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Inverse problem for dirac system with singularities in interior points

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Abstract We study the non-selfadjoint Dirac system on a finite interval having non-integrable regular singularities in interior points with additional matching conditions at these points. Properties of spectral characteristics are established, and the inverse spectral problem is investigated. We provide a constructive procedure for the solution of the inverse problem, and prove its uniqueness. Moreover, necessary and sufficient conditions for the global solvability of this nonlinear inverse problem are obtained.

Keywords Differential systems · Singularity · Spectral analysis · Inverse problems

Mathematics Subject Classification 34A55 · 34L40 · 34A36 · 47E05

1 Introduction

Consider the boundary value problem $L = L(Q_\omega(x), Q(x), \alpha, \beta)$ for the Dirac system on a finite interval with N regular singularities inside the interval:

$$BY' + \Big(Q_\omega(x) + Q(x)\Big)Y = \lambda Y, \quad 0 < x < \pi, \tag{1}$$

$$(\cos\alpha, \sin\alpha)Y(0) = (\cos\beta, \sin\beta)Y(\pi) = 0, \tag{2}$$

where

$$Y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Q(x) = \begin{pmatrix} q_1(x) & q_2(x) \\ q_2(x) & -q_1(x) \end{pmatrix},$$

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$$Q_\omega(x) = Q_\omega^{\langle k\rangle}(x) = \frac{\mu_k}{x-\gamma_k}\begin{pmatrix}\sin 2\eta_k & \cos 2\eta_k\\ \cos 2\eta_k & \sin 2\eta_k\end{pmatrix} \text{for } x \in \omega_{k+1/2} \cup \gamma_{k+1/2},\ k=\overline{1,N}.$$

Here $0 < \gamma_1 < \gamma_2 < \cdots < \gamma_N < \pi$, $\omega_p = (\gamma_p, \gamma_{p+1})$, $\gamma_{k+1/2} = (\gamma_{k+1} + \gamma_k)/2$, $k = \overline{1, N-1}$, $\gamma_{1/2} = \gamma_0 = 0$, $\gamma_{N+1/2} = \gamma_{N+1} = \pi$, $q_j(x)$ are complex-valued functions, and μ_k are complex numbers. Let for definiteness, α, β, $\eta_k \in [-\pi/2, \pi/2]$, $\mathrm{Re}\,\mu_k > 0$, $\mu_k + 1/2 \notin \mathbb{N}$. Let $q_j(x)$ be absolutely continuous on $[0,\pi]$ and $|q_j(x)| \prod_{k=1}^{N} |x-\gamma_k|^{-2Re\mu_k} \in L(0,\pi)$. If $Q_\omega(x)$, $Q(x)$, α, β satisfy these conditions, we will say that $L \in W$.

In this paper we establish properties of spectral characteristics and investigate the inverse spectral problem of recovering L from the given spectral data. We provide a constructive procedure for the solution of the inverse problem, and prove its uniqueness. Moreover, necessary and sufficient conditions for the global solvability of this nonlinear inverse problem are obtained.

Differential equations with singularities inside the interval play an important role in various areas of mathematics as well as in applications. Moreover, a wide class of differential equations with turning points can be reduced to equations with singularities. For example, such problems appear in electronics for constructing parameters of heterogeneous electronic lines with desirable technical characteristics [1–3]. Boundary value problems with discontinuities in an interior point appear in geophysical models for oscillations of the Earth [4]. Differential equations with turning points arise in various physical and technical problems; see [5] where further references and links to applications can be found. We also note that in different problems of natural sciences we face different kind of matching conditions in singular points.

The case when a singular point lies at the endpoint of the interval was investigated fairly completely for various classes of differential equations in [6–10] and other works. The presence of singularity inside the interval produces essential qualitative modifications in the investigation (see [11]).

A few words on the structure of the paper. In Sect. 2 properties of spectral characteristics are studied. For this we use the results from [12] where special fundamental systems of solutions are constructed with prescribed analytic and asymptotic properties. In Sect. 3 we provide a constructive procedure for the solution of the inverse problem, and prove its uniqueness. Necessary and sufficient conditions for the global solvability of the inverse problem are presented in Sect. 4.

2 Properties of the spectrum

System (1) has non-integrable singularities at the points γ_k, hence it is necessary to require additional matching conditions for solutions on the intervals ω_{k-1} and ω_k. We will do it as follows. It was shown in [12] that for $x \in \omega_{k-1} \cup \omega_k$ there exist a fundamental system of solutions $S^{\langle k\rangle}(x,\lambda) = (S_1^{\langle k\rangle}(x,\lambda),\ S_2^{\langle k\rangle}(x,\lambda))$ such that

$$S_1^{\langle k\rangle}(x,\lambda) \sim (x-\gamma_k)^{-\mu_k}\begin{pmatrix}0\\ c_{01}\end{pmatrix},\ S_2^{\langle k\rangle}(x,\lambda) \sim (x-\gamma_k)^{\mu_k}\begin{pmatrix}c_{02}\\ 0\end{pmatrix} \quad \text{for } x \to \gamma_k.$$

where $c_{01}c_{02} = 1$. Let $Y(x,\lambda) = a_1(\lambda)S_1^{\langle k\rangle}(x,\lambda) + a_2(\lambda)S_2^{\langle k\rangle}(x,\lambda)$ be a solution of system (1) for $x \in \omega_{k-1}$. Then we put by definition

$$Y(x,\lambda) = a_1(\lambda)S_1^{\langle k\rangle}(x,\lambda)A^{\langle k\rangle}(\lambda) + a_2(\lambda)S_2^{\langle k\rangle}(x,\lambda)A^{\langle k\rangle}(\lambda),$$

for $x \in \omega_k$, where $A^{\langle k\rangle}(\lambda)$ is a fixed given transition matrix for γ_k. For example, if $A^{\langle k\rangle}(\lambda) = I$ (I is the identity matrix) and $Q(x)$ is analytic at γ_k, then this continuation of the solution coincides with the analytic continuation through the upper half-plane $Imx > 0$. If $A^{\langle k\rangle}(\lambda) = \begin{pmatrix} e^{2i\pi\mu_k} & 0 \\ 0 & e^{-2i\pi\mu_k}\end{pmatrix}$, then it corresponds to the analytic continuation through the lower half-plane $Imx < 0$.

Let $S(x,\lambda) = (S_1(x,\lambda),\ S_2(x,\lambda))$ be the fundamental matrix for system (1) with the initial condition $S(0,\lambda) = I$ and with the above mentioned matching conditions. For definiteness, everywhere below $A^{\langle k\rangle}(\lambda) = I,\ k = \overline{1, N}$. The construction of this fundamental matrix can be described as follows. If $x \in \omega_0 \cup \omega_1$, then we put $S(x,\lambda) = S^{\langle 1\rangle}(x,\lambda)\left(S^{\langle 1\rangle}(0,\lambda)\right)^{-1}$; moreover, if $x \in \omega_1$, then $S(x,\lambda) = S^{\langle 2\rangle}(x,\lambda)C^{\langle 1\rangle}(\lambda)$. Fix $x_1 \in \omega_1$. Then $S^{\langle 1\rangle}(x_1,\lambda)\left(S^{\langle 1\rangle}(0,\lambda)\right)^{-1} = S^{\langle 2\rangle}(x_1,\lambda)C^{\langle 1\rangle}(\lambda)$, i.e.

$$S(x,\lambda) = S^{\langle 2\rangle}(x,\lambda)\left(S^{\langle 2\rangle}(x_1,\lambda)\right)^{-1}S^{\langle 1\rangle}(x_1,\lambda)\left(S^{\langle 1\rangle}(0,\lambda)\right)^{-1},\ x \in \omega_1.$$

Analogously, one gets for $x \in \omega_k$:

$$S(x,\lambda) = S^{\langle k+1\rangle}(x,\lambda)\left(\prod_{j=1}^{k}\left(S^{\langle j+1\rangle}(x_j,\lambda)\right)^{-1}S^{\langle j\rangle}(x_j,\lambda)\right)(S^{\langle 1\rangle}(0,\lambda))^{-1}, x_j \in \omega_j. \tag{3}$$

Lemma 1 *For $x \in \omega_k$ and $|\lambda(x-\gamma_k)| \geq 1$,*

$$S(x,\lambda) = \frac{1}{2i}\left(e^{i\lambda x}\begin{bmatrix} i & -1 \\ 1 & i\end{bmatrix}_{\langle k\rangle} + e^{-i\lambda x}\begin{bmatrix} i & 1 \\ -1 & i\end{bmatrix}_{\langle k\rangle}\right) + \sum_{j=1}^{k}\sin\pi\mu_j e^{-il\lambda(x-2\gamma_j)+2il\eta_j}\begin{bmatrix} -i & l \\ l & i\end{bmatrix}_{\langle k\rangle},$$

$\left[\left(a_{ij}\right)_{i,j=1}^{n,m}\right]_{\langle k\rangle} := \left(a_{ij} + O\left(|\lambda(x-\gamma_k)|^{-\nu}\right)\right)_{i,j=1}^{n,m}$, $\nu = \min\{1,\ 2\mathrm{Re}\mu_1,\ 2\mathrm{Re}\mu_2, \ldots, 2\mathrm{Re}\mu_N\}$, $l = \begin{cases} 1, & \arg\lambda \in \Pi_{-1} \cup \Pi_1, \\ -1, & \arg\lambda \in \Pi_0,\end{cases}$, $\Pi_k = \left\{\lambda \;\middle|\; \arg\lambda \in \left(\pi\frac{5k-3}{6-2k}, \pi\frac{5k+3}{6+2k}\right]\right\}$, $k = 0,\ \pm 1$.

We prove the lemma by induction. According to [12], the matrix $S^{\langle k\rangle}(x,\lambda)$ can be represented by $S^{\langle k\rangle}(x,\lambda)=E^{\langle k\rangle}(x,\lambda)\beta^{\langle k\rangle}(\lambda)$, where $E^{\langle k\rangle}(x,\lambda)$ is the Birkhoff-type fundamental matrix, and $\beta^{\langle k\rangle}(\lambda)$ are Stockes multipliers.

Let $x\in\omega_0$. Then $S(x,\lambda)=S^{\langle 1\rangle}(x,\lambda)(S^{\langle 1\rangle}(0,\lambda))^{-1}$ and (see [12])

$$S(x,\lambda)=\frac{1}{[2i]_{\langle 1\rangle}}\begin{pmatrix} e^{i\lambda x)}[i]_{\langle 1\rangle}+e^{-i\lambda x)}[i]_{\langle 1\rangle} & e^{i\lambda x)}[-1]_{\langle 1\rangle}+e^{-i\lambda x)}[1]_{\langle 1\rangle} \\ e^{i\lambda x)}[1]_{\langle 1\rangle}+e^{-i\lambda x)}[-1]_{\langle 1\rangle} & e^{i\lambda x)}[i]_{\langle 1\rangle}+e^{-i\lambda x)}[i]_{\langle 1\rangle}\end{pmatrix}.$$

Suppose that the assertion of the lemma is true for $x\in\omega_{k-1}$. Let us prove it for $x\in\omega_k$. It follows from (3) that for $x\in\omega_k$,

$$S(x,\lambda)=S^{\langle k\rangle}(x,\lambda)(S^{\langle k\rangle}(x_{k-1},\lambda))^{-1}S(x_{k-1},\lambda). \tag{4}$$

We find the asymptotics for $S^{\langle k\rangle}(x,\lambda)(S^{\langle k\rangle}(x_{k-1},\lambda))^{-1}$, using the asymptotics from [12]. Denote $l^{+}=l^{\langle k\rangle},\ m^{+}=m^{\langle k\rangle}$ for $x>\gamma_k$, and $l^{-}=l^{\langle k\rangle},\ m^{-}=m^{\langle k\rangle}$ for $x<\gamma_k$. One has

$$\begin{aligned}
&S^{\langle k\rangle}(x,\lambda)B\Big(S^{\langle k\rangle}(x_{k-1},\lambda)\Big)^T B^T\\
&=\left(e^{-i\lambda(x-\gamma_k)+i\eta_k}\begin{bmatrix}-i & -i\\ 1 & 1\end{bmatrix}_{\langle k\rangle}+e^{i\lambda(x-\gamma_k)-i\eta_k}\begin{bmatrix} i & -i\\ 1 & -1\end{bmatrix}_{\langle k\rangle}H(e^{i\pi\mu_k l^{+}})\right)\\
&\quad\times H(e^{2i\pi\mu_k m^{+}})H(\lambda^{-\mu_k})\beta^{\langle k\rangle}B\beta^{\langle k\rangle}H(\lambda^{-\mu_k})H(e^{2i\pi\mu_k m^{-}})\\
&\quad\times\Bigg(e^{-i\lambda(x_{k-1}-\gamma_k)+i\eta_k}\begin{bmatrix}-i & 1\\ -i & 1\end{bmatrix}_{\langle k\rangle}\\
&\quad+e^{i\lambda(x_{k-1}-\gamma_k)-i\eta_k}H(e^{i\pi\mu_k l^{-}})\begin{bmatrix} i & 1\\ -i & -1\end{bmatrix}_{\langle k\rangle}\Bigg)B^T,
\end{aligned}$$

where $H(z)=\begin{pmatrix} z^{-1} & 0\\ 0 & z\end{pmatrix}$, and T is the sign for the transposition. Since $BH(z)B^T=H(z^{-1})$ and $\beta^{\langle k\rangle}B\beta^{\langle k\rangle}B^T=\beta_1^{\langle k\rangle}\beta_2^{\langle k\rangle}$, it follows that

$$\begin{aligned}
&S^{\langle k\rangle}(x,\lambda)B\Big(S^{\langle k\rangle}(x_{k-1},\lambda)\Big)^T B^T\\
&=\beta_1^{\langle k\rangle}\beta_2^{\langle k\rangle}\Bigg(e^{-i\lambda(x-\gamma_k)+i\eta_k}\begin{bmatrix}-i & -i\\ 1 & 1\end{bmatrix}_{\langle k\rangle}H(e^{2i\pi\mu_k(m^{+}-m^{-})})\\
&\quad+e^{2i\lambda(x-\gamma_k)-i\eta_k}\begin{bmatrix} i & -i\\ 1 & -1\end{bmatrix}_{\langle k\rangle}H(e^{i\pi\mu_k(l^{+}+2m^{+}-2m^{-})})\Bigg)\\
&\quad\times\left(e^{-i\lambda(x_{k-1}-\gamma_k)+i\eta_k}\begin{bmatrix}1 & i\\ -1 & -i\end{bmatrix}_{\langle k\rangle}+e^{i\lambda(x_{k-1}-\gamma_k)-i\eta_k}H(e^{-i\pi\mu_k l^{-}})\begin{bmatrix}-1 & i\\ -1 & i\end{bmatrix}_{\langle k\rangle}\right).
\end{aligned}$$

Taking the relation $\beta_1^{\langle k\rangle}\beta_2^{\langle k\rangle} = (4i\cos\pi\mu_k)^{-1}$ into account, we calculate

$$
\begin{aligned}
&S^{\langle k\rangle}(x,\lambda)\Big(S^{\langle k\rangle}(x_{k-1},\lambda)\Big)^{-1}\\
&= \frac{1}{4i\cos\pi\mu_k}\, e^{-i\lambda(x+x_{k-1}-2\gamma_k)+2i\eta_k}\left[2i\sin\Big(2\pi\mu_k(m^- - m^+)\Big)\begin{pmatrix}-i & 1\\ 1 & i\end{pmatrix}\right]_{\langle k\rangle}\\
&\quad+\frac{1}{4i\cos\pi\mu_k}\, e^{-i\lambda(x-x_{k-1})}\left[2\cos\Big(\pi\mu_k(l^- + 2m^- - 2m^+)\Big)\begin{pmatrix}i & 1\\ -1 & i\end{pmatrix}\right]_{\langle k\rangle}\\
&\quad+\frac{1}{4i\cos\pi\mu_k}\, e^{i\lambda(x-x_{k-1})}\left[2\cos\Big(\pi\mu_k(l^+ + 2m^+ - 2m^-)\Big)\begin{pmatrix}i & -1\\ 1 & i\end{pmatrix}\right]_{\langle k\rangle}\\
&\quad+\frac{1}{4i\cos\pi\mu_k}\, e^{i\lambda(x+x_{k-1}-2\gamma_k)-2i\eta_k}\\
&\quad\times\left[2\sin\Big(\pi\mu_k(l^- - l^+ + 2m^- - 2m^+)\Big)\begin{pmatrix}-i & -1\\ -1 & i\end{pmatrix}\right]_{\langle k\rangle}.
\end{aligned}
$$

Consider three cases:

1. If $\lambda \in \Pi_1$, then $m^- = 1,\ m^+ = 0,\ l^- = -1,\ l^+ = 1$, and

$$
\begin{aligned}
&S^{\langle k\rangle}(x,\lambda)\Big(S^{\langle k\rangle}(x_{k-1},\lambda)\Big)^{-1}\\
&= \sin(\pi\mu_k)e^{-i\lambda(x+x_{k-1}-2\gamma_k)+2i\eta_k}\begin{bmatrix}-i & 1\\ 1 & i\end{bmatrix}_{\langle k\rangle} + \frac{1}{2i}\, e^{-i\lambda(x-x_{k-1})}\begin{bmatrix}i & 1\\ -1 & i\end{bmatrix}_{\langle k\rangle}\\
&\quad+\frac{1}{2i}\, e^{i\lambda(x-x_{k-1})}\begin{bmatrix}i & -1\\ 1 & i\end{bmatrix}_{\langle k\rangle} + e^{i\lambda(x+x_{k-1}-2\gamma_k)-2i\eta_k}\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}_{\langle k\rangle}.
\end{aligned}
$$

2. If $\lambda \in \Pi_{-1}$, then $m^- = 0, m^+ = -1, l^- = -1, l^+ = 1$, and

$$
\begin{aligned}
&S^{\langle k\rangle}(x,\lambda)\Big(S^{\langle k\rangle}(x_{k-1},\lambda)\Big)^{-1}\\
&= \sin(\pi\mu_k)e^{-i\lambda(x+x_{k-1}-2\gamma_k)+2i\eta_k}\begin{bmatrix}-i & 1\\ 1 & i\end{bmatrix}_{\langle k\rangle} + \frac{1}{2i}\, e^{-i\lambda(x-x_{k-1})}\begin{bmatrix}i & 1\\ -1 & i\end{bmatrix}_{\langle k\rangle}\\
&\quad+\frac{1}{2i}\, e^{i\lambda(x-x_{k-1})}\begin{bmatrix}i & -1\\ 1 & i\end{bmatrix}_{\langle k\rangle} + e^{i\lambda(x+x_{k-1}-2\gamma_k)-2i\eta_k}\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}_{\langle k\rangle}.
\end{aligned}
$$

3. If $\lambda \in \Pi_0$, then $m^- = 0, m^+ = 0, l^- = 1, l^+ = -1$, and

$$
\begin{aligned}
&S^{\langle k\rangle}(x,\lambda)\Big(S^{\langle k\rangle}(x_{k-1},\lambda)\Big)^{-1} = e^{-i\lambda(x+x_{k-1}-2\gamma_k)+2i\eta_k}\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}_{\langle k\rangle}\\
&\quad+\frac{1}{2i}\, e^{-i\lambda(x-x_{k-1})}\begin{bmatrix}i & 1\\ -1 & i\end{bmatrix}_{\langle k\rangle}\\
&\quad+\frac{1}{2i}\, e^{i\lambda(x-x_{k-1})}\begin{bmatrix}i & -1\\ 1 & i\end{bmatrix}_{\langle k\rangle} + \sin(\pi\mu_k)e^{i\lambda(x+x_{k-1}-2\gamma_k)-2i\eta_k}\begin{bmatrix}-i & -1\\ -1 & i\end{bmatrix}_{\langle k\rangle}.
\end{aligned}
$$

Since $x_{k-1} < \gamma_k,\ x > \gamma_k$, it follows that $x + x_{k-1} - 2\gamma_k = x - x_{k-1} + 2(x_{k-1} - \gamma_k) < x - x_{k-1}$, $x + x_{k-1} - 2\gamma_k = x_{k-1} - x + 2(x - \gamma_k) > x_{k-1} - x$, and the exponentials $e^{\pm i\lambda(x+x_{k-1}-2\gamma_k)}$ grow not faster than $e^{\pm i\lambda(x-x_{k-1})}$. Thus,

$$S^{\langle k\rangle}(x,\lambda)\Big(S^{\langle k\rangle}(x_{k-1},\lambda)\Big)^{-1} = \frac{1}{2i}\, e^{i\lambda(x-x_{k-1})}\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}_{\langle k\rangle}
+\frac{1}{2i}\, e^{-i\lambda(x-x_{k-1})}\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}_{\langle k\rangle} + \sin(\pi\mu_k)e^{-li\lambda(x+x_{k-1}-2\gamma_k)+2li\eta_k}\begin{bmatrix} -i & l \\ l & i \end{bmatrix}_{\langle k\rangle}.$$

Substituting this asymptotics into (4), we get

$$S(x,\lambda) = \Bigg(\frac{1}{2i}\, e^{i\lambda(x-x_{k-1})}\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}_{\langle k\rangle} + \frac{1}{2i}\, e^{-i\lambda(x-x_{k-1})}\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}_{\langle k\rangle}
+ \sin(\pi\mu_k)e^{-li\lambda(x+x_{k-1}-2\gamma_k)+2li\eta_k}\begin{bmatrix} -i & l \\ l & i \end{bmatrix}_{\langle k\rangle}\Bigg)
\times\Bigg(\frac{1}{2i}\, e^{i\lambda x_{k-1}}\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}_{\langle k-1\rangle} + \frac{1}{2i}\, e^{-i\lambda x_{k-1}}\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}_{\langle k-1\rangle}
+\sum_{j=1}^{k-1}\sin\pi\mu_j e^{-il\lambda(x_{k-1}-2\gamma_j)+2il\eta_j}\begin{bmatrix} -i & l \\ l & i \end{bmatrix}_{\langle k-1\rangle}\Bigg).$$

Since $0 < x_{k-1} < x$, it follows that $0 < 2x_{k-1} < 2x,\ -x < 2x_{k-1} - x < x$, and $e^{\pm i\lambda(2x_{k-1}-x)}$ grow not faster than $e^{\pm i\lambda x}$. Therefore

$$\begin{aligned}
S(x,\lambda) &= \frac{1}{2i}\, e^{i\lambda x}\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}_{\langle k\rangle} + \frac{1}{2i}\, e^{-i\lambda x}\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}_{\langle k\rangle} \\
&+\frac{1}{2i}\sum_{j=1}^{k-1}\sin(\pi\mu_j)e^{i\lambda(x-x_{k-1})-li\lambda(x_{k-1}-2\gamma_j)+2li\eta_j}\left[(1-l)\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}\right]_{\langle k\rangle} \\
&+\frac{1}{2i}\sum_{j=1}^{k-1}\sin(\pi\mu_j)e^{-i\lambda(x-x_{k-1})-li\lambda(x_{k-1}-2\gamma_j)+2li\eta_j}\left[(1+l)\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}\right]_{\langle k\rangle} \\
&+\frac{1}{2i}\sin(\pi\mu_k)e^{i\lambda x_{k-1}-li\lambda(x+x_{k-1}-2\gamma_k)+2li\eta_k}\left[(1+l)\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}\right]_{\langle k\rangle} \\
&+\frac{1}{2i}\sin(\pi\mu_k)e^{-i\lambda x_{k-1}-li\lambda(x+x_{k-1}-2\gamma_k)+2li\eta_k}\left[(1-l)\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}\right]_{\langle k\rangle} \\
&+\sum_{j=1}^{k-1}\sin(\pi\mu_k)\sin(\pi\mu_j)e^{-li\lambda(x+2x_{k-1}-2\gamma_k-2\gamma_j)+4li\eta_j}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{\langle k\rangle}.
\end{aligned}$$

Let $l = -1$. Then

$$\begin{aligned} S(x,\lambda) = \frac{1}{2i}\, e^{i\lambda x} \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}_{\langle k\rangle} + \frac{1}{2i}\, e^{-i\lambda x} \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}_{\langle k\rangle} \\ + \sum_{j=1}^{k-1} \sin(\pi\mu_j) e^{i\lambda x - 2i\lambda\gamma_j - 2i\eta_j} \begin{bmatrix} -i & -1 \\ -1 & i \end{bmatrix}_{\langle k\rangle} \\ + \sum_{j=1}^{k-1} \sin(\pi\mu_j) e^{-i\lambda x + 2i\lambda x_{k-1} - 2i\lambda\gamma_j - 2i\eta_j} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{\langle k\rangle} \\ + e^{2i\lambda x_{k-1} + i\lambda x - 2i\lambda\gamma_k - 2i\eta_k} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{\langle k\rangle} \\ + \sin(\pi\mu_k) e^{i\lambda x - 2i\lambda\gamma_k - 2i\eta_k} \begin{bmatrix} 1 & -i \\ -i & -1 \end{bmatrix}_{\langle k\rangle} \\ + \sum_{j=1}^{k-1} \sin(\pi\mu_k)\sin(\pi\mu_j) e^{i\lambda(x + 2x_{k-1} - 2\gamma_k - 2\gamma_j) - 4i\eta_j} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{\langle k\rangle}. \end{aligned}$$

This yields

$$\begin{aligned} S(x,\lambda) = \frac{1}{2i} \left(e^{i\lambda x} \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}_{\langle k\rangle} + e^{-i\lambda x} \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}_{\langle k\rangle} \right) \\ + \sum_{j=1}^{k} \sin(\pi\mu_j) e^{i\lambda x - 2i\lambda\gamma_j - 2i\eta_j} \begin{bmatrix} -i & -1 \\ -1 & i \end{bmatrix}_{\langle k\rangle}. \end{aligned}$$

The case $l = 1$ is treated similarly. Lemma 1 is proved.

The following assertion is proved analogously.

Lemma 2 *For $x \in \omega_k$ and $|\lambda(x - \gamma_k)| \geq 1$*

$$\begin{aligned} \frac{\partial}{\partial\lambda} S(x,\lambda) = \frac{x}{2i} \left(e^{i\lambda x} \begin{bmatrix} -1 & -i \\ i & -1 \end{bmatrix}_{\langle k\rangle} + e^{-i\lambda x} \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}_{\langle k\rangle} \right) \\ + \sum_{j=1}^{k} (x - 2\gamma_j) \sin\pi\mu_j e^{-il\lambda(x - 2\gamma_j) + 2il\eta_j} \begin{bmatrix} -l & -i \\ -i & l \end{bmatrix}_{\langle k\rangle}. \end{aligned}$$

Definition A function $Y(x,\lambda)$ is called the solution of system (1), if there exist constants $C_1(\lambda)$, $C_2(\lambda)$ such that $Y(x,\lambda) = C_1(\lambda)S_1(x,\lambda) + C_2(\lambda)S_2(x,\lambda)$, $x \in (0,\pi)\setminus\bigcup_{k=1}^{N}\{\gamma_k\}$.

We introduce the functions

$$\varphi(x,\lambda) = \Big(\varphi_1(x,\lambda),\ \varphi_2(x,\lambda)\Big) = S(x,\lambda)V(\alpha),$$

$$V(\alpha) = \Big(V_1(\alpha), V_2(\alpha)\Big) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix},$$

$$\Delta(\lambda) = \begin{pmatrix} \Delta_{11}(\lambda) & \Delta_{12}(\lambda) \\ \Delta_{21}(\lambda) & \Delta_{22}(\lambda) \end{pmatrix} = V^T(\beta)S(\pi,\lambda)V(\alpha),$$
$$\psi(x,\lambda) = \Big(\psi_1(x,\lambda),\ \psi_2(x,\lambda)\Big) = S(x,\lambda)S^{-1}(\pi,\lambda)V(\beta).$$

Clearly, $\varphi(x,\lambda)$, $\psi(x,\lambda)$ are fundamental matrices for system (1). Denote $\langle Y, Z\rangle := Y^T BZ$. If $Y(x,\lambda)$, $Z(x,\lambda)$ are solutions of system (1), then $\langle Y(x,\lambda), Z(x,\lambda)\rangle := \det\{Y(x,\lambda), Z(x,\lambda)\}$ is their Wronskian. Obviously,

$$\langle \psi_2(x,\lambda), \varphi_2(x,\lambda)\rangle = -\Delta_{12}(\lambda). \tag{5}$$

A number λ_0 is called an eigenvalue of problem (1)–(2), if there exist constants A_1, A_2 ($|A_1| + |A_2| > 0$) such that the function $A_1S_1(x,\lambda_0) + A_2S_2(x,\lambda_0)$ satisfies the boundary conditions (2).

Lemma 3 *Zeros of $\Delta_{12}(\lambda)$ coincide with the eigenvalues of the boundary value problem (1)–(2). If λ_0 is an eigenvalue, then $\varphi(x,\lambda_0)$ and $\psi(x,\lambda_0)$ are eigenfunctions, and $\psi(x,\lambda_0) = b_0\varphi(x,\lambda_0)$.*

Proof 1. Let λ_0 be a zero of $\Delta_{12}(\lambda)$, i.e. $V_1^T(\beta)S(\pi,\lambda_0)V_2(\alpha) = 0$. Therefore, $\varphi_2(x,\lambda_0) = S(x,\lambda_0)V_2(\alpha)$ is an eigenfunction, and λ_0 is an eigenvalue. It follows from (5) that $\varphi_2(x,\lambda_0)$ and $\psi_2(x,\lambda_0)$ are linear dependent.
2. Let λ_0 be an eigenvalue, and let $Y_0(x)$ be the corresponding eigenfunction. Since $\varphi_1(x,\lambda), \varphi_2(x,\lambda)$ form a fundamental system of solutions, it follows that $Y_0(x) = D_1\varphi_1(x,\lambda_0)+D_2\varphi_2(x,\lambda_0)$. Substituting this relation into the first boundary condition, we obtain $D_1V_1^T(\alpha)V_1(\alpha) + D_2V_1^T(\alpha)V_2(\alpha) = 0$, hence $D_1 = 0$. Using the second boundary condition, we find $D_2V_1^T(\beta)\varphi_2(\pi,\lambda_0) = 0$. Since $Y_0(x) \not\equiv 0$, one has $D_2 \neq 0$, i.e. $V_1^T(\beta)\varphi_2(\pi,\lambda_0) = 0$. Lemma 3 is proved.

We note that the functions $\Delta_{jk}(\lambda)$, $j,k = 1,2$, are the characteristic functions for the boundary value problems L_{jk} foe system (1) with boundary conditions. $V_{3-k}^T(\alpha)Y(0) = V_j^T(\beta)Y(\pi) = 0$. Denote

$$\Phi(x,\lambda) = \Big(\Phi_1(x,\lambda),\ \Phi_2(x,\lambda)\Big), \quad \Phi_1(x,\lambda) = -\frac{1}{\Delta_{12}(\lambda)}\psi_2(x,\lambda),$$
$$\Phi_2(x,\lambda) = \varphi_2(x,\lambda).$$

It follows from (5) that $\det\Phi(x,\lambda) \equiv 1$. The functions $\Phi_1(x,\lambda)$, $\Phi_2(x,\lambda)$ are called the Weyl solutions, and the matrix $\mathfrak{M}(\lambda) := V_2^T(\lambda)\Phi_1(0,\lambda)$ is called the Weyl matrix for the problem (1)–(2).

Lemma 4 *The following relations hold*

$$\Phi(x,\lambda) = \varphi(x,\lambda)M(\lambda), \quad \textit{where } M(\lambda) = \begin{pmatrix} 1 & 0 \\ \mathfrak{M}(\lambda) & 1 \end{pmatrix}, \ \mathfrak{M}(\lambda) = -\frac{\Delta_{11}(\lambda)}{\Delta_{12}(\lambda)}.$$

Only formula for $\Phi_1(x,\lambda)$ is needed to be proved. Let $\Phi_1(x,\lambda) = D_1(\lambda)\varphi_1(x,\lambda) + D_2(\lambda)\varphi_2(x,\lambda)$. Then

$$-(\Delta_{12}(\lambda))^{-1}\psi_2(0,\lambda) = D_1(\lambda)V_1(\alpha) + D_2(\lambda)V_2(\alpha). \tag{6}$$

Multiplying (6) by $V_1^T(\alpha)$, we infer $-(\Delta_{12}(\lambda))^{-1}V_1^T(\alpha)\psi_2(0,\lambda) = D_1(\lambda)$. Since

$$V_1^T(\alpha)\psi_2(0,\lambda) = V_1^T(\alpha)B^T S^T(\pi,\lambda)BV_2(\beta),$$

it follows that $V_1^T(\alpha)\psi_2(0,\lambda) = -V_1^T(\beta)S(\pi,\lambda)V_2(\alpha) = -\Delta_{12}(\lambda)$, i.e. $D_1(\lambda) = 1$. Multiplying (6) by $V_2^T(\alpha)$, we find $D_2(\lambda) = V_2^T(\alpha)\Phi_1(0,\lambda) = \mathfrak{M}(\lambda)$. Taking the relation $V_2^T(\alpha)\psi_2(0,\lambda) = \Delta_{11}(\lambda)$ into account, we obtain the assertion of the lemma.

Thus, $\mathfrak{M}(\lambda)$ is a meromorphic function; its poles coincide with the eigenvalues of L, and its zeros coincide with the eigenvalues of L_{11}.

Lemma 5 *For* $x \in \omega_k$ *and* $|\lambda(x-\gamma_k)| \geq 1$, $|\lambda(x-\gamma_{k+1})| \geq 1$, *one has*

$$\varphi(x,\lambda) = \frac{1}{2i}\left(e^{i\lambda x + i\alpha}\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}_{\langle k\rangle} + e^{-i\lambda x - i\alpha}\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}_{\langle k\rangle}\right)$$
$$+\sum_{j=1}^{k}\sin\pi\mu_j e^{-il\lambda(x-2\gamma_j)+2il\eta_j-il\alpha}\begin{bmatrix} -i & l \\ l & i \end{bmatrix}_{\langle k\rangle}, \tag{7}$$
$$\psi(x,\lambda) = -\frac{1}{2i}\left(e^{i\lambda(\pi-x)-i\beta}\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}_{\langle k+1\rangle} + e^{-i\lambda(\pi-x)+i\beta}\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}_{\langle k+1\rangle}\right)$$
$$-\sum_{j=k+1}^{N}\sin\pi\mu_j e^{-il\lambda(x+\pi-2\gamma_j)+2il\eta_j-il\beta}\begin{bmatrix} -i & -l \\ -l & i \end{bmatrix}_{\langle k+1\rangle}. \tag{8}$$

Indeed, since $\varphi(x,\lambda) = S(x,\lambda)V(\alpha)$, relation (7) follows from Lemma 1. To prove (8) we make the substitution $x \to \pi - x$ and repeat the arguments.

Taking the relation $\Delta(\lambda) = V(\beta)\varphi(\pi,\lambda)$ into account we arrive at the following assertion.

Corollary 1 *For the characteristic function* $\Delta_{12}(\lambda)$, *the following asymptotics holds*

$$\Delta_{12}(\lambda) = \frac{1}{2i}e^{-i(\lambda\pi+\alpha-\beta)}[1] - \frac{1}{2i}e^{i(\lambda\pi+\alpha-\beta)}[1]$$
$$+\sum_{j=1}^{N}\sin\pi\mu_j e^{-il\lambda(\pi-2\gamma_j)+il(2\eta_j-\alpha-\beta)}[l], \tag{9}$$

where $\left[\left(a_{kj}\right)_{k,j=1}^{n,m}\right] := \left(a_{kj} + O\left(|\lambda|^{-\nu}\right)\right)_{k,j=1}^{n,m}$ *for* $|\lambda| \to \infty$.

By the well-known method (see for example [13–15]) one obtains the following properties:

1. $\Delta_{12}(\lambda) = O(e^{\pi|Im\lambda|})$.
2. All eigenvalues $\lambda_k,\ k \in \mathbb{Z}$ of the problem (1)–(2) lie in the strip $|Im\lambda| \le h$.
3. Let N_a be a number of eigenvalues in the rectangle $\Big\{\lambda \mid Re\lambda \in [a, a+1),\ |Im\lambda| \le h\Big\}$. Then N_a is uniformly bounded.
4. Denote $G_\delta = \{\lambda : |\lambda - \lambda_k| \ge \delta\ \forall k\}$. Then $|\Delta_{12}(\lambda)| \ge C_\delta e^{\pi|Im\lambda|}$ for $\lambda \in G_\delta$.
5. For sufficiently small δ, there exists a sequence $R_n \to \infty$ such that the circles $\Gamma_n = \Big\{\lambda : |\lambda| = R_n\Big\}$ lie in G_δ.
6. Let $\{\lambda_k^0\}_{k=-\infty}^{\infty}$ be zeros of the function
$$\Delta_{12}^0(\lambda) = \frac{1}{2i}e^{-i(\lambda\pi+\alpha-\beta)} - \frac{1}{2i}e^{i(\lambda\pi+\alpha-\beta)} + l\sum_{j=1}^{N} \sin\pi\mu_j e^{-il\lambda(\pi-2\gamma_j)+il(2\eta_j-\alpha-\beta)}. \tag{10}$$
Then $\lambda_k = \lambda_k^0 + O(|\lambda_k^0|^{-\nu})$.

For simplicity, we confine ourselves to the case when all eigenvalues of L are simple, i.e. the function $\Delta_{12}(\lambda)$ has only simple zeros. In particular, it is always true for the self-adjoint case. Denote $a_k := \mathrm{Res}_{\lambda=\lambda_k}\ \mathfrak{M}(\lambda)$. The data $\{a_k,\ \lambda_k\}_{k=-\infty}^{+\infty}$ are called the spectral data for L. The inverse problem is formulated as follows.

Inverse Problem 1 Given $\{a_k, \lambda_k\}_{k=-\infty}^{+\infty}$, construct L, i.e. $Q(x)$, $Q_\omega(x)$, α, β.

In Sects. 3 and 4 we give an algorithm for the global solution of this nonlinear inverse problem and provide necessary and sufficient conditions for its solvability.

Lemma 6 *Let $\mathfrak{M}^0(\lambda)$ be the Weyl function for the problem L^0 of the form (1)–(2) but with the zero potential $Q(x) \equiv 0$. Then*
$$\mathfrak{M}(\lambda) = \mathfrak{M}^0(\lambda) + \sum_{k=-\infty}^{+\infty}\left(\frac{a_k}{\lambda-\lambda_k} - \frac{a_k^0}{\lambda-\lambda_k^0}\right), \quad \sum_{k=-\infty}^{+\infty} = \lim_{n\to\infty}\sum_{|\lambda_k|<R_n,\ |\lambda_k^0|<R_n}.$$

Proof Consider the integral $J_n(\lambda) = \frac{1}{2\pi i}\int_{\Gamma_n}\frac{\mathfrak{M}(\xi)-\mathfrak{M}^0(\xi)}{\xi-\lambda}d\xi$, $\lambda \in \mathrm{int}\,\Gamma_n$. Using Lemmas 4–5 and Corollary 1, we obtain $\mathfrak{M}(\xi) - \mathfrak{M}^0(\xi) = O(|\xi|^{-\nu})$ for $\xi \in G_\delta$, and consequently, $J_n(\lambda) \to 0$ as $n \to \infty$. On the other hand, by residue's theorem,
$$J_n(\lambda) = \mathop{\mathrm{Res}}_{\xi=\lambda}\frac{\mathfrak{M}(\xi)-\mathfrak{M}^0(\xi)}{\xi-\lambda} + \sum_{|\lambda|<R_n,\ |\lambda_k^0|<R_n}\left(\mathop{\mathrm{Res}}_{\xi=\lambda_k}\frac{\mathfrak{M}(\xi)}{\xi-\lambda} - \mathop{\mathrm{Res}}_{\xi=\lambda_k^0}\frac{\mathfrak{M}^0(\xi)}{\xi-\lambda}\right),$$
hence
$$J_n(\lambda) = \mathfrak{M}(\lambda) - \mathfrak{M}^0(\lambda) + \sum_{|\lambda|<R_n,\ |\lambda_k^0|<R_n}\left(\frac{a_k}{\lambda_k-\lambda} - \frac{a_k^0}{\lambda_k^0-\lambda}\right).$$
If $n \to \infty$, we arrive at the assertion of the lemma.

Together with L we consider a boundary value problem $\widetilde{L}$ of the same form (1)–(2) but with different $\widetilde{Q}(x)$, $\widetilde{Q}_\omega(x)$, $\widetilde{\alpha}$, $\widetilde{\beta}$. We agree that if a certain symbol v denotes an object related to L, then $\widetilde{v}$ will denote an analogous object related to $\widetilde{L}$.

Lemma 7 *If $\lambda_k = \widetilde{\lambda}_k$ for all k, then $\Delta_{12}(\lambda) \equiv \widetilde{\Delta}_{12}(\lambda)$.*

Proof The functions $\Delta_{12}(\lambda)$ and $\widetilde{\Delta}_{12}(\lambda)$ are entire in λ of exponential type. Using Hadamard's factorization theorem, we get $\Delta_{12}(\lambda) = e^{a\lambda+b}\widetilde{\Delta}_{12}(\lambda)$. Let us show that $a = 0,\ b = 0$. In view of (9),

$$\frac{1}{2i}e^{-i(\lambda\pi+\alpha-\beta)}[1] - \frac{1}{2i}e^{i(\lambda\pi+\alpha-\beta)}[1] + \sum_{j=1}^{N} \sin\pi\mu_j e^{-il\lambda(\pi-2\gamma_j)+il(2\eta_j-\alpha-\beta)}[l]$$
$$= \frac{1}{2i}e^{-i(\lambda\pi+\widetilde{\alpha}-\widetilde{\beta})+a\lambda+b}[1] - \frac{1}{2i}e^{i(\lambda\pi+\widetilde{\alpha}-\widetilde{\beta})+a\lambda+b}[1]$$
$$+ \sum_{j=1}^{N} \sin\pi\widetilde{\mu}_j e^{-il\lambda(\pi-2\widetilde{\gamma}_j)+il(2\widetilde{\eta}_j-\widetilde{\alpha}-\widetilde{\beta})+a\lambda+b}[l]. \tag{11}$$

Let $\lambda = \sigma + i\tau$. If $\tau = 0$ and $\sigma \to +\infty$, then the right-hand side in (11) is bounded; hence $Re\,a \le 0$; for $\tau = 0$ and $\sigma \to -\infty$, we get $Re\,a \ge 0$, i.e. $Re\,a = 0$. Furthermore, the right-hand side in (11) is $O(e^{-Im\lambda\tau})$, but the left-hand side is $O(e^{-Im\lambda\tau - Im\,a Im\lambda})$ for $\tau \le 0$. For $\tau \to -\infty$ we have $Im\,a \le 0$. If $\tau \ge 0$, then it follows from (11) that $O(e^{\pi\tau}) = O(e^{\pi\tau - Im\,a\tau})$. This means that $Im\,a \ge 0$, i.e. $Im\,a = 0$. Thus, $a = 0$. Similarly, one gets that $b = 0$. Lemma is proved.

Corollary 2 *If $\lambda_k = \widetilde{\lambda}_k$ for all k, then $\Delta^0_{12}(\lambda) \equiv \widetilde{\Delta}^0_{12}(\lambda)$, i.e. $\alpha-\beta = \widetilde{\alpha}-\widetilde{\beta}$, $\gamma_k = \widetilde{\gamma}_k$, $\sin\pi\mu_k e^{il(2\eta_k-\alpha-\beta)} = \sin\pi\widetilde{\mu}_k e^{il(2\widetilde{\eta}_k-\widetilde{\alpha}-\widetilde{\beta})}$. Here $\Delta^0_{12}(\lambda)$ is defined by (10).*

Lemma 8 *If $\alpha - \widetilde{\alpha} = \beta - \widetilde{\beta} = \widetilde{\eta}_k - \eta_k$, $\mu_k = \widetilde{\mu}_k$, $\gamma_k = \widetilde{\gamma}_k$, $k = \overline{1, N}$ and $Q(x) = \widetilde{Q}(x)V^2(\widetilde{\alpha} - \alpha)$, then $\mathfrak{M}(\lambda) = \widetilde{\mathfrak{M}}(\lambda)$.*

Proof Denote $\delta := \alpha - \widetilde{\alpha} = \beta - \widetilde{\beta} = \widetilde{\eta}_k - \eta_k$. Let us show that if $Y(x,\lambda)$ is a solution of (1), then $\widetilde{Y}(x,\lambda) = V(-\delta)Y(x,\lambda)$ is a solution of $\widetilde{(1)}$. Indeed, substituting $V(\delta)\widetilde{Y}(x,\lambda)$ into (1), we obtain

$$BV(\delta)\widetilde{Y}'(x,\lambda) + \Big(Q(x) + Q_\omega(x)\Big)V(\delta)\widetilde{Y}(x,\lambda) = \lambda V(\delta)\widetilde{Y}(x,\lambda).$$

Multiplying by $V^T(\delta) = V(-\delta)$ and taking the relation $V^T(\delta)Q(x) = Q(x)V(\delta)$ into account, we get

$$B\widetilde{Y}'(x,\lambda) + \Big(Q(x) + Q_\omega(x)\Big)V^2(\delta)\widetilde{Y}(x,\lambda) = \lambda\widetilde{Y}(x,\lambda).$$

One has $V(-\delta)S(0,\lambda)V(\delta) = I$. Since the Cauchy problem has the unique solution, we infer $\widetilde{S}(x,\lambda) = V(-\delta)S(x,\lambda)V(\delta)$. Then $\widetilde{\Delta}(\lambda) = V^T(\widetilde{\beta})V(-\delta)S(\pi,\lambda)V(\delta)V(\widetilde{\alpha})$ or

$$\widetilde{\Delta}(\lambda) = V^T(\widetilde{\beta} + \delta)S(\pi, \lambda)V(\delta + \widetilde{\alpha}).$$

This yields $\widetilde{\Delta}_{jk}(\lambda) = \Delta_{jk}(\lambda)$. Lemma is proved.

3 Solution of the inverse problem

Let us first prove the uniqueness theorem.

Theorem 1 *If* $\mathfrak{M}(\lambda) = \widetilde{\mathfrak{M}}(\lambda)$, *then* $\alpha - \widetilde{\alpha} = \beta - \widetilde{\beta} = \widetilde{\eta}_k - \eta_k,\ \mu_k = \widetilde{\mu}_k,\ \gamma_k = \widetilde{\gamma}_k,\ k = \overline{1, N}$ *and* $Q(x) = \widetilde{Q}(x)V^2(\widetilde{\alpha} - \alpha)$.

Proof By virtue of Lemma 8, it is sufficient to prove the theorem for the case $\widetilde{\beta} = \beta = 0$. Consider the function $P(x, \lambda) = \Phi(x, \lambda)\widetilde{\Phi}^{-1}(x, \lambda)$.

Since $\mathfrak{M}(\lambda) = \widetilde{\mathfrak{M}}(\lambda)$, it follows that these functions have the same poles. In view of Lemma 7, one gets $\Delta_{12}(\lambda) = \widetilde{\Delta}_{12}(\lambda)$. By Corollary 2, $\widetilde{\alpha} = \alpha,\ \widetilde{\gamma}_k = \gamma_k,\ \sin \pi \mu_k e^{il(2\eta_k - \alpha)} = \sin \pi \widetilde{\mu}_k e^{il(2\widetilde{\eta}_k - \widetilde{\alpha})}$. This yields

$$\varphi(x, \lambda) - \widetilde{\varphi}(x, \lambda) = O(e^{|Im\lambda|x}|\lambda|^{-\nu}), \quad \psi(x, \lambda) - \widetilde{\psi}(x, \lambda) = O(e^{|Im\lambda|(\pi - x)}|\lambda|^{-\nu}). \tag{12}$$

Since $\Phi_1(x, \lambda) = -(\Delta_{12}(\lambda))^{-1}\psi_2(x, \lambda)$, it follows that

$$\Phi_1(x, \lambda) = O(e^{-x|Im\lambda|}), \quad \lambda \in G_\delta. \tag{13}$$

Taking (12) into account, we infer

$$\Phi_1(x, \lambda) - \widetilde{\Phi}_1(x, \lambda) = O(e^{-x|Im\lambda|}|\lambda|^{-\nu}), \quad \lambda \in G_\delta. \tag{14}$$

Obviously, $P(x, \lambda) - I = (\Phi(x, \lambda) - \widetilde{\Phi}(x, \lambda))B\widetilde{\Phi}(x, \lambda)B^T$. Using (12)–(14), we obtain for $\lambda \in G_\delta$:

$$\begin{aligned} P(x, \lambda) - I &= |\lambda|^{-\nu} \begin{pmatrix} O(e^{-|Im\lambda|x}) & O(e^{|Im\lambda|x}) \\ O(e^{-|Im\lambda|x}) & O(e^{|Im\lambda|x}) \end{pmatrix} \begin{pmatrix} O(e^{|Im\lambda|x}) & O(e^{|Im\lambda|x}) \\ O(e^{-|Im\lambda|x}) & O(e^{-|Im\lambda|x}) \end{pmatrix} \\ &= O(|\lambda|^{-\nu}). \end{aligned} \tag{15}$$

Since $\Phi(x, \lambda) = \varphi(x, \lambda)M(\lambda)$, we get $P(x, \lambda) = \varphi(x, \lambda)M(\lambda)\widetilde{M}^{-1}(\lambda)\widetilde{\varphi}^{-1}(x, \lambda)$, or $P(x, \lambda) = \varphi(x, \lambda)\widetilde{\varphi}^{-1}(x, \lambda)$. Therefore, $P(x, \lambda)$ is entire in λ. Using (15), maximum modulus principle and Liouville's theorem, we conclude that $P(x, \lambda) = I$, i.e. $\widetilde{\Phi}(x, \lambda) = \Phi(x, \lambda)$. Then $Q(x) + Q_\omega(x) = \widetilde{Q}(x) + \widetilde{Q}_{\widetilde{\omega}}(x)$, and consequently, $Q(x) = \widetilde{Q}(x),\ Q_\omega(x) = \widetilde{Q}_{\widetilde{\omega}}(x)$. Theorem is proved.

Corollary 3 *If* $a_k = \widetilde{a}_k,\ \lambda_k = \widetilde{\lambda}_k$ *for all* k, *then* $L = \widetilde{L}$.

Corollary 4 *If* $\lambda_k^{\langle 11\rangle} = \widetilde{\lambda}_k^{\langle 11\rangle},\ \lambda_k = \widetilde{\lambda}_k$ *for all* k, *then* $L = \widetilde{L}$. *Here* $\{\lambda_k^{\langle 11\rangle}\}$ *and* $\{\widetilde{\lambda}_k^{\langle 11\rangle}\}$ *are eigenvalues of* L_{11} *and* $\widetilde{L}_{11}$, *respectively.*

Indeed, according to Lemma 7, $\Delta_{12}(\lambda) = \widetilde{\Delta}_{12}(\lambda)$. Analogously, we obtain $\Delta_{11}(\lambda) = \widetilde{\Delta}_{11}(\lambda)$. By Lemma 4, $\mathfrak{M}(\lambda) = \widetilde{\mathfrak{M}}(\lambda)$.

Let us now go on to constructing the solution of the nonlinear Inverse Problem 1. The central role here is played by the so-called main equation of the inverse problem, which is a linear equation in the corresponding Banach space. Let us derive the main equation.

Let the problem L with a simple spectrum be given. We choose a model boundary value problem $\widetilde{L}$ with a simple spectrum such that $\omega = \tilde{\omega}$, $Q_\omega(x) = \widetilde{Q}_\omega(x)$ and

$$\Lambda := \sum_{k=-\infty}^{+\infty} |\widetilde{a}_k|\xi_k < \infty, \quad \xi_k := |\lambda_k - \widetilde{\lambda}_k| + |\widetilde{a}_k^{-1} a_k - 1|. \tag{16}$$

For definiteness, we assume that $\alpha = \widetilde{\alpha} = 0$. Then $\beta = \widetilde{\beta}$. Denote $\Omega_\varepsilon := \{x : x \in (0,\pi),\ |x - \gamma_k| \geq \varepsilon,\ k = \overline{1, N}\}$, $\lambda_{k0} = \lambda_k$, $\lambda_{k1} = \widetilde{\lambda}_k$, $a_{k0} = a_k$, $a_{k1} = \widetilde{a}_k$,

$$\widetilde{D}^{\langle l\rangle}(x,\lambda,\theta) := \frac{\left\langle \widetilde{\Phi}_l(x,\lambda),\ \widetilde{\varphi}_2(x,\theta)\right\rangle}{\lambda - \theta}, \quad l = 1,2, \qquad \widetilde{D}^{\langle 2\rangle}_{kj}(x,\lambda) = \widetilde{D}^{\langle 2\rangle}(x,\lambda,\lambda_{kj}),$$
$$\widetilde{P}_{ni,kj}(x) = \widetilde{D}^{\langle 2\rangle}(x,\lambda_{ni},\lambda_{kj})a_{kj}, \quad \varphi_{2,kj}(x) = \varphi_2(x,\lambda_{kj}), \quad \widetilde{\varphi}_{2,kj}(x) = \widetilde{\varphi}_2(x,\lambda_{kj}),$$

where $\langle Y, Z\rangle := \det(Y,Z) = Y^T BZ$. Analogously we define $D^{\langle l\rangle}(x,\lambda,\theta)$, $D^{\langle 2\rangle}_{kj}(x,\lambda)$ and $P_{ni,kj}(x)$.

Lemma 9 *For $x \in \Omega_\varepsilon$ and λ on compact sets,*

$$|\widetilde{\varphi}^{(m)}_{2,kj}(x)| \leq C(1 + |\lambda^0_k|)^m, \quad |\widetilde{\varphi}^{(m)}_{2,k1}(x) - \widetilde{\varphi}^{(m)}_{2,k0}(x)| \leq C\xi_k(1+|\lambda^0_k|)^m, \quad m=0,1, \tag{17}$$

$$\left.\begin{aligned} &|\widetilde{D}^{\langle 2\rangle}_{kj}(x,\lambda)| \leq \frac{C}{1+|\lambda-\lambda^0_k|}, \quad |\widetilde{D}^{\langle 2\rangle}_{k0}(x,\lambda)a_{k0} - \widetilde{D}^{\langle 2\rangle}_{k1}(x,\lambda)a_{k1}| \leq \frac{C|a_{k1}|\xi_k}{1+|\lambda-\lambda^0_k|}, \\ &|(\widetilde{D}^{\langle 2\rangle}_{kj}(x,\lambda))'| \leq C, \quad |(\widetilde{D}^{\langle 2\rangle}_{k0}(x,\lambda)a_{k0} - \widetilde{D}^{\langle 2\rangle}_{k1}(x,\lambda)a_{k1})'| \leq C|a_{k1}|\xi_k. \end{aligned}\right\}. \tag{18}$$

The same estimates are valid for $\varphi_{2,kj}(x)$, $D^{\langle 2\rangle}_{kj}(x,\lambda)$.

In order to prove the lemma, we need the following generalization of Schwarz's lemma:

Let the function $f(z)$ be analytic inside the circle $|z - z_0| \leq R$ and continuous in the whole circle. Moreover, $|f(z)| \leq C$ on the boundary, and $f(z_0) = 0$. Then $|f(z)| \leq C|z - z_0|/R$ in the circle $|z - z_0| \leq R$.

1. It follows from (7) that

$$|\widetilde{\varphi}_2(x,\lambda)| \leq Ce^{|Im\lambda|x}, \quad x \in \Omega_\varepsilon. \tag{19}$$

The eigenvalues lie in the strip $|Im\lambda| \leq \max\{h, \widetilde{h}\}$; it follows from (19) that $|\widetilde{\varphi}^{(m)}_{2,kj}(x)| \leq C(1 + |\lambda_{kj}|)^m$. Using (10), we obtain the first estimate in (17) for $m = 0$.

Applying Schwarz's lemma, we find $|\widetilde{\varphi}_2(x,\lambda)-\widetilde{\varphi}_{2,k1}(x)|\le Ce^{|Im\lambda|x}|\lambda-\lambda_{k1}|$. Hence the second estimate in (17) holds (17) for $m=0$. For $m=1$, the arguments are similar.

2. Since $\widetilde{D}^{\langle 2\rangle}(x,\lambda,\theta)=(\lambda-\theta)^{-1}(\widetilde{\varphi}_2(x,\lambda))^T B\widetilde{\varphi}_2(x,\theta)$, it follows from (19) for $\lambda\neq\theta$, $|\lambda|\le R$, $|\theta|\le R$ that $|\widetilde{D}(x,\lambda,\theta)|\le C|\lambda-\theta|^{-1}$. If $\lambda=\theta$, then $\widetilde{D}(x,\lambda,\lambda)=(\widetilde{\varphi}_2(x,\lambda))^T B\dot{\widetilde{\varphi}}_2(x,\lambda)$, where $\dot{\widetilde{\varphi}}_2(x,\lambda)=\dfrac{\partial}{\partial\lambda}\widetilde{\varphi}_2(x,\lambda)$. Using Lemma 2, we obtain $|\dot{\widetilde{\varphi}}_2(x,\lambda)|\le Cxe^{|Im\lambda|x}$ for $x\in\Omega_\varepsilon$. Then $|\widetilde{D}(x,\lambda,\lambda)|\le C$ for $|\lambda|\le R$. Thus,

$$|\widetilde{D}(x,\lambda,\theta)|\le\frac{C}{1+|\lambda-\theta|},\quad x\in\Omega_\varepsilon,\ |\lambda|\le R,\ |\theta|\le R. \tag{20}$$

Furthermore, $\langle\widetilde{\Phi}_j(x,\lambda),\widetilde{\varphi}_2(x,\theta)\rangle'=(\widetilde{\Phi}_j^T(x,\lambda))'B\widetilde{\varphi}_2(x,\theta)+\widetilde{\Phi}_j^T(x,\lambda)B\widetilde{\varphi}'_2(x,\theta)$. Then

$$\langle\widetilde{\Phi}_j(x,\lambda),\widetilde{\varphi}_2(x,\theta)\rangle'=-(B\widetilde{\Phi}'_j(x,\lambda))^T\widetilde{\varphi}_2(x,\theta)+\widetilde{\Phi}_j^T(x,\lambda)B\widetilde{\varphi}'_2(x,\theta).$$

Since $\widetilde{\Phi}_j,\widetilde{\varphi}_2$ are solutions of the system, it follows that $\langle\widetilde{\Phi}_j(x,\lambda),\widetilde{\varphi}_2(x,\theta)\rangle'=(\theta-\lambda)\widetilde{\Phi}_j^T(x,\lambda)\widetilde{\varphi}_2(x,\theta)$. This yields

$$(\widetilde{D}^{\langle j\rangle}(x,\lambda,\theta))'=-\widetilde{\Phi}_j^T(x,\lambda)\widetilde{\varphi}_2(x,\theta). \tag{21}$$

Taking (21) and (19) into account, we arrive at the third estimate in (18).

Using Schwarz's lemma and (20), we infer $|\widetilde{D}_{k0}^{\langle 2\rangle}(x,\lambda)-\widetilde{D}_{k1}^{\langle 2\rangle}(x,\lambda)|\le\dfrac{C|\lambda_{k0}-\lambda_{k1}|}{1+|\lambda-\lambda_k^0|}$. Since

$$\begin{aligned}|\widetilde{D}_{k0}^{\langle 2\rangle}(x,\lambda)a_{k0}-\widetilde{D}_{k1}^{\langle 2\rangle}(x,\lambda)a_{k1}|&\le|\widetilde{D}_{k0}^{\langle 2\rangle}(x,\lambda)(a_{k0}-a_{k1})|\\&+|(\widetilde{D}_{k0}^{\langle 2\rangle}(x,\lambda)-\widetilde{D}_{k1}^{\langle 2\rangle}(x,\lambda))a_{k1}|,\end{aligned}$$

one gets the second estimate in (18). Other estimates are obtained analogously. Lemma is proved.

Similarly one can prove the following assertion.

Lemma 10 *For $x\in\Omega_\varepsilon$ and λ on compact sets,*

$$\begin{aligned}&|\widetilde{P}_{ni,kj}(x)|\le\frac{C|a_{k1}|}{1+|\lambda_n^0-\lambda_k^0|},\quad|\widetilde{P}'_{ni,kj}(x)|\le C|a_{k1}|,\\&|\widetilde{P}_{ni,k1}(x)-\widetilde{P}_{ni,k0}(x)|\le\frac{C|a_{k1}|\xi_k}{1+|\lambda_n^0-\lambda_k^0|},\quad|\widetilde{P}'_{ni,k1}(x)-\widetilde{P}'_{ni,k0}(x)|\le C|a_{k1}|\xi_k,\\&|\widetilde{P}_{n1,kj}(x)-\widetilde{P}_{n0,kj}(x)|\le\frac{C|a_{k1}|\xi_n}{1+|\lambda_n^0-\lambda_k^0|},\quad|\widetilde{P}'_{n1,kj}(x)-\widetilde{P}'_{n0,kj}(x)|\le C|a_{k1}|\xi_n,\end{aligned}$$

$$|\widetilde{P}_{n1,k1}(x) - \widetilde{P}_{n1,k0}(x) - \widetilde{P}_{n0,k1}(x) + \widetilde{P}_{n0,k0}(x)| \le \frac{C|a_{k1}|\xi_k\xi_n}{1 + |\lambda_n^0 - \lambda_k^0|},$$
$$|\widetilde{P}'_{n1,k1}(x) - \widetilde{P}'_{n1,k0}(x) - \widetilde{P}'_{n0,k1}(x) + \widetilde{P}'_{n0,k0}(x)| \le C|a_{k1}|\xi_k\xi_n.$$

Moreover, if $\lambda \in G_\delta = \Big\{\lambda \,:\, |\lambda - \widetilde{\lambda}_k| \ge \delta,\ k \in \mathbb{Z}\Big\}$*, then*

$$|\widetilde{D}^{\langle 1\rangle}_{kj}(x,\lambda)| \le \frac{C_\delta}{|\lambda - \lambda_{kj}|}, \quad |(\widetilde{D}^{\langle 1\rangle}_{kj}(x,\lambda))'| \le C_\delta,$$
$$|\widetilde{D}^{\langle 1\rangle}_{k0}(x,\lambda)a_{k0} - \widetilde{D}^{\langle 1\rangle}_{k1}(x,\lambda)a_{k1}| \le C_\delta|a_{k1}|\xi_k\left(\frac{1}{|\lambda - \lambda_{k0}|} + \frac{1}{|\lambda - \lambda_{k1}|}\right),$$
$$|(\widetilde{D}^{\langle 1\rangle}_{k0}(x,\lambda)a_{k0} - \widetilde{D}^{\langle 1\rangle}_{k1}(x,\lambda)a_{k1})'| \le C_\delta|a_{k1}|\xi_k,$$

where C *and* C_δ *depend on* ε*. The same estimates are valid for* $D^{\langle l\rangle}_{kj}(x,\lambda),\ P_{ni,kj}(x)$*.*

Lemma 11 *The following relations hold*

$$\Phi_j(x,\lambda) = \widetilde{\Phi}_j(x,\lambda) + \sum_{k=-\infty}^{+\infty}\Big(\widetilde{D}^{\langle j\rangle}_{k0}(x,\lambda)a_{k0}\varphi_{2,k0}(x) - \widetilde{D}^{\langle j\rangle}_{k1}(x,\lambda)a_{k1}\varphi_{2,k1}(x)\Big),$$
$$j = 1, 2, \tag{22}$$

the series converge absolutely and uniformly for $x \in \Omega_\varepsilon$ *and* λ *on compact sets without the spectra of* L *and* $\widetilde{L}$*.*

Proof Consider the function $P(x,\lambda) = \Phi(x,\lambda)\widetilde{\Phi}^{-1}(x,\lambda)$. Denote

$$J_n(x,\lambda) = \frac{1}{2\pi i}\int_{\Gamma_n}\frac{1}{\xi - \lambda}\Big(P(x,\xi) - I\Big)d\xi, \quad \Gamma_n := \Big\{\lambda \,:\, |\lambda| = R_n\Big\}.$$

The functions $\Phi(x,\lambda)$ and $\widetilde{\Phi}(x,\lambda)$ have the same main term in the asymptotics. Therefore, for a fixed $x \ne \gamma_k$, one has $P(x,\xi) - I = O(|\xi|^{-\nu})$, and $J_n(x,\lambda) \to 0$ as $n \to \infty$ uniformly in λ on the compact sets. Integration on Γ_n is divided into integration on the contours $\Gamma_n^{\langle 1\rangle} = \Gamma_n^{\langle 3\rangle} \bigcup \Gamma_n^{\langle 5\rangle}$, $\Gamma_n^{\langle 2\rangle} = \Gamma_n^{\langle 4\rangle} \bigcup \Gamma_n^{\langle 5\rangle}$ (with counterclockwise circuit), where $\Gamma_n^{\langle 3\rangle} = \{\lambda \,:\, |Im\lambda| \le h\} \bigcap \Gamma_n$, $\Gamma_n^{\langle 4\rangle} = \Gamma_n \backslash \Gamma_n^{\langle 3\rangle} = \{\lambda \,:\, |Im\lambda| > h\} \bigcap \Gamma_n$, $\Gamma_n^{\langle 5\rangle} = \{\lambda \,:\, |Im\lambda| = h\} \bigcap \operatorname{int}\Gamma_n$. Let $\lambda \in \operatorname{int}\Gamma_n^{\langle 2\rangle}$. By the Cauchy integral formula,

$$\frac{1}{2\pi i}\int_{\Gamma_n^{\langle 2\rangle}}\frac{1}{\xi - \lambda}\Big(P(x,\xi) - I\Big)d\xi = P(x,\lambda) - I.$$

Clearly,

$$\frac{1}{2\pi i}\int_{\Gamma_n^{\langle 1\rangle}}\frac{1}{\xi - \lambda}I\,d\xi = 0.$$

Then

$$P(x,\lambda) = I + \frac{1}{2\pi i}\int_{\Gamma_n^{\langle 1\rangle}} \frac{1}{\lambda-\xi} P(x,\xi)d\xi - J_n(x,\lambda). \tag{23}$$

Since $\Phi(x,\lambda) = \varphi(x,\lambda)M(\lambda)$, it follows that

$$\begin{aligned} P(x,\xi) &= \varphi(x,\xi)M(\xi)\widetilde{M}^{-1}(\xi)\widetilde{\varphi}^{-1}(x,\xi) \\ &= \varphi(x,\xi)\widetilde{\varphi}^{-1}(x,\xi) - (\mathfrak{M}(\xi) - \widetilde{\mathfrak{M}}(\xi))\varphi(x,\xi)B_{\langle 1\rangle}\widetilde{\varphi}^{-1}(x,\xi), \\ B_{\langle 1\rangle} &= \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}. \end{aligned}$$

The function $\varphi(x,\xi)\widetilde{\varphi}^{-1}(x,\xi)$ is entire in ξ. Therefore,

$$\frac{1}{2\pi i}\int_{\Gamma_n^{\langle 1\rangle}} \varphi(x,\xi)\widetilde{\varphi}^{-1}(x,\xi)\frac{d\xi}{\lambda-\xi} = 0,$$

since λ is outside $\Gamma_n^{\langle 1\rangle}$. Thus, it follows from (23) that

$$P(x,\lambda) = I - \frac{1}{2\pi i}\int_{\Gamma_n^{\langle 1\rangle}} (\mathfrak{M}(\xi) - \widetilde{\mathfrak{M}}(\xi))\varphi(x,\xi)B_{\langle 1\rangle}\widetilde{\varphi}^{-1}(x,\xi)\frac{d\xi}{\lambda-\xi} - J_n(x,\lambda).$$

One has $\Phi(x,\lambda) = P(x,\lambda)\widetilde{\Phi}(x,\lambda)$, hence $\Phi_j(x,\lambda) = P(x,\lambda)\widetilde{\Phi}_j(x,\lambda)$. Then

$$\begin{aligned} &\Phi_j(x,\lambda) = \widetilde{\Phi}_j(x,\lambda) \\ &\quad -\frac{1}{2\pi i}\int_{\Gamma_n^{\langle 1\rangle}} (\mathfrak{M}(\xi) - \widetilde{\mathfrak{M}}(\xi))\varphi(x,\xi)B_{\langle 1\rangle}\widetilde{\varphi}^{-1}(x,\xi)\widetilde{\Phi}_j(x,\lambda)\frac{d\xi}{\lambda-\xi} + \varepsilon_n(x,\lambda), \end{aligned}$$

and $\varepsilon_n(x,\lambda) \to 0$ as $n \to \infty$ uniformly for $x \in \Omega_\varepsilon$. Furthermore,

$$B_{\langle 1\rangle}\widetilde{\varphi}^{-1}(x,\xi)\widetilde{\Phi}_j(x,\lambda) = \left(\widetilde{\varphi}_{12}(x,\xi)\widetilde{\Phi}_{2j}(x,\lambda) - \widetilde{\varphi}_{22}(x,\xi)\widetilde{\Phi}_{1j}(x,\lambda)\right)\begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and consequently,

$$\begin{aligned} \Phi_j(x,\lambda) = {} & \widetilde{\Phi}_j(x,\lambda) - \frac{1}{2\pi i}\int_{\Gamma_n^{\langle 1\rangle}} (\mathfrak{M}(\xi) - \widetilde{\mathfrak{M}}(\xi))\langle\widetilde{\Phi}_j(x,\lambda), \widetilde{\varphi}_2(x,\xi)\rangle\varphi_2(x,\xi)\frac{d\xi}{\lambda-\xi} \\ & + \varepsilon_n(x,\lambda). \end{aligned}$$

Calculating the integral by residue's theorem and taking $n \to \infty$, we arrive at (22). Lemma is proved.

Consider (22) for $j = 2$ and $\lambda = \lambda_{ni}$:

$$\widetilde{\varphi}_{m2,ni}(x) = \varphi_{m2,ni}(x) - \sum_{k=-\infty}^{+\infty} \Big(\widetilde{P}_{ni,k0}(x)\varphi_{m2,k0}(x) - \widetilde{P}_{ni,k1}(x)\varphi_{m2,k1}(x)\Big),$$
$$m = 1, 2, \tag{24}$$

where $\varphi_{2,kj}(x) = \begin{pmatrix} \varphi_{12,kj}(x) \\ \varphi_{22,kj}(x) \end{pmatrix}$. The last relation is not convenient for our purpose, since the series converges only "with brackets". We transform (24) as follows:

$$\begin{aligned}
&\widetilde{\varphi}_{m2,n0}(x) - \widetilde{\varphi}_{m2,n1}(x) = \varphi_{m2,n0}(x) - \varphi_{m2,n1}(x) \\
&\quad - \sum_{k=-\infty}^{+\infty} \Big((\widetilde{P}_{n0,k0}(x) - \widetilde{P}_{n1,k0}(x))(\varphi_{m2,k0}(x) - \varphi_{m2,k1}(x)) \\
&\quad + (\widetilde{P}_{n0,k0}(x) - \widetilde{P}_{n1,k0}(x) - \widetilde{P}_{n0,k1}(x) + \widetilde{P}_{n1,k1}(x))\varphi_{m2,k1}(x)\Big), \\
&\widetilde{\varphi}_{m2,n1}(x) = \varphi_{m2,n1}(x) - \sum_{k=-\infty}^{+\infty} \Big(\widetilde{P}_{n1,k0}(x)(\varphi_{m2,k0}(x) - \varphi_{m2,k1}(x)) \\
&\qquad\qquad + (\widetilde{P}_{n1,k0}(x) - \widetilde{P}_{n1,k1}(x))\varphi_{m2,k1}(x)\Big).
\end{aligned}$$

Denote

$$\begin{aligned}
&\Psi_{n0}^{\langle m \rangle}(x) = \chi_n \Big(\varphi_{m2,n0}(x) - \varphi_{m2,n1}(x)\Big), \\
&\qquad \chi_n = \begin{cases} 0, & \xi_n = 0, \\ \xi_n^{-1}, & \xi_n \neq 0, \end{cases} \quad \Psi_{n1}^{\langle m \rangle}(x) = \varphi_{m2,n1}(x), \\
&\widetilde{H}_{n0,k0}(x) = (\widetilde{P}_{n0,k0}(x) - \widetilde{P}_{n1,k0}(x))\chi_n \xi_k, \\
&\widetilde{H}_{n0,k1}(x) = (\widetilde{P}_{n0,k0}(x) - \widetilde{P}_{n1,k0}(x) - \widetilde{P}_{n0,k1}(x) + \widetilde{P}_{n1,k1}(x))\chi_n, \\
&\widetilde{H}_{n1,k0}(x) = \widetilde{P}_{n1,k0}(x)\xi_k, \ \widetilde{H}_{n1,k1}(x) = \widetilde{P}_{n1,k0}(x) - \widetilde{P}_{n1,k1}(x).
\end{aligned}$$

Then

$$\widetilde{\Psi}_{ni}^{\langle m \rangle}(x) = \Psi_{ni}^{\langle m \rangle}(x) - \sum_{k=-\infty}^{+\infty} \Big(\widetilde{H}_{ni,k0}(x)\Psi_{k0}^{\langle m \rangle}(x) + \widetilde{H}_{ni,k1}(x)\Psi_{k1}^{\langle m \rangle}(x)\Big). \tag{25}$$

Using Lemmas 9 and 10, we obtain the estimates

$$\left.\begin{aligned}
&|\widetilde{\Psi}_{ni}^{\langle m \rangle}(x)| \le C, \ |(\widetilde{\Psi}_{ni}^{\langle m \rangle}(x))'| \le C(1 + |\lambda_k^0|), \\
&|\widetilde{H}_{ni,kj}(x)| \le \frac{C|a_{k1}|\xi_k}{1 + |\lambda_n^0 - \lambda_k^0|}, \ |\widetilde{H}'_{ni,kj}(x)| \le C|a_{k1}|\xi_k.
\end{aligned}\right\} \tag{26}$$

The same estimates are valid for $\Psi_{ni}^{\langle m\rangle}(x),\ \ H_{ni,kj}(x)$. Denote

$$\Psi^{\langle m\rangle}(x)=\begin{pmatrix}\Psi_{n0}^{\langle m\rangle}(x)\\ \Psi_{n1}^{\langle m\rangle}(x)\end{pmatrix}_{n=-\infty}^{+\infty}$$
$$=\Big(\dots,\Psi_{-1,1}^{\langle m\rangle}(x),\quad \Psi_{00}^{\langle m\rangle}(x),\Psi_{01}^{\langle m\rangle}(x),\quad \Psi_{10}^{\langle m\rangle}(x),\dots\Big)^T.$$

Similarly we define the block-matrix

$$\widetilde{H}(x)=\begin{pmatrix}\widetilde{H}_{n0,k0}(x) & \widetilde{H}_{n0,k1}(x)\\ \widetilde{H}_{n1,k0}(x) & \widetilde{H}_{n1,k1}(x)\end{pmatrix}_{n,k=-\infty}^{+\infty}.$$

Then we rewrite (25) as follows

$$\widetilde{\Psi}^{\langle m\rangle}(x)=(I-\widetilde{H}(x))\Psi^{\langle m\rangle}(x),\quad m=1,2, \tag{27}$$

where I is the identity operator. It follows from (26) that $\Psi^{\langle m\rangle}(x),\ \widetilde{\Psi}^{\langle m\rangle}(x)\in\mathbf{m}$ for each fixed $x\neq\gamma_k,\ k=\overline{1,N}$, where $\mathbf{m}$ is the Banach space of bounded sequences. The operator $\widetilde{H}(x)$, acting from $\mathbf{m}$ to $\mathbf{m}$, is a linear bounded operator, and

$$\|\widetilde{H}(x)\|_{m\to m}\le C\sup_n\sum_{k=-\infty}^{+\infty}\frac{|a_{k1}|\xi_k}{1+|\lambda_n^0-\lambda_k^0|}\le C\sum_{k=-\infty}^{+\infty}|a_{k1}|\xi_k<\infty.$$

For each fixed x, relation (27) is a linear equation in $\mathbf{m}$ with respect to $\Psi^{\langle m\rangle}(x)$. This equation is called the main equation of the inverse problem.

Lemma 12 *The following relation holds*

$$Q(x)=\widetilde{Q}(x)+B\text{æ}(x)-\text{æ}(x)B, \tag{28}$$

where

$$\text{æ}(x)=\sum_{k=-\infty}^{+\infty}\Big(a_{k0}\widetilde{\varphi}_{2,k0}(x)\varphi_{2,k0}^T(x)-a_{k1}\widetilde{\varphi}_{2,k1}(x)\varphi_{2,k1}^T(x)\Big), \tag{29}$$

and the series converges uniformly for $x\in\Omega_\varepsilon$.

Proof Differentiating (22), we calculate

$$\Phi_j'(x,\lambda)=\widetilde{\Phi}_j'(x,\lambda)+\sum_{k=-\infty}^{+\infty}\Big((\widetilde{D}_{k0}^{\langle j\rangle}(x,\lambda))'a_{k0}\varphi_{2,k0}(x)+\widetilde{D}_{k0}^{\langle j\rangle}(x,\lambda)a_{k0}\varphi'_{2,k0}(x)$$
$$-(\widetilde{D}_{k1}^{\langle j\rangle}(x,\lambda))'a_{k1}\varphi_{2,k1}(x)-\widetilde{D}_{k1}^{\langle j\rangle}(x,\lambda)a_{k1}\varphi'_{2,k1}(x)\Big).$$

Multiplying this relation by B and using (21), we obtain

$$\Big(\lambda I - Q(x) - Q_\omega(x)\Big)\Phi_j(x,\lambda) = \Big(\lambda I - \widetilde{Q}(x) - \widetilde{Q}_\omega(x)\Big)\widetilde{\Phi}_j(x,\lambda)$$
$$+ \sum_{k=-\infty}^{+\infty}\Big(- \widetilde{\Phi}_j^T(x,\lambda)\widetilde{\varphi}_{2,k0}(x)a_{k0}B\varphi_{2,k0}(x)$$
$$+\widetilde{D}_{k0}^{\langle j\rangle}(x,\lambda)a_{k0}\Big(\lambda_{k0}I - Q(x) - Q_\omega(x)\Big)\varphi_{2,k0}(x) + \widetilde{\Phi}_j^T(x,\lambda)\widetilde{\varphi}_{2,k1}(x)a_{k1}B\varphi_{2,k1}(x)$$
$$-\widetilde{D}_{k1}^{\langle j\rangle}(x,\lambda)a_{k1}\Big(\lambda_{k1}I - Q(x) - Q_\omega(x)\Big)\varphi_{2,k1}(x)\Big),$$

and consequently,

$$(Q(x) - \widetilde{Q}(x))\widetilde{\Phi}_j(x,\lambda) + \sum_{k=-\infty}^{+\infty}(-\widetilde{\Phi}_j^T(x,\lambda)\widetilde{\varphi}_{2,k0}(x)a_{k0}B\varphi_{2,k0}(x)$$
$$+\widetilde{D}_{k0}^{\langle j\rangle}(x,\lambda)a_{k0}(\lambda_{k0} - \lambda)\varphi_{2,k0}(x)$$
$$+\widetilde{\Phi}_j^T(x,\lambda)\widetilde{\varphi}_{2,k1}(x)a_{k1}B\varphi_{2,k1}(x) - \widetilde{D}_{k1}^{\langle j\rangle}(x,\lambda)a_{k1}(\lambda_{k1} - \lambda)\varphi_{2,k1}(x)) = 0.$$

Since $\widetilde{D}_{ni}^{\langle j\rangle}(x,\lambda) = \dfrac{\widetilde{\Phi}_j^T(x,\lambda)B\widetilde{\varphi}_{2,ni}(x)}{\lambda - \lambda_{ni}}$, it follows that

$$(Q(x) - \widetilde{Q}(x))\widetilde{\Phi}_j(x,\lambda) + \sum_{k=-\infty}^{+\infty}\Big(- \{\widetilde{\Phi}_j^T(x,\lambda)\widetilde{\varphi}_{2,k0}(x)a_{k0}\}B\varphi_{2,k0}(x)$$
$$-\{\widetilde{\Phi}_j^T(x,\lambda)B\widetilde{\varphi}_{2,k0}(x)a_{k0}\}\varphi_{2,k0}(x) + \{\widetilde{\Phi}_j^T(x,\lambda)\widetilde{\varphi}_{2,k1}(x)a_{k1}\}B\varphi_{2,k1}(x)$$
$$+\{\widetilde{\Phi}_j^T(x,\lambda)B\widetilde{\varphi}_{2,k1}(x)a_{k1}\}\varphi_{2,k1}(x)\Big) = 0.$$

The matrices $Q(x)$ and $\widetilde{Q}(x)$ are symmetrical. Then

$$\widetilde{\Phi}^T(x,\lambda)\Big\{Q(x) - \widetilde{Q}(x) + \sum_{k=-\infty}^{+\infty}\Big(\Big(a_{k0}\widetilde{\varphi}_{2,k0}(x)\varphi_{2,k0}^T(x) - a_{k1}\widetilde{\varphi}_{2,k1}(x)\varphi_{2,k1}^T(x)\Big)B$$
$$-B\Big(a_{k0}\widetilde{\varphi}_{2,k0}(x)\varphi_{2,k0}^T(x) - a_{k1}\widetilde{\varphi}_{2,k1}(x)\varphi_{2,k1}^T(x)\Big)\Big)\Big\} = 0.$$

Multiplying by $(\widetilde{\Phi}^T(x,\lambda))^{-1}$, we arrive at (28). It follows from the estimate

$$\begin{aligned}&|a_{k0}\widetilde{\varphi}_{2,k0}(x)\varphi_{2,k0}^T(x) - a_{k1}\widetilde{\varphi}_{2,k1}(x)\varphi_{2,k1}^T(x)|\\ &\quad\le |\widetilde{\varphi}_{2,k0}(x)\varphi_{2,k0}^T(x) - \widetilde{\varphi}_{2,k1}(x)\varphi_{2,k1}^T(x)|\cdot|a_{k1}|\\ &\qquad+|\widetilde{\varphi}_{2,k1}(x)\varphi_{2,k1}^T(x)|\cdot|a_{k0} - a_{k1}| \le C|a_{k1}|\xi_k\end{aligned}$$

that the series in (29) converges uniformly. Lemma is proved.

Let us now study the solvability of the main equation. For this purpose we need the following assertion.

Lemma 13 *The following relation holds*

$$D^{\langle 2\rangle}(x,\lambda,\theta)=\widetilde{D}^{\langle 2\rangle}(x,\lambda,\theta)+\sum_{k=-\infty}^{+\infty}(\widetilde{D}_{k0}^{\langle 2\rangle}(x,\lambda)D_{k0}^{\langle 2\rangle}(x,\theta)a_{k0}$$
$$-\widetilde{D}_{k1}^{\langle 2\rangle}(x,\lambda)D_{k1}^{\langle 2\rangle}(x,\theta)a_{k1}), \tag{30}$$

and the series converges uniformly for $x\in\Omega_\varepsilon$ and λ on compact sets.

Proof According to (23) we have for $\lambda,\ \theta\in\Gamma_n^{\langle 2\rangle}$:

$$P(x,\lambda)-P(x,\theta)=\frac{1}{2\pi i}\int_{\Gamma_n^{\langle 1\rangle}}\left(\frac{1}{\lambda-\xi}-\frac{1}{\theta-\xi}\right)P(x,\xi)d\xi+J_n(x,\lambda,\theta),$$

where $J_n(x,\lambda,\theta)\to 0$ as $n\to\infty$ uniformly for $x\in\Omega_\varepsilon$ and $\lambda,\ \theta$ on compact sets. Therefore,

$$\frac{1}{\lambda-\theta}\Big(P^T(x,\lambda)-P^T(x,\theta)\Big)=\frac{1}{2\pi i}\int_{\Gamma_n^{\langle 1\rangle}}\frac{1}{(\lambda-\xi)(\xi-\theta)}P^T(x,\xi)d\xi$$
$$+J_n^1(x,\lambda,\theta). \tag{31}$$

Since $P(x,\xi)=\Phi(x,\xi)\widetilde{\Phi}^{-1}(x,\xi)=-\Phi(x,\lambda)B\widetilde{\Phi}^T(x,\xi)B$, it follows that

$$\widetilde{\varphi}_2^T(x,\lambda)P^T(x,\xi)B\varphi_2(x,\theta)=-\widetilde{\varphi}_2^T(x,\lambda)B\widetilde{\Phi}(x,\xi)B\Phi^T(x,\xi)B\varphi_2(x,\theta).$$

One has $\langle y,z\rangle=y^TBz$, and consequently,

$$\widetilde{\varphi}_2^T(x,\lambda)P^T(x,\xi)B\varphi_2(x,\theta)=\langle\widetilde{\varphi}_2(x,\lambda),\widetilde{\varphi}_2(x,\xi)\rangle\Big\langle\Phi_1(x,\xi),\varphi_2(x,\theta)\Big\rangle$$
$$-\Big\langle\widetilde{\varphi}_2(x,\lambda),\widetilde{\Phi}_1(x,\xi)\Big\rangle\langle\varphi_2(x,\xi),\varphi_2(x,\theta)\rangle. \tag{32}$$

Since $\Big\langle\Phi_1(x,\lambda),\varphi_2(x,\lambda)\Big\rangle\equiv 1,\Big\langle\varphi_2(x,\lambda),\varphi_2(x,\lambda)\Big\rangle\equiv 0$, we infer

$$\widetilde{\varphi}_2^T(x,\lambda)P^T(x,\lambda)B\varphi_2(x,\theta)=\Big\langle\varphi_2(x,\lambda),\varphi_2(x,\theta)\Big\rangle,$$
$$\widetilde{\varphi}_2^T(x,\lambda)P^T(x,\theta)B\varphi_2(x,\theta)=\Big\langle\widetilde{\varphi}_2(x,\lambda),\widetilde{\varphi}_2(x,\theta)\Big\rangle.$$

Multiplying (31) by $\widetilde{\varphi}_2^T(x,\lambda)$ from the left, and by $B\varphi_2(x,\theta)$ from the right, and using (32), we calculate

$$\frac{\Big\langle \varphi_2(x,\lambda), \varphi_2(x,\theta)\Big\rangle}{\lambda-\theta} - \frac{\Big\langle \widetilde{\varphi}_2(x,\lambda), \widetilde{\varphi}_2(x,\theta)\Big\rangle}{\lambda-\theta}$$
$$= \frac{1}{2\pi i}\int_{\Gamma_n^{\langle 1\rangle}} \Big(\frac{\langle \widetilde{\varphi}_2(x,\lambda), \widetilde{\varphi}_2(x,\xi)\rangle \Big\langle \Phi_1(x,\xi), \varphi_2(x,\theta)\Big\rangle}{(\lambda-\xi)(\xi-\theta)}$$
$$- \frac{\Big\langle \widetilde{\varphi}_2(x,\lambda), \widetilde{\Phi}_1(x,\xi)\Big\rangle \langle \varphi_2(x,\xi), \varphi_2(x,\theta)\rangle}{(\lambda-\xi)(\xi-\theta)}\Big) d\xi + J_n^2(x,\lambda,\theta).$$

By Lemma 4, $\Phi_1(x,\xi) = \varphi_1(x,\xi) + \mathfrak{M}(\xi)\varphi_2(x,\xi)$. This yields

$$\frac{\Big\langle \varphi_2(x,\lambda), \varphi_2(x,\theta)\Big\rangle}{\lambda-\theta} - \frac{\Big\langle \widetilde{\varphi}_2(x,\lambda), \widetilde{\varphi}_2(x,\theta)\Big\rangle}{\lambda-\theta}$$
$$= \frac{1}{2\pi i}\int_{\Gamma_n^{\langle 1\rangle}} \frac{\Big\langle \widetilde{\varphi}_2(x,\lambda), \widetilde{\varphi}_2(x,\xi)\Big\rangle \Big\langle \varphi_2(x,\xi), \varphi_2(x,\theta)\Big\rangle}{(\lambda-\xi)(\xi-\theta)} \Big(\mathfrak{M}(\xi) - \widetilde{\mathfrak{M}}(\xi)\Big) d\xi$$
$$+ J_n^2(x,\lambda,\theta),$$

since the integrals from analytic functions are equal to zero. Calculating the integral by residue's theorem and taking $n \to \infty$, we arrive at (30) firstly for $|\lambda| \geq h$, and by analytic continuation for all λ. Lemma is proved.

Taking $\lambda = \lambda_{ni}, \quad \theta = \lambda_{lj}$ in (30) and multiplying by a_{lj}, we obtain

$$P_{ni,lj}(x) - \widetilde{P}_{ni,lj}(x) - \sum_{k=-\infty}^{+\infty} \Big(\widetilde{P}_{ni,k0}(x) P_{k0,lj}(x) - \widetilde{P}_{ni,k1}(x) P_{k1,lj}(x) \Big) = 0. \quad (33)$$

Symmetrically, one has

$$P_{lj,ni}(x) - \widetilde{P}_{lj,ni}(x) - \sum_{k=-\infty}^{+\infty} \Big(P_{lj,k0}(x) \widetilde{P}_{k0,ni}(x) - P_{lj,k1}(x) \widetilde{P}_{k1,ni}(x) \Big) = 0. \quad (34)$$

It follows from (33)–(34) that

$$H_{ni,lj}(x) - \widetilde{H}_{ni,lj}(x) - \sum_{k=-\infty}^{+\infty} \Big(\widetilde{H}_{ni,k0}(x) H_{k0,lj}(x) - \widetilde{H}_{ni,k1}(x) H_{k1,lj}(x) \Big) = 0, \quad (35)$$
$$H_{ni,lj}(x) - \widetilde{H}_{ni,lj}(x) - \sum_{k=-\infty}^{+\infty} \Big(H_{ni,k0}(x) \widetilde{H}_{k0,lj}(x) - H_{ni,k1}(x) \widetilde{H}_{k1,lj}(x) \Big) = 0. \quad (36)$$

We rewrite relations (35) and (36) in the matrix form

$$H(x) - \widetilde{H}(x) - \widetilde{H}(x)H(x) = 0, \quad H(x) - \widetilde{H}(x) - H(x)\widetilde{H}(x) = 0$$

or $(I-\widetilde{H}(x))(I+H(x))=I,\ (I+H(x))(I-\widetilde{H}(x))=I$. Thus, we have proved the following assertion.

Theorem 2 *For each fixed* x $(x\neq\gamma_k,\ k=\overline{1,N})$, *the linear bounded operator* $I-\widetilde{H}(x)$, *acting from* **m** *to* **m**, *has the unique inverse operator, and the main equation* (27) *is uniquely solvable in* **m**.

The solution of Inverse Problem 1 can be constructed by the following algorithm.

Algorithm 1 Given the spectral data $\{\lambda_k,\ a_k\}_{k=-\infty}^{+\infty}$ of the problem L.

1. Choose a model boundary value problem $\tilde{L}$, for example, with the zero potential.
2. Construct $\widetilde{\Psi}^{\langle m\rangle}(x)$ and $\widetilde{H}(x)$.
3. Solving the linear main Eq. (27), find $\Psi^{\langle m\rangle}(x)$, and then calculate $\varphi_{2,kj}(x)$.
4. Construct $Q(x)$ by (28), and $\alpha=\widetilde{\alpha},\quad \beta=\widetilde{\beta}$.

4 Necessary and sufficient conditions for the solvability of the inverse problem

Theorem 3 *For numbers* $\{\lambda_k,\ a_k\}_{k=-\infty}^{+\infty}$, $a_k\neq 0$, $\lambda_k\neq\lambda_n$, $(k\neq n)$, *to be the spectral data for a certain problem* $L\in W$, *it is necessary and sufficient that the following conditions hold*

1. (*Asymptotics*): *There exists* $\widetilde{L}\in W$ *such that* (16) *holds*;
2. (*Condition S*): *For each fixed* $x\neq\gamma_k,\ k=\overline{1,N}$, *the linear bounded operator* $I-\tilde{H}(x)$ *has the unique inverse operator*;
3. $(B\text{æ}(x)-\text{æ}(x)B)|x-\gamma_k|^{-2Re\mu_k}\in L(w_{k+1/2})$, *where* $\text{æ}(x)$ *is constructed by* (29).

Under these conditions the potential $Q(x)$ *is constructed by* (28) *and* $\alpha=\widetilde{\alpha}$, $\beta=\widetilde{\beta}$.

The necessity part of the theorem was proved above. Let us prove the sufficiency. Let numbers $\{\lambda_k,\ a_k\}_{k=-\infty}^{+\infty}$ be given such that $a_k\neq 0$ and $\lambda_k\neq\lambda_n$, $(k\neq n)$. Let $\widetilde{L}=L(Q_\omega(x),\widetilde{Q}(x),0,\beta)\in W$ be chosen such that (16) holds. Let $\{\Psi_{ni}^{\langle m\rangle}(x)\}$ be the solution of the main equation (25). The following assertion is proved in [14].

Lemma 14 *Consider the equations*

$$(I+A_0)y_0=f_0,\quad (I+A)y=f,$$

in a Banach space $\mathfrak{B}$, *where* A_0, A *are linear bounded operators, acting from* $\mathfrak{B}$ *to* $\mathfrak{B}$, *and* I *is the identity operator. Suppose that there exists the linear bounded operator* $R_0:=(I+A_0)^{-1}$. *If* $\|A-A_0\|\leq(2\|R_0\|)^{-1}$, *then there exists the linear bounded operator* $R=(I+A)^{-1}$, *and* $\|R\|\leq 2\|R_0\|$, $\|R-R_0\|\leq 2\|R_0\|^2\|A-A_0\|$.

Lemma 15 *The following relations hold*

$$\begin{aligned}&\Psi_{ni}^{\langle m\rangle}(x)\in C(\Omega_\varepsilon),\quad |\Psi_{ni}^{\langle m\rangle}(x)|\leq C_\varepsilon,\\ &|\Psi_{ni}^{\langle m\rangle}(x)-\widetilde{\Psi}_{ni}^{\langle m\rangle}(x)|\leq C_\varepsilon\Lambda\theta_n,\end{aligned}\qquad(37)$$

$$\theta_n = \left(\sum_{k=-\infty}^{+\infty} \frac{1}{(1+|\lambda_n^0 - \lambda_k^0|)^2 (1+|\lambda_k^0|)^2} \right)^{1/2}, \; x \in \Omega_\varepsilon, \tag{38}$$

$$|(\Psi_{ni}^{\langle m \rangle}(x))'| \le C_\varepsilon (1+|\lambda_k^0|) \quad x \in \Omega_\varepsilon,$$
$$|(\Psi_{ni}^{\langle m \rangle}(x))' - (\widetilde{\Psi}_{ni}^{\langle m \rangle}(x))'| \le C_\varepsilon \Lambda, \quad x \in \Omega_\varepsilon. \tag{39}$$

Proof Using (26), we infer

$$|\widetilde{H}_{ni,kj}(x) - \widetilde{H}_{ni,kj}(x_0)| \le |\widetilde{H}'_{ni,kj}(\xi)|\,|x - x_0| \le C_\varepsilon |a_{k1}|\xi_k |x - x_0|, \quad x, x_0, \xi \in \Omega_\varepsilon,$$

hence

$$\|\widetilde{H}(x) - \widetilde{H}(x_0)\| \le C_\varepsilon \sum_{k=-\infty}^{+\infty} |a_{k1}|\xi_k |x - x_0| \le C_\varepsilon \Lambda |x - x_0|.$$

Choose $\delta_0 > 0$ such that $\|\widetilde{H}(x) - \widetilde{H}(x_0)\| \le (2\|\widetilde{R}(x_0)\|)^{-1}$ for $|x - x_0|\delta_0$, where $\widetilde{R}(x) = (I - \widetilde{H}(x))^{-1}$. By Lemma 14, $\|\widetilde{R}(x) - \widetilde{R}(x_0)\| \le |\widetilde{R}(x_0)\| C_\varepsilon |x - x_0|, \quad x \in \Omega_\varepsilon$. Therefore, the function $f(x) = \|\widetilde{R}(x)\|$ is continuous and bounded on Ω_ε. Then

$$\|\widetilde{R}(x)\| \le C_\varepsilon, \quad \|\widetilde{R}(x) - \widetilde{R}(x_0)\| \le C_\varepsilon |x - x_0|, \qquad x \in \Omega_\varepsilon,$$

where the constant C_ε does not depend on x, x_0. Since $\Psi^{\langle m \rangle}(x) = \widetilde{R}(x)\widetilde{\Psi}^{\langle m \rangle}(x)$, it follows that $\|\Psi^{\langle m \rangle}(x)\| \le \|\widetilde{R}(x)\|\,\|\widetilde{\Psi}^{\langle m \rangle}(x)\| \le C_\varepsilon$. Thus, (37) is proved.

Taking (25)–(26) into account, we obtain

$$|\Psi_{ni}^{\langle m \rangle}(x) - \widetilde{\Psi}_{ni}^{\langle m \rangle}(x)| \le \sum_{k=-\infty}^{+\infty} \sum_{j=0}^{1} |\widetilde{H}_{ni,kj}(x)|\,|\Psi_{kj}^{\langle m \rangle}(x)| \le C_\varepsilon \sum_{k=-\infty}^{+\infty} \frac{|a_{k1}|\xi_k}{1+|\lambda_k^0 - \lambda_n^0|},$$

and consequently,

$$|\Psi_{ni}^{\langle m \rangle}(x) - \widetilde{\Psi}_{ni}^{\langle m \rangle}(x)| \le C_\varepsilon \theta_n \sum_{k=-\infty}^{+\infty} \Big(|a_{k1}|\xi_k (1+|\lambda_k^0|) \Big)^2,$$

i.e. (38) holds. Estimates (39) are proved similarly. Lemma is proved.

Construct the functions $\varphi_{2,ni}(x) = \Big(\varphi_{12,ni}(x),\ \varphi_{22,ni}(x)\Big)^T$ by the formula

$$\varphi_{m2,n0}(x) = \Psi_{n0}^{\langle m \rangle}(x)\xi_n + \Psi_{n1}^{\langle m \rangle}(x), \quad \varphi_{m2,n1}(x) = \Psi_{n1}^{\langle m \rangle}(x), \qquad m = 1, 2. \tag{40}$$

It follows from (40) and Lemma 15 that

$$|\varphi_{2,ni}^{(m)}(x)| \le C_\varepsilon (1+|\lambda_n^0|)^m, \quad |\varphi_{2,n0}^{(m)}(x) - \varphi_{2,n1}^{(m)}(x)| \le C_\varepsilon \xi_n (1+|\lambda_n^0|)^m, \quad m = 0, 1, \tag{41}$$

$$|\widetilde{\varphi}_{2,ni}(x) - \varphi_{2,ni}(x)| \le C_\varepsilon \Lambda \theta_n, \quad |\widetilde{\varphi}'_{2,ni}(x) - \varphi'_{2,ni}(x)| \le C_\varepsilon \Lambda. \tag{42}$$

Lemma 16 *The function* $Q(x)$, *constructed by* (28), *is absolutely continuous on* $[0, \pi]$.

Proof In view of (41), the series in (29) converges uniformly on Ω_ε. According to Lemma 15, the functions $\varphi_{2,ni}(x)$ are continuous, and consequently, $æ(x)$ is continuous on Ω_ε. One has

$$æ(x) = A_1(x) + A_2(x), \quad A_1(x) = \sum_{k=-\infty}^{+\infty} (a_{k0} - a_{k1})\widetilde{\varphi}_{2,k0}(x)\varphi_{2,k0}^T(x),$$

$$A_2(x) = \sum_{k=-\infty}^{+\infty} a_{k1}\Big(\widetilde{\varphi}_{2,k0}(x)\varphi_{2,k0}^T(x) - \widetilde{\varphi}_{2,k1}(x)\varphi_{2,k1}^T(x)\Big).$$

Taking (41) and (17) into account, we infer $|(a_{k0} - a_{k1})\widetilde{\varphi}_{2,k0}(x)\varphi_{2,k0}^T(x)| \le |a_{k1}|\xi_k C_\varepsilon(1 + |\lambda_k^0|)$. This yields that the series for $A_1(x)$ converges uniformly on Ω_ε and $A_1'(x) \in L(0, \pi)$. Since

$$\begin{aligned}(\widetilde{\varphi}_{2,k0}(x)\varphi_{2,k0}^T(x) - \widetilde{\varphi}_{2,k1}(x)\varphi_{2,k1}^T(x))' &= (\widetilde{\varphi}'_{2,k0}(x) - \widetilde{\varphi}'_{2,k1}(x))\varphi_{2,k0}^T(x)\\ &+ \widetilde{\varphi}'_{2,k1}(x)(\varphi_{2,k0}(x) - \varphi_{2,k1}(x))^T + \widetilde{\varphi}_{2,k0}(x)(\varphi'_{2,k0}(x) - \varphi'_{2,k1}(x))^T\\ &+ (\widetilde{\varphi}_{2,k0}(x) - \widetilde{\varphi}_{2,k1}(x))(\varphi_{2,k1}^T(x))',\end{aligned}$$

it follows from (41) and (17) that $|a_{k1}(\widetilde{\varphi}_{2,k0}(x)\varphi_{2,k0}^T(x) - \widetilde{\varphi}_{2,k1}(x)\varphi_{2,k1}^T(x))'| \le C_\varepsilon|a_{k1}|\xi_k(1 + |\lambda_k^0|)$. This yields $A_2'(x) \in L(0, \pi)$. Thus, $æ(x)$ is absolutely continuous on $[0, \pi]$ and $æ'(x) \in L(0, \pi)$. Lemma is proved.

Let us now show that the given numbers $\{\lambda_k\}_{k=-\infty}^{+\infty}$ are eigenvalues of the constructed boundary value problem $L(Q_\omega(x), Q(x), 0, \beta)$.

Lemma 17 *The following relations hold*

$$\ell\varphi_{2,kj}(x) = \lambda_{kj}\varphi_{2,kj}(x), \quad \ell\Phi_j(x, \lambda) = \lambda\Phi_j(x, \lambda), \tag{43}$$

$$\Phi_2(0, \lambda) = V_2(0), \quad V_1^T(0)\Phi_1(0, \lambda) = 1, \quad V_1^T(\beta)\Phi_1(\pi, \lambda) = 0, \quad \Delta_{12}(\lambda_k)=0. \tag{44}$$

Proof 1. We construct $\Phi_j(x, \lambda)$ by (22). In view of (41)–(42), the series in (22) converges uniformly in Ω_ε. Moreover, differentiating (22) and taking (21) into account, we obtain

$$\begin{aligned}\Phi'_j(x, \lambda) = \widetilde{\Phi}'_j(x, \lambda) + \sum_{k=-\infty}^{+\infty} \Big(\widetilde{D}_{k0}^{\langle j\rangle}(x, \lambda)a_{k0}\varphi'_{2,k0}(x) - \widetilde{D}_{k1}^{\langle j\rangle}(x, \lambda)a_{k1}\varphi'_{2,k1}(x)\Big)\\ - \sum_{k=-\infty}^{+\infty} \Big(\widetilde{\Phi}_j^T(x, \lambda)\widetilde{\varphi}_{2,k0}(x)a_{k0}\varphi_{2,k0}(x) - \widetilde{\Phi}_j^T(x, \lambda)\widetilde{\varphi}_{2,k1}(x)a_{k1}\varphi_{2,k1}(x)\Big),\end{aligned}$$

and consequently,

$$(\Phi'_j(x,\lambda))^T = (\widetilde{\Phi}'_j(x,\lambda))^T - \widetilde{\Phi}^T_j(x,\lambda)æ(x) + \sum_{k=-\infty}^{+\infty}\Big(\widetilde{D}^{\langle j\rangle}_{k0}(x,\lambda)a_{k0}(\varphi'_{2,k0}(x))^T - \widetilde{D}^{\langle j\rangle}_{k1}(x,\lambda)a_{k1}(\varphi'_{2,k1}(x))^T\Big).$$

This yields

$$\Big(B\Phi'_j(x,\lambda)+Q(x)\Phi_j(x,\lambda)\Big)^T = (\widetilde{\Phi}'_j(x,\lambda))^T + \widetilde{\Phi}^T_j(x,\lambda)\Big(Q(x)+æ(x)B\Big) + \sum_{k=-\infty}^{+\infty}\Big(\widetilde{D}^{\langle j\rangle}_{k0}(x,\lambda)a_{k0}\Big(B\varphi'_{2,k0}(x)+Q(x)\varphi_{2,k0}(x)\Big)^T - \widetilde{D}^{\langle j\rangle}_{k1}(x,\lambda)a_{k1}\Big(B\varphi'_{2,k1}(x)+Q(x)\varphi_{2,k1}(x)\Big)^T\Big).$$

Since $Q(x)=\widetilde{Q}(x)+Bæ(x)-æ(x)B$, $\widetilde{\Phi}^T_j(x,\lambda)B\widetilde{\varphi}_{2,ni}(x)=\widetilde{D}^{\langle j\rangle}_{ni}(x,\lambda)(\lambda-\lambda_{ni})$, it follows that

$$\widetilde{\Phi}^T_j(x,\lambda)Bæ(x) = \sum_{k=-\infty}^{+\infty}\Big(\widetilde{D}^{\langle j\rangle}_{k0}(x,\lambda)(\lambda-\lambda_{k0})a_{k0}\varphi^T_{2,k0}(x) - \widetilde{D}^{\langle j\rangle}_{k1}(x,\lambda)(\lambda-\lambda_{k1})a_{k1}\varphi^T_{2,k1}(x)\Big),$$

hence

$$\Big(B\Phi'_j(x,\lambda)+Q(x)\Phi_j(x,\lambda)\Big)^T = \Big(\widetilde{\Phi}'_j(x,\lambda)+\widetilde{Q}(x)\widetilde{\Phi}_j(x,\lambda)\Big)^T + \sum_{k=-\infty}^{+\infty}\Big(\widetilde{D}^{\langle j\rangle}_{k0}(x,\lambda)a_{k0}\Big(B\varphi'_{2,k0}(x)+Q(x)\varphi_{2,k0}(x)+(\lambda-\lambda_{k0})\varphi_{2,k0}(x)\Big)^T - \widetilde{D}^{\langle j\rangle}_{k1}(x,\lambda)a_{k1}\Big(B\varphi'_{2,k1}(x)+Q(x)\varphi_{2,k1}(x)+(\lambda-\lambda_{k1})\varphi_{2,k1}(x)\Big)^T\Big).$$

Taking (22) into account, we calculate

$$\ell\Phi_j(x,\lambda)-\lambda\Phi_j(x,\lambda) = \sum_{k=-\infty}^{+\infty}\Big(\widetilde{D}^{\langle j\rangle}_{k0}(x,\lambda)a_{k0}\Big(\ell\varphi_{2,k0}(x)-\lambda_{k0}\varphi_{2,k0}(x)\Big) - \widetilde{D}^{\langle j\rangle}_{k1}(x,\lambda)a_{k1}\Big(\ell\varphi_{2,k1}(x)-\lambda_{k1}\varphi_{2,k1}(x)\Big)\Big). \tag{45}$$

Consider (45) for $j = 2$ and $\lambda = \lambda_{ni}$:

$$z_{ni}(x) - \sum_{k=-\infty}^{+\infty} \Big(\widetilde{P}_{ni,k0}(x) z_{k0}(x) - \widetilde{P}_{ni,k1}(x) z_{k1}(x)\Big) = 0,$$

where $z_{ni}(x) = \ell\varphi_{2,ni}(x) - \lambda_{ni}\varphi_{2,ni}(x)$, or

$$Z_{ni}^{\langle m\rangle}(x) - \sum_{k,j} \widetilde{H}_{ni,kj}(x) Z_{kj}^{\langle m\rangle}(x) = 0, \tag{46}$$

where $Z_{n0}^{\langle m\rangle} = \Big(z_{m,n0}(x) - z_{m,n1}(x)\Big)\chi_n$, $Z_{n1}^{\langle m\rangle}(x) = z_{m,n1}(x)$, $z_{ni}(x) = (z_{1,ni}(x),\ z_{2,ni}(x))^T$, $m = 1, 2$. Taking (41) into account, we get $|Z_{ni}^{\langle m\rangle}(x)| \le C_\varepsilon(1 + |\lambda_n^0|)$. Using (46) and (26), we infer

$$|Z_{ni}^{\langle m\rangle}(x)| \le C_\varepsilon \sum_{k=-\infty}^{+\infty} \frac{|a_{k1}|\xi_k(|\lambda_k^0| + 1)}{1 + |\lambda_n^0 - \lambda_k^0|} \le C_\varepsilon \Lambda,$$

and consequently, $\{Z_{ni}^{\langle m\rangle}(x)\} \in \mathfrak{m}$. Equation (46) has only trivial solution, i.e. $Z_{ni}^{\langle m\rangle}(x) = 0$, hence $\ell\varphi_{2,ni}(x) - \lambda_{ni}\varphi_{2,ni}(x) = 0$. The second relation (43) follows now from (45).

2. Since $\widetilde{D}_{kj}^{\langle 2\rangle}(x, \lambda) = \dfrac{1}{\lambda - \lambda_{kj}} \det(\widetilde{\varphi}_2(x, \lambda), \widetilde{\varphi}_{2,kj}(x))$, it follows that

$$\widetilde{D}_{kj}^{\langle 2\rangle}(0, \lambda) = \frac{1}{\lambda - \lambda_{kj}} \det(V_2(0), V_2(0)) = 0.$$

Using (22), we find $\Phi_2(0, \lambda) = \widetilde{\Phi}_2(0, \lambda) = V_2(0)$. Furthermore, taking $j = 1$, $x = 0$ in (22) and multiplying by $V_1^T(0)$, we calculate

$$V_1^T(0)\Phi_1(0, \lambda) = V_1^T(0)\widetilde{\Phi}_1(0, \lambda)$$
$$+ \sum_{k=-\infty}^{+\infty} \Big(\widetilde{D}_{k0}^{\langle 1\rangle}(0, \lambda) a_{k0} V_1^T(0)\varphi_{2,k0}(0) - \widetilde{D}_{k1}^{\langle 1\rangle}(0, \lambda) a_{k1} V_1^T(0)\varphi_{2,k1}(0)\Big).$$

Since $V_1^T(0)\varphi_{2,kj}(0) = V_1^T(0)V_2(0) = 0$, one gets $V_1^T(0)\Phi_1(0, \lambda) = V_1^T(0)\widetilde{\Phi}_1(0, \lambda) = 1$. Thus, $\Phi_2(x, \lambda) = \varphi_2(x, \lambda)$ is a solution of (1) with the initial condition $\varphi_2(0, \lambda) = V_2(0)$. Then $\Delta_{12}(\lambda) = V_1^T(\beta)\varphi_2(\pi, \lambda)$. Let us show that $\Delta_{12}(\lambda_{n0}) = 0$, i.e. $\{\lambda_n\}_{n=-\infty}^{+\infty}$ are eigenvalues of L. For this purpose we take $j = 2$, $x = \pi$ in (22) and multiply by $V_1^T(\beta)$. This yields

$$\Delta_{12}(\lambda) = \widetilde{\Delta}_{12}(\lambda) + \sum_{k=-\infty}^{+\infty} \Big(\widetilde{D}_{k0}^{\langle 2\rangle}(\pi, \lambda) a_{k0} \Delta_{12}(\lambda_{k0}) - \widetilde{D}_{k1}^{\langle 2\rangle}(\pi, \lambda) a_{k1} \Delta_{12}(\lambda_{k1})\Big),$$

and consequently,

$$\Delta_{12}(\lambda_{ni}) = \widetilde{\Delta}_{12}(\lambda_{ni}) + \sum_{k=-\infty}^{+\infty} \Big(\widetilde{P}_{ni,k0}(\pi)\Delta_{12}(\lambda_{k0}) - \widetilde{P}_{ni,k1}(\pi)\Delta_{12}(\lambda_{k1}) \Big). \quad (47)$$

By definition $\Delta(\lambda) = V^T(\beta)\varphi(\pi,\lambda)$, then $\det \Delta(\lambda) \equiv 1$, or

$$\Delta_{11}(\lambda)\Delta_{22}(\lambda) - \Delta_{12}(\lambda)\Delta_{21}(\lambda) \equiv 1. \quad (48)$$

Furthermore, $\langle \widetilde{\varphi}_2(\pi,\lambda), \widetilde{\varphi}_{2,kj}(\pi) \rangle = \widetilde{\varphi}_2^T(\pi,\lambda) B \widetilde{\varphi}_{2,kj}(\pi)$. Since $V(\beta)V^T(\beta) = I$, it follows that $\Big\langle \widetilde{\varphi}_2(\pi,\lambda), \widetilde{\varphi}_{2,kj}(\pi) \Big\rangle = \Big(V^T(\beta)\varphi_2(\pi,\lambda) \Big)^T B V^T(\beta)\varphi_2(\pi,\lambda_{kj})$, and consequently,

$$\Big\langle \widetilde{\varphi}_2(\pi,\lambda), \widetilde{\varphi}_{2,kj}(\pi) \Big\rangle = \widetilde{\Delta}_{12}(\lambda)\widetilde{\Delta}_{22}(\lambda_{kj}) - \widetilde{\Delta}_{12}(\lambda_{kj})\widetilde{\Delta}_{22}(\lambda).$$

Thus,

$$\widetilde{D}_{kj}^{\langle 2 \rangle}(\pi,\lambda) = \frac{1}{\lambda - \lambda_{kj}} \Big(\widetilde{\Delta}_{12}(\lambda)\widetilde{\Delta}_{22}(\lambda_{kj}) - \widetilde{\Delta}_{12}(\lambda_{kj})\widetilde{\Delta}_{22}(\lambda) \Big). \quad (49)$$

From (49) for $n \neq k$, we find

$$\widetilde{P}_{n1,k1}(\pi) = \frac{a_{k1}}{\lambda_{n1} - \lambda_{k1}} \Big(\widetilde{\Delta}_{12}(\lambda_{n1})\widetilde{\Delta}_{22}(\lambda_{k1}) - \widetilde{\Delta}_{12}(\lambda_{k1})\widetilde{\Delta}_{22}(\lambda_{n1}) \Big) = 0.$$

For $n = k$, one has $\widetilde{P}_{n1,n1}(\pi) = a_{n1}\dot{\widetilde{\Delta}}_{12}(\lambda_{n1})\widetilde{\Delta}_{22}(\lambda_{n1})$, where $\dot{\widetilde{\Delta}}_{12}(\lambda) := \frac{d}{d\lambda}\widetilde{\Delta}_{12}(\lambda)$. Since $a_{n1} = \mathrm{Res}_{\lambda=\lambda_{n1}} \widetilde{\mathfrak{M}}(\lambda) = -(\widetilde{\Delta}_{11}(\lambda_{n1}))(\dot{\widetilde{\Delta}}_{12}(\lambda_{n1}))^{-1}$, it follows that $\widetilde{P}_{n1,n1}(\pi) = -\widetilde{\Delta}_{11}(\lambda_{n1})\widetilde{\Delta}_{22}(\lambda_{n1})$. From (48) for $\lambda = \lambda_{n1}$ we infer $\widetilde{P}_{n1,n1}(\pi) = -1$. Thus,

$$\widetilde{P}_{n1,k1}(\pi) = -\delta_{nk}, \quad (50)$$

where δ_{nk} is the Kronecker symbol. From (49) for $\lambda_{n0} \neq \lambda_{k1}$ one has

$$\begin{aligned} \widetilde{P}_{n0,k1}(\pi) &= \frac{a_{k1}}{\lambda_{n0} - \lambda_{k1}} \Big(\widetilde{\Delta}_{12}(\lambda_{n0})\widetilde{\Delta}_{22}(\lambda_{k1}) - \widetilde{\Delta}_{12}(\lambda_{k1})\widetilde{\Delta}_{22}(\lambda_{n0}) \Big) \\ &= a_{k1}\frac{\widetilde{\Delta}_{12}(\lambda_{n0})\widetilde{\Delta}_{22}(\lambda_{k1})}{\lambda_{n0} - \lambda_{k1}}. \end{aligned}$$

By virtue of (48), $\widetilde{\Delta}_{22}(\lambda_{k1}) = (\widetilde{\Delta}_{11}(\lambda_{k1}))^{-1}$. Moreover, $\widetilde{P}_{n0,k1}(\pi) = -1$ for $\lambda_{n0} = \lambda_{k1}$. Thus,

$$\widetilde{P}_{n0,k1}(\pi) = -1 \text{ for } \lambda_{n0} = \lambda_{k1}, \quad \widetilde{P}_{n0,k1}(\pi) = -\frac{\widetilde{\Delta}_{12}(\lambda_{n0})}{\dot{\widetilde{\Delta}}_{12}(\lambda_{k1})(\lambda_{n0} - \lambda_{k1})} \quad \text{for } \lambda_{n0} \neq \lambda_{k1}. \quad (51)$$

Consider the function $Z(\lambda) = (\Delta_{12}(\lambda) - \widetilde{\Delta}_{12}(\lambda))(\widetilde{\Delta}_{12}(\lambda))^{-1}$. Since $Q_\omega(x) = \widetilde{Q}_\omega(x), \alpha = \widetilde{\alpha} = 0, \beta = \widetilde{\beta} = 0$, it follows that $\Delta_{12}(\lambda) - \widetilde{\Delta}_{12}(\lambda) = O\left(e^{\pi|Im\lambda|}|\lambda|^{-\nu}\right)$, hence $|Z(\lambda)| \le C_\delta|\lambda|^{-\nu}$ for $\lambda \in \widetilde{G}_\delta$. In particular, this yields $\int_{|\xi|=\widetilde{R}_n} \frac{Z(\xi)}{\xi-\lambda} d\xi \to 0$ as $n \to \infty$. Calculating the integral by residue's theorem, we obtain for $n \to \infty$,

$$\Delta_{12}(\lambda) = \widetilde{\Delta}_{12}(\lambda) + \sum_{k=-\infty}^{+\infty} \frac{\widetilde{\Delta}_{12}(\lambda)}{(\lambda - \lambda_{k1})\dot{\widetilde{\Delta}}_{12}(\lambda_{k1})} \Delta_{12}(\lambda_{k1}).$$

Putting $\lambda = \lambda_{n0}$ and taking (51) into account, we infer

$$\Delta_{12}(\lambda_{n0}) = \widetilde{\Delta}_{12}(\lambda_{n0}) - \sum_{k=-\infty}^{+\infty} \widetilde{P}_{n0,k1}(\pi)\Delta_{12}(\lambda_{k1}).$$

Together with (47) and (50) this yields

$$\sum_{k=-\infty}^{+\infty} \widetilde{P}_{ni,k0}(\pi)\Delta_{12}(\lambda_{k0}) = 0,$$

and consequently, $\Delta_{12}(\lambda_{n0}) = 0$. Now we take $j = 1, x = \pi$ in (22) and multiply by $V_1^T(\beta)$. Then

$$V_1^T(\beta)\Phi_1(\pi, \lambda) = V_1^T(\beta)\widetilde{\Phi}_1(\pi, \lambda) + \sum_{k=-\infty}^{+\infty} \left(\widetilde{D}_{k0}^{\langle 1\rangle}(\pi, \lambda)a_{k0}\Delta_{12}(\lambda_0) - \widetilde{D}_{k1}^{\langle 1\rangle}(\pi, \lambda)a_{k1}\Delta_{12}(\lambda_1)\right).$$

Furthermore, $\widetilde{D}_{k1}^{\langle 1\rangle}(\pi, \lambda) = \dfrac{1}{\lambda - \lambda_{k1}}(V^T(\beta)\widetilde{\Phi}_1(\pi, \lambda))^T BV^T(\beta)\widetilde{\varphi}_2(\pi, \lambda_{k1})$ or

$$\widetilde{D}_{k1}^{\langle 1\rangle}(\pi, \lambda) = \frac{1}{\lambda - \lambda_{k1}}\left(V_1^T(\beta)\widetilde{\Phi}_1(\pi, \lambda)\widetilde{\Delta}_{22}(\lambda_{k1}) - V_2^T(\beta)\widetilde{\Phi}_1(\pi, \lambda)\widetilde{\Delta}_{12}(\lambda_{k1})\right),$$

hence, $\widetilde{D}_{k1}^{\langle 1\rangle}(\pi, \lambda) = 0$. Thus, $V_1^T(\beta)\Phi_1(\pi, \lambda) = V_1^T(\beta)\widetilde{\Phi}_1(\pi, \lambda) = 0$, and all relations (44) are valid. Lemma 17 is proved.

It remains to show that $a_k = \mathrm{Res}_{\lambda=\lambda_k}\mathfrak{M}(\lambda)$, where $\mathfrak{M}(\lambda) = V_2^T(0)\Phi_1(0, \lambda)$. Taking in (22) $j = 1, x = 0$ and multiplying by $V_2^T(0)$, we obtain

$$\mathfrak{M}(\lambda) = \widetilde{\mathfrak{M}}(\lambda) + \sum_{k=-\infty}^{+\infty} \Big(\widetilde{D}_{k0}^{\langle 1\rangle}(0, \lambda)a_{k0}V_2^T(0)\varphi_{2,k0}(0) - \widetilde{D}_{k1}^{\langle 1\rangle}(0, \lambda)a_{k1}V_2^T(0)\varphi_{2,k1}(0)\Big). \tag{52}$$

Using Lemma 17, we calculate $V_2^T(0)\varphi_{2,kj}(0) = V_2^T(0)V_2(0) = 1$. Since

$$\widetilde{D}_{kj}^{\langle 1\rangle}(0,\lambda) = \frac{1}{\lambda-\lambda_{kj}}\widetilde{\Phi}_1^T(0,\lambda)B\widetilde{\varphi}_{2,kj}(0), \quad \widetilde{\varphi}_{2,kj}(0) = V_2(0),$$

it follows that $\widetilde{D}_{kj}^{\langle 1\rangle}(0,\lambda) = \dfrac{1}{\lambda-\lambda_{kj}}$. Thus, (52) takes the form

$$\mathfrak{M}(\lambda) = \widetilde{\mathfrak{M}}(\lambda) + \sum_{k=-\infty}^{+\infty}\left(\frac{a_{k0}}{\lambda-\lambda_{k0}} - \frac{a_{k1}}{\lambda-\lambda_{k1}}\right).$$

With the help of Lemma 6, this yields $a_k = \text{Res}_{\lambda=\lambda_k}\mathfrak{M}(\lambda)$. Theorem 3 is proved.

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