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On the stability of solitary waves of a generalized Ostrovsky equation

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Abstract We consider the stability of ground state solitary waves of the generalized Ostrovsky equation $(u_t - \beta u_{xxx} + f(u)_x)_x = \gamma u$, with homogeneous nonlinearities of the form $f(u) = a_e|u|^p + a_o|u|^{p-1}u$. We obtain bounds on the function d whose convexity determines the stability of the solitary waves. These bounds imply that, when $2 \leq p < 5$ and $a_o < 0$, solitary waves are stable for c near $c_* = 2\sqrt{\beta\gamma}$. These bounds also imply that, for $\gamma > 0$ small, solitary waves are stable when $2 \leq p < 5$ and unstable when $p > 5$. We also numerically compute the function d, and thereby determine precise regions of stability and instability, for several nonlinearities.

1 Introduction

The focus of this paper is on the stability of solitary wave solutions of the generalized Ostrovsky equation

$$(u_t - \beta u_{xxx} + f(u)_x)_x = \gamma u, \tag{1.1}$$

where f is a C^2 function that is homogeneous of degree $p \geq 2$. Also known as the rotation modified KdV equation, this equation was originally proposed by Ostrovsky [10], with $f(u) = u^2$, as a model for the unidirectional propagation of weakly nonlinear long surface and internal waves of small amplitude in a rotating fluid, where the parameter $\gamma > 0$ is a measure of rotational effects due to the Coriolis force. Equation (1.1) has also been derived, with $f(u) = -u^3$, as a model for the propagation of short

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pulses in nonlinear media [3]. Setting $\gamma = 0$ in the Ostrovsky equation and integrating, one obtains the generalized KdV equation

$$u_t - \beta u_{xxx} + f(u)_x = 0. \tag{1.2}$$

In [14], Varlamov and Liu considered the Cauchy problem for the Ostrovsky equation. They proved that (1.1) is well-posed in the space X_s for $s > 3/2$. In [9] they also showed that solutions of the Ostrovsky equation converge to solutions of the KdV equation on finite time intervals, in the sense that if $\psi \in X_s$ with $s > 3/2$, v is the solution of (1.2) with $v(0) = \psi$ and u_n is the solution of (1.1) with $u_n(0) = \psi$ and $\gamma_n \to 0$, then $\sup_{0 \le t \le T} \|u_n(t) - v(t)\|_{L^2} \to 0$ for fixed $T > 0$. Tsugawa [13] extended the results of Varlamov and Liu by proving well-posedness in X_s for $s > -3/4$.

Existence of solitary waves was considered in [6,15] for the quadratic nonlinearity, and in [7] for more general homogeneous nonlinearities. Using variational methods, it was shown that solitary waves exist in the space X_1 provided $\beta > 0$, $\gamma > 0$ and $c < c_* = 2\sqrt{\beta\gamma}$. In [3], existence was shown for pure power nonlinearities and small $\gamma > 0$ by considering the Ostrovsky equation as a perturbation of the KdV equation.

The behavior of the solitary waves of the Ostrovsky equation as $\gamma \to 0$ was studied in [6,7], where it was shown that solitary waves of the Ostrovsky equation converge in H^1 to solitary waves of the KdV equation. This is somewhat surprising, as the Ostrovsky solitary waves have zero mass, while the KdV solitary waves have nonzero mass.

Stability of solitary waves was considered in [6,7,9]. It was shown that the stability of solitary waves is determined by the convexity or concavity of the function d defined in equation (3.4). Although there are no known explicit expressions for d, by using the scaling identity satisfied by d, together with numerically computed solitary wave solutions, it is possible to obtain numerical approximations of d that determine the regions of stability and instability in terms of the parameters β, c and γ. This was done in [6,7] for the class of pure power nonlinearities $f(u) = (-u)^p$.

The purpose of this paper is twofold. We first investigate the behavior of d near the boundary of its domain. In particular, when $p < 5$ we obtain bounds that imply convexity of d, and hence stability, for c near $c_* = 2\sqrt{\beta\gamma}$. We also show that the scaling identity implies stability for $p < 5$ and instability for $p > 5$, for γ near zero. Secondly, we extend the numerical results of [7] to a broader class of nonlinearities. The main results of the paper can be summarized as follows.

Main Results

(i) Suppose $f(s) = a_e|s|^p + a_o|s|^{p-1}s$ where $2 \le p < 5$ and $a_o < 0$. Then solitary waves are stable for c near c_*. (Corollary 5.1.)
(ii) If $p < 5$ solitary waves are stable for small $\gamma > 0$. (Corollary 5.4, part (i).)
(iii) If $p > 5$ solitary waves are unstable for small $\gamma > 0$. (Corollary 5.4, part (ii).)
(iv) If $p > 5 + 4\sqrt{2}$ and $c < \left(\frac{p^2-10p-7}{(p-1)^2}\right) c_*$, solitary waves are unstable. (Theorem 3.5, part (ii).)
(v) Precise regions of stability and instability are given for the nonlinearity $f(s) = \cos(\theta)|s|^p + \sin(\theta)|s|^{p-1}s$ for several values of θ and p. (Table 1.)

The stability results (i) and (ii) are new. The instability result (iii) was proved in [7], but is proved here using a different method, and the instability result (iv) is an improvement on the previously known result. The numerical results (v) extend those of [7].

The paper is organized as follows. In Sects. 2 and 3 we summarize the known results concerning the properties of solitary waves of the Ostrovsky equation and their stability. In Sect. 4 we discuss the class of nonlinearities that will be considered. In Sect. 5 we derive bounds on d near the boundary of its domain. The main results are Theorem 5.1 and Theorem 5.2. Together with Theorem 3.3, these results provide information about the stability of solitary waves for parameter values close to the boundary of the domain of d. Finally, in Sect. 6 we present numerical results for several different homogeneous nonlinearities.

Notation

We shall use $\mathscr{F}(u)$ or $\hat{u}$ to denote the Fourier transform of u with respect to the spatial variable x, and $\mathscr{F}^{-1}(u)$ or $\check{u}$ to denote the inverse Fourier transform. We denote by ∂_x^{-1} the operator defined by

$$\partial_x^{-1} u = \mathscr{F}^{-1}\left((i\xi)^{-1}\mathscr{F}u\right).$$

The space X_s is defined by

$$X_s = \{u \in H^s(\mathbf{R}) \mid \partial_x^{-1} u \in H^s(\mathbf{R})\}$$

with norm

$$\|u\|_{X_s} = \|u\|_{H^s} + \|\partial_x^{-1} u\|_{H^s}.$$

2 Solitary waves

In this section we summarize the known properties of solitary waves of Eq. (1.1). By a solitary wave we mean a solution of (1.1) of the form $u(x,t) = \varphi(x - ct)$. The profile φ must then satisfy the stationary equation

$$\beta\varphi_{xx} + c\varphi + \gamma D_x^{-2}\varphi = f(\varphi), \tag{2.1}$$

or equivalently

$$\beta\varphi_{xxxx} + c\varphi_{xx} + \gamma\varphi = f(\varphi)_{xx}. \tag{2.2}$$

We will restrict attention to the case $\beta > 0$, $\gamma > 0$ and $c < c_* = 2\sqrt{\beta\gamma}$, when this equation is elliptic. In all other cases, the results of Zhang and Liu [17] and Liu [8] imply non-existence of solutions of (2.1) in X_1. Existence of solutions of (2.1) in the space X_1 was established in [7] (and in [9] for the quadratic nonlinearity) by considering the following variational problem. Let

$$I(u) = I(u; \beta, c, \gamma) = \int_{\mathbf{R}} \beta u_x^2 - cu^2 + \gamma(\partial_x^{-1} u)^2 \, dx, \tag{2.3}$$

$$K(u) = -(p+1) \int_{\mathbf{R}} F(u) \, dx, \tag{2.4}$$

where $F' = f$ and $F(0) = 0$. Let $\lambda > 0$ and suppose there exist $u \in X_1$ such that $K(u) = \lambda$. Then define

$$M_\lambda = \inf\{I(u) : u \in X_1, K(u) = \lambda\}.$$

Then since I is coercive over X_1, it follows that $M_\lambda > 0$. A minimizing sequence is then defined to be a sequence $\psi_k \in X_1$ such that

$$K(\psi_k) \to \lambda, \qquad I(\psi_k) \to M_\lambda$$

as $k \to \infty$. We then have the following result [7].

Theorem 2.1 *Let $\beta > 0$, $\gamma > 0$ and $c < c_*$. Let ψ_k be a minimizing sequence for some $\lambda > 0$. Then there exists a subsequence (renamed ψ_k), scalars $y_k \in \mathbf{R}$ and $\psi \in X_1$ such that $\psi_k(\cdot + y_k) \to \psi$ in X_1. The function ψ achieves the minimum $I(\psi) = M_\lambda$ subject to the constraint $K(\psi) = \lambda$.*

The minimizer ψ then satisfies the equation

$$\beta\psi_{xx} + c\psi + \gamma D_x^{-2}\psi = \mu f(\psi), \tag{2.5}$$

for some $\mu \in \mathbf{R}$. Multiplying by ψ and integrating yields $M_\lambda = I(\psi) = \mu K(\psi) = \mu\lambda$, so $\mu > 0$, and thus $\varphi = \mu^{\frac{1}{p-1}} \psi$ is a solution of (2.1). Such solutions are referred to as ground states. The homogeneity of F implies that ground state solitary waves achieve the minimum

$$m = m(\beta, c, \gamma) = \inf \left\{ \frac{I(u; \beta, c, \gamma)}{(K(u))^{\frac{2}{p+1}}} : u \in X_1, K(u) > 0 \right\}. \tag{2.6}$$

Multiplying the solitary wave equation (2.1) by φ and integrating gives $I(\varphi) = K(\varphi)$, and therefore the set of all ground state solutions of (2.1) can be described as follows.

$$\mathscr{G}(\beta, c, \gamma) = \{\varphi \in X_1 : I(\varphi) = K(\varphi) = m(\beta, c, \gamma)^{\frac{p+1}{p-1}}\} \tag{2.7}$$

The regularity and decay of solutions of (2.1) with quadratic nonlinearity were considered by Zhang and Liu [17], where they proved the following.

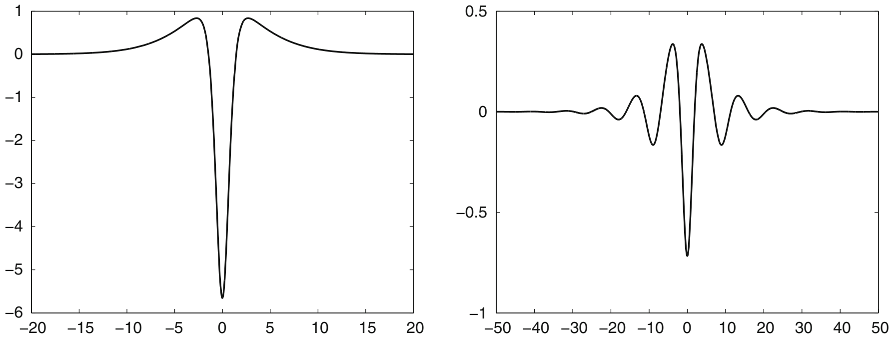

Fig. 1 Solitary waves of the Ostrovsky equation with nonlinearity $f(u) = u^2$. Here $\beta = 1$ and $\gamma = \frac{1}{4}$, so $c_* = 1$. In the figure on the *left* $c = -2 < -c_*$ and in the figure on the *right* $c = 0.9$

Theorem 2.2 *Let $\beta > 0$, $\gamma > 0$ and $c < c_*$, and suppose φ is a solution of* (2.1) *with $f(u) = u^2$. Then $\varphi \in C^\infty(\mathbf{R}) \cap H^\infty(\mathbf{R})$ and*

$$\partial_x^{-1}\varphi(x), \varphi(x), \varphi_x(x) = O\left(e^{-\alpha|x|}\right)$$

where α is a positive constant depending only on β, c and γ.

As can be seen from the behavior of solutions of the linear equation

$$\beta u_{xxxx} + c u_{xx} + \gamma u = 0 \tag{2.8}$$

solitary waves have exponentially decaying, oscillatory tails when $-c_* < c < c_*$, while they have exponentially decaying, non-oscillatory tails when $c < -c_*$. See Fig. 1. In the latter case, Zhang and Liu also proved the following uniqueness result for the quadratic nonlinearity.

Theorem 2.3 *Let $\beta > 0$, $\gamma > 0$ and $c < -c_*$. Then, up to translation, Eq.* (2.1) *with $f(u) = u^2$ has a unique, symmetric (even), solution $\varphi \in X_1$.*

The behavior of solitary waves of the Ostrovsky equation as the rotation parameter γ approaches zero was studied in [6,7]. Although the solitary waves of the Ostrovsky equation have zero mass, while the KdV solitary waves have non-zero mass, it was shown that the Ostrovsky solitary waves converge to the KdV solitary waves, in the following sense.

Theorem 2.4 *Fix $\beta > 0$, $c < 0$ and consider any sequence $\gamma_k \to 0^+$. For each k, choose $\varphi_k \in \mathcal{G}(\beta, c, \gamma_k)$. Then there exists a subsequence (renamed γ_k) and translations y_k so that*

$$\varphi_k(\cdot + y_k) \to \varphi_0 \tag{2.9}$$

in $H^1(\mathbf{R})$ as $\gamma_k \to 0^+$.

3 Stability

In this section we summarize the known results concerning the stability of solitary waves of the Ostrovsky equation. In light of the well-posedness result of Tsugawa, it now makes sense to use the following definition of stability, which removes the X_s, $s > 3/2$ assumption.

Definition 3.1 We say that a set $\mathscr{S} \subset X_1$ is stable with respect to the Ostrovsky equation (1.1) if for any $\epsilon > 0$ there exists some $\delta > 0$ such that, for any $u_0 \in X_1$,

$$\inf\{\|u_0 - v\|_{X_1} : v \in \mathscr{S}\} < \delta,$$

it follows that the solution u of (1.1) with $u(0) = u_0$ satisfies

$$\inf\{\|u(t) - v\|_{X_1} : v \in \mathscr{S}\} < \epsilon$$

for all $t > 0$. Otherwise we say $\mathscr{S}$ is unstable.

The conserved quantities of the Ostrovsky equation,

$$E(u) = \int_{\mathbf{R}} \frac{\beta}{2} u_x^2 + \frac{\gamma}{2} |\partial_x^{-1} u|^2 + F(u)\, dx \tag{3.1}$$

and

$$V(u) = \frac{1}{2} \int_{\mathbf{R}} u^2\, dx, \tag{3.2}$$

play a critical role in the analysis of the stability of solitary waves. With respect to E and V, solutions of the solitary wave equation (2.1) are critical points of the action functional L defined by

$$L(u) = E(u) - cV(u), \tag{3.3}$$

in the sense that $L'(\varphi) = 0$. This fact motivates considering the function $d(c)$ defined by

$$d(c) = d(\beta, c, \gamma) = E(\varphi) - cV(\varphi), \tag{3.4}$$

where φ is any element of $\mathscr{G}(\beta, c, \gamma)$. Using the relation

$$E(u) - cV(u) = \frac{1}{2} I(u) - \frac{1}{p+1} K(u), \tag{3.5}$$

which holds for all $u \in X_1$, it follows that

$$d(\beta, c, \gamma) = \frac{p-1}{2(p+1)} I(\varphi) = \frac{p-1}{2(p+1)} K(\varphi) = \frac{p-1}{2(p+1)} \left(m(\beta, c, \gamma)\right)^{\frac{p+1}{p-1}}. \tag{3.6}$$

This implies that d is well-defined, and its properties may be investigated by studying the function m. Using this approach, the following results were obtained in [7].

Theorem 3.1 *On the domain $\beta > 0$, $\gamma > 0$, $c < c_*$, the function d is continuous, strictly increasing in γ and β and strictly decreasing in c. For each fixed $\beta > 0$ and $\gamma > 0$, the partial derivative $\partial d/\partial c(\beta, c, \gamma)$ exists for all but countably many $c < c_*$. Similarly, $\partial d/\partial\beta$ and $\partial d/\partial\gamma$ exist for all but countably many β and γ, respectively. At points where these partial derivatives exist,*

$$\frac{\partial d}{\partial\beta} = \frac{1}{2}\int (\varphi_x)^2\,dx$$
$$\frac{\partial d}{\partial c} = -\frac{1}{2}\int \varphi^2\,dx$$
$$\frac{\partial d}{\partial\gamma} = \frac{1}{2}\int (\partial_x^{-1}\varphi)^2\,dx.$$

Remark 3.1 The uniqueness result in Theorem 2.3 implies that, in the case of the quadratic nonlinearity $f(u) = u^2$, d is differentiable for *all* (β, c, γ) satisfying $c < -c_*$.

The scaling identity satisfied by d is also a consequence of the relation between d and m.

Theorem 3.2 *Let $\beta > 0$, $\gamma > 0$ and $c < c_*$. For any $r > 0$ and $s > 0$ we have*

$$d(rs^2\beta, rc, rs^{-2}\gamma) = r^{\frac{p+1}{p-1}}sd(\beta, c, \gamma).$$

In terms of the function d, the main stability results of [7] may be summarized as follows.

Theorem 3.3 *Fix $\beta > 0$, $\gamma > 0$ and $c < c_*$. Then*

(i) *If $d_{cc}(\beta, c, \gamma) > 0$ then $\mathscr{G}(\beta, c, \gamma)$ is stable.*
(ii) *If $d_{cc}(\beta, c, \gamma) < 0$, then $\mathscr{O}_\varphi = \{\varphi(\cdot - y) : y \in \mathbf{R}\}$ is unstable for any $\varphi \in \mathscr{G}(\beta, c, \gamma)$.*

Part (i) of this result follows from a variational argument due to Cazenave and Lions [2], while part (ii) of this result is actually a consequence of a more general instability criterion. Using techniques of Goncalves-Ribeiro [4], the following was proved in [7].

Theorem 3.4 *Fix $\beta > 0$, $\gamma > 0$ and $c < c_*$. Let $\varphi \in \mathscr{G}(\beta, c, \gamma)$. Suppose there exists $\psi \in L^2(\mathbf{R})$ such that $\psi_x \in X_s$, $s > 3/2$, $\psi_{xx} \in X_1$ and the following conditions are satisfied.*

(i) $\langle V'(\varphi), \psi_x\rangle = 0.$
(ii) $\langle L''(\varphi)\psi_x, \psi_x\rangle < 0.$

Then $\mathscr{O}_\varphi$ is unstable.

Part (ii) of Theorem 3.3 follows by showing that when $d_{cc}(\beta, c, \gamma) < 0$, there exists a function ψ satisfying the hypotheses of Theorem 3.4. With the choice $\psi_x = \varphi + 2x\varphi_x$, one also obtains the following result.

Theorem 3.5 *Fix $\beta > 0$, $\gamma > 0$ and $c < c_*$. Then $\mathscr{O}_\varphi$ is unstable if either of the following conditions hold.*

(i) *$p > 5$, $c < 0$ and $0 < \gamma < \gamma_0$ for some small $\gamma_0 > 0$.*
(ii) *$p > 5 + 4\sqrt{2} \approx 10.657$ and $c < \left(\frac{p^2-10p-7}{(p-1)^2}\right) c_*$.*

Proof Part (i), and a slightly weaker version of part (ii), were proved in [7]. By Lemma 4.11 in [7], $\psi = \partial_x^{-1}(\varphi + 2x\varphi_x)$ satisfies the hypotheses of Theorem 3.4, and

$$\left\langle L''(\varphi)\psi_x, \psi_x \right\rangle = \frac{(p-1)(5-p)}{p+1} K(\varphi) + 16\gamma \int_{\mathbf{R}} (\partial_x^{-1}\varphi)^2 \, dx.$$

Using the identity $I(\varphi) = K(\varphi)$ and the Pohozaev identity

$$\int_{\mathbf{R}} \beta\varphi_x^2 + c\varphi^2 - 3\gamma(\partial_x^{-1}\varphi)^2 \, dx = -\frac{2}{p+1} K(\varphi) \tag{3.7}$$

one obtains

$$c \int_{\mathbf{R}} \varphi^2 \, dx - 2\gamma \int_{\mathbf{R}} (\partial_x^{-1}\varphi)^2 \, dx = -\frac{p+3}{2(p+1)} K(\varphi),$$

from which it follows that

$$\left\langle L''(\varphi)\psi_x, \psi_x \right\rangle = \frac{-p^2 + 10p + 7}{p+1} K(\varphi) + 8c \int_{\mathbf{R}} \varphi^2 \, dx.$$

By the elementary inequality $I(\varphi) \geq (c_* - c) \int_{\mathbf{R}} \varphi^2 \, dx$ it then follows that

$$\left\langle L''(\varphi)\psi_x, \psi_x \right\rangle \leq \left(\frac{-p^2 + 10p + 7}{p+1} + \frac{8c}{c_* - c} \right) K(\varphi).$$

Since $-p^2 + 10p + 7 < 0$ when $p > 5 + 4\sqrt{2}$, the term in parentheses is negative precisely when condition (ii) holds. □

Remark 3.2 As p increases, the region described in part (ii) of Theorem 3.5 approaches the entire region of existence $c < c_*$. We note that this result is not at all sharp since, in light of the numerical calculations of d_{cc}, it appears that we have instability for all $c < c_*$ even when $p > 5$.

As mentioned in the introduction, although Theorem 3.3 provides necessary and sufficient conditions for stability in terms of d, there are no known explicit formulas for d, and thus it is difficult to directly apply this result. However, since d and d_c are defined in terms of φ by Eq. (3.4) and Theorem (3.1), respectively, it is possible to obtain numerical approximations of d and d_c, and hence d_{cc}, via numerical approximations of the solitary waves. By doing so, the following conclusions were drawn in [6,7].

Observation 3.1 *Let* $f(u) = (-u)^p$.

(i) *When* $p = 2$ *or* $p = 3$*, solitary waves are stable for all* $c < c_*$.
(ii) *When* $p = 4$ *there exists* $\alpha_0(\approx 0.88)$ *such that solitary waves are stable for* $\frac{c}{c_*} < \alpha_0$ *and unstable for* $\alpha_0 < \frac{c}{c_*} < 1$.
(iii) *When* $p \geq 5$*, solitary waves are unstable for all* $c < c_*$.

While it is is some sense natural to consider the family of nonlinearities $(-u)^p$ for integer p, this family includes both odd and even functions, and the behavior of the function d differs depending on whether p is even or odd, particularly for c near c_*, where the solitary waves become highly oscillatory. It is thus more natural to consider the families of odd and even nonlinearities separately, as we do in the following section.

4 Homogeneous nonlinearities

In this section we discuss the class of nonlinearities that will be considered. The existence and stability results discussed in Sects. 2 and 3 apply to Eq. (1.1) where f is any homogeneous function with the property that

$$K(u) = -(p+1)\int_{\mathbf{R}} F(u)\,dx > 0$$

for some $u \in X_1$. Let $\mathscr{H}_p(\mathbf{R})$ denote the set of all functions f on $\mathbf{R}$ that are homogeneous of degree p, in the sense that for any $\lambda > 0$ we have $f(\lambda s) = \lambda^p f(s)$ for all $s \in \mathbf{R}$.

Lemma 4.1 $\mathscr{H}_p(\mathbf{R})$ *is a two-dimensional vector space.*

Proof Define $f_+(s) = (\max\{s, 0\})^p$ and $f_-(s) = (\max\{-s, 0\})^p$. Then $f_+, f_- \in \mathscr{H}_p(\mathbf{R})$ are clearly linearly independent, and given any $f \in \mathscr{H}_p(\mathbf{R})$, we have

$$f(s) = f(1)f_+(s) + f(-1)f_-(s)$$

for all $s \in \mathbf{R}$. □

It is somewhat more convenient to think of functions in $\mathscr{H}_p(\mathbf{R})$ in terms of the even and odd functions $f_e(s) = |s|^p$ and $f_o(s) = |s|^{p-1}s$. If we write

$$\mathscr{H}_p(\mathbf{R}) = \{a_e f_e + a_o f_o : a_e, a_o \in \mathbf{R}\}$$

then we have the following result.

Theorem 4.1 *Fix* $p \geq 2$*,* $\beta > 0$*,* $\gamma > 0$ *and* $c < c_*$*. Let* $f = a_e f_e + a_o f_o$*. Then there exist solutions of* (2.1) *in* X_1 *if and only if* $a_o < |a_e|$ *(equivalently,* $f(1) < 0$ *or* $f(-1) > 0$*).*

Proof First suppose $a_o < |a_e|$. Then

$$K(u) = -(p+1)\int_{\mathbf{R}} F(u)\,dx = \int_{\mathbf{R}} -(a_e|u|^p u + a_o|u|^{p+1})\,dx.$$

Without loss of generality, suppose $a_e < 0$. Then we have $a_e + a_o < 0$ and thus

$$K(u) = -(a_e + a_o)\int_{u>0} |u|^{p+1}\,dx + (a_e - a_o)\int_{u<0} |u|^{p+1}\,dx.$$

We claim that there exists $v \in X_1$ such that $K(v) > 0$. The existence result then follows from Theorem 2.1. To prove the claim, let ψ be any C^1 function that vanishes outside (0, 1) and is positive on (0, 1). Then for $A > 0$, set

$$v(x) = \begin{cases} A\psi(x) & 0 \le x \le 1 \\ -\psi((x-1)/A) & 1 \le x \le A+1 \\ 0 & \text{otherwise} \end{cases}$$

Then $\int_{\mathbf{R}} v(x)\,dx = 0$ so $v \in X_1$. On the other hand

$$K(v) = [-(a_e + a_o)A^{p+1} + (a_e - a_o)A]\int_0^1 \psi(x)^{p+1}\,dx.$$

Since $-(a_e + a_o) > 0$ and $\psi > 0$ on (0, 1) it follows that $K(v) > 0$ for A sufficiently large. This proves the claim.

Now suppose $a_o \ge |a_e|$. Then $a_e|s|^p u + a_o|s|^{p+1} \ge 0$ for all $s \in \mathbf{R}$, so $K(u) \le 0$ for all $u \in X_1$. If a solution $\varphi \in X_1$ of (2.1) existed, then multiplying by φ and integrating would yield $0 < I(\varphi) = K(\varphi) \le 0$, a contradiction. $\square$

We next observe that, without loss of generality, it suffices to consider only $f \in \mathscr{H}_p(\mathbf{R})$ for which $a_e \ge 0$, since if u solves (2.1) with $f = a_e f_e + a_o f_o$, then $v = -u$ solves (2.1) with $f = -a_e f_e + a_o f_o$. Also, by re-scaling $v = \alpha u$ for $\alpha > 0$ it suffices to consider only $f \in \mathscr{H}_p(\mathbf{R})$ for which $a_e^2 + a_o^2 = 1$. Hence the set of homogeneous nonlinearities of degree $p \ge 2$ that we shall consider can be parametrized by

$$f_\theta = \cos(\theta) f_e + \sin(\theta) f_o \tag{4.1}$$

where $-\pi/2 \le \theta < \pi/4$. See Fig. 2.

5 Boundary behavior of d

In this section we consider the behavior of d near the boundary of its domain. We first derive bounds on d as c approaches $c_* = 2\sqrt{\beta\gamma}$. The lower bound is given in the following result.

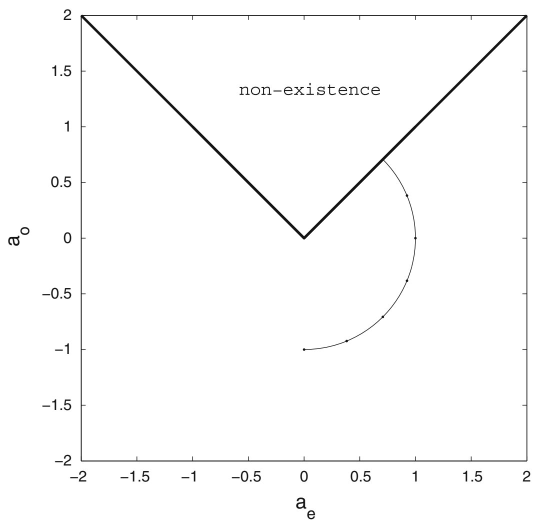

Fig. 2 The set of nonlinearities $f(s) = a_e|s|^p + a_o|s|^{p-1}s$ that will be considered

Lemma 5.1 *Fix* $p \geq 2$*,* $\beta > 0$ *and* $\gamma > 0$*. Then there exists a constant* $C > 0$ *such that for* $0 < c < c_*$ *we have*

$$d(c) \geq C\,(c_* - c)^{\frac{p+1}{p-1}}\,.$$

Proof We give two proofs of this fact. First, for $c > 0$ and any $u \in X_1$ we have

$$I(u) \geq C_{\beta,c,\gamma} \int\limits_{\mathbf{R}} u_x^2 + (\partial_x^{-1} u)^2\, dx = C_{\beta,c,\gamma}\|u\|_{X_1}^2,$$

where

$$C_{\beta,c,\gamma} = \frac{4\beta\gamma - c^2}{2(\beta + \gamma + \sqrt{(\beta-\gamma)^2 + c^2})}.$$

Observe that $C_{\beta,c,\gamma} \geq C(c_* - c)$ for c near c_*. Since $|K(u)| \leq C\|u\|_{L^{p+1}}^{p+1} \leq C\|u\|_{x_1}^{p+1}$ it then follows that

$$\frac{I(u)}{K(u)^{\frac{2}{p+1}}} \geq C(c_* - c)$$

for all $u \in X_1$ with $K(u) > 0$. Therefore $m(c) \geq C(c_* - c)$ and the lemma follows from relation (3.6).

Alternately, using Theorem 3.1 and (3.6), we have

$$d'(c) = -\frac{1}{2}\int_{\mathbf{R}} \varphi^2\, dx \geq \frac{-\frac{1}{2}I(\varphi)}{c_* - c} = -\frac{p+1}{p-1}\cdot\frac{d(c)}{c_* - c}, \tag{5.1}$$

so integrating yields

$$d(c) \geq d(0)\left(1 - \frac{c}{c_*}\right)^{\frac{p+1}{p-1}}.$$

□

This implies that, as c approaches c_*, d at best vanishes to order $(c_* - c)^{\frac{p+1}{p-1}}$. We next show that d does in fact vanish near $c = c_*$. Our main result is the following.

Theorem 5.1 *Suppose $p \geq 2$ and $f = a_e f_e + a_o f_o$ where $a_o < 0$ (equivalently $f(1) < f(-1)$). Then*

$$d(c) = O\left((c_* - c)^{\frac{p+3}{2(p-1)}}\right)$$

as c approaches c_*.

Corollary 5.1 *Suppose $2 \leq p < 5$ and $f = a_e f_e + a_o f_o$ where $a_o < 0$ ($f(1) < f(-1)$). Fix $\beta > 0$ and $\gamma > 0$. Then there exist c arbitrarily close to c_* for which $\mathscr{G}(\beta, c, \gamma)$ is stable.*

Proof Using (5.1), it follows that

$$0 > d'(c) \geq -\frac{p+1}{p-1}\cdot\frac{d(c)}{c_* - c} \geq -O\left((c_* - c)^{\frac{5-p}{2(p-1)}}\right).$$

Thus if $2 \leq p < 5$, $d'(c) \to 0$ as $c \to c_*$. Since $d'(c) < 0$ for all $c < c_*$, this implies that there must exist c arbitrarily close to c_* such that $d''(c) > 0$. The result then follows from part (i) of Theorem 3.3. □

To prove Theorem 5.1, we select trial functions to obtain upper bounds on the value of the quotient that defines m in expression (2.6). To motivate the choice of trial function, observe that if we ignore the nonlinear term in (2.1) and differentiate twice we obtain the linear equation

$$\beta u_{xxxx} + cu_{xx} + \gamma u = 0. \tag{5.2}$$

This equation has solutions $u = e^{rx}$ where $\beta r^4 + cr^2 + \gamma = 0$. When $c > 0$, the roots of this equation are $r = \pm(a \pm bi)$ where

$$a = \frac{1}{2}\left(\frac{c_* - c}{\beta}\right)^{1/2} \quad \text{and} \quad b = \frac{1}{2}\left(\frac{c_* + c}{\beta}\right)^{1/2}.$$

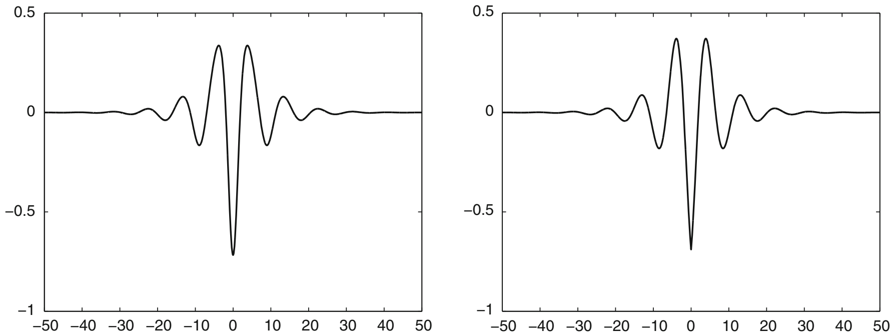

Fig. 3 The trial function u given by equation (5.3) (*right*) together with the numerically computed solitary wave (*left*) for $\beta = 1$, $\gamma = 1/4$ and $c = 0.9$, and $f(u) = u^2$

If we then make the choice

$$u = -\partial_x \left(e^{-a|x|} \sin(bx)\right) = -e^{-a|x|}(b\cos(bx) - a\sin(b|x|)), \tag{5.3}$$

it follows that $u \in X_1$ and satisfies the above linear equation for $x \neq 0$. As one can see from Fig. 3, the function u is a reasonable approximation of the exact solitary wave, the main difference being that u is not differentiable at $x = 0$.

A direct calculation reveals that

$$I(u) = \frac{1}{4}(c_* + c)(c_* - c)^{1/2} = O(c_* - c)^{1/2} \tag{5.4}$$

as c approaches c_*. We next need a lower bound on $K(u)$. This depends on the nonlinear term. We consider separately the purely even and purely odd nonlinearities.

Lemma 5.2 *Fix $\beta > 0$ and $\gamma > 0$. There exists some positive constants C_1 and C_2 such that for $c > 0$ near c_* we have*

(i) $K(u) \geq C_1(c_* - c)^{1/2}$ *if* $f = f_e$,
(ii) $K(u) \geq C_2(c_* - c)^{-1/2}$ *if* $f = -f_o$,

where u is the trial function given by (5.3).

Proof First consider $f = f_e$. Since $a \geq C(c_* - c)^{1/2}$, part (i) will follow once we show that $K(u) \geq Ca$ for c near c_*. If we write $b\cos(bx) - a\sin(bx) = \sqrt{a^2 + b^2}\cos(bx + \phi)$ where $\phi = \arctan(a/b)$ then we have

$$K(u) = -2\int_0^\infty |u|^p u\, dx = +2(a^2 + b^2)^{\frac{p+1}{2}} \int_0^\infty e^{-a(p+1)x} |\cos(bx + \phi)|^p \cos(bx + \phi)\, dx,$$

and after the change of variable $y = bx + \phi$ this becomes

$$\frac{2e^{a(p+1)\phi/b}}{b}(a^2 + b^2)^{\frac{p+1}{2}} \int_\phi^\infty e^{-a(p+1)y/b} |\cos(y)|^p \cos(y)\, dy.$$

As c approaches c_* the term outside the integral approaches $2(\gamma/\beta)^{p/4} > 0$, so we will henceforth ignore this term. We now write the integral as

$$\int_{\phi}^{0} e^{-a(p+1)y/b}|\cos(y)|^p \cos(y)\,dy + \sum_{k=0}^{\infty} \int_{k\pi}^{(k+1)\pi} e^{-a(p+1)y/b}|\cos(y)|^p \cos(y)\,dy.$$

The first term is negative, but bounded below by $-\phi = -\arctan(a/b) \geq -a/b$. In each term of the summation we make the change of variable $z = y - k\pi$ and then sum the resulting geometric series to obtain

$$\frac{1}{1 + e^{-a(p+1)\pi/b}} \int_0^{\pi} e^{-a(p+1)z/b}|\cos(z)|^p \cos(z)\,dz.$$

The remaining integral we rewrite as

$$\int_0^{\pi/2} e^{-a(p+1)z/b}|\cos(z)|^p \cos(z)\,dz + \int_{\pi/2}^{\pi} e^{-a(p+1)z/b}|\cos(z)|^p \cos(z)\,dz.$$

Making the change of variable $y = \pi - z$ in the second integral and combining the two integrals gives

$$\int_0^{\pi/2} (e^{-a(p+1)z/b} - e^{-a(p+1)(\pi-z)/b}) \cos(z)^{p+1}\,dz.$$

Since

$$\lim_{a\to 0} \frac{e^{-a(p+1)z/b} - e^{-a(p+1)(\pi-z)/b}}{a} = \frac{p+1}{b} \cdot (\pi - 2z)$$

uniformly in z on $[0, \pi/2]$ the integral approaches

$$\frac{a}{b} \int_0^{\pi/2} (p+1)(\pi - 2z)|\cos(z)|^p \cos(z)\,dz$$

as $c \to c_*$. Since

$$\int_0^{\pi/2} (p+1)(\pi - 2z) \cos(z)^{p+1}\,dz > \pi(p+1) \int_0^{\pi/2} \left(1 - \frac{2z}{\pi}\right)^{p+2} dx = \frac{(p+1)\pi^2}{2(p+3)}$$
$$> \frac{\pi^2}{4}$$

for all $p > 1$, it follows that as $c \to c_*$ we have

$$\frac{1}{1 + e^{-a(p+1)\pi/b}} \int_0^{\pi} e^{-a(p+1)z/b} |\cos(z)|^p \cos(z)\, dz \geq \frac{1}{2} \cdot \frac{1}{4}\pi^2 \cdot \frac{a}{b},$$

and therefore

$$\int_{\phi}^{\infty} e^{-a(p+1)y/b} |\cos(y)|^p \cos(y)\, dy \geq \frac{1}{2} \left(\frac{1}{4}\pi^2 - 2 \right) \frac{a}{b},$$

which implies that

$$K(u) \geq Ca$$

as claimed.

Next suppose $f = -f_o$. Using the calculations above, we have

$$\begin{aligned} K(u) = + \int_{\mathbf{R}} |u|^{p+1}\, dx &= \frac{2e^{a(p+1)\phi/b}}{b} (a^2 + b^2)^{\frac{p+1}{2}} \int_{\phi}^{\infty} e^{-a(p+1)y/b} |\cos(y)|^{p+1}\, dy \\ &\geq \frac{2e^{a(p+1)\phi/b}}{b} (a^2 + b^2)^{\frac{p+1}{2}} \int_{\pi/2}^{\infty} e^{-a(p+1)y/b} |\cos(y)|^{p+1}\, dy. \end{aligned}$$

Writing the integral as

$$\sum_{k=1}^{\infty} \int_{(k-\frac{1}{2})\pi}^{(k+\frac{1}{2})\pi} e^{-a(p+1)y/b} |\cos(y)|^{p+1}\, dy,$$

and making the change of variable $z = y - k\pi$, this becomes

$$\begin{aligned} &\sum_{k=1}^{\infty} e^{-a(p+1)\pi k/b} \int_{-\pi/2}^{\pi/2} e^{-a(p+1)z/b} \cos(z)^{p+1}\, dz \\ &= \frac{e^{a(p+1)\pi/b}}{e^{a(p+1)\pi/b} - 1} \int_{-\pi/2}^{\pi/2} e^{-a(p+1)z/b} \cos(z)^{p+1}\, dz. \end{aligned}$$

For small a this is approximately

$$\frac{b}{a(p+1)\pi}\int_{-\pi/2}^{\pi/2}\cos(z)^{p+1}\,dz \geq C'a^{-1} = C(c_* - c)^{-1/2}.$$

□

Proof of Theorem 5.1 Let u be given by (5.3), and define

$$K_e(u) = -\int_{\mathbf{R}} |u|^p u\,dx$$
$$K_o(u) = \int_{\mathbf{R}} |u|^{p+1}\,dx.$$

By part (ii) of Lemma 5.2 we have $K_o(u) \geq C_2(c_* - c)^{-1/2}$, and it follows by the same calculations used to prove part (i) of Lemma 5.2 that $|K_e(u)| \leq C_3(c_* - c)^{1/2}$ for some constant C_3. Thus if $f = a_e f_e + a_o f_o$ where $a_o < 0$, then we have

$$\begin{aligned} K(u) = a_e K_e(u) - a_o K_o(u) &\geq (-a_o)C_2(c_* - c)^{-1/2} - |a_e|C_3(c_* - c)^{1/2} \\ &\geq C_4(c_* - c)^{-1/2} \end{aligned}$$

for c near c_*. Thus

$$m(\beta, c, \gamma) \leq \frac{I(u)}{K(u)^{\frac{2}{p+1}}} \leq \frac{\frac{1}{4}(c_* + c)(c_* - c)^{1/2}}{[C(c_* - c)^{-1/2}]^{\frac{2}{p+1}}} = O(c_* - c)^{\frac{p+3}{2(p+1)}}$$

and it follows from (3.6) that

$$d(c) = \frac{p-1}{2(p+1)} m(\beta, c, \gamma)^{\frac{p+1}{p-1}} = O(c_* - c)^{\frac{p+3}{2(p-1)}},$$

as claimed. □

Remark 5.1 In view of the numerical results of Sect. 6, it appears that the bound in Theorem 5.1 is sharp. See Fig. 4.

As a consequence of the first part of Lemma 5.2 we have the following bound on d for even nonlinearities.

Corollary 5.2 *Suppose $f = kf_e$ for some $k \neq 0$. Then*

$$d(c) = O\left((c_* - c)^{1/2}\right)$$

as c approaches c_*.

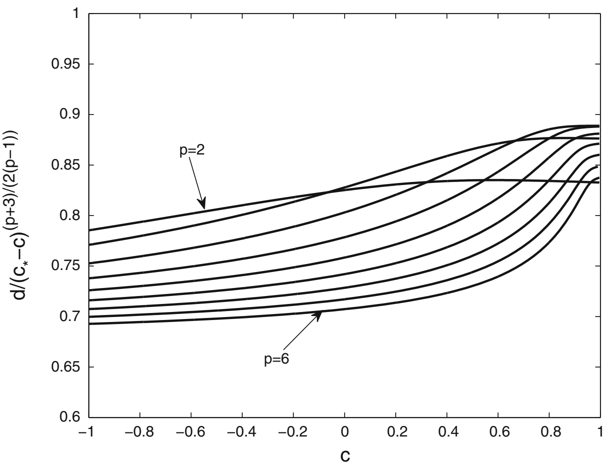

Fig. 4 Plots of $d(c)/(c_* - c)^{\frac{p+3}{2(p-1)}}$ for the odd nonlinearity $f(u) = -|u|^{p-1}u$. Here $\beta = 1$ and $\gamma = 1/4$, so $c_* = 1$. Plots for other nonlinearities with $a_o < 0$ are similar. These illustrate the sharpness of the bound in Theorem 5.1

Proof For $k < 0$, Lemma 5.2 implies that, for u defined by (5.3), we have

$$m(\beta, c, \gamma) \leq \frac{I(u)}{K(u)^{\frac{2}{p+1}}} \leq \frac{\frac{(c_*+c)(c_*-c)^{1/2}}{2\sqrt{\beta}}}{[C(c_* - c)]^{\frac{1}{p+1}}} = O(c_* - c)^{\frac{p-1}{2(p+1)}}$$

and it follows from (3.6) that

$$d(c) = \frac{p-1}{2(p+1)} m(\beta, c, \gamma)^{\frac{p+1}{p-1}} = O(c_* - c)^{\frac{1}{2}}.$$

For $k > 0$, the same estimate follows by using the trial function $-u$. □

Remark 5.2 The numerical evidence obtained by the methods of Sect. 6 suggest that the bound in Corollary 5.2 is *not* sharp. Rather, it appears that the optimal bound is

$$d(c) = O\left((c_* - c)^{\frac{p+1}{2(p-1)}}\right).$$

See Fig. 5. This bound would imply stability for c near c_* when $2 \leq p < 3$. However, it is not clear what choice of trial function is required to prove this bound.

Remark 5.3 When $0 < a_0 < |a_e|$, the behavior of d as c approaches c_* is less clear. In fact the function u defined by (5.3) is no longer a valid trial function for c near c_* since $K(u)$ becomes negative as c approaches c_*. Numerical results seem to indicate that when $p > 2$, $d''(c)$ is negative for c near c_*.

We next derive bounds on d as γ approaches zero. The idea once again is to use appropriately chosen trial functions. By Theorem 2.4, the Ostrovsky solitary waves

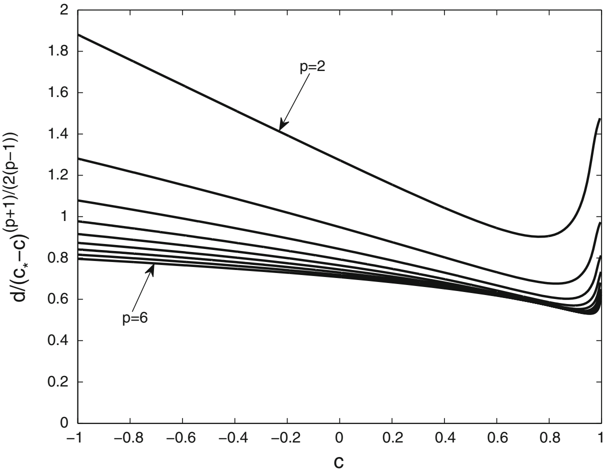

Fig. 5 Plots of $d(c)/(c_* - c)^{\frac{p+1}{2(p-1)}}$ for the even nonlinearity $f(u) = |u|^p$. Here $\beta = 1$ and $\gamma = 1/4$, so $c_* = 1$

converge in H^1 to the KdV solitary waves, so it would be natural to use the KdV solitary waves as trial functions. However, since the KdV solitary waves have nonzero mass, they are inadmissible as trial functions in the Ostrovsky variational problem. To remedy this, we use a trial function obtained by truncating a KdV solitary wave and giving it a tail that satisfies the linear equation (2.8) in such a way that the resulting function has zero mass. We thereby obtain the following result.

Theorem 5.2 *Fix $\beta > 0$ and $c < 0$. Then*

$$d(\beta, c, \gamma) = d(\beta, c, 0) + O(\sqrt{\gamma})$$

as γ approaches zero.

Proof For $c < 0$ and $\gamma > 0$ small, solutions of the linear equation take the form $e^{\pm\lambda_1 x}$, $e^{\pm\lambda_2 x}$ where

$$\lambda_1 = \left(\frac{-c+\sqrt{c^2-4\beta\gamma}}{2\beta}\right)^{1/2}$$
$$\lambda_2 = \left(\frac{-c-\sqrt{c^2-4\beta\gamma}}{2\beta}\right)^{1/2}.$$

As $\gamma \to 0$ we have

$$\lambda_1 = \sqrt{-c/\beta} + O(\gamma)$$
$$\lambda_2 \quad = O(\sqrt{\gamma}).$$

Next let φ_0 denote the ground state solution of the KdV equation

$$\beta\varphi_{xx} + c\varphi = f(\varphi)$$

and define

$$\psi_0(x) = \int_0^x \varphi_0(y)\,dy$$

for $x \geq 0$. Then for $R > 0$ we set

$$v_R(x) = \begin{cases} \psi_0(x) & x \leq R \\ a_1(R)e^{-\lambda_1 x} + a_2(R)e^{-\lambda_2 x} & x \geq R \end{cases} \tag{5.5}$$

where $a_1(R)$ and $a_2(R)$ are chosen so as to make v_R differentiable on $(0, \infty)$. A straightforward calculation reveals that

$$\begin{aligned} a_1(R) &= -\tfrac{e^{\lambda_1 R}}{\lambda_1 - \lambda_2}\left(\lambda_2\psi_0(R) + \varphi_0(R)\right) \\ a_2(R) &= \tfrac{e^{\lambda_2 R}}{\lambda_1 - \lambda_2}\left(\lambda_1\psi_0(R) + \varphi_0(R)\right). \end{aligned}$$

Next we set $u_R(x) = v_R'(x)$ for $x > 0$ and extend u_R to be even on $\mathbf{R}$. Then for each $R > 0$ we have $u_R \in X_1$. Now we compute

$$\begin{aligned} I(u_R; \beta, c, \gamma) &= 2\int_0^\infty \beta(u_R)_x^2 - cu_R^2 + \gamma(v_R)^2\,dx \\ &= 2\int_0^R \beta(\varphi_0)_x^2 - c\varphi_0^2 + \gamma(v_R)^2\,dx + 2\int_R^\infty \beta[(u_R)_x]^2 - cu_R^2 + \gamma(v_R)^2\,dx \\ &= I(\varphi_0; \beta, c, 0) - 2\int_R^\infty \beta(\varphi_0)_x^2 - c\varphi_0^2\,dx + 2\int_0^R \gamma(v_R)^2\,dx \\ &\quad + 2\int_R^\infty \beta[(v_R)_{xx}]^2 - c[(v_R)_x]^2 + \gamma(v_R)^2\,dx \end{aligned}$$

Observe that for $0 < x < R$, $v_R(x)$ is bounded independently of R, so

$$I_1 \equiv \int_0^R \gamma(v_R)^2\,dx \leq C\gamma R \tag{5.6}$$

for some constant C. Next, since φ_0 and $(\varphi_0)_x$ decay like $e^{-x\sqrt{-c/\beta}}$ as $x \to \infty$, we have

$$I_2 \equiv \int_R^\infty \beta(\varphi_0)_x^2 - c\varphi_0^2\, dx \le Ce^{-2R\sqrt{-c/\beta}} \tag{5.7}$$

for any $R > 0$. Finally, since v_R satisfies the linear equation (2.8) for $x > R$, we have

$$\begin{aligned} I_3 &\equiv \int_R^\infty \beta[(v_R)_{xx}]^2 - c[(v_R)_x]^2 + \gamma(v_R)^2\, dx \\ &= -\beta v_R'(R)v_R''(R) + \beta v_R(R)v_R'''(R) + cv_R(R)v_R'(R). \end{aligned}$$

Using the definition of v_R and the values of $a_1(R)$ and $a_2(R)$ above, this becomes

$$\begin{aligned} I_3 = \beta[&\lambda_1\lambda_2(\lambda_1+\lambda_2)\psi_0^2(R) + (\lambda_1+\lambda_2)^2\psi_0(R)\varphi_0(R) + (\lambda_1+\lambda_2)\varphi_0^2(R)] \\ &+c\psi_0(R)\varphi_0(R). \end{aligned} \tag{5.8}$$

Recalling that $\lambda_1 = O(1)$ and $\lambda_2 = O(\sqrt{\gamma})$ as $\gamma \to 0$ and that $\psi_0(R)$ is bounded independently of R, it follows that the first term in (5.8) is $O(\sqrt{\gamma})$. By (5.6) we will have $I_1 = O(\sqrt{\gamma})$ if we choose $R = \gamma^{-1/2}$. Since φ_0 decays exponentially, it follows from (5.7) that $I_2 = O(\sqrt{\gamma})$, and that the remaining terms in (5.8) are $O(\sqrt{\gamma})$. Hence $I_1 + I_2 + I_3 = O(\sqrt{\gamma})$, so that

$$I(u_R; \beta, c, \gamma) = I(\varphi_0; \beta, c, 0) + O(\sqrt{\gamma}).$$

Next, we compute

$$K(u_R) - K(\varphi_0) = 2\int_R^\infty F(u_R) - F(\varphi_0)\, dx.$$

We bound the first term by

$$\begin{aligned} \left|\int_R^\infty F(u_R)\, dx\right| &\le C\int_R^\infty \left|\lambda_1 a_1(R)e^{-\lambda_1 x}\right|^{p+1} + \left|\lambda_2 a_2(R)e^{-\lambda_2 x}\right|^{p+1} dx \\ &\le C\left(\lambda_1^p|a_1(R)e^{-\lambda_1 R}|^{p+1} + \lambda_2^p|a_2(R)e^{-\lambda_2 R}|^{p+1}\right) \\ &\le \frac{C}{\lambda_1 - \lambda_2}\left(\lambda_1^p|\lambda_2\psi_0(R) + \varphi_0(R)|^{p+1} + \lambda_2^p|\lambda_1\psi_0(R) + \varphi_0(R)|^{p+1}\right). \end{aligned}$$

For $R = \gamma^{-1/2}$, it then follows that this term is $O(\gamma^{p/2})$. The exponential decay of φ_0 implies that

$$\left| \int_R^\infty F(\varphi_0)\,dx \right| = O(\gamma^{p/2})$$

as well. Hence we have

$$K(u_R) \geq K(\varphi_0) - O(\gamma^{p/2}),$$

and thus

$$K(u_R)^{\frac{2}{p+1}} \geq K(\varphi_0)^{\frac{2}{p+1}} - O(\gamma^{p/2}).$$

Combining this with the estimate on $I(u_R)$, we have

$$\begin{aligned} d(\beta, c, \gamma) \leq \frac{I(u_R)}{K(u_R)^{\frac{2}{p+1}}} \leq \frac{I(\varphi_0) + O(\sqrt{\gamma})}{K(\varphi_0)^{\frac{2}{p+1}} - O(\gamma^{p/2})} &= \frac{I(\varphi_0)}{K(\varphi_0)^{\frac{2}{p+1}}} + O(\sqrt{\gamma}) \\ &= d(\beta, c, 0) + O(\sqrt{\gamma}). \end{aligned}$$

Since d is increasing in γ, we have $d(\beta, c, \gamma) \geq d(\beta, c, 0)$ and the proof is complete. □

Corollary 5.3 *Fix $\beta > 0$ and $c < 0$. Then*

$$d_\gamma(\beta, c, \gamma) = O(\gamma^{-1/2})$$

as γ approaches zero.

Proof For $\gamma > 0$ we have

$$\begin{aligned} d(\beta, c, \gamma) - d(\beta, c, 0) &= \frac{p-1}{2(p+1)} \big(I(\varphi_\gamma; \beta, c, \gamma) - I(\varphi_0; \beta, c, 0) \big) = \frac{p-1}{2(p+1)} \\ &\quad \times \left(I(\varphi_\gamma; \beta, c, 0) - I(\varphi_0; \beta, c, 0) + \gamma \int_{\mathbf{R}} (\partial_x^{-1} \varphi_\gamma)^2\,dx \right) \end{aligned}$$

where $\varphi_\gamma \in \mathcal{G}(\beta, c, \gamma)$ and $\varphi_0 \in \mathcal{G}(\beta, c, 0)$. Since d is strictly increasing in γ, we have

$$K(\varphi_0) = \frac{2(p+1)}{p-1} d(\beta, c, 0) < \frac{2(p+1)}{p-1} d(\beta, c, \gamma) = K(\varphi_\gamma),$$

so $K(\alpha\varphi_\gamma) = K(\varphi_0)$ where $\alpha = \left(\frac{d(\beta,c,0)}{d(\beta,c,\gamma)}\right)^{\frac{1}{p+1}} < 1$. By the variational characterization of φ_0, this implies $I(\varphi_0; \beta, c, 0) < I(\varphi_\gamma; \beta, c, 0)$, and thus

$$d(\beta, c, \gamma) - d(\beta, c, 0) > \frac{p-1}{2(p+1)}\gamma \int_{\mathbf{R}} (\partial_x^{-1}\varphi_\gamma)^2\, dx = \frac{p-1}{p+1}\gamma d_\gamma(\beta, c, \gamma).$$

The result then follows from Theorem 5.2. □

Corollary 5.4 *Let $p \geq 2$ and fix $\beta > 0$ and $c < 0$. Suppose d is twice differentiable on its domain.*

(i) *If $p < 5$, then there exist $\gamma > 0$ arbitrarily close to zero such that $\mathscr{G}(\beta, c, \gamma)$ is stable.*
(ii) *If $p > 5$, then there exist $\gamma > 0$ arbitrarily close to zero such that $\mathscr{O}_\varphi$ is unstable for any $\varphi \in \mathscr{G}(\beta, c, \gamma)$.*

Proof First fix $c_0 < 0$. Then for $c < 0$, Theorem 3.2 with $s^{-2} = r = c_0/c$ implies

$$d(\beta, c, \gamma) = (c/c_0)^{\frac{p+3}{2(p-1)}} d(\beta, c_0, (c_0/c)^2\gamma).$$

Setting $q = \frac{p+3}{2(p-1)}$ and differentiating with respect to c then gives

$$d_c(\beta, c, \gamma) = -\frac{2\gamma}{c_0}(c/c_0)^{q-3} d_\gamma(\beta, c_0, (c_0/c)^2\gamma) + \frac{1}{c_0} q (c/c_0)^{q-1} d(\beta, c_0, (c_0/c)^2\gamma)$$

and

$$\begin{aligned} d_{cc}(\beta, c, \gamma) = &\frac{4\gamma^2}{c_0^2}(c/c_0)^{q-6} d_{\gamma\gamma}(\beta, c_0, (c_0/c)^2\gamma) \\ &-\frac{2\gamma}{c_0^2}(2q-3)(c/c_0)^{q-4} d_\gamma(\beta, c_0, (c_0/c)^2\gamma) \\ &+\frac{1}{c_0^2} q(q-1)(c/c_0)^{q-2} d(\beta, c_0, (c_0/c)^2\gamma) \end{aligned}$$

Setting $c = c_0$, this reduces to

$$d_{cc}(\beta, c, \gamma) = \frac{1}{c^2}\left(4\gamma^2 d_{\gamma\gamma}(\beta, c, \gamma) - 2\gamma(2q-3) d_\gamma(\beta, c, \gamma) + q(q-1) d(\beta, c, \gamma)\right). \tag{5.9}$$

We now let γ approach zero. By Theorem 5.2 and Corollary 5.3 we have

$$\lim_{\gamma \to 0} -2\gamma(2q-3) d_\gamma(\beta, c, \gamma) + q(q-1) d(\beta, c, \gamma) = q(q-1) d(\beta, c, 0).$$

To handle the $\gamma^2 d_{\gamma\gamma}(\beta, c, \gamma)$ term we first define

$$g(\gamma) = \begin{cases} \gamma^2 d_\gamma(\beta, c, \gamma) & \gamma > 0 \\ 0 & \gamma = 0. \end{cases}$$

Then g is differentiable for $\gamma > 0$, and by Corollary 5.3, g is continuous for $\gamma \geq 0$. Therefore, by the mean value theorem, for each $j \in \mathbf{N}$ there exists $\gamma_j \in (0, 1/j)$ such that

$$g'(\gamma_j) = \frac{g(1/j)}{1/j} = \frac{1}{j} d_\gamma(\beta, c, 1/j).$$

By Corollary 5.3 it follows that $\lim_{j\to\infty} g'(\gamma_j) = 0$. But since $g'(\gamma) = \gamma^2 d_{\gamma\gamma} + 2\gamma d_\gamma$ it then follows that

$$\lim_{j\to\infty} \gamma_j^2 d_{\gamma\gamma}(\beta, c, \gamma_j) = \lim_{j\to\infty} g'(\gamma_j) - 2\gamma_j d_\gamma(\beta, c, \gamma_j) = 0.$$

Thus, for j sufficiently large, $d_{cc}(\beta, c, \gamma_j)$ has the same sign as $q(q-1)d(\beta, c, 0)$. Since $d(\beta, c, 0) > 0$ and $q > 0$, this is determined by the sign of $q - 1$, which is positive when $p < 5$ and negative when $p > 5$. This completes the proof. □

6 Numerical computations

In this section we present numerical results which extend those in [7] to the class of general homogeneous nonlinearities discussed in Sect. 4. We employ the same strategy here as in [5,7]. Because the values of d and d_c are given in terms of φ by (3.4) and Theorem 3.1, by numerically approximating the solitary waves we obtain numerical approximations of d, d_c, and hence d_{cc} by varying c. By the scaling property of d, the sign of d_{cc} is constant within $\Gamma_r = \{(\beta, c, \gamma) : c = r\sqrt{\beta\gamma}\}$ for each fixed $r < 2$. It therefore suffices to fix $\beta = 1$ and perform these computations over the segments

$$\begin{aligned} S_1 &= \{(c, \gamma) : -1 \leq c < 1, \gamma = 1/4\} \\ S_2 &= \{(c, \gamma) : c = -1, 0 < \gamma \leq 1/4\}. \end{aligned}$$

See Fig. 6.

On S_1 we approximate d_{cc} directly in terms of the numerically computed values of d_c using a centered difference, while on S_2 we do the same for $d_{\gamma\gamma}$ in terms of d_γ, and then apply formula (5.9) to compute d_{cc}. Two methods were used to compute the solitary wave profiles.

Spectral Method

For most of the solitary wave computations we used following spectral method due to Petviashvili. Using the form (2.2) of the solitary wave equation and setting $\psi_{xx} = \varphi$ we have

$$\beta\psi_{xxxx} + c\psi_{xx} + \gamma\psi = f(\psi_{xx}) = f(\varphi). \tag{6.1}$$

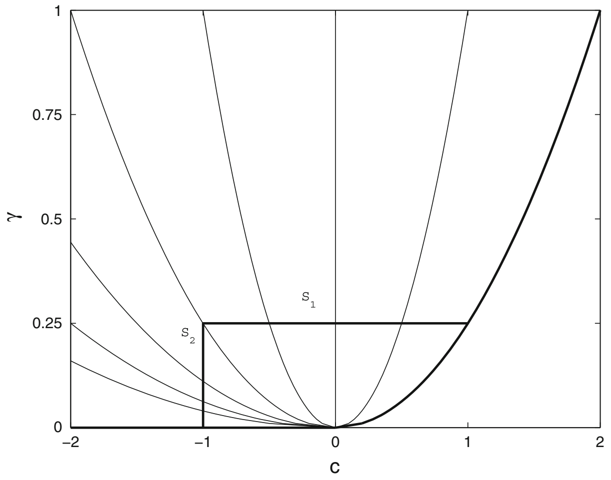

Fig. 6 The domain of d and the segments S_1 and S_2 over which the numerical computations were performed

Taking the Fourier transform gives

$$\hat{\psi} = \frac{\widehat{f(\varphi)}}{\beta\xi^4 - c\xi^2 + \gamma}.$$

We thus perform the iteration

$$\hat{\psi}_{k+1} = M_k^{\alpha} \frac{\widehat{f(\varphi_k)}}{\beta\xi^4 - c\xi^2 + \gamma}, \qquad \varphi_{k+1} = \partial_x^2 \psi_{k+1},$$

where

$$M_k = \frac{\int_{\mathbf{R}} (\beta\xi^4 - c\xi^2 + \gamma)\hat{\psi}_k \overline{\hat{\varphi}_k}\, d\xi}{\int_{\mathbf{R}} \widehat{f(\varphi_k)} \overline{\hat{\varphi}_k}\, d\xi}$$

is the stabilizing factor introduced by Petviashvili. The convergence properties of this algorithm were studied in [12], where it was shown that the rate of convergence is fastest when $\alpha = p/(p-1)$. In most cases, this method converged quite rapidly. However, when $f(s) = a_e|s|^p + a_o|s|^{p-1}s$ with $a_o > 0$ and c near c_* the algorithm failed to converge. In these cases we then resorted to the following shooting method to compute the solitary wave profiles.

Shooting Method

This is a modification of the method used in [5,7] to compute solitary waves of the fifth-order KdV equation and Ostrovsky equation, respectively. Using the form (6.1) of the solitary wave equation, and setting $\vec{y} = (\psi, \psi', \psi'', \psi''')$, we obtain the system

$$\frac{d\vec{y}}{dx} = A\vec{y} + \vec{g}(\vec{y}) \tag{6.2}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\gamma/\beta & 0 & -c\beta & 0 \end{pmatrix}, \qquad \vec{g}(\vec{y}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ f(y_3) \end{pmatrix}.$$

Multiplying (6.1) by ψ''' and integrating gives

$$\frac{\beta}{2}(\psi''')^2 + \frac{c}{2}(\psi'')^2 + \gamma\psi''\psi - \frac{\gamma}{2}(\psi')^2 = F(\psi'').$$

Without loss of generality we may assume φ has a local extremum at $x = 0$, so $\psi'''(0) = \varphi'(0) = 0$. Then setting $\alpha_1 = \psi'(0)$ and $\alpha_2 = \psi''(0)$, we have

$$\psi(0) = \frac{F(\alpha_2) - \frac{c}{2}\alpha_2^2 + \frac{\gamma}{2}\alpha_1^2}{\gamma\alpha_2} = \frac{1}{\gamma}\left(\frac{f(\alpha_2)}{p+1} - \frac{c}{2} + \frac{\gamma\alpha_1^2}{2\alpha_2}\right).$$

We then use MATLAB's Runge–Kutta–Fehlberg solver to solve (6.2) over a spatial domain $[-X, X]$ with initial data

$$\vec{y}_0 = \left(\frac{1}{\gamma}\left(\frac{f(\alpha_2)}{p+1} - \frac{c}{2} + \frac{\gamma\alpha_1^2}{2\alpha_2}\right), \alpha_1, \alpha_2, 0\right).$$

If we denote by E_s and E_u the stable and unstable subspaces associated with the linear system $\frac{d\vec{y}}{dx} = A\vec{y}$, then there exist matrices A_s and A_u such that for any $\vec{y} \in \mathbf{R}^4$ we have $\vec{y} = A_s\vec{y} + A_u\vec{u}$, where $A_s\vec{y} \in E_s$ and $A_u\vec{y} \in E_u$. By the stable manifold theorem, a solution $\vec{y}(x)$ of (6.2) satisfies $A_u\vec{y}(x) \to 0$ as $x \to \infty$ and $A_s\vec{y}(x) \to 0$ as $x \to -\infty$. We therefore define the shooting function by

$$S(\alpha_1, \alpha_2) = \|A_u\vec{y}(X)\|^2 + \|A_s\vec{y}(-X)\|^2.$$

To apply Newton's method to the function S, we also need to solve for $\frac{d\vec{y}}{d\alpha_1}$ and $\frac{d\vec{y}}{d\alpha_2}$ via the auxiliary system

$$\frac{d\vec{w}}{dx} = A\vec{w} + \vec{h}(\vec{y}, \vec{w}), \qquad \vec{h}(\vec{y}, \vec{w}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ f'(y_3)w_3 \end{pmatrix},$$

with initial data

$$\left(\frac{\alpha_1}{\alpha_2}, 1, 0, 0\right) \quad \text{and} \quad \left(\frac{1}{\gamma}\left(\frac{f'(\alpha_2)}{p+1} - \frac{c}{2} - \frac{\gamma\alpha_1^2}{2\alpha_2^2}\right), 0, 1, 0\right).$$

Table 1 Regions of instability for the nonlinearity $f(s) = \cos(\theta)|s|^p + \sin(\theta)|s|^{p-1}s$

p	$\theta = -\pi/2$	$\theta = -3\pi/8$	$\theta = -\pi/4$	$\theta = -\pi/8$	$\theta = 0$	$\theta = \pi/8$
2	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
2.5	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$0.98 < \frac{c}{c_*} < 1$
3	$\emptyset$	$\emptyset$	$\emptyset$	$0.95 < \frac{c_*}{c} < 0.97$	$0.94 < \frac{c}{c_*} < 1$	$0.95 < \frac{c}{c_*} < 1$
3.5	$\emptyset$	$\emptyset$	$\emptyset$	$0.93 < \frac{c}{c_*} < 0.98$	$0.92 < \frac{c}{c_*} < 1$	$0.92 < \frac{c}{c_*} < 1$
4	$\emptyset$	$\emptyset$	$0.89 < \frac{c}{c_*} < 0.98$	$0.88 < \frac{c}{c_*} < 0.99$	$0.88 < \frac{c}{c_*} < 1$	$0.89 < \frac{c}{c_*} < 1$
4.5	$0.55 < \frac{c}{c_*} < 0.9$	$0.65 < \frac{c}{c_*} < 0.98$	$0.71 < \frac{c}{c_*} < 0.99$	$0.74 < \frac{c}{c_*} < 0.995$	$0.77 < \frac{c}{c_*} < 1$	$0.81 < \frac{c}{c_*} < 1$
5	$\frac{c}{c_*} < 1$	$\frac{c}{c_*} < 1$	$\frac{c}{c_*} < 1$	$\frac{c}{c_*} < 1$	$\frac{c}{c_*} < 1$	$\frac{c}{c_*} < 1$
5.5	$\frac{c}{c_*} < 1$	$\frac{c}{c_*} < 1$	$\frac{c}{c_*} < 1$	$\frac{c}{c_*} < 1$	$\frac{c}{c_*} < 1$	$\frac{c}{c_*} < 1$
6	$\frac{c}{c_*} < 1$	$\frac{c}{c_*} < 1$	$\frac{c}{c_*} < 1$	$\frac{c}{c_*} < 1$	$\frac{c}{c_*} < 1$	$\frac{c}{c_*} < 1$

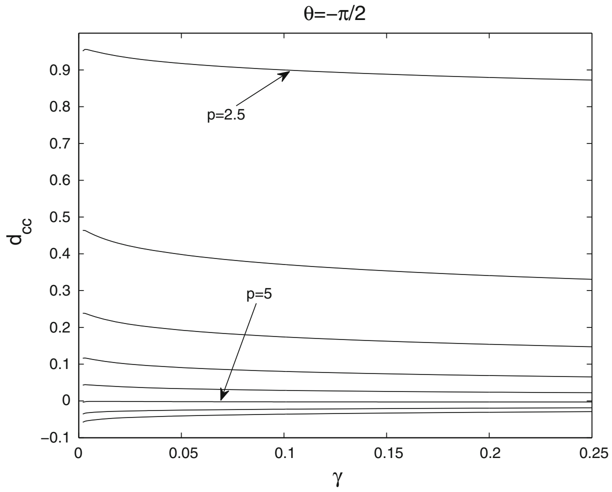

Fig. 7 Plots of d_{cc} for the odd nonlinearity $f(s) = -|s|^{p-1}s$ over the segment S_2, where p varies from 2 to 6 by multiples of $\frac{1}{2}$. Observe that when $p < 5$, d_{cc} is positive on S_2 and when $p \geq 5$, d_{cc} is negative on S_2

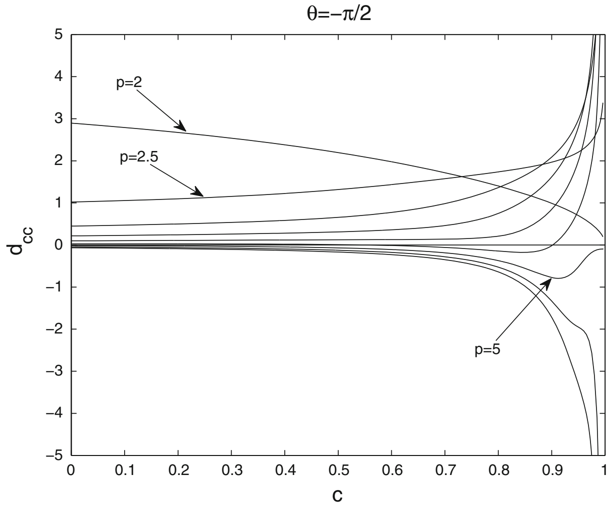

Fig. 8 Plots of d_{cc} for the odd nonlinearity $f(s) = -|s|^{p-1}s$ over S_1, where p varies from 2 to 6 by multiples of $\frac{1}{2}$. When $p \leq 4$, $d_{cc} > 0$ everywhere, and when $p \geq 5$, $d_{cc} < 0$ everywhere. When $p = 4.5$, d_{cc} changes sign twice

The method then proceeds by applying Newton's method to S, while incrementing X, until $\|\vec{y}(X)\| + \|\vec{y}(-X)\|$ is sufficiently small.

Results The computations described above were performed for the general homogeneous nonlinearity $f(u) = \cos(\theta)|u|^p + \sin(\theta)|u|^{p-1}u$ where p varied from 2 to 6 by multiples of 0.5 and θ varied from $-\pi/2$ to $\pi/8$ by multiples of $\pi/8$. The choice of θ was made in order to include nonlinearities with $a_o < 0$, $a_o = 0$ (the purely

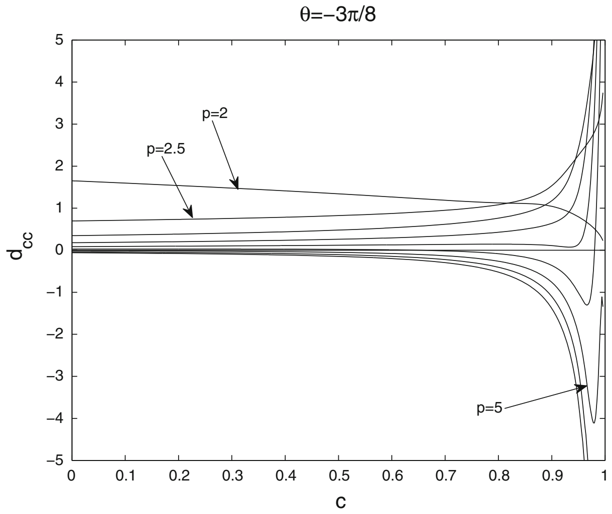

Fig. 9 Plots of d_{cc} over S_1 for the nonlinearity given by (4.1) with $\theta = -\frac{3}{8}\pi$, where p varies from 2 to 6 by multiples of $\frac{1}{2}$. When $p \leq 4$, $d_{cc} > 0$ everywhere, and when $p \geq 5$, $d_{cc} < 0$ everywhere. When $p = 4.5$, d_{cc} changes sign twice

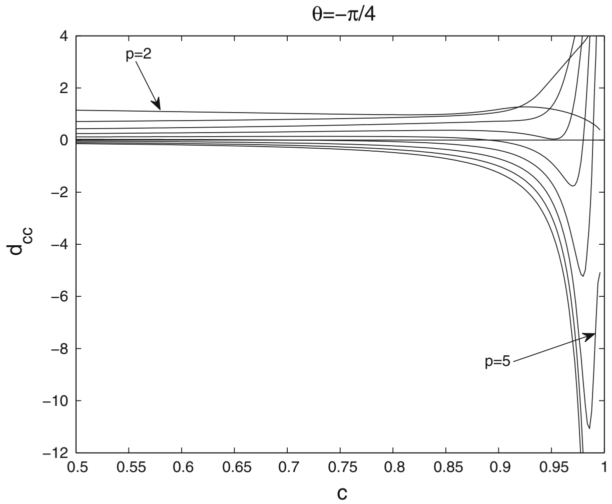

Fig. 10 Plots of d_{cc} over S_1 for the nonlinearity given by (4.1) with $\theta = -\frac{1}{4}\pi$ (equivalently $\sqrt{2}f_{-}$), where p varies from 2 to 6 by multiples of $\frac{1}{2}$. When $p \leq 3.5$, $d_{cc} > 0$ everywhere, and when $p \geq 5$, $d_{cc} < 0$ everywhere. When $p = 4$ and $p = 4.5$, d_{cc} changes sign twice

even nonlinearities) and $a_o > 0$, while the choice of p was made to illustrate how the regions of instability grow as p increases. The results are shown in Figs. 7 through 13 and summarized in Table 1.

Computations over S_2. Figure 7 shows d_{cc} over S_2 for the odd nonlinearities $f(s) = |s|^{p-1}s$. We note that d_{cc} is positive on S_2 when $p < 5$ and negative on

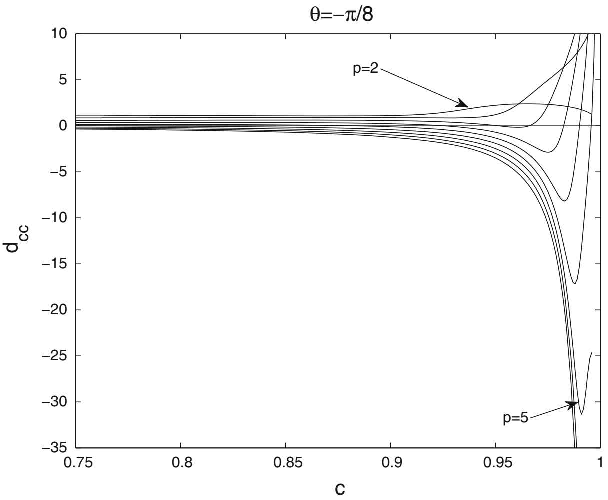

Fig. 11 Plots of d_{cc} over S_1 for the nonlinearity given by (4.1) with $\theta = -\frac{1}{8}\pi$, where p varies from 2 to 6 by multiples of $\frac{1}{2}$. When $p \leq 2.5$, $d_{cc} > 0$ everywhere, and when $p \geq 5$, $d_{cc} < 0$ everywhere. When $p = 3$, 3.5, 4 and $p = 4.5$, d_{cc} changes sign twice

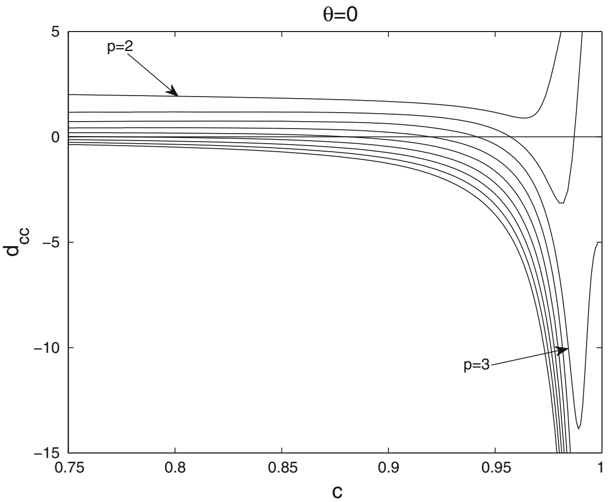

Fig. 12 Plots of d_{cc} over S_1 for the nonlinearity $f(s) = |s|^p$, where p varies from 2 to 6 by multiples of $\frac{1}{2}$. When $p = 2$, $d_{cc} > 0$ everywhere, and when $p \geq 5$, $d_{cc} < 0$ everywhere. When $2.5 \leq p \leq 4.5$, d_{cc} changes sign twice

S_2 when $p \geq 5$, in agreement with the conclusions of Corollary 5.4. The same was true for plots of d_{cc} over S_2 for all other nonlinearities considered, so the others were omitted.

Nonlinearities with $a_o < 0$. Figures 8, 9, 10 and 11 show d_{cc} over S_1 for nonlinearities with $\theta < 0$, i.e. $a_o < 0$. The features common to this group are:

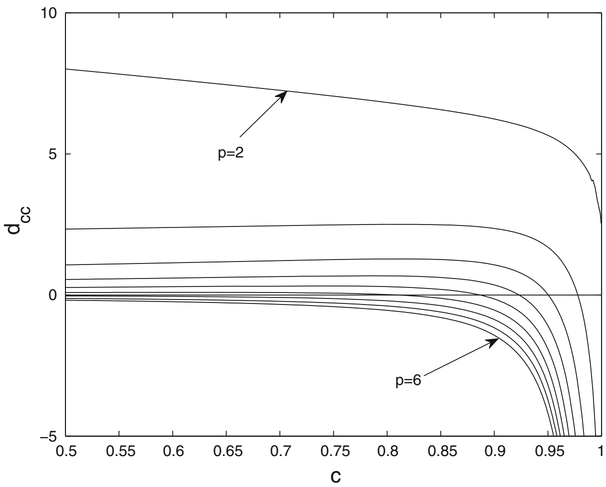

Fig. 13 Plots of d_{cc} over S_1 for the nonlinearity given by (4.1) with $\theta = +\frac{1}{8}\pi$, where p varies from 2 to 6 by multiples of $\frac{1}{2}$. It appears that when $p = 2$, $d_{cc} > 0$ everywhere, and when $p \geq 5$, $d_{cc} < 0$ everywhere. When $2.5 \leq p \leq 4.5$, d_{cc} changes sign once

- when $p \geq 5$, $d_{cc} < 0$ everywhere on S_1.
- when $p < 5$, $d_{cc} > 0$ for c near c_*, in agreement with Corollary 5.1.
- if d_{cc} changes sign, it does so twice.

Even Nonlinearities Figure 12 shows d_{cc} over S_1 for the family purely even nonlinearities $f(s) = |s|^p$. Features shared by this family include:

- when $p \geq 5$, $d_{cc} < 0$ everywhere on S_1.
- when $p < 3$, $d_{cc} > 0$ for c near c_*, and when $p \geq 3$, $d_{cc} < 0$ for c near c_*.
- if d_{cc} changes sign, it does so twice.

Nonlinearities with $a_o > 0$. Finally, Fig. 13 shows d_{cc} over S_1 for $\theta = \pi/8$. This is an example of a case where $a_o > 0$ and we have no analytical bound on d near c_*. Here we observe that:

- when $p = 2$, $d_{cc} > 0$ everywhere on S_1.
- when $p > 2$, $d_{cc} < 0$ for c near c_*.
- when $p \geq 5$, $d_{cc} < 0$ everywhere on S_1.
- d_{cc} changes sign at most once.

7 Conclusion

While the stability criteria given in Theorems 3.3 and 3.4 are quite useful, they have thus far only provided analytic proof of stability and instability in cases where (a) γ is near zero, (b) c is near c_*, or (c) p is very large (> 10). The numerical results of the previous section provide a much more detailed picture of the stability situation for the nonlinearities considered, but do not provide analytical proof of stability or instability. Based on these numerical results we make the following conjecture.

Conjecture 7.1 *Let* $f(s) = a_e|s|^p + a_o|s|^{p-1}s$ *with* $a_o < |a_e|$.

(i) *If* $p = 2$, $\mathscr{G}(\beta, c, \gamma)$ *is stable for all* $c < c_*$.
(ii) *If* $p \geq 5$, $\mathscr{O}_\varphi$ *is unstable for* $\varphi \in \mathscr{G}(\beta, c, \gamma)$ *for all* $c < c_*$.

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