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Optimal Positional Strategies in Differential Games for Neutral-Type Systems

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Abstract
The paper deals with a zero-sum differential game for a dynamical system described by neutral-type functional-differential equations in Hale's form with initial conditions determined by piecewise continuous functions. It is proved that the differential game has a value and optimal positional (feedback) players' strategies. If the value functional satisfies certain smoothness conditions, the optimal strategies are constructed based on its gradient. In the general case, such strategies are described using quasi-gradient constructions. The fact that the quasi-gradients under consideration require looking for extremum points only on a finite-dimensional set is the crucial contribution of this paper.

Keywords Differential games · Optimal strategies · Neutral-type equations

Mathematics Subject Classification 34K40 · 49L20 · 49N35 · 49N70

1 Introduction

In differential games for dynamical systems described by ordinary differential equations satisfying the Isaacs condition [11] (or the saddle point condition in a small game in other terminology [14]), it is known (see, e.g., [14, 25]) that the value of the game exists and can be achieved by positional (feedback) players' strategies. If the value function is continuously differentiable, then the optimal positional strategies can be obtained utilizing its gradient. If the differentiability of this function is not assumed, they can be constructed by various regularizing tools [2, 6, 14, 22, 25, 27]. In particular, under fairly general conditions, the optimal positional strategies can utilize the quasi-gradients of the value function [25]. This paper aims to describe optimal positional strategies for more general differential games in which dynamical systems are described by a functional-differential equation of neutral-type in Hale's from [10].

Such equations represent a fairly general class of functional-differential systems which contain not only a delay in the state vector but also a delay in its derivative. They arise in studying, for example, transmission line nonlinear oscillators [4, 5], torsional motions of

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driven drill strings [1] and other applications (see [13]). As a quality index for the differential games under consideration, we choose a Boltz cost functional, which is quite a typical choice for the differential games [14, 25] and natural for applications. In a particular and often used case, it estimates the distance from the target point at the terminal time and the integral of the players' control cost.

Note that the usage of the previously developed optimal positional strategies in differential games for both time-delay [15–17, 23] and neutral-type [9, 18] functional-differential systems is problematic since their application requires looking for extremum points on infinite-dimensional sets of continuous functions (possible histories of system motions). However, recent papers [20, 21] devoted to optimal control problems and differential games for time-delay systems established that it is possible to look for extremum points on finite-dimensional sets if motion histories are piecewise continuous functions. Therefore, starving to obtain a similar result in differential games for neutral-type systems, we also consider a motion history space with jumps. Namely, following [24], we choose the space of piecewise Lipschitz continuous functions. In particular, paper [24] established the uniqueness of a generalized (minimax and viscosity) solution of the Hamilton-Jacobi equations arising from control problems for neural-type systems in this space. Thus, proving that the upper and lower values of the game in the class of non-anticipatory strategies are the viscosity solution of the corresponding Hamilton-Jacobi equation, we obtain the value of the game exists (Theorem 1).

Next, we describe the smoothness conditions under which optimal positional strategies can be constructed based on the gradient of the value functional (Theorem 2). Note that, in contrast to time-delay systems (see [16, 17]), we cannot use the ci-smoothness condition since, even in the simplest case of neutral-type systems, the gradient is not continuous (see [8]). Instead, we use conditions considering possible discontinuities of the gradient following [8]. However, the choice of the motion history space with jumps allows us to obtain more general smoothness conditions than in [8].

Finally, thanks to such a choice of the motion history space, for the general case when the value functional is not smooth, we prove the optimality of positional strategies based, in fact, on the classical quasi-gradient definition [25] (Theorem 3). Let us emphasize again that, unlike previous papers [9, 18] devoted to optimal positional strategies for neutral-type systems, such quasi-gradient constructions requires looking for extremum points only on a finite-dimensional set, which is the crucial contribution of this paper.

Note also that the usage of the motion history space with jumps creates various additional difficulties in proofs. Namely, motions of neutral-type systems on such space have a certain periodicity of jumps during the control interval (see Remark 1), which is not typical for time-delay systems, for example. In addition, the value functional has rather specific continuity properties (see condition (ρ_1) and (ρ_2)) that are different from [9, 18], where Lipschitz motion histories were considered. Nonetheless, the accounting of this specificity in the proofs allows us to obtain the above results.

2 Results

2.1 Functional Spaces

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$. A function $x(\cdot)\colon [a, b) \mapsto \mathbb{R}^n$ (or $x(\cdot)\colon [a, b] \mapsto \mathbb{R}^n$) is called piecewise Lipschitz continuous if there exist points $a = \xi_0 < \xi_1 < \ldots < \xi_{I+1} = b$ such that the function $x(\cdot)$ is Lipschitz continuous on the interval $[\xi_i, \xi_{i+1})$ for each $i \in \overline{0, I}$. Note that such a function $x(\cdot)$ is right continuous on $[a, b)$ and has a finite left limit $x(\xi - 0)$ for any $\xi \in (a, b]$. Denote by $\mathrm{PLip}([a, b), \mathbb{R}^n)$ and $\mathrm{Lip}([a, b), \mathbb{R}^n)$ the linear spaces of piecewise Lipschitz and Lipschitz continuous functions $x(\cdot)\colon [a, b) \mapsto \mathbb{R}^n$, respectively.

Let $\vartheta, h > 0$. Without loss of generality of the results presented below, we can suppose the existence of $J \in \mathbb{N}$ such that $\vartheta = Jh$. For the sake of brevity, we set $\mathrm{PLip} = \mathrm{PLip}([-h, 0), \mathbb{R}^n)$ and, for any $w(\cdot) \in \mathrm{PLip}$, we denote

$$\|w(\cdot)\|_1 = \int_{-h}^{0} \|w(\xi)\| \mathrm{d}\xi, \quad \|w(\cdot)\|_\infty = \sup_{\xi \in [-h, 0)} \|w(\xi)\|, \quad w(-0) = w(0 - 0).$$

Following [24], define the space PLip_* of functions $w(\cdot) \in \mathrm{PLip}$ continuously differentiable on $[-h, -h + \delta_w]$ for some $\delta_w > 0$ and the spaces

$$\mathbb{G} = [0, \vartheta] \times \mathbb{R}^n \times \mathrm{PLip}, \quad \mathbb{G}_* = \cup_{j=0}^{J-1} (jh, (j+1)h) \times \mathbb{R}^n \times \mathrm{PLip}_*. \tag{1}$$

2.2 Differential Game

For each $(\tau, z, w(\cdot)) \in \mathbb{G}$, consider a zero-sum differential game for a dynamical system described by the neutral-type differential equation in Hale's form [10]

$$\begin{gathered} \frac{d}{dt}\Big(x(t) - g(t, x(t-h))\Big) = f(t, x(t), x(t-h), u(t), v(t)), \quad t \in [\tau, \vartheta], \\ x(t) \in \mathbb{R}^n, \quad u(t) \in \mathbb{U} \subset \mathbb{R}^l, \quad v(t) \in \mathbb{V} \subset \mathbb{R}^m, \end{gathered} \tag{2}$$

with the initial condition

$$x(\tau) = z, \quad x(t) = w(t - \tau), \quad t \in [\tau - h, \tau), \tag{3}$$

and the quality index

$$\gamma = \sigma(x(\vartheta), x_\vartheta(\cdot)) + \int_{\tau}^{\vartheta} f^0(t, x(t), x(t-h), u(t), v(t)) dt, \tag{4}$$

Here t is the time variable; $x(t)$ is the state vector at the time t; $u(t)$ and $v(t)$ are control actions of the first and second players, respectively; $\mathbb{U}$ and $\mathbb{V}$ are compact sets. Hereinafter, the symbol $x_t(\cdot)$ denotes the function on the interval $[-h, 0)$ defined by $x_t(\xi) = x(t + \xi)$, $\xi \in [-h, 0)$.

In this differential game, the first player aims to minimize γ, while the second player aims to maximize it.

We assume that the following conditions hold:

- (g_1) The function g is continuously differentiable.
- (g_2) There exists a constant $c_g > 0$ such that

$$\|g(t, x)\| \le c_g\big(1 + \|x\|\big), \quad (t, x) \in [0, \vartheta] \times \mathbb{R}^n.$$

- (f_1) The functions f and f^0 are continuous.
- (f_2) There exists a constant $c_f > 0$ such that

$$\big\|f(t, x, y, u, v)\big\| + \big|f^0(t, x, y, u, v)\big| \le c_f\big(1 + \|x\| + \|y\|\big)$$

for any $t \in [0, \vartheta]$, $x, y \in \mathbb{R}^n$, $u \in \mathbb{U}$, and $v \in \mathbb{V}$.

(f_3) For every $\alpha > 0$, there exists a number $\lambda_f = \lambda_f(\alpha) > 0$ such that

$$\big\| f(t,x,y,u,v) - f(t,x',y',u,v)\big\| + \big| f^0(t,x,y,u,v) - f^0(t,x',y',u,v)\big| \\ \leq \lambda_f\big(\|x-x'\| + \|y-y'\|\big)$$

for any $t \in [0,\vartheta]$, $x, y, x', y' \in B(\alpha) = \{x \in \mathbb{R}^n \colon \|x\| \leq \alpha\}$, $u \in \mathbb{U}$, and $v \in \mathbb{V}$.

(f_4) The equality

$$\min_{u\in\mathbb{U}} \max_{v\in\mathbb{V}} \chi(t,x,y,u,v,s) = \max_{v\in\mathbb{V}} \min_{u\in\mathbb{U}} \chi(t,x,y,u,v,s)$$

holds for any $t \in [0,\vartheta]$ and $x, y, s \in \mathbb{R}^n$, where

$$\chi(t,x,y,u,v,s) = \langle f(t,x,y,u,v), s\rangle + f^0(t,x,y,u,v). \tag{5}$$

(σ_1) For every $\alpha > 0$, there exists $\lambda_\sigma = \lambda_\sigma(\alpha) > 0$ such that

$$\big|\sigma(x,r(\cdot)) - \sigma(x',r'(\cdot))\big| \leq \lambda_\sigma\big(\|x-x'\| + \|r(\cdot) - r'(\cdot)\|_1\big)$$

for any $(x,r(\cdot)), (x',r'(\cdot)) \in P(\alpha)$, where

$$P(\alpha) = \big\{(x,r(\cdot)) \in \mathbb{R}^n \times \mathrm{PLip} \colon \|x\| \leq \alpha,\ \|r(\cdot)\|_\infty \leq \alpha\big\}. \tag{6}$$

(σ_2) There exists $c_\sigma > 0$ such that

$$\big|\sigma(x,r(\cdot))\big| \leq c_\sigma\big(1 + \|x\| + \|r(\cdot)\|_\infty\big), \quad (x,r(\cdot)) \in \mathbb{R}^n \times \mathrm{PLip}.$$

Note that these conditions are quite typical for differential games theory [11, 14, 25]. In particular, condition (f_4), called the Isaacs's condition [11] or the saddle point condition in a small game in other terminology [14, 25], is crucial for proving the existence of a value (see Theorem 1 below).

Define the set of piecewise Lipschitz continuous right extensions from the point $(\tau, z, w(\cdot))$ as follows:

$$\Lambda(\tau,z,w(\cdot)) = \big\{x(\cdot) \in \mathrm{PLip}([\tau-h,\vartheta],\mathbb{R}^n) \colon x(\tau) = z,\ x_\tau(\cdot) = w(\cdot)\big\}.$$

By admissible control realizations of the first and second players, we mean Lebesgue measurable functions $u(\cdot)\colon [\tau,\vartheta] \mapsto \mathbb{U}$ and $v(\cdot)\colon [\tau,\vartheta] \mapsto \mathbb{V}$, respectively. Denote by $\mathcal{U}_\tau$ and $\mathcal{V}_\tau$ the sets of admissible control realizations of the first and second players. Under conditions (g_1) and (f_1) $-$ (f_3), following, for example, the scheme from [7, Section 7] (see also [12, Section 4.2]), one can show that each pair of realizations $u(\cdot) \in \mathcal{U}_\tau$ and $v(\cdot) \in \mathcal{V}_\tau$ uniquely generates the motion $x(\cdot) = x(\cdot \mid \tau, z, w(\cdot), u(\cdot), v(\cdot))$ of system (2), (3) that is the function from $\Lambda(\tau,z,w(\cdot))$ such that the function $x(t) - g(t, x(t-h))$, $t \in [\tau,\vartheta]$ is Lipschitz continuous and $x(\cdot)$ satisfies Eq. (2) almost everywhere.

Remark 1 Note that the motions of system (2), (3) have a certain structure of Lipschitz continuous pieces (and discontinuity points). Namely, if $-h = \xi_0 < \xi_1 < \ldots < \xi_{I+1} = 0$ such that the function $w(\cdot)$ is Lipschitz continuous on $[\xi_i, \xi_{i+1})$, $i \in \overline{0, I}$ then the motion $x(\cdot) = x(\cdot \mid \tau, z, w(\cdot), u(\cdot), v(\cdot))$ is Lipschitz continuous on the intervals $[\tau + \xi_i + jh, \tau + \xi_{i+1} + jh) \cap [\tau, \vartheta]$, $i \in \overline{0, I}$, $j \in \overline{0, J}$. This fact can be proved similar to Proposition 8.

We first consider differential game (2)–(4) in classes of non-anticipative strategies of players (see, e.g. [2, Chapter VIII, Section 1]) or quasi-strategies in another terminology (see, e.g. [25, Chapter III, Section 14.2]).

By a non-anticipative strategy of the first player, we mean a mapping $Q^u_\tau \colon \mathcal{V}_\tau \mapsto \mathcal{U}_\tau$ such that, for each $v(\cdot), v'(\cdot) \in \mathcal{V}_\tau$ and $t \in [\tau, \vartheta]$, if the equality $v(\xi) = v'(\xi)$ is valid for a.e. $\xi \in [\tau, t]$ then the equality $Q^u_\tau[v(\cdot)](\xi) = Q^u_\tau[v'(\cdot)](\xi)$ holds for a.e. $\xi \in [\tau, t]$.

A non-anticipative strategy of the first player Q^u_τ and a control realization of the second player $v(\cdot) \in \mathcal{V}_\tau$ define the control realization of the first player $u(\cdot) = Q^u_\tau[v(\cdot)](\cdot)$, the motion $x(\cdot) = x(\cdot \mid \tau, z, w(\cdot), u(\cdot), v(\cdot))$ and the value $\gamma = \gamma(\tau, z, w(\cdot), Q^u_\tau, v(\cdot))$ of quality index (4). The lower value of differential game (2)–(4) is defined by

$$\rho^u(\tau, z, w(\cdot)) = \inf_{Q^u_\tau} \sup_{v(\cdot)\in\mathcal{V}_\tau} \gamma(\tau, z, w(\cdot), Q^u_\tau, v(\cdot)). \tag{7}$$

The functional $\mathbb{G} \ni (\tau, z, w(\cdot)) \mapsto \rho^u = \rho^u(\tau, z, w(\cdot)) \in \mathbb{R}$ is the lower value functional of differential game (2)–(4).

Similarly, a non-anticipative strategy of the second player is a mapping $Q^v_\tau : \mathcal{U}_\tau \mapsto \mathcal{V}_\tau$ such that, for each $u(\cdot), u'(\cdot) \in \mathcal{U}_\tau$ and $t \in [\tau, \vartheta]$, if the equality $u(\xi) = u'(\xi)$ is valid for a.e. $\xi \in [\tau, t]$ then the equality $Q^v_\tau[u(\cdot)](\xi) = Q^v_\tau[u'(\cdot)](\xi)$ holds for a.e. $\xi \in [\tau, t]$. Such a non-anticipative strategy together with $u(\cdot) \in \mathcal{U}_\tau$ define the realization $v(\cdot) = Q^v_\tau[u(\cdot)](\cdot)$, the motion $x(\cdot) = x(\cdot \mid \tau, z, w(\cdot), u(\cdot), v(\cdot))$, and the quality index $\gamma = \gamma(\tau, z, w(\cdot), u(\cdot), Q^v_\tau)$. The upper value of differential game (2)–(4) is

$$\rho^v(\tau, z, w(\cdot)) = \sup_{Q^v_\tau} \inf_{u(\cdot)\in\mathcal{U}_\tau} \gamma(\tau, z, w(\cdot), u(\cdot), Q^v_\tau). \tag{8}$$

The functional $\mathbb{G} \ni (\tau, z, w(\cdot)) \mapsto \rho^v = \rho^v(\tau, z, w(\cdot)) \in \mathbb{R}$ is the upper value functional of differential game (2)–(4).

Note that the functionals ρ^u and ρ^v satisfy the following conditions:

(ρ_0) The equality $\rho(\vartheta, z, w(\cdot)) = \sigma(z, w(\cdot))$, $(z, w(\cdot)) \in \mathbb{R}^n \times \mathrm{PLip}$ holds.

(ρ_1) For each pair $(\tau, w(\cdot)) \in [0, \vartheta] \times \mathrm{Lip}([-h, 0), \mathbb{R}^n)$, the function $\hat{\rho}(t) = \rho(t, w(-0), w(\cdot))$ is continuous on $[\tau, \vartheta]$.

(ρ_2) For every $\alpha > 0$, there exists $\lambda_\rho = \lambda_\rho(\alpha) > 0$ such that

$$\left|\rho(\tau, z, w(\cdot)) - \rho(\tau, z', w'(\cdot))\right| \le \lambda_\rho \upsilon(\tau, z - z', w(\cdot) - w'(\cdot)) \tag{9}$$

for any $\tau \in [0, \vartheta]$ and $(z, w(\cdot)), (z', w'(\cdot)) \in P(\alpha)$, where

$$\upsilon(\tau, z, w(\cdot)) = \|z\| + \|w(\cdot)\|_1 + \|w(-h)\| + \|w(jh - \tau)\| \tag{10}$$

in which $j \in \overline{-1, J-1}$ is such that $\tau \in (jh, (j+1)h]$.

(ρ_3) For every $(\tau, z, w(\cdot)) \in \mathbb{G}$, $\tau < \vartheta$, $\zeta > 0$, $t_* \in (\tau, \vartheta]$, and $v(\cdot) \in \mathcal{V}_\tau$, there exists $u(\cdot) \in \mathcal{U}_\tau$ such that the motion $x(\cdot) = x(\cdot \mid \tau, z, w(\cdot), u(\cdot), v(\cdot))$ satisfies the estimate

$$\rho(t_*, x(t_*), x_{t_*}(\cdot)) + \int_\tau^{t_*} f^0(t, x(t), x(t-h), u(t), v(t))dt \le \rho(\tau, z, w(\cdot)) + \zeta.$$

(ρ_4) For every $(\tau, z, w(\cdot)) \in \mathbb{G}$, $\tau < \vartheta$, $\zeta > 0$, $t_* \in (\tau, \vartheta]$, and $u(\cdot) \in \mathcal{U}_\tau$, there exists $v(\cdot) \in \mathcal{V}_\tau$ such that the motion $x(\cdot) = x(\cdot \mid \tau, z, w(\cdot), u(\cdot), v(\cdot))$ satisfies the estimate

$$\rho(t_*, x(t_*), x_{t_*}(\cdot)) + \int_\tau^{t_*} f^0(t, x(t), x(t-h), u(t), v(t))dt \ge \rho(\tau, z, w(\cdot)) - \zeta.$$

The fulfillment of terminal condition (ρ_0) follows directly from definitions (7) and (8) of value functionals ρ^u and ρ^v. Conditions (ρ_1), (ρ_2) naturally generalize the continuity properties of value functionals to the case of neural-type systems considered on a set $\mathbb{G}$ (see [24, Remark 2]) and are proved in Proposition 10. Conditions (ρ_3), (ρ_4) describe the dynamic programming principle for the functionals ρ^u and ρ^v and can be proved following the scheme from [2, Chapter VIII, Theorem 1.9] if we take into account that any non-anticipative strategy Q^u_τ defines a rule—for every $v(\cdot) \in \mathcal{V}_\tau$ there exists $u(\cdot) \in \mathcal{U}_\tau$ and similarly for Q^v_τ. Note also the fact that system (2) has a delay does not in any way affect the proof scheme.

2.3 Hamilton–Jacobi Equation

In this section, we consider the corresponding to differential game (2)–(4) Hamilton-Jacobi equation with coinvariant derivatives to prove the existence of the value and other auxiliary properties.

Following [24] (see also [12, 15]), a functional $\rho\colon \mathbb{G} \mapsto \mathbb{R}$ is called coinvariantly (ci-) differentiable at a point $(\tau, z, w(\cdot)) \in \mathbb{G}$, $\tau < \vartheta$ if there exist $\partial^{ci}_{\tau,w}\rho(\tau, z, w(\cdot)) \in \mathbb{R}$ and $\nabla_z \rho(\tau, z, w(\cdot)) \in \mathbb{R}^n$ such that, for every $t \in [\tau, \vartheta]$, $y \in \mathbb{R}^n$, and $x(\cdot) \in \Lambda(\tau, z, w(\cdot))$, the relation below holds

$$\begin{aligned} \rho(t, y, x_t(\cdot)) - \rho(\tau, z, w(\cdot)) = (t-\tau)\partial^{ci}_{\tau,w}\rho(\tau, z, w(\cdot)) \\ +\langle y - z, \nabla_z \rho(\tau, z, w(\cdot))\rangle + o(|t-\tau| + \|y - z\|), \end{aligned} \tag{11}$$

where the value $o(\delta)$ can depend on $x(\cdot)$ and $o(\delta)/\delta \to 0$ as $\delta \downarrow 0$. Then $\partial^{ci}_{\tau,w}\rho(\tau, z, w(\cdot))$ is called the ci-derivative of ρ with respect to $\{\tau, w(\cdot)\}$ and $\nabla_z \rho(\tau, z, w(\cdot))$ is the gradient of ρ with respect to z.

Denote

$$\begin{aligned} G(t, x, y) &= \partial g(t, x)/\partial t + \nabla_x g(t, x) y \\ H(t, x, y, s) &= \min_{u \in \mathbb{U}} \max_{v \in \mathbb{V}} \chi(t, x, y, u, v, s) \end{aligned} \quad t \in [0, \vartheta], \quad x, y, s \in \mathbb{R}^n, \tag{12}$$

and consider the Cauchy problem for the Hamilton-Jacobi equation

$$\begin{aligned} \partial^{ci}_{\tau,w}\rho(\tau, z, w(\cdot)) + \langle G(\tau, w(-h), d^+ w(-h)/d\xi), \nabla_z \rho(\tau, z, w(\cdot))\rangle \\ +H(\tau, z, w(-h), \nabla_z \rho(\tau, z, w(\cdot))) = 0, \quad (\tau, z, w(\cdot)) \in \mathbb{G}_*, \end{aligned} \tag{13}$$

and the terminal condition

$$\rho(\vartheta, z, w(\cdot)) = \sigma(z, w(\cdot)), \quad (z, w(\cdot)) \in \mathbb{R}^n \times \mathrm{PLip}, \tag{14}$$

where $d^+ w(-h)/d\xi$ is the right derivative of the function $w(\xi)$, $\xi \in [-h, 0)$ at the point $\xi = -h$. The properties and singularities of such Cauchy problems were studied in [24]. In particular, this paper proves the existence and uniqueness of the generalized (minimax or viscosity) solution of such a problem. Thus, showing in Proposition 11 that the both functionals ρ^u and ρ^v are the viscosity solution of problem (13), (14), we obtain the following statement.

Theorem 1 *Differential game (2)–(4) has the value functional*

$$\rho^\circ(\tau, z, w(\cdot)) = \rho^u(\tau, z, w(\cdot)) = \rho^v(\tau, z, w(\cdot)), \quad (\tau, z, w(\cdot)) \in \mathbb{G}. \tag{15}$$

Note also that from this theorem and [24, Theorem 5], we get the following auxiliary property which directly connects ρ° and equation (13):

(ρ_5) Let the value functional ρ° be ci-differentiable on $\mathbb{G}_*$. Then ρ° satisfies Hamilton-Jacobi equation (13) for any $(\tau, z, w(\cdot)) \in \mathbb{G}_*$.

2.4 Optimal Positional Strategies

In this section, we introduce the concept of positional (feedback) players' strategies following the scheme from [14] (see also [16] for time-delay systems). Note that the positional strategies are a particular case of non-anticipative strategies, and therefore, their guaranteed results cannot be better than values (7) and (8). Below, we present positional strategies capable of providing precisely these values (i.e. optimal strategies), which is the main result of this paper. Namely, following the differential game theory [25], firstly, we describe optimal strategies based on the value functional gradient if it satisfies certain smoothness conditions, are secondly, we present strategies utilizing quasi-gradients for the general case.

By a positional strategy of the first player, we mean an arbitrary function $U : \mathbb{G} \mapsto \mathbb{U}$. Let us fix $(\tau, z, w(\cdot)) \in \mathbb{G}$ and a partition of the interval $[\tau, \vartheta]$:

$$\Delta_\delta = \{t_j : t_1 = \tau,\ 0 < t_{j+1} - t_j \leq \delta,\ j = \overline{1, k},\ t_{k+1} = \vartheta\}. \tag{16}$$

The pair $\{U, \Delta_\delta\}$ defines a control law that forms a piecewise constant function $u(\cdot) \in \mathcal{U}_\tau$ according to the following step-by-step rule:

$$u(t) = U(t_j, x(t_j), x_{t_j}(\cdot)), \quad t \in [t_j, t_{j+1}), \quad j = \overline{1, k}. \tag{17}$$

This control law together with any function $v(\cdot) \in \mathcal{V}_\tau$ uniquely determine the motion $x(\cdot) = x(\cdot \mid \tau, z, w(\cdot), u(\cdot), v(\cdot))$ and the quality index $\gamma = \gamma(t, z, w(\cdot), U, \Delta_\delta, v(\cdot))$ of quality index (4). The guaranteed result of the strategy U is defined by

$$\rho_u(\tau, z, w(\cdot), U) = \lim_{\delta \downarrow 0} \sup_{\Delta_\delta} \sup_{v(\cdot)} \gamma(\tau, z, w(\cdot), U, \Delta_\delta, v(\cdot)). \tag{18}$$

Similarly, with the corresponding changes, for the second player, we define a positional control strategy $V : \mathbb{G} \mapsto \mathbb{V}$, control law $\{V, \Delta_\delta\}$ that forms a function $v(\cdot) \in \mathcal{V}_\tau$ by

$$v(t) = V(t_j, x(t_j), x_{t_j}(\cdot)), \quad t \in [t_j, t_{j+1}), \quad j = \overline{1, k},$$

the guaranteed result of the strategy V

$$\rho_v(\tau, z, w(\cdot), V) = \lim_{\delta \downarrow 0} \inf_{\Delta_\delta} \inf_{u(\cdot)} \gamma(\tau, z, w(\cdot), u(\cdot), V, \Delta_\delta). \tag{19}$$

Theorem 2 *Let the value functional $\rho^\circ = \rho^\circ(t, z, w(\cdot))$ is differentiable by z on $\mathbb{G}$ and ci-differentiable on $\mathbb{G}_*$ (see (1)). Let the functional $\nabla_z \rho^\circ = \nabla_z \rho^\circ(\tau, z, w(\cdot))$ satisfy the condition*

(ρ_1^*) *For each $(\tau, w(\cdot)) \in [0, \vartheta] \times \mathrm{Lip}$, the function $\hat{\rho}(t) = \nabla_z \rho^\circ(t, w(-0), w(\cdot))$ is continuous on $[\tau, \vartheta] \cap (jh, (j+1)h)$ for each $j \in \overline{0, J-1}$.*

and condition (ρ_2). Then, for every $(\tau, z, w(\cdot)) \in \mathbb{G}$, the players' strategies

$$U(t, x, r(\cdot)) \in \operatorname*{argmin}_{u \in \mathbb{U}} \max_{v \in \mathbb{V}} \chi\big(t, x, r(-h), u, v, \nabla_z \rho^\circ(t, x, r(\cdot))\big)$$
$$V(t, x, r(\cdot)) \in \operatorname*{argmax}_{v \in \mathbb{V}} \min_{u \in \mathbb{U}} \chi\big(t, x, r(-h), u, v, \nabla_z \rho^\circ(t, x, r(\cdot))\big)$$

provide the equalities

$$\rho_u(\tau, z, w(\cdot), U) = \rho^\circ(\tau, z, w(\cdot)) = \rho_v(\tau, z, w(\cdot), V). \tag{20}$$

Thus, the positional strategies U and V described above are optimal.

The simplest example of a differential game in which the conditions of Theorem 2 are satisfied can be found in [8]. This example also forces us to use a weaker condition (ρ_1^*) than (ρ_1) in this theorem because we can see that even in such straightforward cases, condition (ρ_1) for the value functional gradient does not hold.

Next, determine the optimal strategies for the general case following [25]. For every $\lambda, \varepsilon > 0$, denote

$$\begin{aligned} \eta^{\lambda,\varepsilon}(t, x) = \theta^{\lambda,\varepsilon}(t)\mu^\varepsilon(x), \quad \theta^{\lambda,\varepsilon}(t) = (e^{-\lambda t} - \varepsilon)/\varepsilon, \\ \mu^\varepsilon(x) = \sqrt{\varepsilon^4 + \|x\|^2}, \end{aligned} \quad (t, x) \in [0, \vartheta] \times \mathbb{R}^n \tag{21}$$

and consider the functionals

$$\begin{aligned} p^{\lambda,\varepsilon}(t,x,r(\cdot)) &= \operatorname*{argmin}_{p\in\mathbb{R}^n} \big(\rho^\circ(t,p,r(\cdot)) + \eta^{\lambda,\varepsilon}(t,x-p)\big), \\ q^{\lambda,\varepsilon}(t,x,r(\cdot)) &= \operatorname*{argmax}_{q\in\mathbb{R}^n} \big(\rho^\circ(t,q,r(\cdot)) - \eta^{\lambda,\varepsilon}(t,x-q)\big), \end{aligned} \tag{22}$$

where $(t,x,r(\cdot)) \in \mathbb{G}$. Proposition 14 proves the argmin and argmax values are archived for sufficiently small ε and therefore, these functionals are well defined. Let us consider the quasi-gradients

$$\begin{aligned} \nabla_z^{\lambda,\varepsilon,+}\rho^\circ(t,x,r(\cdot)) &= \nabla_z\eta^{\lambda,\varepsilon}(t,x-p^{\lambda,\varepsilon}(t,x,r(\cdot))), \\ \nabla_z^{\lambda,\varepsilon,-}\rho^\circ(t,x,r(\cdot)) &= \nabla_z\eta^{\lambda,\varepsilon}(t,x-q^{\lambda,\varepsilon}(t,x,r(\cdot))), \end{aligned} \tag{23}$$

and describe the optimal strategies that does not require additional smoothness conditions for the value functional.

Theorem 3 *For every $(\tau,z,w(\cdot)) \in \mathbb{G}$ and $\zeta > 0$, there exist $\lambda > 0$ such that the players' strategies*

$$\begin{aligned} U^{\lambda,\varepsilon}(t,x,r(\cdot)) &\in \operatorname*{argmin}_{u\in\mathbb{U}} \max_{v\in\mathbb{V}} \chi\big(t,x,r(-h),u,v,\nabla_z^{\lambda,\varepsilon,+}\rho^\circ(t,x,r(\cdot))\big) \\ V^{\lambda,\varepsilon}(t,x,r(\cdot)) &\in \operatorname*{argmax}_{v\in\mathbb{V}} \min_{u\in\mathbb{U}} \chi\big(t,x,r(-h),u,v,\nabla_z^{\lambda,\varepsilon,-}\rho^\circ(t,x,r(\cdot))\big) \end{aligned} \tag{24}$$

provide the equities

$$\overline{\lim_{\varepsilon\downarrow 0}}\, \rho_u(\tau,z,w(\cdot),U^{\lambda,\varepsilon}) = \rho^\circ(\tau,z,w(\cdot)) = \underline{\lim_{\varepsilon\downarrow 0}}\, \rho_v(\tau,z,w(\cdot),V^{\lambda,\varepsilon}). \tag{25}$$

Note that, in contrast to Theorem 2, Theorem 3 describes strategies providing $\rho^\circ(\tau,z,w(\cdot))$ only in the limit. Nonetheless, this result is typical for the theory of differential games (see, e.g., [25, Section 12.2]) since, even for ordinary differential equations, universal positional strategies (i.e. positional strategies independent of a particular initial condition) that provide the value without ε may not exist [26].

3 Proof

3.1 Properties of the Dynamical System

In this section, we give some properties of dynamical system (2). Proposition 4 follows directly from condition (g_1). Proposition 6 was proved in [24, Lemma 1]. The proofs of the remaining propositions are given below.

Proposition 4 *For every $\alpha > 0$, there exists $\lambda_g = \lambda_g(\alpha) > 0$ such that*

$$\big|g(t,x) - g(t',x')\big| \le \lambda_g\big(|t-t'| + \|x-x'\|\big)$$

for any $t,t' \in [0,\vartheta]$ and $x,x' \in B(\alpha) := \{x \in \mathbb{R}^n : \|x\| \le \alpha\}$.

Proposition 5 *There exists $c_X > 0$ such that, for every $(\tau,z,w(\cdot)) \in \mathbb{G}$, $u(\cdot) \in \mathcal{U}_\tau$, and $v(\cdot) \in \mathcal{V}_\tau$, the motion $x(\cdot) = x(\cdot \mid \tau,z,w(\cdot),u(\cdot),v(\cdot))$ satisfies the estimate*

$$\|x(t)\| \le c_X\big(1 + \|z\| + \|w(\cdot)\|_\infty\big), \quad t \in [\tau,\vartheta]. \tag{26}$$

Proof Define $c_g, c_f > 0$ according to conditions (g_2), (f_2). Denote $c_* = 1 + 2c_g + 2\vartheta$ and put $c_X = c_*^{J+1} e^{c_f \vartheta}$, where J is from (1). Then, due to (2), we have

$$\|x(t)\| \le c_g(1+\|x(t-h)\|)+\|z\|+c_g(1+\|w(-h)\|)+c_f \int\limits_{\tau}^{t} \big(1+\|x(\xi)\|+\|x(\xi-h)\|\big)d\xi$$

for any $t \in [\tau, \vartheta]$. Define the function $\kappa(t) = 1 + \max\{\|x(\xi)\| \mid \xi \in [\tau - h, t]\}$, $t \in [\tau, \vartheta]$. Denote $t_j = \min\{\tau + jh, \vartheta\}$, $j \in \overline{0, J}$. Then, we derive

$$\kappa(t) \le c_* \kappa(t_j) + c_f \int\limits_{t_j}^{t} \kappa(\xi) d\xi, \quad t \in [t_j, t_{j+1}], \quad j \in \overline{0, J-1}.$$

From this estimate, applying the method of mathematical induction and Gronwall–Bellman Lemma (see, e.g., [3, p. 31]), we obtain the estimate

$$\kappa(t) \le c_*^{j+1} \kappa(\tau) e^{c_f(t-\tau)}, \quad t \in [t_j, t_{j+1}], \quad j \in \overline{0, J-1}. \tag{27}$$

which implies (26). □

Proposition 6 *For every $\alpha > 0$, there exist $\alpha_X = \alpha_X(\alpha) > 0$, $\alpha_X^g = \alpha_X^g(\alpha) > 0$, and $\lambda_X^g = \lambda_X^g(\alpha) > 0$ such that, for each $\tau \in [0, \vartheta]$, $(z, w(\cdot)) \in P(\alpha)$ (see (6)), $u(\cdot) \in \mathcal{U}_\tau$, and $v(\cdot) \in \mathcal{V}_\tau$, the motion $x(\cdot) = x(\cdot \mid \tau, z, w(\cdot), u(\cdot), v(\cdot))$ and the function $x^g(t) = x(t) - g(t, x(t-h))$, $t \in [\tau, \vartheta]$, satisfy the relations*

$$(x(t), x_t(\cdot)) \in P(\alpha_X), \quad \|x^g(t)\| \le \alpha_X^g, \quad \|x^g(t) - x^g(t')\| \le \lambda_X^g |t - t'|, \quad t, t' \in [\tau, \vartheta].$$

Proposition 7 *For every $\alpha > 0$, there exist $\lambda_{XX} = \lambda_{XX}(\alpha) > 0$ such that, for every $\tau \in [0, \vartheta]$, $(z, w(\cdot)), (p, r(\cdot)) \in P(\alpha)$, $u(\cdot) \in \mathcal{U}_\tau$, and $v(\cdot) \in \mathcal{V}_\tau$, the motions $x(\cdot) = x(\cdot|\tau, z, w(\cdot), u(\cdot), v(\cdot))$ and $y(\cdot) = x(\cdot|\tau, p, r(\cdot), u(\cdot), v(\cdot))$ satisfy the inequality*

$$\|x(t) - y(t)\| + \int_{\tau-h}^{t} \|x(\xi) - y(\xi)\| d\xi \le \lambda_{XX} \upsilon(\tau, z - p, w(\cdot) - r(\cdot)), \quad t \in [\tau, \vartheta].$$

Proof Let $\alpha > 0$. According to Propositions 4, 6 and condition (f_3), define the numbers $\alpha_X = \alpha_X(\alpha) > 0$, $\lambda_g = \lambda_g(\alpha_X) > 1$, and $\lambda_f = \lambda_f(\alpha_X) > 0$. Define also the numbers $\lambda_0^g = 1$, $\lambda_0^s = 1$, and

$$\lambda_j^g = (1 + \lambda_f(2 + \lambda_g)\lambda_{j-1}^s) e^{\lambda_f h}, \quad \lambda_j^s = 1 + jh\lambda_j^g + \lambda_g \lambda_{j-1}^s, \quad j \in \overline{1, J}. \tag{28}$$

Put $\lambda_{XX} = (J\lambda_g^J \lambda_J^g + \lambda_J^s)\lambda_g$.

Let $\tau \in [0, \vartheta]$, $(z, w(\cdot)), (p, r(\cdot)) \in P(\alpha)$, $u(\cdot) \in \mathcal{U}_\tau$, and $v(\cdot) \in \mathcal{V}_\tau$. Define the motions $x(\cdot) = x(\cdot|\tau, z, w(\cdot), u(\cdot), v(\cdot))$ and $y(\cdot) = x(\cdot|\tau, p, r(\cdot), u(\cdot), v(\cdot))$. Denote

$$s(t) = x(t) - y(t), \quad t \in [\tau - h, \vartheta], \quad s^g(t) = s(t) - g(t, x(t-h) + g(t, y(t-h)), \quad t \in [\tau, \vartheta].$$

Let us prove the estimates

$$\|s^g(t)\| \le \lambda_j^g \big(\|s^g(\tau)\| + \|s_\tau(\cdot)\|_1\big), \quad \int_{\tau-h}^{t} \|s(\xi)\| d\xi \le \lambda_j^s \big(\|s^g(\tau)\| + \|s_\tau(\cdot)\|_1\big) \tag{29}$$

for any $t \in [\tau, \tau + jh] \cap [\tau, \vartheta]$ and $j \in \overline{0, J}$. Note that these estimates hold for $j = 0$. Following the method of mathematical induction, we assume that these estimates are proved for $j-1$ and prove them for j. Due to the definitions of $x(\cdot)$, $y(\cdot)$, $s(\cdot)$, and λ_g, λ_f, using the second estimate in (29) for $j-1$, we derive

$$\|s^g(t)\| \le \|s^g(\tau)\| + \lambda_f \int\limits_{\tau+(j-1)h}^{t} \|s^g(\xi)\|d\xi + \lambda_f(2+\lambda_g) \int\limits_{\tau-h}^{\tau+(j-1)h} \|s(\xi)\|d\xi$$
$$\le (1+\lambda_f(2+\lambda_g)\lambda^s_{j-1})\big(\|s^g(\tau)\| + \|s_\tau(\cdot)\|_1\big) + \lambda_f \int\limits_{\tau+(j-1)h}^{t} \|s^g(\xi)\|d\xi.$$

Then, applying Gronwall–Bellman Lemma (see, e.g., [3, p. 31]), we get the first estimate in (29) on $[\tau+(j-1)h, \tau+jh]$. Since $\lambda^g_j > \lambda^g_i$ for any $i \in \overline{0, j-1}$, this estimate also holds on $[\tau, \tau+jh]$. The second estimate for j follows from the relations

$$\int_{\tau-h}^{t} \|s(\xi)\|d\xi \le \|s_\tau(\cdot)\|_1 + \int_{\tau}^{t} \|s^g(\xi)\|d\xi + \lambda_g \int_{\tau-h}^{t-h} \|s(\xi)\|d\xi$$
$$\le \|s_\tau(\cdot)\|_1 + jh\lambda^g_j\big(\|s^g(\tau)\| + \|s_\tau(\cdot)\|_1\big) + \lambda_g\lambda^s_{j-1}\big(\|s^g(\tau)\| + \|s_\tau(\cdot)\|_1\big),$$

in which, we use the choice of λ_g, the first estimate in (29) for j, and the second estimate in (29) for $j-1$. Thus, we have proved (29) for any $j \in \overline{0, J}$.

Let $t \in [\tau, \vartheta]$. Let $j_t \in \overline{0, J}$ be such that $t-(j_t+1)h \in [\tau-h, \tau)$. Then, applying (29) for $i \in \overline{0, j_t}$, taking into account the choice of λ_g and λ_{XX}, we conclude

$$\|s(t)\| + \int_{\tau-h}^{t} \|s(\xi)\|d\xi \le \sum_{i=0}^{j_t} \lambda^i_g \|s^g(t-ih)\| + \lambda^{j_t+1}_g \|s^g(t-(j_*+1)h)\| + \int_{\tau-h}^{t} \|s(\xi)\|d\xi$$
$$\le (J\lambda^J_g\lambda^g_J + \lambda^s_J)\big(\|s^g(\tau)\| + \|s_\tau(\cdot)\|_1\big) + \lambda^J_g\|s^g(\vartheta-(j+1)h)\| \le \lambda_{XX}\upsilon(\tau, s(\tau), s_\tau(\cdot))$$

and, hence, prove the proposition. □

Let $(\tau, w(\cdot)) \in [0, \vartheta] \times \mathrm{PLip}$. Let $-h = \xi_0 < \xi_1 < \ldots < \xi_{I+1} = 0$ be such that the function $w(\cdot)$ is Lipschitz continuous on $[\xi_i, \xi_{i+1})$ for each $i \in \overline{0, I}$. Denote $j_\tau \in \overline{0, J}$ such that $j_\tau h - \tau \in [-h, 0)$. Then, without loss of generality, we can assume that $\xi_i = j_\tau h - \tau$ for some $i \in \overline{0, I}$. Denote

$$\Theta_\nu(\tau, w(\cdot)) = \big\{[\tau+\xi_i+jh, \tau+\xi_{i+1}+jh-\nu) \cap [\tau, \vartheta) \colon i \in \overline{0, I},\ j \in \overline{1, J}\big\}, \quad (30)$$

where $\nu \in [0, \nu_*)$ and $\nu_* = \min\{(\xi_{i+1}-\xi_i) | i \in \overline{0, I}\}/2$.

Proposition 8 *For each $w(\cdot) \in \mathrm{PLip}$, there exists $\lambda_X = \lambda_X(w(\cdot)) > 0$ such that, for every $\tau \in [0, \vartheta]$, $z \in \mathbb{R}^n$, $u(\cdot) \in \mathcal{U}_\tau$, and $v(\cdot) \in \mathcal{V}_\tau$, the motion $x(\cdot) = x(\cdot \mid \tau, z, w(\cdot), u(\cdot), v(\cdot))$ satisfies the inequality*

$$\|x(t) - x(t')\| + \|x(t-h) - x(t'-h)\| \le \lambda_X|t-t'|, \quad t, t' \in \theta, \quad \theta \in \Theta_0(\tau, w(\cdot)).$$

Proof Due to the inclusion $x_\tau(\cdot) = w(\cdot) \in \mathrm{PLip}$, there exists $\lambda_w > 0$ such that

$$\|x(t-h) - x(t'-h)\| = \|w(t-\tau-h) - w(t'-\tau-h)\| \le \lambda_w|t-t'|$$

for any $t, t' \in [\xi_i, \xi_{i+1})$ and $i \in \overline{0, I}$. Let $\alpha > 0$. Taking $\alpha_X = \alpha_X(\alpha) > 0$, $\lambda^g_X = \lambda^g_X(\alpha) > 0$, and $\lambda_g = \lambda_g(\alpha_X) > 0$ from Propositions 6 and 4, respectively, define

$$\lambda_0 = \lambda_w, \quad \lambda_{j+1} = \lambda^g_X + (1+\lambda_g)\lambda_j, \quad j \in \overline{0, J}, \quad \lambda_X = \lambda_J.$$

Then, denoting $x^g(t) = x(t) - g(t, x(t-h))$, we derive

$$\begin{aligned} &\|x(t) - x(t')\| + \|x(t-h) - x(t'-h)\| \\ &\leq \|x^g(t) + x^g(t)\| + (1+\lambda_g)\|x(t-h) - x(t'-h)\| \leq \lambda_X^g + (1+\lambda_g)\lambda_j = \lambda_{j+1} \end{aligned}$$

for any $t, t' \in [\tau + \xi_i + jh, \tau + \xi_{i+1} + jh)$, $i \in \overline{0, I}$, and $j \in \overline{1, J}$. Since $\lambda_X = \lambda_J > \lambda_j$, $j \in \overline{0, J}$, from this inequalty, we get the statement of the proposition. □

3.2 Property of the Value Functional

Proposition 9 *Let ρ satisfy (ρ_1^*) and (ρ_2). Let $(\tau, z, w(\cdot)) \in \mathbb{G}$ and $\nu \in (0, \nu_*)$. Then, for every $\zeta > 0$, there exists $\delta > 0$ such that, for every $u(\cdot) \in \mathcal{U}_\tau$ and $v(\cdot) \in \mathcal{V}_\tau$, the motion $x(\cdot) = x(\cdot|\tau, z, w(\cdot), u(\cdot), v(\cdot))$ satisfies the inequality*

$$\left|\rho(t, x(t), x_t(\cdot)) - \rho(t', x(t'), x_{t'}(\cdot))\right| \leq \zeta$$

for any $t, t' \in \theta$: $|t - t'| \leq \delta$, and $\theta \in \Theta_\nu(\tau, w(\cdot))$.

Proof For the sake of contradiction, we assume the existence of $\zeta > 0$, $u^m(\cdot) \in \mathcal{U}_\tau$, $v^m(\cdot) \in \mathcal{V}_\tau$, $\theta_m \in \Theta$, and $t_m, t'_m \in \theta_m$ such that $|t_m - t'_m| \leq 1/m$ and the motion $x^m(\cdot) = x(\cdot|\tau, z, w(\cdot), u^m(\cdot), v^m(\cdot))$ satisfies

$$\left|\rho(t_m, x^m(t_m), x^m_{t_m}(\cdot)) - \rho(t'_m, x^m(t'_m), x^m_{t'_m}(\cdot))\right| > \zeta. \tag{31}$$

Without loss of generality, taking into account definition (30) of $\Theta_\nu(\tau, w(\cdot))$, we can assume the existence of $t_* \in [\tau, \vartheta]$ such that $|t_* - t_m| + |t_* - t'_m| \leq 1/m$, $m \in \mathbb{N}$ and $t_m, t'_m, t_* \in \theta_*$ for some $\theta_* \in \Theta_\nu(\tau, w(\cdot))$. Moreover, in accordance with (10) and Proposition 8, we can assume the existence of $x^*(\cdot) \in \mathrm{Lip}([\tau - h, T], \mathbb{R}^n)$ such that

$$\upsilon(t, x^m(t) - x^*(t), x^m_t(\cdot) - x^*_t(\cdot)) \leq 1/m, \quad t \in \theta_*, \quad m \geq\in N.$$

Let λ_* be a Lipschitz constant of $x^*(\cdot)$. Then, due to (10) and Proposition 8, we have

$$\begin{aligned} &\upsilon(t_m, x^*(t_m) - x^*(t_*), x^*_{t_m}(\cdot) - x^*_{t_*}(\cdot)) = \|x^*(t_m) - x^*(t_*)\| \\ &\quad + \|x^*(t_m - h) - x^*(t_* - h)\| + \|x^*_{t_m}(\cdot) - x^*_{t_*}(\cdot)\|_1 \leq (2+h)\lambda_*/m, \end{aligned} \quad m \in \mathbb{N}.$$

According to (ρ_1^*), there exists $M_* > 0$ such that

$$\|\rho(t_m, x^*(t_*), x^*_{t_*}(\cdot)) - \rho(t_*, x^*(t_*), x^*_{t_*}(\cdot))\| \leq \zeta/4, \quad m \geq M_*.$$

Thus, defining $\alpha = \max\{\|z\|, \|w(\cdot)\|_\infty\}$ and taking $\alpha_X = \alpha_X(\alpha)$ and $\lambda_\rho = \lambda_\rho(\alpha_X)$ in accordance with Propositions 6 and condition (ρ_2), we derive

$$\left\|\rho(t_m, x^m(t_m), x^m_{t_m}(\cdot)) - \rho(t_*, x^*(t_*), x^*_{t_*}(\cdot))\right\| \leq \zeta/4 + \lambda_\rho(1 + (2+h)\lambda_*)/m \leq \zeta/2$$

for any $m > M_1 := \max\{M_*, 4\lambda_\rho(1 + (2+h)\lambda_*)/\zeta\}$.

By the same way, we can find $M_2 > 0$ such that

$$\left\|\rho(t'_m, x^m(t'_m), x^m_{t'_m}(\cdot)) - \rho(t_*, x^*(t_*), x^*_{t_*}(\cdot))\right\| \leq \zeta/2, \quad m > M_2,$$

and, thus, obtain the contradiction with (31). □

Proposition 10 *The functionals ρ^u and ρ^v satisfy conditions (ρ_1) and (ρ_2).*

Proof We prove the statement only for ρ^u since it is proved similarly for the ρ^v.

First, we prove that ρ^u satisfies condition (ρ_2). Let $\alpha > 0$. According to Propositions 6, 7 and conditions (f_3), (σ_1), define $\alpha_X = \alpha_X(\alpha) > 0$, $\lambda_{XX} = \lambda_{XX}(\alpha_X) > 0$, $\lambda_f = \lambda_f(\alpha_X) > 0$, and $\lambda_\sigma = \lambda_\sigma(\alpha_X) > 0$. Put $\lambda_\rho = (\lambda_\sigma + 2\lambda_f)\lambda_{XX}$. To prove that ρ^u satisfies condition (ρ_2), it suffices to show the inequality

$$\rho^u(\tau, p, r(\cdot)) - \rho^u(\tau, z, w(\cdot)) \leq \lambda_\rho \upsilon(\tau, z - p, w(\cdot) - r(\cdot)) + \zeta,$$

for any $\tau \in [0, \vartheta]$, $(z, w(\cdot)), (p, r(\cdot)) \in P(\alpha)$, and $\zeta > 0$.

Let us take $\tau \in [0, \vartheta]$, $(z, w(\cdot)), (p, r(\cdot)) \in P(\alpha)$, and $\zeta > 0$. According to definition (18) of ρ^u, there exists $\hat{Q}^u_\tau$ such that

$$\sup_{v(\cdot)\in\mathcal{V}_\tau} \gamma(\tau, z, w(\cdot), \hat{Q}^u_\tau, v(\cdot)) \leq \rho^u(\tau, z, w(\cdot)) + \zeta/2,$$

there exists $\hat{v}(\cdot) \in \mathcal{V}_\tau$ such that

$$\sup_{v(\cdot)\in\mathcal{V}_\tau} \gamma(\tau, p, r(\cdot), \hat{Q}^u_\tau, v(\cdot)) \leq \gamma(\tau, p, r(\cdot), \hat{Q}^u_\tau, \hat{v}(\cdot)) + \zeta/2,$$

and therefore we have

$$\rho^u(\tau, p, r(\cdot)) - \rho^u(\tau, z, w(\cdot)) \leq \gamma(\tau, p, r(\cdot), \hat{Q}^u_\tau, \hat{v}(\cdot)) - \gamma(\tau, z, w(\cdot), \hat{Q}^u_\tau, \hat{v}(\cdot)) + \zeta.$$

Define $\hat{u}(\cdot) = \hat{Q}^u_\tau[\hat{v}(\cdot)](\cdot)$, the motions $x(\cdot) = x(\cdot \,|\, \tau, z, w(\cdot), \hat{u}(\cdot), \hat{v}(\cdot))$ and $y(\cdot) = x(\cdot \,|\, \tau, p, r(\cdot), \hat{u}(\cdot), \hat{v}(\cdot))$, and the function $s(t) = x(t) - y(t)$, $t \in [\tau - h, \vartheta]$. Then, due to the definition (4) of γ and the choice of the numbers λ_σ, λ_f, and λ_{XX}, we derive

$$\begin{aligned}\gamma(\tau, p, r(\cdot), \hat{Q}^u_\tau, \hat{v}(\cdot)) - \gamma(\tau, z, w(\cdot), \hat{Q}^u_\tau, \hat{v}(\cdot)) &\leq \lambda_\sigma \|s(\vartheta)\| + (\lambda_\sigma + 2\lambda_f)\int_{\tau-h}^{\vartheta} \|s(t)\| dt \\ &\leq \lambda_\rho \upsilon(\tau, z - p, w(\cdot) - r(\cdot)).\end{aligned}$$

Thus, we has shown that ρ^u satisfies (ρ_2).

Now, let us prove that the functional ρ^u satisfies condition (ρ_1). Let $(\tau, w(\cdot)) \in [0, \vartheta] \times \mathrm{Lip}$. Let us show the function $\hat{\rho}^u(t) = \rho^u(t, w(-0), w(\cdot))$ is uniformly continuous on $[\tau, \vartheta]$. Let $\zeta > 0$. According to Proposition 6 and conditions (f_2) and (ρ_2) proved above, define $\alpha_X(\alpha_0) > 0$, $c_f > 0$, and $\lambda_\rho(\alpha_X) > 0$, where $\alpha_0 = \|w(\cdot)\|_\infty$. Note also that, since $w(\cdot) \in \mathrm{Lip}$ due to [19, Lemma 3], there exists $\lambda_X = \lambda_X(w(\cdot)) > 0$ such that, for every $t \in [\tau, \vartheta]$, $u(\cdot) \in \mathcal{U}_t$, and $v(\cdot) \in \mathcal{V}_t$, the motion $x(\cdot) = x(\cdot \,|\, t, w(-0), w(\cdot), u(\cdot), v(\cdot))$ satisfies

$$\left\|x(\xi') - x(\xi)\right\| \leq \lambda_X |\xi' - \xi|, \quad \xi, \xi' \in [t - h, \vartheta].$$

Put $\delta = \zeta/3\max\{\lambda_\rho(2 + h)\lambda_X, c_f(1 + 2\alpha_X)\}$.

Let $t, t_* \in [\tau, \vartheta]$ be such that $|t - t_*| \leq \delta$. Without loss of generality, suppose that $t \leq t_*$. Let $v(\cdot) \in \mathcal{V}_\tau$. According to (ρ_3), there exists $u(\cdot) \in \mathcal{U}_\tau$ such that, for the motion $x(\cdot) = x(\cdot \,|\, t, w(-0), w(\cdot), u(\cdot), v(\cdot))$, we have

$$\rho^u(t_*, x(t_*), x_{t_*}(\cdot)) + \int_t^{t_*} f^0(\xi, x(\xi), x_\xi(\cdot), u(\xi), v(\xi)) d\xi \leq \rho^u(t, w(-0), w(\cdot)) + \zeta/3.$$

Due to the choice of λ_ρ, λ_X, and δ, we derive

$$\begin{aligned}&|\rho^u(t_*, x(t_*), x_{t_*}(\cdot)) - \rho^u(t_*, w(-0), w(\cdot))| \\ &\quad \leq \lambda_\rho \upsilon\big(t_*, x(t_*) - w(-0), x_{t_*}(\cdot) - w(\cdot)\big) \leq \lambda_\rho(2 + h)\lambda_X \delta \leq \zeta/3.\end{aligned}$$

According to the choice of α_X, c_f, and δ, we get

$$\int_t^{t_*} \big|f^0(\xi, x(\xi), x_\xi(\cdot), u(\xi), v(\xi))\big| d\xi \le c_f(1+2\alpha_X)(t_*-t) \le \zeta/3.$$

Thus, we obtain $\rho^u(t_*, w(-0), w(\cdot)) - \rho^u(t, w(-0), w(\cdot)) \le \zeta$. The inequality $\rho^u(t_*, w(-0), w(\cdot)) - \rho^u(t, w(-0), w(\cdot)) \ge -\zeta$ can be proved in the similar way, using (ρ_4) instead of (ρ_3). □

The subdifferential of a functional $\rho\colon \mathbb{G} \mapsto \mathbb{R}$ at a point $(\tau, z, w(\cdot)) \in \mathbb{G}$, $\tau < \vartheta$ is a set, denoted by $D^-(\tau, z, w(\cdot))$, of pairs $(p_0, p) \in \mathbb{R} \times \mathbb{R}^n$ such that

$$\lim_{\delta\to 0} \inf_{(t,x)\in O^+_\delta(\tau,z)} \frac{\varphi(t, x, \kappa_t(\cdot)) - \varphi(\tau, z, w(\cdot)) - (t-\tau)p_0 - \langle x - z, p\rangle}{|t-\tau| + \|x - z\|} \ge 0, \tag{32}$$

where $\kappa(t) = w(t-\tau)$, $t \in [\tau - h, \tau)$, $\kappa(t) = z$, $t \in [\tau, \vartheta]$ and $O^+_\delta(\tau, z) = \{(t, x) \in [\tau, \vartheta] \times \mathbb{R}^n : t \in [\tau, \tau+\delta], \|x - z\| \le \delta\}$. The superdifferential of a functional $\varphi\colon \mathbb{G} \mapsto \mathbb{R}$ at a point $(\tau, z, w(\cdot)) \in \mathbb{G}$, $\tau < \vartheta$ is a set, denoted by $D^+(\tau, z, w(\cdot))$, of pairs $(q_0, q) \in \mathbb{R}\times\mathbb{R}^n$ such that

$$\lim_{\delta\to 0} \sup_{(t,x)\in O^+_\delta(\tau,z)} \frac{\varphi(t, x, \kappa_t(\cdot)) - \varphi(\tau, z, w(\cdot)) - (t-\tau)q_0 - \langle x - z, q\rangle}{|t-\tau| + \|x - z\|} \le 0. \tag{33}$$

Proposition 11 *Let a functional $\rho\colon \mathbb{G} \mapsto \mathbb{R}$ satisfy conditions (ρ_2)–(ρ_4). Then, for every $(\tau, z, w(\cdot)) \in \mathbb{G}_*$, the following inequalities holds:*

$$\begin{aligned} &p_0 + \langle G(\tau, w(-h), d^+w(-h)/d\xi), p\rangle \\ &\quad + H(\tau, z, w(-h), p) \le 0, \quad (p_0, p) \in D^-\rho(\tau, z, w(\cdot)) \\ &q_0 + \langle G(\tau, w(-h), d^+w(-h)/d\xi), q\rangle \\ &\quad + H(\tau, z, w(-h), q) \ge 0, \quad (q_0, q) \in D^+\rho(\tau, z, w(\cdot)) \end{aligned} \tag{34}$$

Proof We prove only the first inequality from (34) since the second one can be shown similarly. Let $(\tau, z, w(\cdot)) \in \mathbb{G}_*$ and $(p_0, p) \in D^-\rho(\tau, z, w(\cdot))$. Note that the estimate

$$p_0 + \langle G(\tau, w(-h), \mathrm{d}^+w(-h)/\mathrm{d}\xi), p\rangle + H(\tau, z, w(-h), p) \le \zeta \tag{35}$$

for any $\zeta > 0$ implies the first inequality from (34).

Let $\zeta > 0$. According to Proposition 8, define $\lambda_X = \lambda_X(w(\cdot)) > 0$. Due to definition (32) of $D^-\rho(\tau, z, w(\cdot))$, there exists $\delta > 0$ such that

$$\frac{\rho(t, x, \kappa_t(\cdot)) - \rho(\tau, z, w(\cdot)) - (t-\tau)p_0 - \langle x - z, p\rangle}{|t-\tau| + \|x - z\|} \ge -\frac{\zeta}{3(1+\lambda_X)} \tag{36}$$

for every $(t, x) \in O^+_\delta(\tau, z)$. Since $w(\cdot) \in \mathrm{PLip}_*$ (see (1)), taking into account (5), (12), and (ρ_2), there exists $t \in (\tau, \tau + \delta/(1+\lambda_X))$ such that, for every $u(\cdot) \in \mathcal{U}_\tau$ and $v(\cdot) \in \mathcal{V}_\tau$, the motion $x(\cdot) = x(\cdot \mid \tau, z, w(\cdot), u(\cdot), v(\cdot))$ satisfies the estimates

$$\begin{aligned} &\big|\chi(\tau, z, w(-h), u, v) - \chi(\xi, x(\xi), x(\xi-h), u, v)\big| \le \zeta/6, \\ &\big|G(\tau, w(-h), d^+w(-h)/d\xi) - G(\xi, x(\xi-h), d^+x(\xi-h)/d\xi)\big| \le \zeta/6, \\ &\big|\rho(\xi, x(\xi), \kappa_\xi(\cdot)) - \rho(\xi, x(\xi), x_\xi(\cdot))\big| \le (\xi-\tau)\zeta/3 \end{aligned} \tag{37}$$

for any $\xi \in [\tau, t]$, $u \in \mathbb{U}$, and $v \in \mathbb{V}$. Define

$$v_* \in \operatorname*{argmax}_{v\in\mathbb{V}} \min_{u\in\mathbb{U}} \chi\big(\tau, z, w(-h), u, v, p\big). \tag{38}$$

Due to condition (ρ_3), there exists $u(\cdot) \in \mathcal{U}_\tau$ such that the motion $x(\cdot) = x(\cdot \mid \tau, z, w(\cdot), u(\cdot), v(\cdot) = v_*)$ satisfies

$$\rho(t, x(t), x_t(\cdot)) + \int_\tau^t f^0(\xi, x(\xi), x(\xi), u(\xi), v_*)d\xi \leq \rho(\tau, z, w(\cdot)) + (t-\tau)\zeta/2. \quad (39)$$

Thus, due to (2), (5), (12), (37), and (38), we derive

$$\begin{aligned}&\langle G(\tau, w(-h), \mathrm{d}^+ w(-h)/\mathrm{d}\xi), p\rangle + H(\tau, z, w(-h), p)\\ &\leq \frac{1}{t-\tau}\int_\tau^t \left(\frac{d}{d\xi}\Big(g(\xi, x(\xi-h)\Big) + \chi(\xi, x(\xi), x(\xi-h), u(\xi), v_*, p)\right)d\xi + \zeta/3\\ &= \frac{\langle x(t)-z, p\rangle}{t-\tau} + \frac{1}{t-\tau}\int_\tau^t f^0(\xi, x(\xi), x(\xi), u(\xi), v_*)d\xi + \zeta/3.\end{aligned}$$

From this inequality, using (36), (37), and (39), we obtain (35).

Proposition 12 *There exists $c_\rho > 0$ such that ρ° satisfies the equality*

$$\big|\rho^\circ(\tau, z, w(\cdot))\big| \leq c_\rho\big(1 + \|z\| + \|w(\cdot)\|_\infty\big), \quad (\tau, z, w(\cdot)) \in \mathbb{G}. \quad (40)$$

Proof Let us take c_f, c_σ, and c_X from conditions (f_2), (σ_2) and the Proposition 6. Then, putting $c_\rho = c_\sigma(1 + c_X(1+h)) + c_f\vartheta(1+2c_X)$, for every $(\tau, z, w(\cdot)) \in \mathbb{G}$, $u(\cdot) \in \mathcal{U}_\tau$, and $v(\cdot) \in \mathcal{V}_\tau$, the motion $x(\cdot) = x(\cdot \mid \tau, z, w(\cdot), u(\cdot), v(\cdot))$ satisfies

$$\begin{aligned}&\sigma(x(\vartheta), x_\vartheta(\cdot)) + \int_\tau^\vartheta f^0(t, x(t), x(t-h), u(t), v(t))dt\\ &\leq c_\sigma\big(1 + \|x(\vartheta)\| + \|x_\vartheta(\cdot)\|_\infty\big) + c_f\int_\tau^\vartheta \big(1 + \|x(t)\| + \|x(t-h)\|\big)dt\\ &\leq c_\sigma(1 + c_X(1+h)) + c_f\vartheta(1+2c_X) + c_X(c_\sigma(1+h) + 2c_f\vartheta)\alpha_0 \leq c_\rho\big(1+\alpha_0\big)\end{aligned}$$

where $\alpha_0 = \max\{\|z\|, \|w(\cdot)\|_\infty\}$. Thus, from (4), (7), and Theorem 1, we obtain (40). □

Proposition 13 *Let the value functional $\rho^\circ = \rho^\circ(t, z, w(\cdot))$ be differentiable by z. Then, for every $\alpha > 0$, there exists $\lambda_\rho = \lambda_\rho(\alpha) > 0$ such that*

$$\big\|\nabla_z \rho^\circ(\tau, z, w(\cdot))\big\| \leq \lambda_\rho, \quad \tau \in [0, \vartheta], \quad (z, w(\cdot)) \in P(\alpha). \quad (41)$$

Proof Due to Theorem 1 and Proposition 10, there exists $\lambda_\rho = \lambda_\rho(\alpha) > 0$ such that

$$\big|\rho^\circ(\tau, z, w(\cdot)) - \rho^\circ(\tau, p, w(\cdot))\big| \leq \lambda_\rho\|z - p\|, \quad \tau \in [0, \vartheta], \quad (z, w(\cdot)) \in P(\alpha).$$

Since ρ° is differentiable by z, from this estimate, we obtain (41). □

Proposition 14 *Let $\lambda > 0$. Let $\varepsilon_* = \varepsilon_*(\lambda) > 0$ be such that $\theta^{\lambda,\varepsilon}(t) > 2c_\rho$ for any $t \in [0, \vartheta]$ and $\varepsilon \in (0, \varepsilon_*)$, where $\theta^{\lambda,\varepsilon}$ and c_ρ are from (21) and Proposition 12, respectively. Then, the argmin and argmax values in (22) are achieved.*

Proof Let us prove the statement for the argmin value. Let $(t, x, r(\cdot)) \in \mathbb{G}$. Consider the function $\varphi(p) = \rho^\circ(t, p, r(\cdot)) + \eta^{\lambda,\varepsilon}(t, x - p)$, $p \in \mathbb{R}^n$. According to Theorem 1 and Proposition 10, this function is continuous. Due to the choice of ε_* and c_ρ, we derive

$$\varphi(p) \geq -c_\rho\big(1 + \|p\| + \|r(\cdot)\|_\infty\big) + \theta^{\lambda,\varepsilon}(t)\|x - p\| \geq c_\rho\big(1 + \|p\| - 2\|x\| + \|r(\cdot)\|_\infty\big).$$

Hence, the function $\varphi(p)$ is bounded below and $\varphi(p) \to +\infty$ as $p \to \infty$. Thus, the minimum of $\varphi(p)$ is achieved. □

3.3 Proof of Theorem 2

Proof The proof is carried out by the scheme from [16] (see also [8]).

Let us fix $(\tau, z, w(\cdot)) \in \mathbb{G}$. According to Proposition 6, define $\alpha_* = \alpha_X(\alpha_X(\alpha_0))$ and $\lambda_*^g = \lambda_X^g(\alpha_X(\alpha_0))$, where $\alpha_0 = \max\{\|z\|, \|w(\cdot)\|_\infty\}$. Then, for every $u(\cdot) \in \mathcal{U}_\tau$, $v(\cdot) \in \mathcal{V}_\tau$, $t_* \in [\tau, \vartheta]$, and for every $r(\cdot) \in$ PLip such that $\|r(\cdot)\|_\infty \leq \|x_{t_*}(\cdot)\|_\infty$, where $x(\cdot) = x(\cdot \mid \tau, z, w(\cdot), u(\cdot), v(\cdot))$, the motion $y(\cdot) = x(\cdot \mid t_*, x(t_*), r(\cdot), u(\cdot), v(\cdot))$ and the function $y^g(t) = y(t) - g(t, y(t-h))$, $t \in [t_*, \vartheta]$ satisfy the relations

$$(y(t), y_t(\cdot)) \in P(\alpha_*), \quad \left|y^g(t) - y^g(t')\right| \leq \lambda_*^g |t - t'|, \quad t, t' \in [t_*, \vartheta]. \tag{42}$$

Moreover, due to condition (f_2) and Proposition 13, there exist $\beta_f, \beta_\nabla > 0$ such that

$$\begin{gathered} \left\| f(t, y(t), y(t-h), u(t), v(t)) \right\| + \left| f^0(t, y(t), y(t-h), u(t), v(t)) \right| \leq \beta_f, \\ \left\| \nabla_z \rho^\circ(t, y(t), y_t(\cdot)) \right\| \leq \beta_\nabla, \quad t \in [t_*, \vartheta]. \end{gathered} \tag{43}$$

Note that (42) and (43) are also valid for $x(\cdot)$ if we take $t_* = \tau$ and $r(\cdot) = w(\cdot)$.

Since both equalities in (20) are proved similarly, we present only the proof of the first equality which, according to (18), will be proved if we show that, for every $\zeta > 0$, there exists $\delta > 0$ such that, for every partition Δ_δ (see (16)) and every $v(\cdot) \in \mathcal{V}_\tau$, if $u(\cdot) \in \mathcal{U}_\tau$ satisfies the relation

$$u(t) = u_j \in \underset{u \in \mathbb{U}}{\operatorname{argmin}} \max_{v \in \mathbb{V}} \chi\left(t_j, x(x_j), x(t_j - h), u, v, \nabla_z \rho^\circ(t_j, x(t_j), x_{t_j}(\cdot))\right) \tag{44}$$

for any $t \in [t_j, t_{j+1})$ and $j \in \overline{1, k}$, then the motion $x(\cdot) = x(\cdot \mid \tau, z, w(\cdot), u(\cdot), v(\cdot))$ satisfies the estimate

$$\sigma(x(\vartheta), x_\vartheta(\cdot)) + \int_\tau^\vartheta f^0(t, x(t), x(t-h), u(t), v(t)) dt \leq \rho^\circ(\tau, z, w(\cdot)) + \zeta. \tag{45}$$

Let $\zeta > 0$. According to the definition (30) of the set $\Theta_\nu(\tau, w(\cdot))$, the number of intervals in the set $[\tau, \vartheta] \setminus \Theta_\nu(\tau, w(\cdot))$ does not depend on $\nu \in (0, \nu_*)$. Denote this number as l_*. Due to Proposition 10, define $\lambda_\rho = \lambda_\rho(\alpha_*)$. Set

$$\nu = \min\left\{\frac{\zeta}{6\beta_* l_*}, \frac{\nu_*}{6}\right\}, \quad \beta_* = \max\left\{4\beta_f, 4\lambda_\rho\left(4\lambda_*^g + 2h\alpha_*\right)\right\}, \quad \zeta_* = \frac{\zeta}{2(\vartheta - \tau)}. \tag{46}$$

Due to condition (f_1), Propositions 8, 9, and (43), there exists $\delta \in (0, \min\{\nu, h\})$ such that, for every $u(\cdot) \in \mathcal{U}_\tau$, $v(\cdot) \in \mathcal{V}_\tau$, $\theta \in \Theta_\nu(\tau, w(\cdot))$, $t, t' \in \theta$: $|t - t'| \leq \delta$, and $u \in \mathbb{U}$, $v \in \mathbb{V}$, the motion $x(\cdot) = x(\cdot \mid \tau, z, w(\cdot), u(\cdot), v(\cdot))$ satisfies the inequality

$$\begin{aligned} &\big|\chi(t, x(t), x(t-h), u, v, \nabla_z \rho^\circ(t, x(t), x_t(\cdot))) \\ &\quad - \chi(t', x(t'), x(t'-h), u, v, \nabla_z \rho^\circ(t', x(t'), x_{t'}(\cdot)))\big| \leq \zeta_*/16. \end{aligned} \tag{47}$$

and, taking into account (12), as a consequence, the estimate

$$\begin{aligned} &\big|H(t, x(t), x(t-h), \nabla_z \rho^\circ(t, x(t), x_t(\cdot))) \\ &\qquad - H(t', x(t'), x(t'-h), \nabla_z \rho^\circ(t', x(t'), x_{t'}(\cdot)))\big| \leq \zeta_*/16. \end{aligned} \tag{48}$$

Let us take a partition Δ_δ and a realization $v(\cdot) \in \mathcal{V}_\tau$. Define the index sets

$$K_1 = \left\{j \in \overline{0, k} \,\middle|\, \exists \theta \in \Theta_\nu(\tau, w(\cdot)) \colon [t_j, t_{j+1}] \subset \theta\right\}, \quad K_2 = \left\{j \in \overline{0, k} \,\middle|\, j \notin K_1\right\}. \tag{49}$$

Then, according to the choice of the numbers ν, l_* and δ, we have

$$\sum_{j \in K_1} (t_{j+1} - t_j) \leq \vartheta - \tau, \quad \sum_{j \in K_2} (t_{j+1} - t_j) \leq 3\nu l_* \leq \zeta/(2\beta_*).$$

Thus, to show (45), we need to prove the inequality

$$\begin{aligned}&\rho^\circ(t_{j+1}, x(t_{j+1}), x_{t_{j+1}}(\cdot)) + \int_{t_j}^{t_{j+1}} f^0(\xi, x(\xi), x(\xi-h), u(\xi), v(\xi))d\xi \\ &\quad \le \rho^\circ(t_j, x(t_j), x_{t_j}(\cdot)) + (t_{j+1}-t_j)\zeta_*, \quad j \in K_1,\end{aligned} \tag{50}$$

and the inequality

$$\begin{aligned}&\rho^\circ(t_{j+1}, x(t_{j+1}), x_{t_{j+1}}(\cdot)) + \int_{t_j}^{t_{j+1}} f^0(\xi, x(\xi), x(\xi-h), u(\xi), v(\xi))d\xi \\ &\quad \le \rho^\circ(t_j, x(t_j), x_{t_j}(\cdot)) + (t_{j+1}-t_j)\beta_*, \quad j \in K_2.\end{aligned} \tag{51}$$

Let us prove (50). Let $j \in K_1$. Then, due to Proposition 8, the function $x(\cdot)$ is Lipschitz continuous on $[t_j - h, t_{j+1} - h]$. One can take a sequence of continuously differentiable on $[-h, -h + t_{j+1} - t_j]$ functions $r^m(\cdot) \in \mathrm{PLip}$, $m \in \mathbb{N}$ such that

$$\|r^m(\cdot)\|_\infty \le \|x_{t_j}(\cdot)\|_\infty, \quad \|x_{t_j}(\cdot) - r^m(\cdot)\|_\infty \le 1/m, \quad m \in \mathbb{N}.$$

Define the sequence of motions $y^m(\cdot) = x(\cdot|t_j, x(t_j), r^m(\cdot), u(\cdot), v(\cdot))$. Note that, according to Proposition 7 and (10), there exists $\lambda_{XX} = \lambda_{XX}(\alpha_*) > 0$ such that

$$\|x(t) - y^m(t)\| \le \lambda_{XX}(2+h)/m, \quad m \in \mathbb{N}.$$

Then, due to condition (f_3), Proposition 10, and relations (42), (43), we can take $y(\cdot) \in \{y^m(\cdot) \mid m \in \mathbb{N}\}$ such that

$$\begin{aligned}&\big|\rho^\circ(t, x(t), x_t(\cdot)) - \rho^\circ(t, y(t), y_t(\cdot))\big| \le (t_{j+1}-t_j)\zeta_*/4, \\ &\big\|f^0(t, x(t), x(t-h), u(t), v(t)) - f^0(t, y(t), y(t-h), u(t), v(t))\big\| \le \zeta_*/4, \\ &\big|\chi(t, x(t), x(t-h), u(t), v(t), \nabla_z\rho^\circ(t, x(t), x_t(\cdot))) \\ &-\chi(t, y(t), y(t-h), u(t), v(t), \nabla_z\rho^\circ(t, y(t), y_t(\cdot)))\big| \le \zeta_*/16\end{aligned} \tag{52}$$

for any $t \in [t_j, t_{j+1}]$, and, taking into account (12), as a consequence

$$\begin{aligned}&\big|H(t, x(t), x(t-h), \nabla_z\rho^\circ(t, x(t), x_t(\cdot))) \\ &\quad -H(t, y(t), y(t-h), \nabla_z\rho^\circ(t, y(t), y_t(\cdot)))\big| \le \zeta_*/16\end{aligned} \tag{53}$$

for any $t \in [t_j, t_{j+1}]$. Thus, in order to conclude (50), it is necessary to prove

$$\begin{aligned}&\rho^\circ(t_{j+1}, y(t_{j+1}), y_{t_{j+1}}(\cdot)) + \int_{t_j}^{t_{j+1}} f^0(\xi, y(\xi), y(\xi-h), u(\xi), v(\xi))d\xi \\ &\quad \le \rho^\circ(t_j, y(t_j), y_{t_j}(\cdot)) + (t_{j+1}-t_j)\zeta_*/4.\end{aligned} \tag{54}$$

Let us consider the function

$$\omega(t) = \rho^\circ(t, y(t), y_t(\cdot)) + \int_{t_j}^{t} f^0(\xi, y(\xi), y(\xi-h), u(\xi), v(\xi))dt, \quad t \in [t_j, t_{j+1}]. \tag{55}$$

Since $y(\cdot)$ is continuously differentiable on $[t_j - h, t_{j+1} - h]$, we have $y_t(\cdot) \in \mathrm{PLip}_*$ for any $t \in [t_j, t_{j+1})$. Then, taking into account definition (1) of the set $\mathbb{G}_*$ we derive $(t, y(t), y_t(\cdot)) \in \mathbb{G}_*$ for almost every $t \in [t_j, t_{j+1})$. Since $y(\cdot)$ is Lipschitz continuous on $[t_j, t_{j+1}]$, there exists $dy(t)/dt$ for almost every $[t_j, t_{j+1}]$. Then, from the coinvariant differentiability of ρ° on $\mathbb{G}_*$, we obtain

$$\begin{aligned}\frac{d}{dt}\omega(t) &= \partial^{ci}_{\tau,w}\rho^\circ(t, y(t), y_t(\cdot)) + \langle \frac{d}{dt}y(t), \nabla_z\rho^\circ(t, y(t), y_t(\cdot))\rangle \\ &\quad + f^0(t, y(t), y(t-h), u(t), v(t))\end{aligned}$$

for almost every $t \in [t_j, t_{j+1}]$. Then, due to property (ρ_5), taking into account (5) and (12), we have

$$\frac{d}{dt}\omega(t) = \chi(t, y(t), y(t-h), u(t), v(t), \nabla_z \rho^\circ(t, y(t), y_t(\cdot))) \\ -H(t, y(t), y(t-h), \nabla_z \rho^\circ(t, y(t), y_t(\cdot))) \tag{56}$$

for almost every $t \in [t_j, t_{j+1}]$. Next, in accordance with (44), (47), and (52), we derive

$$\begin{aligned} &\chi(t, y(t), y(t-h), u(t), v(t), \nabla_z \rho^\circ(t, y(t), y_t(\cdot))) \\ &\quad \le \chi(t, x(t), x(t-h), u(t), v(t), \nabla_z \rho^\circ(t, x(t), x_t(\cdot))) + \zeta_*/16 \\ &\quad \le \chi(t_j, x(t_j), x(t_j-h), u(t), v(t), \nabla_z \rho^\circ(t_j, x(t_j), x_{t_j}(\cdot))) + \zeta_*/8 \\ &\quad \le \max_{v \in \mathbb{V}} \chi(t_j, x(t_j), x(t_j-h), u_j, v, \nabla_z \rho^\circ(t_j, x(t_j), x_{t_j}(\cdot))) + \zeta_*/8 \\ &\quad = \min_{u \in \mathbb{U}} \max_{v \in \mathbb{V}} \chi(t_j, x(t_j), x(t_j-h), u, v, \nabla_z \rho^\circ(t_j, x(t_j), x_{t_j}(\cdot))) + \zeta_*/8. \end{aligned}$$

Finally, according to (12), (48), and (53), we get

$$\begin{aligned} &\min_{u \in \mathbb{U}} \max_{v \in \mathbb{V}} \chi(t_j, x(t_j), x(t_j-h), u, v, \nabla_z \rho(t_j, x(t_j), x_{t_j}(\cdot))) \\ &\quad \le H(t, x(t), x(t-h), \nabla_z \rho^\circ(t, x(t), x_t(\cdot))) + \zeta_*/16 \\ &\quad \le H(t, y(t), y(t-h), \nabla_z \rho^\circ(t, y(t), y_t(\cdot))) + \zeta_*/8. \end{aligned}$$

Thus, we obtain $d\omega(t)/dt \le \zeta_*/4$ and conclude (54) which proves (50).

Let us prove (51). Let $j \in K_2$. According to condition (ρ_3), there exists $u_j(\cdot) \in \mathcal{U}_{t_j}$ such that the motion $y(\cdot) = x(\cdot \mid t_j, x(t_j), x_{t_j}(\cdot), u_j(\cdot), v(\cdot))$ satisfies the estimate

$$\rho^\circ(t_{j+1}, y(t_{j+1}), y_{t_{j+1}}(\cdot)) + \int_{t_j}^{t_{j+1}} f^0(\xi, y(\xi), y(\xi-h), u(\xi), v(\xi)) d\xi \\ \le \rho^\circ(t_j, x(t_j), x_{t_j}(\cdot)) + (t_{j+1} - t_j)\beta_*/4. \tag{57}$$

Define the function $s(t) = x(t) - y(t)$, $t \in [t_j - h, \vartheta]$. Then, since $t_{j+1} - t_j \le \delta < h$ and according to (42), we have

$$s(t) = 0, \quad t \in [t_j - h, t_j], \quad \|s(t)\| \le 2\alpha_*, \\ \|s(t)\| = \|x(t) - g(t, x(t-h)) - y(t) + g(t, x(t-h))\| \le 2\lambda_*^g (t_{j+1} - t_j) \tag{58}$$

for any $t \in [t_j, t_{j+1}]$. Hence, according to the choice of λ_ρ and β_* (see (46)), we derive

$$\left|\rho(t_{j+1}, x(t_{j+1}), x_{t_{j+1}}(\cdot)) - \rho(t_{j+1}, y(t_{j+1}), y_{t_{j+1}}(\cdot))\right| \\ \le \lambda_\rho \big(\|s(t_{j+1})\| + \|s(lh)\| + \|s_{j+1}(\cdot)\|_1\big) \le (t_{j+1} - t_j)\beta_*/4, \tag{59}$$

where $l \in \overline{-1, J-1}$ satisfies $lh \in [t_{j+1} - h, t_{j+1})$. Due to (43) and (46), we have

$$\int_{t_j}^{t_{j+1}} f^0(\xi, x(\xi), x(\xi-h), u(\xi), v(\xi)) d\xi \le (t_{j+1} - t_j)\beta_*/4, \\ \int_{t_j}^{t_{j+1}} f^0(\xi, y(\xi), y(\xi-h), u_j(\xi), v(\xi)) d\xi \le (t_{j+1} - t_j)\beta_*/4. \tag{60}$$

From (57)–(60), we obtain (51). Thus, we have proved (57), (51), and the theorem. □

3.4 Proof of Theorem 3

Define the functional

$$\varphi^{\lambda,\varepsilon}(t, x, r(\cdot)) = \min_{p \in \mathbb{R}^n} \big(\rho^\circ(t, p, r(\cdot)) - \eta^{\lambda,\varepsilon}(t, x - p)\big), \quad (t, x, r(\cdot)) \in \mathbb{G}, \tag{61}$$

where $\eta^{\lambda,\varepsilon}$ is from (21).

Proposition 15 *Let $(\tau, z, w(\cdot)) \in \mathbb{G}$ and $\lambda, \zeta_1 > 0$. There exists $\varepsilon_1 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_1]$, $u(\cdot) \in \mathcal{U}_\tau$, and $v(\cdot) \in \mathcal{V}_\tau$, the motion $x(\cdot) = x(\cdot \mid \tau, z, w(\cdot), u(\cdot), v(\cdot))$ satisfies the estimates*

$$\left\| p^{\lambda,\varepsilon}(t, x(t), x_t(\cdot)) - x(t) \right\| \leq \zeta_1, \quad t \in [\tau, \vartheta], \tag{62}$$

where the functional $p^{\lambda,\varepsilon}$ is defined in (22).

Proof Define $\alpha_X = \alpha_X(\alpha_0)$, where $\alpha_0 = \max\{\|z\|, \|w(\cdot)\|_\infty\}$, and c_ρ according to Propositions 6 and 12. Due to definitions (21), we can take $\varepsilon_1 > 0$ so that

$$\theta^{\lambda,\varepsilon}(t) \geq c_\rho + 3\beta/\zeta_1, \quad \eta^{\lambda,\varepsilon}(t, 0) \leq \beta, \quad \beta = c_\rho(1 + (1+h)\alpha_X), \quad t \in [0, \vartheta], \quad \varepsilon \in (0, \varepsilon_1].$$

Let $u(\cdot) \in \mathcal{U}_\tau$, $v(\cdot) \in \mathcal{V}_\tau$, $\varepsilon \in (0, \varepsilon_1]$, and $t \in [\tau, \vartheta]$. Denote $p = p^{\lambda,\varepsilon}(t, x(t), x_t(\cdot))$. Then, we have

$$\begin{aligned}
3\beta/\zeta_1 \|x(t) - p\| - \beta &\leq \big(\theta^{\lambda,\varepsilon}(t) - c_\rho\big)\left\|x(t) - p\right\| - c_\rho\big(1 + \|x(t)\| + \|x_t(\cdot)\|_\infty\big) \\
&\leq \theta^{\lambda,\varepsilon}(t)\|x(t) - p\| - c_\rho\big(1 + \|p\| + \|x_t(\cdot)\|_\infty\big) \leq \rho^\circ(t, p, x_t(\cdot)) + \eta^{\lambda,\varepsilon}(t, x(t) - p) \\
&= \varphi^{\lambda,\varepsilon}(t, x(t), x_t(\cdot)) \leq \rho^\circ(t, x(t), x_t(\cdot)) + \eta^{\lambda,\varepsilon}(t, 0) \leq 2\beta.
\end{aligned}$$

Thus, we obtain (62). □

Proposition 16 *Let $(\tau, z, w(\cdot)) \in \mathbb{G}$ and $\lambda, \zeta_2 > 0$. There exists $\varepsilon_2 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_2]$, $u(\cdot) \in \mathcal{U}_\tau$, and $v(\cdot) \in \mathcal{V}_\tau$, the motion $x(\cdot \mid \tau, z, w(\cdot), u(\cdot), v(\cdot))$ satisfies the estimates*

$$\left\|\varphi^{\lambda,\varepsilon}(t, x(t), x_t(\cdot)) - \rho^\circ(t, x(t), x_t(\cdot))\right\| \leq \zeta_2, \quad t \in [\tau, \vartheta]. \tag{63}$$

Proof In accordance with Proposition 6, define $\alpha_X = \alpha_X(\alpha_0) > 0$, where $\alpha_0 = \max\{\|z\|, \|w(\cdot)\|_\infty\}$. Due to Theorem 1 and Proposition 10, the value functional ρ° satisfies condition (ρ_2). According to this condition, determine $\lambda_\rho = \lambda_\rho(\alpha_X) > 0$. Due to Proposition 15, putting $\zeta_1 = \zeta_2/\lambda_\rho$, define $\varepsilon_1 > 0$. Set $\varepsilon_2 = \min\{\varepsilon_1, \zeta_2\}$. Then, in particular, due to (21), we have $\eta^{\lambda,\varepsilon}(t, 0) \leq \zeta_2$, $t \in [0, \vartheta]$, $\varepsilon \in (0, \varepsilon_2]$. Thus, taking into account (61), for every $\varepsilon \in (0, \varepsilon_2]$ and $t \in [\tau, \vartheta]$, we obtain

$$\begin{aligned}
-\zeta_2 &\leq -\eta^{\lambda,\varepsilon}(t, 0) \leq \rho^\circ(t, x(t), x_t(\cdot)) - \varphi^{\lambda,\varepsilon}(t, x(t), x_t(\cdot)) \\
&= \rho^\circ(t, x(t), x_t(\cdot)) - \rho^\circ(t, p, r(\cdot)) - \eta^{\lambda,\varepsilon}(t, x - p) \leq \lambda_\rho \|x(t) - p\| \leq \zeta_2,
\end{aligned}$$

where we denote $p = p^{\lambda,\varepsilon}(t, x(t), x_t(\cdot))$. □

Proposition 17 *Let $(\tau, z, w(\cdot)) \in \mathbb{G}$ and $\zeta_3 > 0$. There exists $\lambda, \varepsilon_3 > 0$ with the following properties. For every $\varepsilon \in (0, \varepsilon_3]$, there exists $\delta > 0$ such that, for every partition Δ_δ and every $v(\cdot) \in \mathcal{V}_\tau$, if $u(\cdot) \in \mathcal{U}_\tau$ is defined according to the strategy $U^{\lambda,\varepsilon}$ (see (24)), then the motion $x(\cdot) = x(\cdot \mid \tau, z, w(\cdot), u(\cdot), v(\cdot))$ satisfies the estimate*

$$\varphi^{\lambda,\varepsilon}(\vartheta, x(\vartheta), x_\vartheta(\cdot)) + \int_\tau^\vartheta f^0(\xi, x(\xi), x(\xi - h), u(\xi), v(\xi)) \leq \varphi^{\lambda,\varepsilon}(\tau, z, w(\cdot)) + \zeta_3.$$

Proof Let $(\tau, z, w(\cdot)) \in \mathbb{G}$. According to Propositions 6 and 8, define $\alpha_* = \alpha_X(\alpha_X(\alpha_0) + 1)$, $\lambda_*^g = \lambda_X^g(\alpha_X(\alpha_0) + 1)$, and $\lambda_* = \lambda_X(\alpha_X(\alpha_0) + 1)$, where $\alpha_0 = \max\{\|z\|, \|w(\cdot)\|_\infty\}$. Then, for every $u(\cdot) \in \mathcal{U}_\tau$, $v(\cdot) \in \mathcal{V}_\tau$, and $t_* \in [\tau, \vartheta]$, for every $p \in \mathbb{R}^n$ such that $\|p\| \leq \|x(t_*)\| + 1$, where $x(\cdot) = x(\cdot \mid \tau, z, w(\cdot), u(\cdot), v(\cdot))$, the motion $y(\cdot) = x(\cdot \mid t_*, p, x_{t_*}(\cdot), u(\cdot), v(\cdot))$ and the function $y^g(t) = y(t) - g(t, y(t - h))$, $t \in [t_*, \vartheta]$ satisfy the relations

$$(y(t), y_t(\cdot)) \in P(\alpha_*), \quad \|y^g(t) - y^g(t')\| \leq \lambda_*^g |t - t'|, \quad \|y(t) - y(t')\| \leq \lambda_* |t - t'| \tag{64}$$

for any $t \in [t_*, \vartheta]$. Moreover, due to condition (f_2), there exist $\beta_f > 0$ such that

$$\big\| f(t, y(t), y(t-h), u(t), v(t)) \big\| + \big| f^0(t, y(t), y(t-h), u(t), v(t)) \big| \le \beta_f \tag{65}$$

for any $t \in [t_*, \vartheta]$. Note that relations (64) and (65) are also valid for the motion $x(\cdot)$ and the function $x^g(t) = x(t) - g(t, x(t-h))$ if we take $t_* = \tau$ and $p = z$. In accordance with condition (f_3), put

$$\lambda = \lambda_f(\alpha_*). \tag{66}$$

Let $\zeta_3 > 0$. Denote

$$\zeta_* = \zeta_3/(2(\vartheta - \tau)). \tag{67}$$

In accordance with condition (ρ_2), define $\lambda_\rho = \lambda_\rho(\alpha_*)$. Due to Propositions 15 and 16, there exists $\varepsilon_3 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_3]$, $u(\cdot) \in \mathcal{U}_\tau$, $v(\cdot) \in \mathcal{V}_\tau$, and $t \in [\tau, \vartheta]$, the motion $x(\cdot) = x(\cdot \mid \tau, z, w(\cdot), u(\cdot), v(\cdot))$ satisfies the estimates

$$\big\| p^{\lambda,\varepsilon}(t, x(t), x_t(\cdot)) - x(t) \big\| \le \zeta_*/(12\lambda_\rho). \tag{68}$$

Let us take $\varepsilon \in (0, \varepsilon_3]$. Recall that, according to (30), the number l_* of intervals in the set $[\tau, \vartheta] \setminus \Theta_\nu(\tau, w(\cdot))$ does not depend on $\nu \in (0, \nu_*)$. Set

$$\nu = \zeta_3/(6 l_* \beta_*), \quad \beta_* = 2\lambda\alpha_*(1 + 1/\varepsilon). \tag{69}$$

Using condition (f_1), Proposition 8, and (43), one can show the existence of

$$\delta \in (0, \delta_*], \quad \delta_* = \min\big\{\nu, h, \zeta_*/(12\lambda_\rho\lambda_*^g)\big\} \tag{70}$$

such that, for each $u(\cdot) \in \mathcal{U}_\tau$ and $v(\cdot) \in \mathcal{V}_\tau$, the motion $x(\cdot) = x(\cdot \mid \tau, z, w(\cdot), u(\cdot), v(\cdot))$ satisfies the inequality

$$\begin{aligned} \big|\chi(t, x(t), x(t-h), u, v, \nabla_x \eta^{\lambda,\varepsilon}(t, s(t))) \\ - \chi(t', x(t'), x(t'-h), u, v, \nabla_x \eta^{\lambda,\varepsilon}(t', s(t')))\big| \le \zeta_*/6 \end{aligned} \tag{71}$$

for any $t, t' \in \theta$: $|t - t'| \le \delta$, $\theta \in \Theta_\nu(\tau, w(\cdot))$, any $u \in \mathbb{U}$, $v \in \mathbb{V}$, and any function $s(\cdot)$, satisfying

$$\big|s(t) - s(t')\big| \le 2\lambda_*^g |t - t'|, \quad t, t' \in \theta, \quad \theta \in \Theta_\nu(\tau, w(\cdot)). \tag{72}$$

Let us take Δ_δ and $v(\cdot) \in \mathcal{V}_\tau$. Let $x(\cdot) = x(\cdot \mid \tau, z, w(\cdot), u(\cdot), v(\cdot))$ be the motion with $u(\cdot) \in \mathcal{U}_\tau$ defined by the strategy $U^{\lambda,\varepsilon}$. In accordance with (17), it means that

$$u(t) = u_j \in \mathop{\mathrm{argmin}}_{u \in \mathbb{U}} \max_{v \in \mathbb{V}} \chi\big(t_j, x(t_j), x(t_j - h), u, v, \nabla_z \eta^{\lambda,\varepsilon}(t, x(t_j) - p_j)\big) \tag{73}$$

for any $t \in [t_j, t_{j+1})$ and $j \in \overline{1, k}$, where we denote $p_j = p^{\lambda,\varepsilon}(t_j, x(t_j), x_{t_j}(\cdot))$. Let us consider the index sets K_1 and K_2 from (49) and prove the inequalities

$$\begin{aligned} &\varphi^{\lambda,\varepsilon}(t_{j+1}, x(t_{j+1}), x_{t_{j+1}}(\cdot)) + \int_{t_j}^{t_{j+1}} f^0(\xi, x(\xi), x(\xi - h), u(\xi), v(\xi)) d\xi \\ &\le \varphi^{\lambda,\varepsilon}(t_j, x(t_j), x_{t_j}(\cdot)) + (t_{j+1} - t_j)\zeta_*, \quad j \in K_1, \end{aligned} \tag{74}$$

and

$$\begin{aligned} &\varphi^{\lambda,\varepsilon}(t_{j+1}, x(t_{j+1}), x_{t_{j+1}}(\cdot)) + \int_{t_j}^{t_{j+1}} f^0(\xi, x(\xi), x(\xi - h), u(\xi), v(\xi)) d\xi \\ &\le \varphi^{\lambda,\varepsilon}(t_j, x(t_j), x_{t_j}(\cdot)) + (t_{j+1} - t_j)\beta_*, \quad j \in K_2. \end{aligned} \tag{75}$$

Let $j \in \overline{1, k}$. From (22) and (61), we have

$$\varphi^{\lambda,\varepsilon}(t_j, x(t_j), x_{t_j}(\cdot)) = \rho^\circ(t_j, p_j, x_{t_j}(\cdot)) + \eta^{\lambda,\varepsilon}(t_j, x(t_j) - p_j). \tag{76}$$

Define

$$v_j(t) = v_j \in \operatorname*{argmax}_{v\in\mathbb{V}} \min_{u\in\mathbb{U}} \chi(t_j, x(t_j), x(t_j - h), u, v, \nabla_x \eta^{\lambda,\varepsilon}(t_j, x(t_j) - p_j)) \tag{77}$$

for any $t \in [t_j, \vartheta]$. Then, due to condition (ρ_3), there exists $u_j(\cdot) \in \mathcal{U}_{t_j}$ such that the motion $y(\cdot) = x(\cdot \mid t_j, p_j, x_{t_j}(\cdot), u_j(\cdot), v_j(\cdot))$ satisfies the inequality

$$\begin{aligned}&\rho^\circ(t_{j+1}, y(t_{j+1}), y_{t_{j+1}}(\cdot)) + \int_{t_j}^{t_{j+1}} f^0(\xi, y(\xi), y(\xi - h), u_j(\xi), v_j(\xi))d\xi \\ &\leq \rho^\circ(t_j, p_j, x_{t_j}(\cdot)) + (t_{j+1} - t_j)\zeta_*/3.\end{aligned} \tag{78}$$

Denote $s(t) = x(t) - y(t)$, $t \in [t_j - h, \vartheta]$. Due to the inequality $t_{j+1} - t_j \leq \delta < h$ and (64), the function $s(\cdot)$ satisfies estimates (58). Then, due to the choice of λ_ρ, the inclusion $j \in K_1$, and the relations (70), we have

$$\begin{aligned}&\left|\rho^\circ(t_{j+1}, y(t_{j+1}), y_{t_{j+1}}(\cdot)) - \rho^\circ(t_{j+1}, y(t_{j+1}), x_{t_{j+1}}(\cdot))\right| \leq \lambda_\rho \|s_{t_{j+1}}(\cdot)\|_1 \\ &\leq \lambda_\rho \int_{t_j}^{t_j+1} \|s(\xi) - s(t_j)\| d\xi + 2\lambda_\rho(t_{j+1} - t_j)\|x(t_j) - p_j\| \leq (t_{j+1} - t_j)\zeta_*/3.\end{aligned} \tag{79}$$

Denote

$$\begin{aligned}\kappa(t) = \eta^{\lambda,\varepsilon}(t, s(t)) + \int_{t_j}^{t} \Big(&f^0(\xi, x(\xi), x(\xi - h), u(\xi), v(\xi)) \\ &- f^0(\xi, y(\xi), y(\xi - h), u_j(\xi), v_j(\xi))\Big)d\xi.\end{aligned}$$

Note that, due to (21), we have

$$\partial\eta^{\lambda,\varepsilon}(t, x)/\partial t = -\lambda(\theta^{\lambda,\varepsilon}(t) + 1)\mu^\varepsilon(x), \quad \nabla_x \eta^{\lambda,\varepsilon}(t, x) = (\theta^{\lambda,\varepsilon}(t)/\mu^\varepsilon(x))x \tag{80}$$

Then, taking (58) into account, the function $s(\cdot)$ and, as a consequence the function $\kappa(\cdot)$ are Lipschitz continuous on $[t_j, t_{j+1}]$, and, in accordance with (5), we obtain

$$\begin{aligned}d\kappa(t)/dt = \partial\eta^{\lambda,\varepsilon}(t, s(t))/\partial t &+ \chi(t, x(t), x(t - h), u(t), v(t), \nabla_x \eta^{\lambda,\varepsilon}(t, s(t))) \\ &- \chi(t, y(t), y(t - h), u_j(t), v_j(t), \nabla_x \eta^{\lambda,\varepsilon}(t, s(t))).\end{aligned} \tag{81}$$

Now, let us consider the case $j \in K_1$. Due to (58), estimate (72) holds for the function $s(t)$. Then, using (12), (71), (73), and (77), we derive

$$\begin{aligned}&\chi\big(t, x(t), x(t - h), u(t), v(t), \nabla_x \eta^{\lambda,\varepsilon}(t, s(t))\big) \\ &\quad\leq \max_{v\in\mathbb{V}} \chi\big(t_j, x(t_j), x(t_j - h), u_j, v, \nabla_x \eta^{\lambda,\varepsilon}(t_j, s(t_j))\big) + \zeta_*/6 \\ &\quad= H\big(t_j, x(t_j), x(t_j - h), \nabla_x \eta^{\lambda,\varepsilon}(t_j, s(t_j))\big) + \zeta_*/6 \\ &\quad\leq \min_{u\in\mathbb{U}} \chi\big(t_j, x(t_j), x(t_j - h), u, v_j, \nabla_x \eta^{\lambda,\varepsilon}(t_j, s(t_j))\big) + \zeta_*/6 \\ &\quad\leq \chi\big(t, x(t), x(t - h), u_j(t), v_j(t), \nabla_x \eta^{\lambda,\varepsilon}(t, s(t))\big) + \zeta_*/3\end{aligned}$$

and, hence, taking into account the inclusion in (64), choice (66) of λ, and equalities (58), (80), we obtain

$$d\kappa(t)/dt \leq \partial\eta^{\lambda,\varepsilon}(t, s(t))/\partial t + \lambda_f \|s(t)\|\big(1 + \nabla_x \eta^{\lambda,\varepsilon}(t, s(t))\big) + \zeta_*/3 \leq \zeta_*/3. \tag{82}$$

Thus, due to (61), (76), (78), (79), and (82), we conclude (74).

In the case of $j \in K_2$, according to (21), (58), (64), (66), (69), (80), (81), we get

$$d\kappa(t)/dt \leq \lambda\|s(t)\|\big(1 + \nabla_x \eta^{\lambda,\varepsilon}(t, s(t))\big) \leq 2\lambda\alpha_*(1 + 1/\varepsilon) \leq \beta_*.$$

and, taking into account (61), (76), and (78), we obtain (75).

From the inequalities (74), (75), and definitions in (69), we conclude the statement of the lemma. □

Proof Let us prove the first equality in Theorem 3. Let $(\tau, z, w(\cdot)) \in \mathbb{G}$ and $\zeta > 0$. According to Proposition 17, taking $\zeta_3 = \zeta/3$, define $\lambda_3, \varepsilon_3 > 0$. Due to Proposition 16, taking $\lambda = \lambda_3$ and $\zeta_2 = \zeta/3$, define $\varepsilon_2 > 0$. Put $\varepsilon_* = \min\{\varepsilon_2, \varepsilon_3\}$. Then, for every $\varepsilon \in (0, \varepsilon_*]$, there exists $\delta > 0$ such that, for every partition Δ_δ and every $v(\cdot) \in \mathcal{V}_\tau$, if $u(\cdot) \in \mathcal{U}_\tau$ is defined by $U^{\lambda,\varepsilon}$ (see (24)), then the motion $x(\cdot) = x(\cdot \mid \tau, z, w(\cdot), u(\cdot), v(\cdot))$ satisfies the estimate

$$\begin{aligned}
&\rho^\circ(\vartheta, x(\vartheta), x_\vartheta(\cdot)) + \int_\tau^\vartheta f^0(\xi, x(\xi), x(\xi - h), u(\xi), v(\xi))d\xi \\
&\le \varphi^{\lambda,\varepsilon}(\vartheta, x(\vartheta), x_\vartheta(\cdot)) + \int_\tau^\vartheta f^0(\xi, x(\xi), x(\xi - h), u(\xi), v(\xi))) + \zeta/3 \\
&\le \varphi^{\lambda,\varepsilon}(\tau, z, w(\cdot)) + 2\zeta/3 \le \rho^\circ(\tau, z, w(\cdot)) + \zeta.
\end{aligned}$$

According to definition (18), the statement above means the first equality in Theorem 3 holds. The second equality can be proved by the similar way based on the statements symmetrical to Proposition 15–17. □

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