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Small amplitude periodic solutions in time for one-dimensional nonlinear wave equations

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Abstract This paper is devoted to the construction of solutions for one-dimensional wave equations with Dirichlet or Neumann boundary conditions by means of a Nash-Moser iteration scheme, for a large set of frequencies.

Keywords Wave equation · Periodic solution · Small divisors · Nash-Moser iteration

Mathematics Subject Classification 35B10 · 35L70 · 58C15

1 Introduction

We consider one-dimensional nonlinear wave equations like

$$u_{tt} - u_{xx} + mu = \varepsilon g(x, \omega t, u), \quad x \in [0, \pi], \ t \in \mathbb{R}, \tag{1}$$

where $g(x, \cdot, u)$ is a time-periodic external forcing with period 2π, $g(x, t, u) \in C^\kappa([0, \pi] \times \mathbb{R} \times \mathbb{R}; \mathbb{R})$ for some κ large enough, and $g(x, t, 0) = 0$; the mass $m \in \mathbb{R}^+$; $\varepsilon > 0$ is a small amplitude parameter; ω is a frequency parameter; and the displacement $u : [0, \pi] \times \mathbb{R} \to \mathbb{R}$ is the unknown. In the present paper we want to consider both Dirichlet boundary conditions

$$u(0, t) = u(\pi, t) = 0, \quad t \in \mathbb{R}, \tag{2}$$

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and Neumann boundary conditions

$$u_x(0,t) = u_x(\pi,t) = 0, \quad t \in \mathbb{R}. \tag{3}$$

The existence of Cantor families of periodic solutions of the nonlinear wave equations have been studied by many authors, for example, see [2,3,7–13] and the references therein. Recently, Berti and Bolle [11] have proved the existence of Cantor families of spatial periodic solutions for nonlinear wave equations in higher spatial dimensions with periodic boundary conditions of the form

$$\begin{cases} u_{tt} - \Delta u + mu = \varepsilon F(\omega t, x, u), \\ u(t,x) = u(t, x + 2\pi k), \quad \forall k \in \mathbb{Z}^d. \end{cases}$$

Biasco and Gregorio [13] have studied the periodic in time solutions of the one-dimensional autonomous nonlinear wave equation with Dirichlet boundary conditions:

$$\begin{cases} u_{tt} - u_{xx} + \mu u + f(u) = 0, \\ u(t,0) = u(t,\pi) = 0, \end{cases}$$

where $\mu > 0$ is the mass and the nonlinearity f is an odd, real analytic function with $f'(0) = 0,\ f'''(0) \neq 0$.

In addition, there are many other references, but most of them studied on the nonlinear wave equations with Dirichlet boundary conditions. The proofs of all above results rely on the use of the Nash-Moser implicit function theorem, to overcome unavoidable losses of derivatives coming from the small divisors appearing when inverting the linear part of the equation. In order to construct the existence of periodic and quasi-periodic solutions to nonlinear wave equations, this main difficulty, namely the presence of small divisors in the expansion series of the solutions, can be handled by KAM theory (see, e.g., [5,6,14,20]), Lindstedt series method (see, e.g., [15–18]), and Nash-Moser iteration (see, e.g., [1–3,7–11]).

The principle objective here is to look for small amplitude, $2\pi/\omega$-periodic in time solutions of Eq. (1) under Dirichlet boundary conditions (2) or Neumann boundary conditions (3) for all frequencies ω in some set of positive measure. The small divisors problem is overcome thanks to employing Nash-Moser iteration techniques.

The organization of the paper is described as follows. The next section states the main theorem on existence of Cantor families of time-periodic solutions of the system (1)–(2) or (1)–(3). In Sect. 3, we construct the solutions to the systems by making use of suitable Nash-Moser iteration scheme, and give the proof of theorem afterward in Sect. 4. The last section is devoted to showing the invertibility of linearized problem via the eigenvalues technique.

2 Statement of the main theorem

We denote by

$$\{\lambda_i | 0 < \lambda_1 < \lambda_2 \le \cdots \le \lambda_k \le \cdots\}$$

and

$$\{\tilde{\lambda}_j | 0 < \tilde{\lambda}_1 < \tilde{\lambda}_2 \le \cdots \le \tilde{\lambda}_k \le \cdots\}$$

respectively, the eigenvalues of Dirichlet boundary problem

$$\begin{cases} -\varphi_i'' + m\varphi_i = \lambda_i^2 \varphi_i, & x \in (0, \pi), \\ \varphi_i(0) = \varphi_i(\pi) = 0, \end{cases} \tag{4}$$

and the eigenvalues of Neumann boundary problem

$$\begin{cases} -\tilde{\varphi}_j'' + m\tilde{\varphi}_j = \tilde{\lambda}_j^2 \tilde{\varphi}_j, & x \in (0, \pi), \\ \tilde{\varphi}_j'(0) = \tilde{\varphi}_j'(\pi) = 0, \end{cases} \tag{5}$$

where $\varphi_i(x)$ and $\tilde{\varphi}_j(x)$ are the corresponding eigenfunctions, respectively.

Normalizing the period to 2π, (1) can be written to

$$\omega^2 u_{tt} - u_{xx} + mu = \varepsilon g(x, t, u), \quad x \in [0, \pi], \ t \in \mathbb{R}. \tag{6}$$

We look for periodic solutions of (1)–(2) in the Banach spaces

$$X_{\sigma,s} := \left\{ u(x,t) = \sum_{j=1}^{\infty} u_j(t)\varphi_j(x) \,\middle|\, u_j \in H^1(\mathbb{R}, \mathbb{R}), \right.$$
$$\left. \|u\|_{\sigma,s}^2 = \sum_{j=1}^{\infty} \|u_j\|_{H^1}^2 j^{2s} e^{2\sigma j} < +\infty \right\} \tag{7}$$

or look for periodic solutions of (1)–(3) in the Banach spaces as follows

$$\tilde{X}_{\sigma,s} := \left\{ u(x,t) = \sum_{j=1}^{\infty} \tilde{u}_j(t)\tilde{\varphi}_j(x) \,\middle|\, \tilde{u}_j \in H^1(\mathbb{R}, \mathbb{R}), \right.$$
$$\left. \|u\|_{\sigma,s}^2 = \sum_{j=1}^{\infty} \|\tilde{u}_j\|_{H^1}^2 j^{2s} e^{2\sigma j} < +\infty \right\} \tag{8}$$

where $s > 1/2, \ \sigma \ge 0$.

For convenience, the spaces $X_{\sigma,s}$ and $\tilde{X}_{\sigma,s}$, the eigenvalues λ and $\tilde{\lambda}$, the eigenfunctions φ and $\tilde{\varphi}$, are unified into $X_{\sigma,s}$, λ, and φ, respectively. And we write $X_{\sigma,s}$ for X_s, $\|u\|_{\sigma,s}$ for $\|u\|_s$. For $s > 1/2$, X_s is a multiplicative Banach algebra (the proof is as in [4], Appendix 6.5), namely

$$\forall u_1, u_2 \in X_s \Rightarrow u_1 u_2 \in X_s, \quad \text{and} \quad \|u_1 u_2\|_s \leq C(s)\|u_1\|_s \|u_2\|_s.$$

For the two boundary value problems above, we can get the same conclusion as follows.

Theorem 1 *For the fixed* $0 < \bar{\omega}_1 < \bar{\omega}_2$, *there are* $s, \kappa \in \mathbb{N}$, *such that* $\forall g \in C^{\kappa}([0, \pi] \times \mathbb{R} \times \mathbb{R})$, $\forall \gamma \in (0, \lambda_1)$, *there exist* $\varepsilon_0, K, C > 0$, *a map* $\tilde{u} \in C^1([0, \varepsilon_0] \times [\bar{\omega}_1, \bar{\omega}_2]; X_0)$ *with*

$$\|\tilde{u}\|_0 \leq \frac{\varepsilon K}{\gamma}, \qquad \|D_\varepsilon \tilde{u}\|_0 \leq \frac{K}{\gamma}, \qquad \|D_\omega \tilde{u}\|_0 \leq \frac{\varepsilon K}{\gamma^2},$$

and a Cantor like set $A_\infty \subset [0, \varepsilon_0] \times [\bar{\omega}_1, \bar{\omega}_2]$ *such that* $\tilde{u}(\varepsilon, \omega)$ *is a solution of* (1)–(2) *or* (1)–(3).

Moreover, the set A_∞ *satisfies Lebesgue measure property:*

$$|A_\infty| \geq \varepsilon_0(\bar{\omega}_2 - \bar{\omega}_1)(1 - C\gamma),$$

i.e. $\lim_{\gamma \to 0} (|A_\infty| / \varepsilon_0(\bar{\omega}_2 - \bar{\omega}_1)) = 1$.

3 The Nash-Moser iteration scheme

Consider the orthogonal splitting $X_s = X^{(n)} \oplus X^{(n)\perp}$, where

$$X^{(n)} := \left\{ u \in X_s \middle| u(x,t) = \sum_{\lambda_j \leq N_n} u_j(t)\varphi_j(x) \right\},$$

$$X^{(n)\perp} := \left\{ u \in X_s \middle| u(x,t) = \sum_{\lambda_j > N_n} u_j(t)\varphi_j(x) \right\},$$

with $N_n := N_0 2^n$, $n \in \mathbb{N}$, and $N_0 \in \mathbb{N}$ large enough.

The convergence of the Nash-Moser scheme is based on properties (P1)–(P5) below. The first three properties are standard for the composition operator $f : X_s \to X_s$ defined by $f(u)(x,t) := g(x, t, u(x,t))$.

(**P1**) (Regularity) $f \in C^2(X_s; X_s)$ and f, Df, $D^2 f$ are bounded on $\{\|u\|_s \leq 1\}$.
(**P2**) (Tame) $\forall s \leq s' \leq k$, $\forall u \in X_{s'}$ such that $\|u\|_s \leq 1$, $\|f(u)\|_{s'} \leq C(s')(1 + \|u\|_{s'})$.
(**P3**) (Taylor Tame) $\forall s \leq s' \leq k - 2$, $\forall u, h \in X_{s'}$ such that $\|u\|_s, \|h\|_s \leq 1$,

$$\|f(u+h) - f(u) - Df(u)[h]\|_{s'} \leq C(s') \left(\|u\|_{s'} \|h\|_s^2 + \|h\|_s \|h\|_{s'} \right),$$

Where $\mathrm{D}f(u) = \partial_u g(x, t, u(x, t))$. In particular, for $s' = s$,

$$\|f(u+h) - f(u) - \mathrm{D}f(u)[h]\|_s \le C\|h\|_s^2.$$

We refer to the references [19,21] for the proof of (P2). Properties (P1) and (P3) are obtained similarly.
We assume the existence of orthogonal projectors respectively onto $X^{(n)}$ and $X^{(n)\perp}$ denoted by $P_n : X \to X^{(n)}$ and $P_n^\perp : X \to X^{(n)\perp}$.
(**P4**) (Smoothing) For all $s, r > 0$, $\forall n \in \mathbb{N}$, there hold

$$\|P_n u\|_{s+r} \le N_n^r \|u\|_s, \quad \forall u \in X_s, \tag{9}$$

$$\|(I - P_n)u\|_s \le N_n^{-r} \|u\|_{s+r}, \quad \forall u \in X_{s+r}, \tag{10}$$

where I is the identity map.
The key property (P5), proved in Sect. 5, is an invertibility property for the linearized operator

$$\mathcal{L}_n(\varepsilon, \omega, u(\varepsilon, \omega))[h] := L_\omega h - \varepsilon P_n \mathrm{D}f(u(\varepsilon, \omega))h, \tag{11}$$

where $L_\omega := \omega^2\partial_t^2 - \partial_x^2 + m$.
Denote $\mathfrak{J}_n := \{j \in \mathbb{N} \big| 1 < \lambda_j \le N_n\}$, $n = 0, 1, 2, \ldots$.
For $\gamma \in (0, \lambda_1)$, $\tau \ge 3$, we define

$$A_0 := \left\{(\varepsilon, \omega) \in [0, \varepsilon_0] \times [\bar{\omega}_1, \bar{\omega}_2] : |\omega p_l - \lambda_j| > \frac{\gamma}{j^\tau},\ \forall j \in \mathfrak{J}_0,\ l \in \mathbb{N}\right\},$$

$$A_{n+1} := \left\{(\varepsilon, \omega) \in A_n : |\omega p_l - \lambda_j| > \frac{\gamma}{j^\tau},\ \forall j \in \mathfrak{J}_{n+1},\ l \in \mathbb{N}\right\}$$

for every $n = 0, 1, 2, \ldots$, where p_l^2 $(l \in \mathbb{N})$ are eigenvalues of the following problem

$$\begin{cases} y'' + p^2 y = 0, \\ y(t) = y(t + \pi). \end{cases} \tag{12}$$

(**P5**) (Invertibility of $\mathcal{L}_n$) Assume that $u \in X^{(n)}$, $(\varepsilon, \omega) \in A_{n+1}$, there exist positive constants δ_0, K' such that $\varepsilon/\gamma < \delta_0$, then $\mathcal{L}_n$ is invertible and

$$\|\mathcal{L}_n^{-1}(\varepsilon, \omega, u)h\|_0 \le \frac{K'}{\gamma} N_n^\tau \|h\|_0, \qquad \forall h \in X^{(n+1)}. \tag{13}$$

Proof The proof will be given in Sect. 5. □

Lemma 1 (initialization) *For $(\varepsilon, \omega) \in A_0$, there are positive constants K_0, δ_1 such that $\varepsilon/\gamma < \delta_1$, then there exists a solution $u_0 := u_0(\varepsilon, \omega) \in X^{(0)}$ of equation $L_\omega u = \varepsilon P_0 f(u)$ satisfying $\|u_0\|_0 \le \varepsilon K_0/\gamma$.*

Proof Since the eigenvalues of L_ω satisfy

$$|\omega p_l - \lambda_j| > \frac{\gamma}{j^\tau},\ \forall j \in \mathfrak{J}_0,\ l \in \mathbb{N},$$

so L_ω is invertible on $X^{(0)}$ and, for some K_1,

$$\|L_\omega^{-1} h\|_0 \leq \frac{K_1 N_0^\tau}{\gamma}\|h\|_0, \qquad \forall h \in X^{(0)}.$$

By the contraction mapping theorem, using the property (P1), for ε/γ small, there exists a unique solution $u_0 := u_0(\varepsilon, \omega)$ of equation $L_\omega u = \varepsilon P_0 f(u)$ satisfying $\|u_0\|_0 \leq \varepsilon K_0/\gamma$. □

For $(\varepsilon, \omega) \in A_n, n \geq 1$, we construct a sequence $\{u_n\}_{n=0}^\infty$ by

$$u_{n+1} = u_n - \mathcal{L}_{n+1}(u_n)^{-1}\left[L_\omega u_n - \varepsilon P_{n+1} f(u_n)\right], \tag{14}$$

and let $h_0 = u_0,\ h_{n+1} = u_{n+1} - u_n,\ n = 0, 1, 2, \ldots$.

Lemma 2 (induction step) *There exist K_2, $\beta := 4\tau - 2$, and δ_2 small enough. Assume that $h_k \in X^{(k)}$ for all $k = 1, 2, \ldots, n$ satisfy*

$$\|h_k\|_0 < \frac{\varepsilon K_2}{\gamma} N_{k-1}^{-\beta};$$

and u_n defined in (14) solve $L_\omega u = \varepsilon P_n f(u)$ for all $n = 0, 1, 2, \ldots$. If $(\varepsilon, \omega) \in A_{n+1}$ and $\varepsilon/\gamma < \delta_2$, then there exists $h_{n+1} \in X^{(n+1)}$ satisfying

$$\|h_{n+1}\|_0 < \frac{\varepsilon K_2}{\gamma} N_n^{-\beta}. \tag{15}$$

Proof Taking into account $L_\omega u_n = \varepsilon P_n f(u_n)$, for h_{n+1}, we have

$$\mathcal{L}_{n+1}(u_n) h_{n+1} = \varepsilon (P_{n+1} - P_n) f(u_n) + \varepsilon P_{n+1} Q(u_n, h_{n+1}), \tag{16}$$

where

$$Q(u_n, h_{n+1}) = f(u_n + h_{n+1}) - f(u_n) - \mathrm{D} f(u_n) h_{n+1}.$$

Consider the fixed point problem

$$h_{n+1} = \varepsilon \mathcal{L}_{n+1}(u_n)^{-1}[(P_{n+1} - P_n) f(u_n) + P_{n+1} Q] := \mathcal{G}(h_{n+1})$$

for $h_{n+1} \in X^{(n+1)}$. We shall prove that $\mathcal{G}$ is a contraction. By (14) and the properties (P1)–(P4),

$$
\begin{aligned}
\|\mathcal{G}(h_{n+1})\|_0 &\leq \frac{\varepsilon K'}{\gamma}\left(\|(P_{n+1}-P_n)f(u_n)\|_{\tau-1} + \|P_{n+1}Q\|_{\tau-1}\right) \\
&\leq \frac{\varepsilon K'}{\gamma}\left(N_n^{-\beta}\|P_{n+1}f(u_n)\|_{\tau-1+\beta} + C_1\|h_{n+1}\|_{\tau-1}^2\right) \\
&\leq \frac{\varepsilon K'}{\gamma}\left(N_n^{-\beta}C_2(1+\|u_n\|_{\tau-1+\beta}) + C_1\|h_{n+1}\|_{\tau-1}^2\right) \\
&\leq \frac{\varepsilon K'}{\gamma}\left(N_n^{-\beta}C_3 + C_1 N_{n+1}^{2(\tau-1)}\|h_{n+1}\|_0^2\right).
\end{aligned}
$$

If $\|h_{n+1}\|_0 < \rho_{n+1} := (\varepsilon K_2/\gamma)N_n^{-\beta}$, then $\|\mathcal{G}(h_{n+1})\|_0 \leq \rho_{n+1}$ for ε/γ small enough, i.e. $\mathcal{G}(B_{n+1}) \subseteq B_{n+1} := \{\|h\|_0 < \rho_{n+1}\}$. Therefore the lemma follows from the contraction mapping theorem. □

4 Proof of Theorem

The goal of this section is to prove our main result based on Sect. 3.

Lemma 3 (existence of solution) *Suppose that* $A_\infty := \bigcap_{n\geq 0} A_n \neq \varnothing$. *If* $(\varepsilon, \omega) \in A_\infty$ *and* $\varepsilon/\gamma < \delta_3$ *small enough, then the sequence* $\{u_n\}_{n=0}^{\infty}$ *converges in* X_0 *to* $u_\infty := \sum_{n\geq 0} h_n$. u_∞ *is a solution of the equation* (6) *and*

$$
\|u_\infty\|_0 \leq \frac{\varepsilon K}{\gamma} \tag{17}
$$

for some K.

Proof By Lemmas 1 and 2, the series $\sum_{n\geq 0} h_n$ converges, u_n converges to u_∞ in X_0 and (17) holds true. □

Lemma 4 *Assume the hypotheses of Lemma 2, then there exists constant* K_3 *such that*

$$
\|\partial_t^2 h_0\|_0 \leq \frac{\varepsilon K_3}{\gamma\omega^2}, \qquad \|\partial_t^2 h_{n+1}\|_0 \leq \frac{\varepsilon K_3'}{\gamma\omega^2} N_{n+1}^{-\beta}, \quad n = 0, 1, 2, \ldots.
$$

Proof Note that $h_0 = u_0$ solves $L_\omega u = \varepsilon P_0 f(u)$. It implies

$$
\omega^2\partial_t^2 h_0 + \lambda_j h_0 = \varepsilon P_0 f(u).
$$

Thus $\|\partial_t^2 h_0\|_0 \leq \varepsilon K_3/\gamma\omega^2$ for some K_3.

It follows from (11) and (14) that

$$
\omega^2\partial_t^2 h_{n+1} = \varepsilon(P_{n+1}-P_n)f(u_n) + \partial_x^2 h_{n+1} - m h_{n+1} - \varepsilon P_{n+1}\mathrm{D}f(u_n)h_{n+1}.
$$

Hence, by (P2), (9) and (15), there exists some constant K_3' such that

$$\omega^2\|\partial_t^2 h_{n+1}\|_0 \le \varepsilon N_n^{-\beta}\|P_{n+1}f(u_n)\|_\beta + N_{n+1}^2\|h_{n+1}\|_0 + \varepsilon\|\mathrm{D}f(u_n)h_{n+1}\|_0 \\ \le \frac{\varepsilon K_3'}{\gamma}N_{n+1}^{-\beta}$$

for $n = 0, 1, 2, \ldots$. It completes the proof. □

Lemma 5 (estimate of the derivatives) *Assume the hypotheses of Lemma 3, then* $\mathrm{D}_{\varepsilon,\omega}u_n$ *converges to* $\mathrm{D}_{\varepsilon,\omega}u_\infty$ *in* X_0 *satisfying*

$$\|\mathrm{D}_\varepsilon u_\infty\|_0 \le \frac{K}{\gamma}, \qquad \|\mathrm{D}_\omega u_\infty\|_0 \le \frac{\varepsilon K}{\gamma^2}. \tag{18}$$

Proof By the proof of Lemma 1 and the implicit function theorem, $(\varepsilon,\omega) \mapsto u_0(\varepsilon,\omega)$ is in $\mathrm{C}^1(A_0, X_0)$ and $\|\mathrm{D}_{\varepsilon,\omega}u_0\|_0 \le K_0/\gamma$.

Next, assume by induction that $h_n(\varepsilon,\omega)$ is a C^1 map defined in A_n for every $n = 0, 1, 2, \ldots$. We shall prove that $h_{n+1}(\varepsilon,\omega)$ is C^1 too. Recall that h_{n+1} is defined for $(\varepsilon,\omega) \in A_{n+1}$ as a solution in $X^{(n+1)}$ of Eq. (16). We claim that the operator

$$\mathcal{L}_{n+1}(u_{n+1})[z] := L_\omega z - \varepsilon P_{n+1}\mathrm{D}f(u_n + h_{n+1}))[z] \tag{19}$$

is invertible. In fact,

$$\|\left[\mathcal{L}_{n+1}(u_{n+1}) - \mathcal{L}_{n+1}(u_n)\right]h_{n+1}\|_0 \le \varepsilon\|f(u_n)h_{n+1}\|_0 \le \frac{\varepsilon^2 K_2'}{\gamma}N_n^{-\beta},$$

which together with (13) gives

$$\|\mathcal{L}_{n+1}(u_n)^{-1}\left[\mathcal{L}_{n+1}(u_{n+1}) - \mathcal{L}_{n+1}(u_n)\right]\|_0 \le \frac{\varepsilon^2 K' K_2'}{\gamma^2}N_n^{\tau-\beta}.$$

Thus,

$$\|\mathcal{L}_{n+1}(u_{n+1})^{-1}\left[\mathcal{L}_{n+1}(u_{n+1}) - \mathcal{L}_{n+1}(u_n)\right]\|_0 \le \frac{1}{2}$$

provided that ε/γ is appropriate small, while n is appropriate large enough. This shows that $\mathcal{L}_{n+1}(w_{n+1})$ is invertible and

$$\|\mathcal{L}_{n+1}(u_{n+1})^{-1}\|_0 \le \frac{2K'}{\gamma}N_{n+1}^{\tau}.$$

As a consequence, by the implicit function theorem, the map $(\varepsilon,\omega) \mapsto h_{n+1}(\varepsilon,\omega)$ is in $\mathrm{C}^1(A_{n+1}, X^{(n+1)})$.

By (16),

$$L_\omega h_{n+1} - \varepsilon P_{n+1}[f(u_n + h_{n+1}) - f(u_n)] - \varepsilon(P_{n+1} - P_n) f(u_n) = 0. \quad (20)$$

Differentiating the Eq. (20) with respect to ω and utilizing $\mathcal{L}_{n+1}(u_{n+1})^{-1}$, then taking the norm $\|\cdot\|_0$ on both sides, we obtain

$$\begin{aligned}\left\|\partial_\omega h_{n+1}\right\|_0 &\le \frac{2K'}{\gamma} N_{n+1}^{\tau}\Big(\|2\omega(\partial_t^2 h_{n+1})\|_0 + \|\varepsilon(P_{n+1} - P_n)\mathrm{D}f(u_n)\partial_\omega u_n\|_0 \\ &\quad + \|\varepsilon P_{n+1}\left[\mathrm{D}f(u_n + h_{n+1}) - \mathrm{D}f(u_n)\right]\partial_\omega u_n\|_0\Big) \\ &\le \frac{2\varepsilon K'}{\gamma} N_{n+1}^{\tau}\Big(\frac{2K_3'}{\gamma\omega} N_{n+1}^{-\beta} + C_4 N_n^{-\beta}\sum_{i=0}^{n}\|\partial_\omega h_i\|_0 + C_5\rho_{n+1}\sum_{i=0}^{n}\|\partial_\omega h_i\|_0\Big) \\ &\le \frac{\varepsilon K}{\gamma^2} N_{n+1}^{\tau} N_n^{-\beta}.\end{aligned}$$

Hence, we deduce $\|\partial_\omega u_{n+1}\|_0 \le \varepsilon K/\gamma^2$ which implies $\|\mathrm{D}_\omega u_\infty\|_0 \le \varepsilon K/\gamma^2$.

Similarly, differentiating the Eq. (20) with respect to ε gives

$$\begin{aligned}\mathcal{L}_{n+1}(u_{n+1})[\partial_\varepsilon h_{n+1}] &= P_n f(u_n) - P_{n+1} f(u_{n+1}) \\ &\quad + \varepsilon P_n \mathrm{D}f(u_n)\partial_\varepsilon u_n - \varepsilon P_{n+1}\mathrm{D}f(u_{n+1})\partial_\varepsilon u_n,\end{aligned}$$

and then, we can obtain the estimate for $\partial_\varepsilon h_{n+1}$ by using the same method as above. □

Finally we can define, by means of a cut-off function, a C^1-Whitney extension $\tilde{u}_{n+1} \in \mathrm{C}^1(A_0, X^{(n+1)})$ of u_{n+1} as $\tilde{u}_{n+1} := \tilde{u}_n + \tilde{h}_{n+1}$ satisfying

$$\|\tilde{u}\|_0 \le \frac{\varepsilon K}{\gamma}, \qquad \|\mathrm{D}_\varepsilon \tilde{u}\|_0 \le \frac{K}{\gamma}, \qquad \|\mathrm{D}_\omega \tilde{u}\|_0 \le \frac{\varepsilon K}{\gamma^2},$$

where $\tilde{u}_n, \tilde{h}_{n+1}$ are obtained through the corresponding Whitney extension procedures.

Lemma 6 (measure estimate) *For $\tau \ge 3$, there exists $\delta < \min\{\delta_i, i = 0, 1, 2, 3\}$ such that the Cantor set A_∞ has measure property: for every interval $(\bar{\omega}_1, \bar{\omega}_2)$ with $0 < \bar{\omega}_1 < \bar{\omega}_2 < +\infty$, there is a constant C depending on $(\bar{\omega}_1, \bar{\omega}_2)$ such that $|A_\infty| \ge \varepsilon_0(\bar{\omega}_2 - \bar{\omega}_1)(1 - C\gamma)$.*

Proof Given ε, we need to prove that the complementary set $E := \bigcup_{l,j\ge 1}\Omega_{l,j}$ has small measure, where

$$\Omega_{l,j} := \left\{\omega \in (\bar{\omega}_1, \bar{\omega}_2) : |\omega p_l - \lambda_j| \le \frac{\gamma}{j^\tau}\right\},$$

and $\Omega_{0,j} = \varnothing$ for all $j \ge 1$.

Note that $l/4 < \partial_\omega(\omega p_l) < 2\gamma/j^\tau$ provided that $\varepsilon/\omega < \delta$ small enough. In addition,

$$\bar{\omega}_1 l - \frac{\gamma}{j^\tau} < \lambda_j < \bar{\omega}_2 l + \frac{\gamma}{j^\tau},$$

which implies

$$\sharp\{j\} = \frac{1}{\varrho}\left((\bar{\omega}_2 - \bar{\omega}_1)l + \frac{2\gamma}{j^\tau}\right) + 1 < Kl(\bar{\omega}_2 - \bar{\omega}_1),$$

where $\varrho := \inf\{|\lambda_{j+1} - \lambda_j| : \ j \geq 1\}$.

For fixed $0 < \bar{\omega}_1 < \bar{\omega}_2 < +\infty$, if $\Omega_{l,j} \cap (\bar{\omega}_1, \bar{\omega}_2)$ is nonempty, then

$$|E| \leq \sum_{j=1}^{\infty} \frac{8\gamma}{lj^\tau} Kl(\bar{\omega}_2 - \bar{\omega}_1) \leq C\gamma(\bar{\omega}_2 - \bar{\omega}_1)$$

because the series $\sum_{j=1}^{\infty} 1/j^\tau$ converges. Thus

$$|A_\infty(\varepsilon) \cap (\bar{\omega}_1, \bar{\omega}_2)| \geq (\bar{\omega}_2 - \bar{\omega}_1)(1 - C\gamma).$$

Therefore

$$|A_\infty| \geq \int_0^{\varepsilon_0} (\bar{\omega}_2 - \bar{\omega}_1)(1 - C\gamma)\mathrm{d}\varepsilon = \varepsilon_0(\bar{\omega}_2 - \bar{\omega}_1)(1 - C\gamma),$$

and we get the thesis. □

5 Inversion of the linearized operator

In this section, we prove the key property on the inversion of the linearized operator defined in (11). We also write the operator

$$\mathcal{L}_n = \mathcal{D} + \varepsilon\mathcal{M}$$

with

$$\mathcal{D}h := \omega^2 h_{tt} - h_{xx} + mh,$$
$$\mathcal{M}h := P_n(ah), \quad a(x,t) := -\partial_u g(x, t, u(x,t)).$$

Next, it is easy to show the result below.

Lemma 7 *Let p_l^2 and ψ_l ($l \in \mathbb{N}$) be the eigenvalues and eigenfunctions of the problem (12), then the eigenfunctions ψ_l form an orthonormal basis of $H^1([0,\pi])$ with respect to the product $(u,v)_{H^1} = \int_0^\pi [u'v' + uv]\mathrm{d}t$.*

Lemma 8 (inversion of $\mathcal{D}$) *Let $p_l^2 (l \in \mathbb{N})$ be the eigenvalues of the problem (12). For all $j \in \mathfrak{J}_{n+1}$, If (ε, ω) satisfies the conditions*

$$\left|\omega p_l(\varepsilon, \omega) - \lambda_j\right| > \frac{\gamma}{j^\tau}, \qquad l \in \mathbb{N}, \tag{21}$$

then $\mathcal{D}$ is invertible, and

$$\|\mathcal{D}^{-1} h\|_0 \leq \frac{C}{\gamma} \|h\|_\tau, \quad \forall h \in X^{(n+1)} \tag{22}$$

for some positive constant C.

Proof We develop $\mathcal{D}h = \sum D_j h_j(t) \varphi_j(x)$, where

$$D_j z = \omega^2 z'' + \lambda_j^2 z = \sum_{l \in \mathbb{N}} \left(\lambda_j^2 - \omega^2 p_l^2\right) \hat{z}_l \psi_l(t), \quad z = \sum_{l \in \mathbb{N}} \hat{z}_l \psi_l(t).$$

Thus, each D_j is the diagonal with respect to the basis $\psi_l(t)$. By (21), $\forall j \in \mathfrak{J}_{n+1}$, we have D_j is invertible and

$$\left\|D_j^{-1} z\right\|_{H^1} = \sum_{l \in \mathbb{N}} \frac{1}{|\lambda_j^2 - \omega^2 p_l^2|} \|z\|_{H^1} \leq \frac{C j^\tau}{\gamma} \|z\|_{H^1},$$

so that

$$\|\mathcal{D}^{-1} h\|_0^2 \leq \frac{C^2}{\gamma^2} \|h\|_\tau^2$$

for some positive constant C. It implies (22) holds true. □

Lemma 9 *Assume the hypotheses of Lemma 8. Define $|\mathcal{D}|^{-1/2} : X^{(n)} \to X^{(n)}$ obeys*

$$|\mathcal{D}|^{-1/2} h := \sum_{1 \leq \lambda_j \leq N_n} |D_j|^{-1/2} h_j(t) \varphi_j(x),$$

then

$$\left\||\mathcal{D}|^{-1/2} h\right\|_{\sigma,s} \leq \frac{K_4}{\sqrt{\gamma}} \|h\|_{\sigma,s+\frac{\tau}{2}} \leq \frac{K_4}{\sqrt{\gamma}} N_n^{\frac{\tau}{2}} \|h\|_{\sigma,s}, \quad \forall h \in X^{(n)}. \tag{23}$$

Proof Due to $\left\||D_j|^{-1/2} z\right\|_{H^1} \leq (K_4' / \sqrt{\alpha_j}) \|z\|_{H^1}$ for some K_4', where

$$\alpha_j := \min_{l \in \mathbb{N}} \{|\lambda_j^2 - \omega^2 p_l^2|\} > 0,$$

we have

$$\left\| |\mathcal{D}|^{-1/2} h \right\|_{\sigma,s}^2 \le \sum_{1 \le \lambda_j \le N_n} \frac{{K_4'}^2}{\alpha_j} \|h_j\|_{H^1}^2 j^{2s} \mathrm{e}^{2\sigma j}$$
$$\le \sum_{1 \le \lambda_j \le N_n} \frac{{K_4'}^2 j^\tau}{\gamma} \|h_j\|_{H^1}^2 j^{2s} \mathrm{e}^{2\sigma j}$$
$$\le \frac{{K_4'}^2}{\gamma} \|h\|_{\sigma, s+\frac{\tau}{2}}^2, \quad \forall h \in X^{(n)}$$

whence (23) follows. In particular,

$$\left\| |\mathcal{D}|^{-1/2} h \right\|_0 \le \frac{K_4}{\sqrt{\gamma}} \|h\|_{\frac{\tau}{2}} \le \frac{K_4}{\sqrt{\gamma}} N_n^{\frac{\tau}{2}} \|h\|_0.$$

□

6 Proof of (P5)

Proof Let $\mathcal{L}_n(u) = |D|^{1/2}(\mathcal{U} + \varepsilon\mathcal{R})|D|^{1/2}$, where

$$\mathcal{U} := |D|^{-1/2}\mathcal{D}|D|^{-1/2}, \qquad \mathcal{R} := |D|^{-1/2}\mathcal{M}|D|^{-1/2}.$$

It is easily to prove that $\|\mathcal{U}\|_\sigma := \sup_{\|h\|_\sigma \le 1} \|\mathcal{U}h\|_\sigma = 1$.

Noting that

$$\alpha_j = \min_{l \in \mathbb{N}} \left\{ |\omega^2 p_l^2 - \lambda_j^2| \right\} \ge \min_{l \in \mathbb{N}} \left\{ \lambda_j |\omega p_l - \lambda_j| \right\} \ge \frac{\gamma}{j^\tau}$$

for all $j \ge 1$, we get $\alpha_k \alpha_l \ge \gamma^2/(kl)^\tau,\ k, l \ge 1$.

For $h \in X^{(n)}$, $\mathcal{R}h = \sum_{\lambda_k \le N_n} (\mathcal{R}h)_k \mathrm{e}^{\mathrm{i}kt}$ with

$$(\mathcal{R}h)_k = |\mathcal{D}_k|^{-1/2}(\mathcal{M}|\mathcal{D}|^{-1/2}h)_k = |\mathcal{D}_k|^{-1/2}\Big(P_n a \sum_{\lambda_l \le N_n} |\mathcal{D}_l|^{-1/2} h_l\Big)_k,$$

we deduce

$$\|(\mathcal{R}h)_k\|_{H^1} \le C \sum_{\lambda_l \le N_n} \frac{\|a\|_{H^1}}{\sqrt{\alpha_k \alpha_l}} \|h_l\|_{H^1} \le \frac{C}{\gamma} S_k,$$

where

$$S_k := \sum_{\lambda_l \le N_n} \|a\|_{H^1} \Big(\max\{k, l\} \Big)^\tau \|h_l\|_{H^1}.$$

Let $S(t) := \sum_{\lambda_k \le N_n} S_k e^{ikt}$, then

$$\|\mathcal{R}h\|_s^2 = \sum_{\lambda_k \le N_n} \|(\mathcal{R}h)_k\|_{H^1}^2 k^{2s} e^{2\sigma k} \le \frac{C^2}{\gamma^2} \sum_{\lambda_k \le N_n} S_k^2 k^{2s} e^{2\sigma k} = \frac{C^2}{\gamma^2} \|S\|_s^2.$$

It turns out that $S = P_n(bc)$ with

$$b(t) := \sum_{l \in \mathbb{N}} \|a\|_{H^1} \Big(\max\{k, l\} \Big)^{\tau} e^{ilt}$$

and

$$c(t) := \sum_{\lambda_l \le N_n} \|h_l\|_{H^1} e^{ilt}.$$

Hence

$$\|\varepsilon \mathcal{R}h\|_s \le \frac{\varepsilon C}{\gamma} \|b\|_s \|c\|_s \le \frac{\varepsilon C}{\gamma} \|a\|_{s+\tau} \|h\|_s \le \frac{\varepsilon C'}{\gamma} \|h\|_s \le \frac{1}{2} \big\|h\big\|_s$$

provided that we take ε/γ small enough. Then Neumann series $\mathcal{U} + \varepsilon\mathcal{R}$ is invertible in $(X^{(n+1)}, \|\cdot\|_s)$, and $\|(\mathcal{U} + \varepsilon\mathcal{R})^{-1}h\|_s < 2\|h\|_s$. Therefore

$$\|\mathcal{L}_n(u)^{-1}h\|_0 = \Big\| |D|^{-1/2} (\mathcal{U} + \varepsilon\mathcal{R})^{-1} |D|^{-1/2} h \Big\|_0 \le \frac{K'}{\gamma} N_n^{\tau} \|h\|_0.$$

□

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