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A few results on Mourre theory in a two-Hilbert spaces setting

S. Richard · R. Tiedra de Aldecoa

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Abstract We introduce a natural framework for dealing with Mourre theory in an abstract two-Hilbert spaces setting. In particular a Mourre estimate for a pair of self-adjoint operators (H, A) is deduced from a similar estimate for a pair of self-adjoint operators (H_0, A_0) acting in an auxiliary Hilbert space. A new criterion for the completeness of the wave operators in a two-Hilbert spaces setting is also presented.

Keywords Mourre theory · Two-Hilbert spaces · Conjugate operator · Scattering theory

Mathematics Subject Classification (2000) 81Q10 · 47A40 · 46N50 · 47B25 · 47B47

1 Introduction

It is commonly accepted that Mourre theory is a very powerful tool in spectral and scattering theory for self-adjoint operators. In particular, it naturally leads to limiting

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S. Richard (✉)
CNRS, UMR5208, Institut Camille Jordan, Université Lyon 1, Université de Lyon,
43 blvd du 11 novembre 1918, 69622 Villeurbanne-Cedex, France
e-mail: richard@math.univ-lyon1.fr

R. Tiedra de Aldecoa
Facultad de Matemáticas, Pontificia Universidad Católica de Chile,
Av. Vicuña Mackenna, 4860 Santiago, Chile
e-mail: rtiedra@mat.puc.cl

absorption principles which are essential when studying the absolutely continuous part of self-adjoint operators. Since the pioneering work of Mourre [11], a lot of improvements and extensions have been proposed, and the theory has led to numerous applications. However, in most of the corresponding works, Mourre theory is presented in a one-Hilbert space setting and perturbative arguments are used within this framework. In this paper, we propose to extend the theory to a two-Hilbert spaces setting and present some results in that direction. In particular, we show how a Mourre estimate can be deduced for a pair of self-adjoint operators (H, A) in a Hilbert space $\mathcal{H}$ from a similar estimate for a pair of self-adjoint operators (H_0, A_0) in a auxiliary Hilbert space $\mathcal{H}_0$.

The main idea of Mourre for obtaining results on the spectrum $\sigma(H)$ of a self-adjoint operator H in a Hilbert space $\mathcal{H}$ is to find an auxiliary self-adjoint operator A in $\mathcal{H}$ such that the commutator $[iH, A]$ is positive when localised in the spectrum of H. Namely, one looks for a subset $I \subset \sigma(H)$, a number $a \equiv a(I) > 0$ and a compact operator $K \equiv K(I)$ in $\mathcal{H}$ such that

$$E^H(I)[iH, A]E^H(I) \geq aE^H(I) + K, \tag{1.1}$$

where $E^H(I)$ is the spectral projection of H on I. Such an estimate is commonly called a Mourre estimate. In general, this positivity condition is obtained via perturbative technics. Typically, H is a perturbation of a simpler operator H_0 in $\mathcal{H}$ for which the commutator $[iH_0, A]$ is easily computable and the positivity condition easily verifiable. In such a case, the commutator of the formal difference $H - H_0$ with A can be considered as a small perturbation of $[iH_0, A]$, and one can still infer the necessary positivity of $[iH, A]$.

In many other situations one faces the problem that H is not the perturbation of any simpler operator H_0 in $\mathcal{H}$. For example, if H is the Laplace–Beltrami operator on a non-compact manifold, there is no candidate for a simpler operator H_0 in the same Hilbert space. Similarly, for scattering theory with obstacles, one is naturally led to consider two different Hilbert spaces with one operator living in each of these spaces. Alternatively, for multichannel scattering systems, there might exist more than one single candidate for H_0, and one has to take this multiplicity into account. In these situations, it is therefore unclear from the very beginning wether one can find a suitable conjugate operator A for H and how some positivity of $[iH, A]$ can be deduced from a hypothetic similar condition involving a simpler operator H_0. Of course, these interrogations have found positive answers in various situations. Nevertheless, it does not seem to the authors that any general framework has yet been proposed.

The starting point for our investigations is the scattering theory in the two-Hilbert spaces setting. In this setup, one has a self-adjoint operator H in a Hilbert space $\mathcal{H}$, and one looks for a simpler self-adjoint operator H_0 in an auxiliary Hilbert space $\mathcal{H}_0$ and a bounded operator $J : \mathcal{H}_0 \to \mathcal{H}$ such that the strong limits

$$\operatorname{s-lim}_{t\to\pm\infty} \mathrm{e}^{itH} J \mathrm{e}^{-itH_0} \varphi$$

exist for suitable vectors $\varphi \in \mathcal{H}_0$. If such limits exist for enough $\varphi \in \mathcal{H}_0$, then some information on the spectral nature of H can be inferred from similar information on the spectrum of H_0. We refer to the books [4,14] for general presentations of scattering theory in the two-Hilbert spaces setting. Therefore, the following question naturally arises: if A_0 is a conjugate operator for H_0 such that (1.1) holds with (H_0, A_0) instead of (H, A), can we define a conjugate operator A for H such that (1.1) holds? Under suitable conditions, the answer is "yes", and its justification is the content of this paper. In fact, we present a general framework in which a Mourre estimate for a pair (H, A) can be deduced from a similar Mourre estimate for a pair (H_0, A_0). In that framework, we suppose the operators A_0 and A given *a priori*, and then exhibit sufficient conditions on the formal commutators $[iH, A]$ and $[iH_0, A_0]$ guaranteeing the existence of a Mourre estimate for (H, A) if a Mourre estimate for (H_0, A_0) is verified (see the assumptions of Theorem 3.1). We also show how a conjugate operator A for H can be constructed from a conjugate operator A_0 for H_0.

Let us finally sketch the organisation of the paper. In Sect. 2, we recall a few definitions (borrowed from [2, Chap. 7]) in relation with Mourre theory in the usual one-Hilbert space setting. In Sect. 3, we state our main result, Theorem 3.1, on the obtention of a Mourre estimate for (H, A) from a similar estimate for (H_0, A_0). A complementary result on higher order regularity of H with respect to A is also presented. In the second part of Sect. 3, we show how the assumptions of Theorem 3.1 can be checked for short-range type and long-range type perturbations (note that the distinction between short-range type and long-range type perturbations is more subtle here, since H_0 and H do not live in the same Hilbert space). We also show how a natural candidate for A can be constructed from A_0. In Sect. 4, we illustrate our results with the simple example of one-dimensional Schrödinger operator with steplike potential. A more challenging application on manifolds will be presented in [13] (many other applications such as curved quantum waveguides, anisotropic Schrödinger operators, spin models, *etc.* are also conceivable). Finally, in Sect. 5 we prove an auxiliary result on the completeness of the wave operators in the two-Hilbert spaces setting without assuming that the initial sets of the wave operators are equal to the subspace $\mathcal{H}_{ac}(H_0)$ of absolute continuity of H_0 (in [4,14], only that case is presented and this situation is sometimes too restrictive as will be shown for example in [13]).

2 Mourre theory in the one-Hilbert space setting

In this section we recall some definitions related to Mourre theory, such as the regularity condition of H with respect to A, providing a precise meaning to the commutators mentioned in the Introduction. We refer to [2, Sec. 7.2] for more information and details.

Let us consider a Hilbert space $\mathcal{H}$ with scalar product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and norm $\| \cdot \|_{\mathcal{H}}$. Let also H and A be two self-adjoint operators in $\mathcal{H}$, with domains $\mathcal{D}(H)$ and $\mathcal{D}(A)$. The spectrum of H is denoted by $\sigma(H)$ and its spectral measure by $E^H(\cdot)$. For shortness, we also use the notation $E^H(\lambda; \varepsilon) := E^H(\lambda - \varepsilon, \lambda + \varepsilon)$ for all $\lambda \in \mathbb{R}$ and $\varepsilon > 0$.

The operator H is said to be of class $C^1(A)$ if there exists $z \in \mathbb{C} \setminus \sigma(H)$ such that the map

$$\mathbb{R} \ni t \mapsto e^{-itA}(H - z)^{-1} e^{itA} \in \mathscr{B}(\mathcal{H}) \tag{2.1}$$

is strongly of class C^1 in $\mathcal{H}$. In such a case, the set $\mathcal{D}(H) \cap \mathcal{D}(A)$ is a core for H and the quadratic form $\mathcal{D}(H) \cap \mathcal{D}(A) \ni \varphi \mapsto \langle H\varphi, A\varphi\rangle_{\mathcal{H}} - \langle A\varphi, H\varphi\rangle_{\mathcal{H}}$ is continuous in the topology of $\mathcal{D}(H)$. This form extends then uniquely to a continuous quadratic form $[H, A]$ on $\mathcal{D}(H)$, which can be identified with a continuous operator from $\mathcal{D}(H)$ to the adjoint space $\mathcal{D}(H)^*$. Furthermore, the following equality holds:

$$\big[A, (H - z)^{-1}\big] = (H - z)^{-1}[H, A](H - z)^{-1}.$$

This $C^1(A)$-regularity of H with respect to A is the basic ingredient for any investigation in Mourre theory. It is also at the root of the proof of the Virial Theorem (see for example [2, Prop. 7.2.10] or [7]).

Note that if H is of class $C^1(A)$ and if $\eta \in C_c^\infty(\mathbb{R})$ (the set of smooth functions on $\mathbb{R}$ with compact support), then the quadratic form $\mathcal{D}(A) \ni \varphi \mapsto \langle \bar\eta(H)\varphi, A\varphi\rangle_{\mathcal{H}} - \langle A\varphi, \eta(H)\varphi\rangle_{\mathcal{H}}$ also extends uniquely to a continuous quadratic form $[\eta(H), A]$ on $\mathcal{H}$, identified with a bounded operator on $\mathcal{H}$.

We now recall the definition of two very useful functions in Mourre theory described in [2, Sec. 7.2]. For that purpose, we use the following notations: for two bounded operators S and T in a common Hilbert space we write $S \approx T$ if $S - T$ is compact, and we write $S \lesssim T$ if there exists a compact operator K such that $S \le T + K$. If H is of class $C^1(A)$ and $\lambda \in \mathbb{R}$ we set

$$\varrho_H^A(\lambda) := \sup\big\{a \in \mathbb{R} \mid \exists \varepsilon > 0 \text{ s.t. } a\,E^H(\lambda;\varepsilon) \le E^H(\lambda;\varepsilon)[iH, A]E^H(\lambda;\varepsilon)\big\}.$$

A second function, more convenient in applications, is

$$\widetilde\varrho_H^A(\lambda) := \sup\big\{a \in \mathbb{R} \mid \exists \varepsilon > 0 \text{ s.t. } a\,E^H(\lambda;\varepsilon) \lesssim E^H(\lambda;\varepsilon)[iH, A]E^H(\lambda;\varepsilon)\big\}.$$

Note that the following equivalent definition is often useful:

$$\begin{aligned}\widetilde\varrho_H^A(\lambda) = \sup\big\{a \in \mathbb{R} \mid \exists \eta \in C_c^\infty(\mathbb{R}) \text{ real s.t. } \eta(\lambda) \neq 0,\\ a\,\eta(H)^2 \lesssim \eta(H)[iH, A]\eta(H)\big\}.\end{aligned} \tag{2.2}$$

It is commonly said that A is conjugate to H at the point $\lambda \in \mathbb{R}$ if $\widetilde\varrho_H^A(\lambda) > 0$, and that A is strictly conjugate to H at λ if $\varrho_H^A(\lambda) > 0$. Furthermore, the function $\widetilde\varrho_H^A : \mathbb{R} \to (-\infty, \infty]$ is lower semicontinuous and satisfies $\widetilde\varrho_H^A(\lambda) < \infty$ if and only if λ belongs to the essential spectrum $\sigma_{\rm ess}(H)$ of H. One also has $\widetilde\varrho_H^A(\lambda) \ge \varrho_H^A(\lambda)$ for all $\lambda \in \mathbb{R}$, see [2, Prop. 7.2.6].

Another property of the function $\widetilde\varrho$, often used in the one-Hilbert space setting, is its stability under a large class of perturbations: Suppose that H and H' are self-adjoint operators in $\mathcal{H}$ and that both operators H and H' are of class $C_{\rm u}^1(A)$, *i.e.* such that the map (2.1) is C^1 in norm. Assume furthermore that the difference $(H-i)^{-1}-(H'-i)^{-1}$ belongs to $\mathscr{K}(\mathcal{H})$, the algebra of compact operators on $\mathcal{H}$. Then, it is proved in [2, Thm. 7.2.9] that $\widetilde\varrho_{H'}^A = \widetilde\varrho_H^A$, or in other words that A is conjugate to H' at a point $\lambda \in \mathbb{R}$ if and only if A is conjugate to H at λ.

Our first contribution in this paper is to extend such a result to the two-Hilbert spaces setting. But before this, let us recall the importance of the set $\widetilde{\mu}^A(H) \subset \mathbb{R}$ on which $\widetilde{\varrho}_H^A(\,\cdot\,) > 0$: if H is slightly more regular than $C^1(A)$, then H has locally at most a finite number of eigenvalues on $\widetilde{\mu}^A(H)$ (multiplicities counted), and H has no singularly continuous spectrum on $\widetilde{\mu}^A(H)$ (see [2, Thm. 7.4.2] for details).

3 Mourre theory in the two-Hilbert spaces setting

From now on, apart from the triple $(\mathcal{H}, H, A)$ of Sect. 2, we consider a second triple $(\mathcal{H}_0, H_0, A_0)$ and an identification operator $J : \mathcal{H}_0 \to \mathcal{H}$. The existence of two such triples is quite standard in scattering theory, at least for the pairs $(\mathcal{H}, H)$ and $(\mathcal{H}_0, H_0)$ (see for instance the books [4,14]). Part of our goal in what follows is to show that the existence of the conjugate operators A and A_0 is also natural, as was realised in the context of scattering on manifolds [13].

So, let us consider a second Hilbert space $\mathcal{H}_0$ with scalar product $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathcal{H}_0}$ and norm $\|\cdot\|_{\mathcal{H}_0}$. Let also H_0 and A_0 be two self-adjoint operators in $\mathcal{H}_0$, with domains $\mathcal{D}(H_0)$ and $\mathcal{D}(A_0)$. Clearly, the $C^1(A_0)$-regularity of H_0 with respect to A_0 can be defined as before, and if H_0 is of class $C^1(A_0)$ then the definitions of the two functions $\varrho_{H_0}^{A_0}$ and $\widetilde{\varrho}_{H_0}^{A_0}$ hold as well.

In order to compare the two triples, it is natural to require the existence of a map $J \in \mathscr{B}(\mathcal{H}_0, \mathcal{H})$ having some special properties (for example, the ones needed for the completeness of the wave operators, see Sect. 5). But for the time being, no additional information on J is necessary. In the one-Hilbert space setting, the operator H is typically a perturbation of the simpler operator H_0. And as mentioned above, the stability of the function $\widetilde{\varrho}_{H_0}^{A_0}$ is an efficient tool to infer information on H from similar information on H_0. In the two-Hilbert spaces setting, we are not aware of any general result allowing the computation of the function $\widetilde{\varrho}_H^A$ in terms of the function $\widetilde{\varrho}_{H_0}^{A_0}$. The obvious reason for this being the impossibility to consider H as a direct perturbation of H_0 since these operators do not live in the same Hilbert space. Nonetheless, the next theorem gives a result in that direction:

Theorem 3.1 *Let $(\mathcal{H}, H, A)$ and $(\mathcal{H}_0, H_0, A_0)$ be as above, and assume that*

(i) *the operators H_0 and H are of class $C^1(A_0)$ and $C^1(A)$, respectively,*
(ii) *for any $\eta \in C_c^\infty(\mathbb{R})$ the difference of bounded operators $J[iA_0, \eta(H_0)]J^* - [iA, \eta(H)]$ belongs to $\mathscr{K}(\mathcal{H})$,*
(iii) *for any $\eta \in C_c^\infty(\mathbb{R})$ the difference $J\eta(H_0) - \eta(H)J$ belongs to $\mathscr{K}(\mathcal{H}_0, \mathcal{H})$,*
(iv) *for any $\eta \in C_c^\infty(\mathbb{R})$ the operator $\eta(H)(JJ^* - 1)\eta(H)$ belongs to $\mathscr{K}(\mathcal{H})$.*

Then, one has $\widetilde{\varrho}_H^A \geq \widetilde{\varrho}_{H_0}^{A_0}$. In particular, if A_0 is conjugate to H_0 at $\lambda \in \mathbb{R}$, then A is conjugate to H at λ.

Note that with the notations introduced in the previous section, Assumption (ii) reads $J[iA_0, \eta(H_0)]J^* \approx [iA, \eta(H)]$. Furthermore, since the vector space generated by the family of functions $\{(\,\cdot\, - z)^{-1}\}_{z \in \mathbb{C}\setminus\mathbb{R}}$ is dense in $C_0(\mathbb{R})$ and the set $\mathscr{K}(\mathcal{H}_0, \mathcal{H})$ is closed in $\mathscr{B}(\mathcal{H}_0, \mathcal{H})$, the condition $J(H_0 - z)^{-1} - (H - z)^{-1}J \in \mathscr{K}(\mathcal{H}_0, \mathcal{H})$ for all $z \in \mathbb{C}\setminus\mathbb{R}$ implies Assumption (iii) (here, $C_0(\mathbb{R})$ denotes the set of continuous functions on $\mathbb{R}$ vanishing at $\pm\infty$).

Proof Let $\eta \in C_c^\infty(\mathbb{R};\mathbb{R})$, and define $\eta_1, \eta_2 \in C_c^\infty(\mathbb{R};\mathbb{R})$ by $\eta_1(x) := x\,\eta(x)$ and $\eta_2(x) := x\,\eta(x)^2$. Under Assumption (i), it is shown in [2, Eq. 7.2.18] that

$$\eta(H)[iA,H]\eta(H) = [iA,\eta_2(H)] - 2\mathsf{Re}\big\{[iA,\eta(H)]\eta_1(H)\big\}.$$

Therefore, one infers from Assumptions (ii) and (iii) that

$$\begin{aligned}\eta(H)[iA,H]\eta(H) &\approx J[iA_0,\eta_2(H_0)]J^* - 2\mathsf{Re}\big\{J[iA_0,\eta(H_0)]J^*\eta_1(H)\big\}\\ &= J[iA_0,\eta_2(H_0)]J^* - 2\mathsf{Re}\big\{J[iA_0,\eta(H_0)]\eta_1(H_0)J^*\big\}\\ &\quad -2\mathsf{Re}\big\{J[iA_0,\eta(H_0)]\big(J^*\eta_1(H) - \eta_1(H_0)J^*\big)\big\}\\ &\approx J[iA_0,\eta_2(H_0)]J^* - 2J\mathsf{Re}\big\{[iA_0,\eta(H_0)]\eta_1(H_0)\big\}J^*\\ &= J\eta(H_0)[iA_0,H_0]\eta(H_0)J^*,\end{aligned}$$

which means that

$$\eta(H)[iA,H]\eta(H) \approx J\eta(H_0)[iA_0,H_0]\eta(H_0)J^*. \tag{3.1}$$

Furthermore, if $a \in \mathbb{R}$ is such that $\eta(H_0)[iA_0,H_0]\eta(H_0) \gtrsim a\eta(H_0)^2$, then Assumptions (iii) and (iv) imply that

$$J\eta(H_0)[iA_0,H_0]\eta(H_0)J^* \gtrsim aJ\eta(H_0)^2J^* \approx a\eta(H)JJ^*\eta(H) \approx a\eta(H)^2. \tag{3.2}$$

Thus, one obtains $\eta(H)[iA,H]\eta(H) \gtrsim a\eta(H)^2$ by combining (3.1) and (3.2). This last estimate, together with the definition (2.2) of the functions $\widetilde{\varrho}_{H_0}^{A_0}$ and $\widetilde{\varrho}_H^A$ in terms of the localisation function η, implies the claim. □

As mentioned in the previous sections, the $C^1(A)$-regularity of H and the Mourre estimate are crucial ingredients for the analysis of the operator H, but they are in general not sufficient. For instance, the nature of the spectrum of H or the existence and the completeness of the wave operators is usually proved under a slightly stronger $C^{1,1}(A)$-regularity condition of H. It would certainly be valuable if this regularity condition could be deduced from a similar information on H_0. Since we have not been able to obtain such a result, we simply refer to [2] for the definition of this class of regularity and present below a coarser result. Namely, we show that the regularity condition "H is of class $C^n(A)$" can be checked by means of explicit computations involving only H and not its resolvent. For simplicity, we present the simplest, non-perturbative version of the result; more refined statements involving perturbations as in Sects. 3.1 and 3.2 could also be proved.

For that purpose, we first recall that H is of class $C^n(A)$ if the map (2.1) is strongly of class C^n. We also introduce the following slightly more general regularity class: assume that $(\mathcal{G},\mathcal{H})$ is a Friedrichs couple, *i.e.* a pair $(\mathcal{G},\mathcal{H})$ with $\mathcal{G}$ a Hilbert space densely and continuously embedded in $\mathcal{H}$. Assume furthermore that the unitary group $\{e^{itA}\}_{t\in\mathbb{R}}$ leaves $\mathcal{G}$ invariant. Then, the restriction of this group to $\mathcal{G}$ generates a C_0-group in $\mathcal{G}$, and by duality extends to a C_0-group in $\mathcal{G}^*$ (the adjoint space of $\mathcal{G}$). Without ambiguity, the generators of these groups can be denoted by A (see [2, Sec. 6.3] for details). In such a situation, an operator $T \in \mathscr{B}(\mathcal{G}, \mathcal{H})$ is said to belong to $C^n(\mathcal{A}; \mathcal{G}, \mathcal{H})$ if the map

$$\mathbb{R} \ni t \mapsto \mathrm{e}^{-itA} T \mathrm{e}^{itA} \in \mathscr{B}(\mathcal{G}, \mathcal{H})$$

is strongly of class C^n. Similar definitions hold with T in $\mathscr{B}(\mathcal{H}, \mathcal{G})$, $\mathscr{B}(\mathcal{G}, \mathcal{G}^*)$ or in $\mathscr{B}(\mathcal{G}^*, \mathcal{H})$, and one clearly has $C^n(A; \mathcal{G}, \mathcal{H}) \subset C^n(A; \mathcal{G}, \mathcal{G}^*)$.

The next proposition (which improves slightly the result of [10, Lemma 1.2]) is an extension of [2, Thm. 6.3.4.(c)] to higher orders of regularity of H with respect to A. We use for it the notation $\mathcal{G}$ for the domain $\mathcal{D}(H)$ of H endowed with its natural Hilbert space structure. We also recall that if H is of class $C^1(A)$, then $[iH, A]$ can be identified with a bounded operator from $\mathcal{G}$ to $\mathcal{G}^*$.

Proposition 3.2 *Assume that* $\mathrm{e}^{itA}\mathcal{G} \subset \mathcal{G}$ *for all* $t \in \mathbb{R}$ *and that* $H \in C^{n-1}(A; \mathcal{G}, \mathcal{H}) \cap C^n(A; \mathcal{G}, \mathcal{G}^*)$ *for some integer* $n \geq 1$. *Then, H is of class* $C^n(A)$.

Proof We prove the claim by induction on n. For $n = 1$, the claim follows from [2, Thm. 6.3.4.(a)].

Now, assume that the statement is true for $n - 1 \geq 0$, and suppose that $H \in C^{n-1}(A; \mathcal{G}, \mathcal{H}) \cap C^n(A; \mathcal{G}, \mathcal{G}^*)$. Since H is of class $C^1(A)$, one has

$$\big[(H - i)^{-1}, A\big] = -(H - i)^{-1}[H, A](H - i)^{-1}. \tag{3.3}$$

Furthermore, since $(H \pm i) \in \mathscr{B}(\mathcal{G}, \mathcal{H})$ are bijections from $\mathcal{G}$ onto $\mathcal{H}$, one infers from the inclusion $H \in C^{n-1}(A; \mathcal{G}, \mathcal{H})$ and from [2, Prop. 5.1.6.(a)] that $(H \pm i)^{-1} \in C^{n-1}(A; \mathcal{H}, \mathcal{G})$. One also deduces from [2, Prop. 5.1.7] that $(H \mp i)^{-1} \in C^{n-1}(A; \mathcal{G}^*, \mathcal{H})$. Finally, the inclusion $H \in C^n(A; \mathcal{G}, \mathcal{G}^*)$ implies that $[H, A] \in C^{n-1}(A; \mathcal{G}, \mathcal{G}^*)$. So, by taking into account (3.3) and the regularity property for product of operators [2, Prop. 5.1.5], one obtains that $\big[(H - i)^{-1}, A\big] \in C^{n-1}(A)$. This implies the inclusion $(H - i)^{-1} \in C^n(A)$, which proves the statement for n. □

Usually, the regularity of H_0 with respect to A_0 is easy to check. On the other hand, the regularity of H with respect to A is in general rather difficult to establish, and various perturbative criteria have been developed for that purpose in the one-Hilbert space setting. Often, a distinction is made between so-called short-range and long-range perturbations. Roughly speaking, the difference between these types of perturbations is that the two terms of the formal commutator $[A, H - H_0] = A(H - H_0) - (H - H_0)A$ are treated separately in the former situation while the commutator $[A, H - H_0]$ is really computed in the latter situation. In the first case, one usually requires more decay and less regularity, while in the second case more regularity but less decay are imposed. Obviously, this distinction cannot be as transparent in the general two-Hilbert spaces setting presented here. Still, a certain distinction remains, and thus we dedicate to it the following two complementary sections.

3.1 Short-range type perturbations

We show below how the condition "H is of class $C^1(A)$" and the assumptions (ii) and (iii) of Theorem 3.1 can be verified for a class of short-range type perturbations. Our approach is to derive information on H from some equivalent information on H_0, which is usually easier to obtain. Accordingly, our results exhibit some perturbative flavor. The price one has to pay is that a compatibility condition between A_0 and A is necessary. For $z \in \mathbb{C}\setminus\mathbb{R}$, we use the shorter notations $R_0(z) := (H_0 - z)^{-1}$, $R(z) := (H - z)^{-1}$ and

$$B(z) := J R_0(z) - R(z) J \in \mathscr{B}(\mathcal{H}_0, \mathcal{H}). \tag{3.4}$$

Proposition 3.3 *Let H_0 be of class $C^1(A_0)$ and assume that $\mathscr{D} \subset \mathcal{H}$ is a core for A such that $J^*\mathscr{D} \subset \mathcal{D}(A_0)$. Suppose furthermore that for any $z \in \mathbb{C}\setminus\mathbb{R}$*

$$\overline{B(z)A_0 \upharpoonright \mathcal{D}(A_0)} \in \mathscr{B}(\mathcal{H}_0, \mathcal{H}) \quad \textit{and} \quad \overline{R(z)(JA_0J^* - A) \upharpoonright \mathscr{D}} \in \mathscr{B}(\mathcal{H}). \tag{3.5}$$

Then, H is of class $C^1(A)$.

Proof Take $\psi \in \mathscr{D}$ and $z \in \mathbb{C}\setminus\mathbb{R}$. Then, one gets

$$\begin{aligned}
&\langle R(\bar{z})\psi, A\psi\rangle_{\mathcal{H}} - \langle A\psi, R(z)\psi\rangle_{\mathcal{H}} \\
&\quad = \langle R(\bar{z})\psi, A\psi\rangle_{\mathcal{H}} - \langle A\psi, R(z)\psi\rangle_{\mathcal{H}} - \langle \psi, J[R_0(z), A_0]J^*\psi\rangle_{\mathcal{H}} \\
&\qquad + \langle \psi, J[R_0(z), A_0]J^*\psi\rangle_{\mathcal{H}} \\
&\quad = \langle B(\bar{z})A_0J^*\psi, \psi, \rangle_{\mathcal{H}} - \langle \psi, B(z)A_0J^*\psi\rangle_{\mathcal{H}} + \langle \psi, J[R_0(z), A_0]J^*\psi\rangle_{\mathcal{H}} \\
&\qquad + \langle R(\bar{z})(JA_0J^* - A)\psi, \psi\rangle_{\mathcal{H}} - \langle \psi, R(z)(JA_0J^* - A)\psi\rangle_{\mathcal{H}}.
\end{aligned}$$

Now, one has

$$\left|\langle B(\bar{z})A_0J^*\psi, \psi, \rangle_{\mathcal{H}} - \langle \psi, B(z)A_0J^*\psi\rangle_{\mathcal{H}}\right| \le \text{Const.}\, \|\psi\|_{\mathcal{H}}^2$$

due to the first condition in (3.5), and one has

$$\left|\langle R(\bar{z})(JA_0J^* - A)\psi, \psi\rangle_{\mathcal{H}} - \langle \psi, R(z)(JA_0J^* - A)\psi\rangle_{\mathcal{H}}\right| \le \text{Const.}\, \|\psi\|_{\mathcal{H}}^2$$

due to the second condition in (3.5). Furthermore, since H_0 is of class $C^1(A_0)$ one also has

$$\left|\langle \psi, J[R_0(z), A_0]J^*\psi\rangle_{\mathcal{H}}\right| \le \text{Const.}\, \|\psi\|_{\mathcal{H}}^2.$$

Since $\mathscr{D}$ is a core for A, the conclusion then follows from [2, Lemma 6.2.9]. □

We now show how the assumption (ii) of Theorem 3.1 is verified for a short-range type perturbation. Note that the hypotheses of the following proposition are slightly stronger than the ones of Proposition 3.3, and thus H is automatically of class $C^1(A)$.

Proposition 3.4 *Let H_0 be of class $C^1(A_0)$ and assume that $\mathscr{D} \subset \mathcal{H}$ is a core for A such that $J^*\mathscr{D} \subset \mathcal{D}(A_0)$. Suppose furthermore that for any $z \in \mathbb{C}\setminus\mathbb{R}$*

$$\overline{B(z)A_0 \restriction \mathcal{D}(A_0)} \in \mathscr{K}(\mathcal{H}_0, \mathcal{H}) \quad and \quad \overline{R(z)(JA_0J^* - A) \restriction \mathscr{D}} \in \mathscr{K}(\mathcal{H}). \tag{3.6}$$

Then, for each $\eta \in C_c^\infty(\mathbb{R})$ the difference of bounded operators $J[A_0, \eta(H_0)]J^* - [A, \eta(H)]$ belongs to $\mathscr{K}(\mathcal{H})$.

Proof Take $\psi, \psi' \in \mathscr{D}$ and $z \in \mathbb{C}\setminus\mathbb{R}$. Then, one gets from the proof of Proposition 3.3 that

$$\begin{aligned}&\big\langle \psi', J[A_0, R_0(z)]J^*\psi\big\rangle_{\mathcal{H}} - \langle \psi', [A, R(z)]\psi\rangle_{\mathcal{H}}\\&\quad = \big\langle B(\bar{z})A_0J^*\psi', \psi, \big\rangle_{\mathcal{H}} - \big\langle \psi', B(z)A_0J^*\psi\big\rangle_{\mathcal{H}}\\&\qquad + \big\langle R(\bar{z})(JA_0J^* - A)\psi', \psi\big\rangle_{\mathcal{H}} - \big\langle \psi', R(z)(JA_0J^* - A)\psi\big\rangle_{\mathcal{H}}.\end{aligned}$$

By the density of $\mathscr{D}$ in $\mathcal{H}$, one then infers from the hypotheses that $J[A_0, R_0(z)]J^* - [A, R(z)]$ belongs to $\mathscr{K}(\mathcal{H})$.

To show the same result for functions $\eta \in C_c^\infty(\mathbb{R})$ instead of $(\cdot - z)^{-1}$, one needs more refined estimates. Taking the first resolvent identity into account one obtains

$$B(z) = \big\{1 + (z - i)R(z)\big\}B(i)\big\{1 + (z - i)R_0(z)\big\}.$$

Thus, one gets on $\mathscr{D}$ the equalities

$$\begin{aligned}B(z)A_0J^* = \big\{1 + (z - i)R(z)\big\}&B(i)A_0\big\{1 + (z - i)R_0(z)\big\}J^*\\&+\big\{1 + (z - i)R(z)\big\}B(i)(z - i)[R_0(z), A_0]J^*,\end{aligned} \tag{3.7}$$

where

$$[R_0(z), A_0] = \big\{1 + (z - i)R_0(z)\big\}R_0(i)[A_0, H_0]R_0(i)\big\{1 + (z - i)R_0(z)\big\}.$$

Obviously, these equalities extend to all of $\mathcal{H}$ since they involve only bounded operators. Letting $z = \lambda + i\mu$ with $|\mu| \le 1$, one even gets the bound

$$\big\|B(z)A_0J^*\big\|_{\mathscr{B}(\mathcal{H})} \le \text{Const.}\left(1 + \frac{|\lambda + i(\mu - 1)|}{|\mu|}\right)^4.$$

Furthermore, since the first and second terms of (3.7) extend to elements of $\mathscr{K}(\mathcal{H})$, the third term of (3.7) also extends to an element of $\mathscr{K}(\mathcal{H})$. Similarly, the operator on $\mathscr{D}$

$$R(z)(JA_0J^* - A) \equiv \big\{1 + (z - i)R(z)\big\}R(i)(JA_0J^* - A)$$

extends to a compact operator in $\mathcal{H}$, and one has the bound

$$\big\|R(z)(JA_0J^* - A)\big\|_{\mathscr{B}(\mathcal{H})} \le \text{Const.}\left(1 + \frac{|\lambda + i(\mu - 1)|}{|\mu|}\right).$$

Now, observe that for any $\eta \in C_c^\infty(\mathbb{R})$ and any $\psi, \psi' \in \mathscr{D}$ one has

$$\begin{aligned}&\langle \psi', J[A_0, \eta(H_0)]J^*\psi\rangle_{\mathcal{H}} - \langle \psi', [A, \eta(H)]\psi\rangle_{\mathcal{H}}\\ &\quad = \langle \{J\overline{\eta}(H_0) - \overline{\eta}(H)J\}A_0J^*\psi', \psi\rangle_{\mathcal{H}} - \langle \psi', \{J\eta(H_0) - \eta(H)J\}A_0J^*\psi\rangle_{\mathcal{H}}\\ &\qquad + \langle \overline{\eta}(H)(JA_0J^* - A)\psi', \psi\rangle_{\mathcal{H}} - \langle \psi', \eta(H)(JA_0J^* - A)\psi\rangle_{\mathcal{H}}. \end{aligned} \tag{3.8}$$

Then, by expressing the operators $\eta(H_0)$ and $\eta(H)$ in terms of their respective resolvents (using for example [2, Eq. 6.1.18]) and by taking the above estimates into account, one obtains that $\{J\eta(H_0) - \eta(H)J\}A_0J^*$ and $\eta(H)(JA_0J^* - A)$ are equal on $\mathscr{D}$ to a finite sum of norm convergent integrals of compact operators. Since $\mathscr{D}$ is dense in $\mathcal{H}$, these equalities between bounded operators extend continuously to equalities in $\mathscr{B}(\mathcal{H})$, and thus the statement follows by using (3.8). □

Remark 3.5 As mentioned just after Theorem 3.1, the requirement $B(z) \in \mathscr{K}(\mathcal{H}_0, \mathcal{H})$ for all $z \in \mathbb{C}\setminus\mathbb{R}$ implies the assumption (iii) of Theorem 3.1. Since an a priori stronger requirement is imposed in the first condition of (3.6), it is likely that in applications the compactness assumption (iii) will follow from the necessary conditions ensuring the first condition in (3.6).

Before turning to the long-range case, let us reconsider the above statements in the special situation where $A = JA_0J^*$. This case deserves a particular attention since it represents the most natural choice of conjugate operator for H when A_0 is a conjugate operator for H_0. However, in order to deal with a well-defined self-adjoint operator A, one needs the following assumption:

Assumption 3.6 There exists a set $\mathscr{D} \subset \mathcal{D}(A_0J^*) \subset \mathcal{H}$ such that JA_0J^* is essentially self-adjoint on $\mathscr{D}$, with corresponding self-adjoint extension denoted by A.

Assumption 3.6 might be difficult to check in general, but in concrete situations the choice of the set $\mathscr{D}$ can be quite natural (see the examples presented in Sect. 4 and [13, Rem. 4.3]). We now show how the assumptions of the above propositions can easily be checked under Assumption 3.6. Recall that the operator $B(z)$ was defined in (3.4).

Corollary 3.7 *Let H_0 be of class $C^1(A_0)$, suppose that Assumption 3.6 holds for some set $\mathscr{D} \subset \mathcal{H}$, and for any $z \in \mathbb{C}\setminus\mathbb{R}$ assume that*

$$\overline{B(z)A_0 \restriction \mathcal{D}(A_0)} \in \mathscr{B}(\mathcal{H}_0, \mathcal{H}).$$

Then, H is of class $C^1(A)$.

Proof All the assumptions of Proposition 3.3 are verified. □

Corollary 3.8 *Let H_0 be of class $C^1(A_0)$, suppose that Assumption 3.6 holds for some set $\mathscr{D} \subset \mathcal{H}$, and for any $z \in \mathbb{C}\setminus\mathbb{R}$ assume that*

$$\overline{B(z)A_0 \restriction \mathcal{D}(A_0)} \in \mathscr{K}(\mathcal{H}_0, \mathcal{H}). \tag{3.9}$$

Then, for each $\eta \in C_c^\infty(\mathbb{R})$ the difference of bounded operators $J[A_0, \eta(H_0)]J^* - [A, \eta(H)]$ belongs to $\mathscr{K}(\mathcal{H})$.

Proof All the assumptions of Proposition 3.4 are verified.

Remark 3.9 As mentioned above the choice $A = JA_0J^*$ is natural when A_0 is a conjugate operator for H_0. With that respect the second conditions in (3.5) and (3.6) quantify how much one can deviate from this natural choice.

The most important consequence of Mourre theory is the obtention of a limiting absorption principle for H_0 and H. Rather often, the space defined in terms of A_0 (resp. A) in which holds the limiting absorption principle for H_0 (resp. H) is not adequate for applications. In [2, Prop. 7.4.4] a method is given for expressing the limiting absorption principle for H_0 in terms of an auxiliary operator Φ_0 in $\mathcal{H}_0$ more suitable than A_0. Obviously, this abstract result also applies for three operators H, A and Φ in $\mathcal{H}$, but one crucial condition is that $(H-z)^{-1}\mathcal{D}(\Phi) \subset \mathcal{D}(A)$ for suitable $z \in \mathbb{C}$. In the next lemma, we provide a sufficient condition allowing to infer this information from similar information on the operators H_0, A_0 and Φ_0 in $\mathcal{H}_0$. Note that Φ does not need to be of the form $J\Phi_0J^*$ but that such a situation often appears in applications.

Lemma 3.10 *Let $z \in \mathbb{C}\backslash\{\sigma(H_0) \cup \sigma(H)\}$. Suppose that Assumption 3.6 holds for some set $\mathscr{D} \subset \mathcal{H}$. Assume that*

$$\overline{B(\bar{z})A_0 \restriction \mathcal{D}(A_0)} \in \mathscr{B}(\mathcal{H}_0, \mathcal{H}).$$

Furthermore, let Φ_0 and Φ be self-adjoint operators in $\mathcal{H}_0$ and $\mathcal{H}$ satisfying $(H_0 - z)^{-1}\mathcal{D}(\Phi_0) \subset \mathcal{D}(A_0)$ and $J^*(\Phi - i)^{-1} - (\Phi_0 - i)^{-1}J^* = (\Phi_0 - i)^{-1}B$ for some $B \in \mathscr{B}(\mathcal{H}, \mathcal{H}_0)$. Then, one has the inclusion $(H-z)^{-1}\mathcal{D}(\Phi) \subset \mathcal{D}(A)$.

Proof Let $\psi \in \mathscr{D}$ and $\psi' \in \mathcal{H}$. Then, one has

$$\begin{aligned}
&\big\langle A\psi, (H-z)^{-1}(\Phi-i)^{-1}\psi'\big\rangle_{\mathcal{H}} \\
&\quad= \big\langle\big\{(H-\bar{z})^{-1}J - J(H_0-\bar{z})^{-1}\big\}A_0J^*\psi, (\Phi-i)^{-1}\psi'\big\rangle_{\mathcal{H}} \\
&\qquad+ \big\langle J(H_0-\bar{z})^{-1}A_0J^*\psi, (\Phi-i)^{-1}\psi'\big\rangle_{\mathcal{H}} \\
&\quad= -\big\langle B(\bar{z})A_0J^*\psi, (\Phi-i)^{-1}\psi'\big\rangle_{\mathcal{H}} + \big\langle (H_0-\bar{z})^{-1}A_0J^*\psi, (\Phi_0-i)^{-1}J^*\psi'\big\rangle_{\mathcal{H}_0} \\
&\qquad+ \big\langle (H_0-\bar{z})^{-1}A_0J^*\psi, (\Phi_0-i)^{-1}B\psi'\big\rangle_{\mathcal{H}_0}.
\end{aligned}$$

So, $\big|\big\langle A\psi, (H-z)^{-1}(\Phi-i)^{-1}\psi'\big\rangle_{\mathcal{H}}\big| \le \text{Const.}\ \|\psi\|_{\mathcal{H}}$, and thus $(H-z)^{-1}(\Phi - i)^{-1}\psi' \in \mathcal{D}(A)$, since A is essentially self-adjoint on $\mathscr{D}$. □

3.2 Long-range type perturbations

In the case of a long-range type perturbation, the situation is slightly less satisfactory than in the short-range case. One reason comes from the fact that one really has to compute the commutator $[A, H - H_0]$ instead of treating the terms $A(H - H_0)$ and $(H - H_0)A$ separately. However, a rather efficient method for checking that "H is of class $C^1(A)$" has been put into evidence in [9, Lemma. A.2]. We start by recalling this result and then we propose a perturbative type argument for checking the assumption (ii) of Theorem 3.1. Note that there is a missprint in the hypothesis 1 of [9, Lemma A.2]; the meaningless condition $\sup_n \|\chi_n\|_{\mathcal{D}(H)} < \infty$ has to be replaced by $\sup_n \|\chi_n\|_{\mathscr{B}(\mathcal{D}(H))} < \infty$.

Lemma 3.11 (Lemma A.2 of [9]) *Let $\mathscr{D} \subset \mathcal{H}$ be a core for A such that $\mathscr{D} \subset \mathcal{D}(H)$ and $H\mathscr{D} \subset \mathscr{D}$. Let $\{\chi_n\}_{n\in\mathbb{N}}$ be a family of bounded operators on $\mathcal{H}$ such that*

(i) $\chi_n\mathscr{D} \subset \mathscr{D}$ *for each* $n \in \mathbb{N}$, $\operatorname{s-lim}_{n\to\infty} \chi_n = 1$ *and* $\sup_n \|\chi_n\|_{\mathscr{B}(\mathcal{D}(H))} < \infty$,
(ii) *for all* $\psi \in \mathscr{D}$, *one has* $\operatorname{s-lim}_{n\to\infty} A\chi_n\psi = A\psi$,
(iii) *there exists* $z \in \mathbb{C}\setminus\sigma(H)$ *such that* $\chi_n R(z)\mathscr{D} \subset \mathscr{D}$ *and* $\chi_n R(\bar{z})\mathscr{D} \subset \mathscr{D}$ *for each* $n \in \mathbb{N}$,
(iv) *for all* $\psi \in \mathscr{D}$, *one has* $\operatorname{s-lim}_{n\to\infty} A[H, \chi_n]R(z)\psi = 0$ *and one has* $\operatorname{s-lim}_{n\to\infty} A[H, \chi_n]R(\bar{z})\psi = 0$.

Finally, assume that for all $\psi \in \mathscr{D}$

$$\big|\langle A\psi, H\psi\rangle_{\mathcal{H}} - \langle H\psi, A\psi\rangle_{\mathcal{H}}\big| \le \text{Const.}\big(\|H\psi\|^2_{\mathcal{H}} + \|\psi\|^2_{\mathcal{H}}\big).$$

Then, H is of class $C^1(A)$.

In the next statement we provide conditions under which the assumption (ii) of Theorem 3.1 is verified for a long-range type perturbation. One condition is that for each $z \in \mathbb{C}\setminus\mathbb{R}$ the operator $B(z)$ belongs to $\mathscr{K}(\mathcal{H}_0, \mathcal{H})$, which means that the hypothesis (iii) of Theorem 3.1 is also automatically satisfied. We stress that no direct relation between A_0 and A is imposed; the single relation linking A_0 and A only involves the commutators $[H_0, A_0]$ and $[H, A]$. On the other hand, the condition on H_0 is slightly stronger than just the $C^1(A_0)$-regularity.

Proposition 3.12 *Let H_0 be of class $C^1(A_0)$ with $[H_0, A_0] \in \mathscr{B}\big(\mathcal{D}(H_0), \mathcal{H}_0\big)$ and let H be of class $C^1(A)$. Assume that the operator $J \in \mathscr{B}(\mathcal{H}_0, \mathcal{H})$ extends to an element of $\mathscr{B}\big(\mathcal{D}(H_0)^*, \mathcal{D}(H)^*\big)$, and suppose that for each $z \in \mathbb{C}\setminus\mathbb{R}$ the operator $B(z)$ belongs to $\mathscr{K}(\mathcal{H}_0, \mathcal{H})$ and that the difference $J[H_0, A_0]J^* - [H, A]$ belongs to $\mathscr{K}\big(\mathcal{D}(H), \mathcal{D}(H)^*\big)$. Then, for each $\eta \in C_c^\infty(\mathbb{R})$ the difference of bounded operators*

$$J[A_0, \eta(H_0)]J^* - [A, \eta(H)]$$

belongs to $\mathscr{K}(\mathcal{H})$.

Proof By taking the various hypotheses into account one gets for any $z \in \mathbb{C}\setminus\mathbb{R}$ that

$$\begin{aligned}
&J[A_0, R_0(z)]J^* - [A, R(z)]\\
&\quad = JR_0(z)[H_0, A_0]R_0(z)J^* - R(z)[H, A]R(z)\\
&\quad = \big\{JR_0(z) - R(z)J\big\}[H_0, A_0]R_0(z)J^* + R(z)J[H_0, A_0]\big\{R_0(z)J^* - J^*R(z)\big\}\\
&\qquad + R(z)\big\{J[H_0, A_0]J^* - [H, A]\big\}R(z)\\
&\quad = B(z)[H_0, A_0]R_0(z)J^* + R(z)J[H_0, A_0]B(\bar{z})^*\\
&\qquad + R(z)\big\{J[H_0, A_0]J^* - [H, A]\big\}R(z),
\end{aligned}$$

with each term on the last line in $\mathscr{K}(\mathcal{H})$. Now, by taking the first resolvent identity into account, one obtains

$$\begin{aligned}&B(z)[H_0, A_0]R_0(z)J^*\\&\quad= \big\{1 + (z - i)R(z)\big\}B(i)\big\{1 + (z - i)R_0(z)\big\}[H_0, A_0]R_0(i)\big\{1+(z - i)R_0(z)\big\}J^*\end{aligned}$$

and

$$\begin{aligned}&R(z)J[H_0, A_0]B(\bar{z})^*\\&\quad= \big\{1+(z - i)R(z)\big\}R(i)J[H_0, A_0]\big\{1+(z - i)R_0(z)\big\}B(-i)^*\big\{1 + (z - i)R(z)\big\}\end{aligned}$$

as well as

$$\begin{aligned}&R(z)\big\{J[H_0, A_0]J^* - [H, A]\big\}R(z)\\&\quad= \big\{1 + (z - i)R(z)\big\}R(i)\big\{J[H_0, A_0]J^* - [H, A]\big\}R(i)\big\{1 + (z - i)R(z)\big\}.\end{aligned}$$

Thus, by letting $z = \lambda + i\mu$ with $|\mu| \leq 1$, one gets the bound

$$\big\| J[A_0, R_0(z)]J^* - [A, R(z)]\big\|_{\mathscr{B}(\mathcal{H})} \leq \text{Const.}\left(1 + \frac{|\lambda + i(\mu - 1)|}{|\mu|}\right)^3.$$

One concludes as in the proof of Proposition 3.4 by expressing $J[A_0, \eta(H_0)]J^* - [A, \eta(H)]$ in terms of $J[A_0, R_0(z)]J^* - [A, R(z)]$ (using for example [2, Eq. 6.2.16]), and then by dealing with a finite number of norm convergent integrals of compact operators. □

As mentioned before the statement, no direct relation between A_0 and A has been imposed, and thus considering the special case $A = JA_0J^*$ is not really relevant. However, it is not difficult to check how the quantity $J[H_0, A_0]J^* - [H, A]$ looks like in that special case, and in applications such an approach could be of interest. However, since the resulting formulas are rather involved in general, we do not further investigate in that direction.

4 One illustrative example

To illustrate our approach, we present below a simple example for which all the computations can be made by hand (more involved examples will be presented elsewhere, like in [13], where part of the results of the present paper is used). In this model, usually called one-dimensional Schrödinger operator with steplike potential, the choice of a conjugate operator is rather natural, whereas the computation of the ϱ-functions is not completely trivial due to the anisotropy of the potential. We refer to [1,3,5,6,8] for earlier works on that model and to [12] for a n-dimensional generalisation.

So, we consider in the Hilbert space $\mathcal{H} := \mathsf{L}^2(\mathbb{R})$ the Schrödinger operator $H := -\Delta + V$, where V is the operator of multiplication by a function $v \in C(\mathbb{R}; \mathbb{R})$ with finite limits $v_\pm$ at infinity, *i.e.* $v_\pm := \lim_{x\to\pm\infty} v(x) \in \mathbb{R}$. The operator H is self-adjoint on $\mathcal{H}^2(\mathbb{R})$, since V is bounded. As a second operator, we consider in the auxiliary Hilbert space $\mathcal{H}_0 := \mathsf{L}^2(\mathbb{R}) \oplus \mathsf{L}^2(\mathbb{R})$ the operator

$$H_0 := (-\Delta + v_-) \oplus (-\Delta + v_+),$$

which is also self-adjoint on its natural domain $\mathcal{H}^2(\mathbb{R}) \oplus \mathcal{H}^2(\mathbb{R})$. Then, we take a function $j_+ \in C^\infty(\mathbb{R}; [0, 1])$ with $j_+(x) = 0$ if $x \le 1$ and $j_+(x) = 1$ if $x \ge 2$, we set $j_-(x) := j_+(-x)$ for each $x \in \mathbb{R}$, and we define the identification operator $J \in \mathscr{B}(\mathcal{H}_0, \mathcal{H})$ by the formula

$$J(\varphi_-, \varphi_+) := j_-\varphi_- + j_+\varphi_+, \quad (\varphi_-, \varphi_+) \in \mathcal{H}_0.$$

Clearly, the adjoint operator $J^* \in \mathscr{B}(\mathcal{H}, \mathcal{H}_0)$ is given by $J^*\psi = (j_-\psi, j_+\psi)$ for any $\psi \in \mathcal{H}$, and the operator $JJ^* \in \mathscr{B}(\mathcal{H})$ is equal to the operator of multiplication by $j_-^2 + j_+^2$.

Let us now come to the choice of the conjugate operators. For H_0, the most natural choice consists in two copies of the generator of dilations on $\mathbb{R}$, that is, $A_0 := (D, D)$ with D the generator of the group

$$\big(\mathrm{e}^{itD}\psi\big)(x) := \mathrm{e}^{t/2}\psi(\mathrm{e}^t x), \quad \psi \in \mathscr{S}(\mathbb{R}),\ t, x \in \mathbb{R},$$

where $\mathscr{S}(\mathbb{R})$ denotes the Schwartz space on $\mathbb{R}$. In such a case, the map (2.1) with (H, A) replaced by (H_0, A_0) is strongly of class C^∞ in $\mathcal{H}_0$. Moreover, the ϱ-functions can be computed explicitly (see [2, Sec. 8.3.5] for a similar calculation in an abstract setting):

$$\widetilde{\varrho}_{H_0}^{A_0}(\lambda) = \varrho_{H_0}^{A_0}(\lambda) = \begin{cases} +\infty & \text{if } \lambda < \min\{v_-, v_+\} \\ 2\big(\lambda - \min\{v_-, v_+\}\big) & \text{if } \min\{v_-, v_+\} \le \lambda < \max\{v_-, v_+\} \\ 2\big(\lambda - \max\{v_-, v_+\}\big) & \text{if } \lambda \ge \max\{v_-, v_+\}. \end{cases}$$

For the conjugate operator for H, two natural choices exist: either one can use again the generator D of dilations in $\mathcal{H}$, or one can use the (formal) operator JA_0J^* which appears naturally in our framework. Since the latter choice illustrates better the general case, we opt here for this choice and just note that the former choice would also be suitable and would lead to similar results. So, we set $\mathscr{D} := \mathscr{S}(\mathbb{R})$ and $j := j_- + j_+$, and then observe that JA_0J^* is well-defined and equal to

$$JA_0J^* = jDj \tag{4.1}$$

on $\mathscr{D}$. This equality, the fact that j is of class $C^1(D)$, and [2, Lemma 7.2.15], imply that JA_0J^* is essentially self-adjoint on $\mathscr{D}$. We denote by A the corresponding self-adjoint extension.

We are now in a position for applying results of the previous sections such as Theorem 3.1. First, recall that H_0 is of class $C^1(A_0)$ and observe that the assumption

(iv) of Theorem 3.1 is satisfied with the operator J introduced above. Similarly, one easily shows that the assumption (iii) of Theorem 3.1 also holds. Indeed, as mentioned after the statement of Theorem 3.1, the assumption (iii) holds if one shows that $B(z) \in \mathscr{K}(\mathcal{H}_0, \mathcal{H})$ for each $z \in \mathbb{C}\setminus\mathbb{R}$. But, for any $(\varphi_-, \varphi_+) \in \mathcal{H}_0$, a direct calculation shows that $B(z)(\varphi_-, \varphi_+) = B_-(z)\varphi_- + B_+(z)\varphi_+$, with

$$B_\pm(z) := (H-z)^{-1}\big\{[-\Delta, j_\pm] + j_\pm(V - v_\pm)\big\}(-\Delta + v_\pm - z)^{-1} \in \mathscr{K}(\mathcal{H}).$$

So, one readily concludes that $B(z) \in \mathscr{K}(\mathcal{H}_0, \mathcal{H})$.

Thus, one is only left with showing the assumption (ii) of Theorem 3.1 and the $C^1(A)$-regularity of H. We first consider a short-range type perturbation. In such a case, with A defined as above, we know it is enough to check the condition (3.9) of Corollary 3.8. For that purpose, we assume the following stronger condition on v:

$$\lim_{|x|\to\infty} |x|\big(v(x) - v_\pm\big) = 0, \tag{4.2}$$

and observe that for each $(\varphi_-, \varphi_+) \in \mathscr{S}(\mathbb{R}) \oplus \mathscr{S}(\mathbb{R})$ and $z \in \mathbb{C}\setminus\mathbb{R}$ we have the equality

$$B(z)A_0(\varphi_-, \varphi_+) = B_-(z)D\varphi_- + B_+(z)D\varphi_+.$$

Then, taking into account the expressions for $B_-(z)$ and $B_+(z)$ as well as the above assumption on v, one proves easily that $\overline{B_\pm(z)D \restriction \mathcal{D}(D)} \in \mathscr{K}(\mathcal{H})$, which implies (3.9). Collecting our results, we end up with:

Lemma 4.1 (Short-range case) *Assume that* $v \in C(\mathbb{R};\mathbb{R})$ *satisfies* (4.2)*, then the operator* H *is of class* $C^1(A)$ *and one has* $\widetilde{\varrho}_H^A \geq \widetilde{\varrho}_{H_0}^{A_0}$*. In particular,* A *is conjugate to* H *on* $\mathbb{R}\setminus\{v_-, v_+\}$.

We now consider a long-range type perturbation and thus show that the assumptions of Proposition 3.12 hold with A defined as above. For that purpose, we assume that $v \in C^1(\mathbb{R};\mathbb{R})$ and that

$$\lim_{|x|\to\infty} |x|\, v'(x) = 0. \tag{4.3}$$

Then, a standard computation taking the inclusion $(H-z)^{-1}\mathscr{D} \subset \mathcal{D}(A)$ into account shows that H is of class $C^1(A)$ with

$$[A, H] = \big[j(-i\nabla)\,\mathrm{id}_\mathbb{R}\, j, -\Delta\big] - ij^2\,\mathrm{id}_\mathbb{R}\, v' + \frac{i}{2}\big[j^2, -\Delta\big], \tag{4.4}$$

where $\mathrm{id}_\mathbb{R}$ is the function $\mathbb{R} \ni x \mapsto x \in \mathbb{R}$. Then, using (4.3) and (4.4), one infers that $J[H_0, A_0]J^* - [H, A]$ belongs to $\mathscr{K}\big(\mathcal{D}(H), \mathcal{D}(H)^*\big)$. Furthermore, simple considerations show that J extends to an element of $\mathscr{B}\big(\mathcal{D}(H_0)^*, \mathcal{D}(H)^*\big)$. These results, together with the ones already obtained, permit to apply Proposition 3.12, and thus to get:

Lemma 4.2 (Long-range case) *Assume that* $v \in C^1(\mathbb{R};\mathbb{R})$ *satisfies* (4.3)*, then the operator* H *is of class* $C^1(A)$ *and one has* $\widetilde{\varrho}_H^A \geq \widetilde{\varrho}_{H_0}^{A_0}$*. In particular,* A *is conjugate to* H *on* $\mathbb{R}\setminus\{v_-, v_+\}$*.*

5 Completeness of the wave operators

One of the main goal in scattering theory is the proof of the completeness of the wave operators. In our setting, this amounts to show that the strong limits

$$W_\pm(H, H_0, J) := \operatorname{s-lim}_{t\to\pm\infty} \mathrm{e}^{itH} J \mathrm{e}^{-itH_0} P_{\mathrm{ac}}(H_0) \tag{5.1}$$

exist and have ranges equal to $\mathcal{H}_{\mathrm{ac}}(H)$. If the wave operators $W_\pm(H, H_0, J)$ are partial isometries with initial sets $\mathcal{H}_0^\pm$, this implies in particular that the scattering operator

$$S := W_+(H, H_0, J)^* W_-(H, H_0, J)$$

is well-defined and unitary from $\mathcal{H}_0^-$ to $\mathcal{H}_0^+$.

When defining the completeness of the wave operators, one usually requires that $\mathcal{H}_0^\pm = \mathcal{H}_{\mathrm{ac}}(H_0)$ (see for example [4, Def. III.9.24] or [14, Def. 2.3.1]). However, in applications it may happen that the ranges of $W_\pm(H, H_0, J)$ are equal to $\mathcal{H}_{\mathrm{ac}}(H)$ but that $\mathcal{H}_0^\pm \neq \mathcal{H}_{\mathrm{ac}}(H_0)$. Typically, this happens for multichannel type scattering processes. In such situations, the usual criteria for completeness, as [4, Prop. III.9.40] or [14, Thm. 2.3.6], cannot be applied. So, we present below a reformulation of the completeness of the wave operators without assuming that $\mathcal{H}_0^\pm = \mathcal{H}_{\mathrm{ac}}(H_0)$. Its proof is inspired by [14, Thm. 2.3.6]. Note that condition (5.3) below might be the difficult point when a long-range type perturbation is considered.

Proposition 5.1 *Suppose that the wave operators defined in* (5.1) *exist and are partial isometries with initial set projections* $P_0^\pm$*. If there exists* $\widetilde{J} \in \mathscr{B}(\mathcal{H}, \mathcal{H}_0)$ *such that*

$$W_\pm\big(H_0, H, \widetilde{J}\big) := \operatorname{s-lim}_{t\to\pm\infty} \mathrm{e}^{itH_0} \widetilde{J} \mathrm{e}^{-itH} P_{\mathrm{ac}}(H) \tag{5.2}$$

exist and such that

$$\operatorname{s-lim}_{t\to\pm\infty} \big(J\widetilde{J} - 1\big)\mathrm{e}^{-itH} P_{\mathrm{ac}}(H) = 0, \tag{5.3}$$

then the equalities $\mathsf{Ran}\big(W_\pm(H, H_0, J)\big) = \mathcal{H}_{\mathrm{ac}}(H)$ *hold. Conversely, if* $\mathsf{Ran}\big(W_\pm(H, H_0, J)\big) = \mathcal{H}_{\mathrm{ac}}(H)$ *and if there exists* $\widetilde{J} \in \mathscr{B}(\mathcal{H}, \mathcal{H}_0)$ *such that*

$$\operatorname{s-lim}_{t\to\pm\infty} \big(\widetilde{J}J - 1\big)\mathrm{e}^{-itH_0} P_0^\pm = 0, \tag{5.4}$$

then $W_\pm\big(H_0, H, \widetilde{J}\big)$ *exist and* (5.3) *holds.*

Proof (i) By using the chain rule for wave operators [14, Thm. 2.1.7], we deduce from the definitions (5.1)-(5.2) that the limits

$$W_\pm\big(H, H, J\widetilde{J}\big) := \operatorname{s-lim}_{t\to\pm\infty} \mathrm{e}^{itH} J\widetilde{J} \mathrm{e}^{-itH} P_{\mathrm{ac}}(H)$$

exist and satisfy

$$W_\pm\big(H, H, J\widetilde{J}\big) = W_\pm(H, H_0, J)W_\pm\big(H_0, H, \widetilde{J}\big). \tag{5.5}$$

In consequence, the equality

$$\text{s-}\lim_{t\to\pm\infty}\big(\mathrm{e}^{itH}J\widetilde{J}\mathrm{e}^{-itH}P_{\mathrm{ac}}(H) - P_{\mathrm{ac}}(H)\big) = 0,$$

which follow from (5.3), implies that $W_\pm\big(H, H, J\widetilde{J}\big)P_{\mathrm{ac}}(H) = P_{\mathrm{ac}}(H)$. This, together with (5.5) and the equality $W_\pm\big(H_0, H, \widetilde{J}\big) = W_\pm\big(H_0, H, \widetilde{J}\big)P_{\mathrm{ac}}(H)$, gives

$$W_\pm(H, H_0, J)W_\pm\big(H_0, H, \widetilde{J}\big) = W_\pm\big(H, H, J\widetilde{J}\big)P_{\mathrm{ac}}(H) = P_{\mathrm{ac}}(H),$$

which is equivalent to

$$W_\pm\big(H_0, H, \widetilde{J}\big)^* W_\pm(H, H_0, J)^* = P_{\mathrm{ac}}(H).$$

This gives the inclusion $\mathsf{Ker}\big(W_\pm(H, H_0, J)^*\big) \subset \mathcal{H}_{\mathrm{ac}}(H)^\perp$, which together with the fact that the range of a partial isometry is closed imply that

$$\mathcal{H} = \mathsf{Ran}\big(W_\pm(H, H_0, J)\big) \oplus \mathsf{Ker}\big(W_\pm(H, H_0, J)^*\big) \subset \mathcal{H}_{\mathrm{ac}}(H) \oplus \mathcal{H}_{\mathrm{ac}}(H)^\perp = \mathcal{H}.$$

So, one must have $\mathsf{Ran}\big(W_\pm(H, H_0, J)\big) = \mathcal{H}_{\mathrm{ac}}(H)$, and the first claim is proved.

(ii) Conversely, consider $\psi \in \mathcal{H}_{\mathrm{ac}}(H)$. Then we know from the hypothesis $\mathsf{Ran}\big(W_\pm(H, H_0, J)\big) = \mathcal{H}_{\mathrm{ac}}(H)$ that there exist $\psi_\pm \in P_0^\pm\mathcal{H}_0$ such that

$$\lim_{t\to\pm\infty}\big\|\mathrm{e}^{-itH}\psi - J\mathrm{e}^{-itH_0}P_0^\pm\psi_\pm\big\|_{\mathcal{H}} = 0. \tag{5.6}$$

Together with (5.4), this implies that the norm

$$\begin{aligned}
&\big\|\mathrm{e}^{itH_0}\widetilde{J}\mathrm{e}^{-itH}\psi - P_0^\pm\psi_\pm\big\|_{\mathcal{H}_0}\\
&\quad\le \big\|\mathrm{e}^{itH_0}\widetilde{J}\big(\mathrm{e}^{-itH}\psi - J\mathrm{e}^{-itH_0}P_0^\pm\psi_\pm\big)\big\|_{\mathcal{H}_0} + \big\|\mathrm{e}^{itH_0}\widetilde{J}J\mathrm{e}^{-itH_0}P_0^\pm\psi_\pm - P_0^\pm\psi_\pm\big\|_{\mathcal{H}_0}\\
&\quad\le \text{Const.}\,\big\|\mathrm{e}^{-itH}\psi - J\mathrm{e}^{-itH_0}P_0^\pm\psi_\pm\big\|_{\mathcal{H}} + \big\|\big(\widetilde{J}J - 1\big)\mathrm{e}^{-itH_0}P_0^\pm\psi_\pm\big\|_{\mathcal{H}_0}
\end{aligned}$$

converges to 0 as $t \to \pm\infty$, showing that the wave operators (5.2) exist.

For the relation (5.3), observe first that (5.4) gives

$$\text{s-}\lim_{t\to\pm\infty}\big(J\widetilde{J} - 1\big)J\mathrm{e}^{-itH_0}P_0^\pm = \text{s-}\lim_{t\to\pm\infty}J\big(\widetilde{J}J - 1\big)\mathrm{e}^{-itH_0}P_0^\pm = 0.$$

Together with (5.6), this implies that the norm

$$\begin{aligned}
&\big\|\big(J\widetilde{J} - 1\big)\mathrm{e}^{-itH}\psi\big\|_{\mathcal{H}}\\
&\quad\le \big\|\big(J\widetilde{J} - 1\big)\big(J\mathrm{e}^{-itH_0}P_0^\pm\psi_\pm - \mathrm{e}^{-itH}\psi\big)\big\|_{\mathcal{H}} + \big\|\big(J\widetilde{J} - 1\big)J\mathrm{e}^{-itH_0}P_0^\pm\psi_\pm\big\|_{\mathcal{H}}\\
&\quad\le \text{Const.}\,\big\|\mathrm{e}^{-itH}\psi - J\mathrm{e}^{-itH_0}P_0^\pm\psi_\pm\big\|_{\mathcal{H}} + \big\|\big(J\widetilde{J} - 1\big)J\mathrm{e}^{-itH_0}P_0^\pm\psi_\pm\big\|_{\mathcal{H}}
\end{aligned}$$

converges to 0 as $t \to \pm\infty$, showing that (5.3) also holds. □

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