\geq j} \in l_2$, $j = 0, 1$, *by formulae* (21)*;*
(iii) *Find the functions* $W_0(x)$, $W_1(x) \in L_2(0, \pi - a)$ *by the formulae* (22)*;*
(iv) *Construct the function* $q(x)$ *by the formula* (23).

While Theorem 3 gives only *sufficient* conditions for the solvability of Inverse Problem 1, in some cases using it one can obtain *necessary and sufficient* conditions for the solvability. Let us illustrate this for the situation when $a = \pi/2$ and the so-called even subspectra are used, i.e. those consisting of eigenvalues with even indices.

Theorem 4 *Let* $a = \pi/2$. *For arbitrary sequences of complex numbers* $\{\mu_{k,0}\}_{k\geq 1}$ *and* $\{\mu_{k,1}\}_{k\geq 1}$ *to be even subspectra of boundary value problems* $L_0(q)$ *and* $L_1(q)$, *respectively, it is necessary and sufficient to have the asymptotics*

$$\mu_{k,j} = \Big(2k - \frac{j}{2} + \frac{\omega\cos(2k - j/2)a}{2\pi k} + \frac{\kappa_k}{k}\Big)^2, \quad \{\kappa_k\}_{k\geq 1} \in l_2.$$

Proof According to Theorem 3, it is sufficient to show that $\{\cos 2kx\}_{k\geq 0}$ and $\{\sin(2k-1/2)x\}_{k\geq 1}$ are Riesz bases in $L_2(0, \pi/2)$. It is well-known that $\{\cos 2kx\}_{k\geq 0}$ is even an orthogonal bases in $L_2(0, \pi/2)$, while the Riesz basisness of $\{\sin(2k - 1/2)x\}_{k\geq 1}$ has been proven in [25]. □

The following uniqueness theorem is a direct corollary of Theorem 2.

Theorem 5 *The specification of even subspectra* $\{\lambda_{2k,0}\}_{k\geq 1}$ *and* $\{\lambda_{2k,1}\}_{k\geq 1}$ *uniquely determines the potential* $q(x)$.

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Compliance with ethical standards

Conflict of interest The authors declare no conflict of interest.

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