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Evasion Differential Game of One Evader and Many Slow Pursuers

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Abstract

We study a simple motion evasion differential game of m pursuers and one evader. The maximum speed of pursuers is 1, and that of evader is $\sigma > 1$. If for all time the state of the evader doesn't coincide with the state of any pursuer, then evasion is said to be possible. The evader strives to avoid capture. This problem was studied by F.L. Chernous'ko in 1976. We propose a new evasion strategy, which guarantees evasion from any initial positions of players and enables us to estimate the number of approach times from the above by $m(m+1)/2$. Also, it is established that all approach times of each pursuer to the evader may occur only on the time interval associated with the first approach time of the pursuer to the evader.

Keywords Differential game · Evasion · Control · Strategy · Slow pursuers

Mathematics Subject Classification Primary: 91A23 · Secondary: 49N75

1 Introduction

The field of differential games was pioneered by Isaacs [17]. Evasion problem on the infinite time interval $[t_0, \infty)$ was introduced and studied by Pontryagin and Mischenko [24]. Mishchenko et al. [23] proposed a new maneuver for evasion in the game of many pursuers.

A substantial part of the researches study simple motion pursuit or evasion differential games with many players. Chernous'ko [9] studied an evasion game of one faster evader and several pursuers with a state constraint for the evader. The evader must remain in a neighborhood of a given ray during the game. It was proved that the faster evader can escape from the pursuers. This result later on was extended by Chernous'ko and Zak [10, 27–29] to more general differential game problems. Related problems of evasion from a group of pursuers were studied in [7] and [11].

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Blagodatskikh and Petrov [6] obtained necessary and sufficient condition of evasion in a simple motion differential game of a group of pursuers and a group of evaders in $\mathbb{R}^n$ where all evaders use the same control. By definition, pursuit is considered completed if the state of a pursuer coincides with the state of at least one evader. Also, the works [5, 26] related to such games.

In [25], a pursuit-evasion game involving one pursuer and multiple evaders motivated by the seminal "selfish herd" model of Hamilton was considered. The pursuer can freely move in any direction with bounded speed, and evaders move with bounded speed and bounded turning speed. Using Isaacs' heuristic argument, an optimal strategy for the pursuer was constructed and it was concluded that the optimal strategy for the pursuer is to focus on a single evader that can be captured in minimum time. Moreover, "non-targeted" evaders are always able to escape.

The paper of Lee and Bakolas [22] studies a differential game of a heterogeneous group of pursuers and one evader. Pursuers individually attempt to capture the evader, that is, the strategy of the group of pursuers is not cooperative. The evader tries to delay or avoid capture.

Ramana and Mangal [30] studied pursuit-evasion games of multiple pursuers and a high-speed single evader with holonomic constraints in an open domain. Using the idea of Apollonius circle, an escape strategy was developed for the high speed evader. Jin and Qu [18] also apply the Apollonius circles for the evader and each pursuer to study how the evader can form a better strategy to avoid or prolong the capture time provided a successful escape impossible. The work of Awheda and Schwartz [4] also relates to a multi-pursuer pursuit-evasion differential game. Chen at al. [8] studied a simple motion pursuit differential game of many identical pursuers and one faster evader. The evader is captured if the evader becomes in d_c distance from a pursuer. Sufficient conditions of completion of game were obtained. We refer to work [21] for a detailed survey of results on differential games of many players where the controls of players are under geometric constraints.

In the case of integral constraints, evasion differential games of many players were studied in the papers [2, 14–16]. The work [15] is devoted to the differential game of many pursuers and many evaders where the total energy of evaders greater or equal to that of pursuers. Evasion strategies are constructed to avoid from any initial positions of players.

There are few papers on multi pursuer differential games with state constraints. For example, interesting results were obtained by Alexander et al. [1] for a discrete time simple motion differential game in an unbounded region. Kuchkarov et al [20] studied a differential game of many pursuers and one evader on a cylinder, where all the players have equal dynamic capabilities. The paper of Kuchkarov et al [19] is devoted to pursuit and evasion differential games on manifolds with Euclidean metric where necessary and sufficient conditions of evasion are obtained. The optimal number of pursuers in the differential games on the 1-skeleton of orthoplex was found [3] to capture a single faster evader by many pursuers.

In the present paper, we study a simple motion evasion differential game of m pursuers and one evader. The maximum speed of pursuers is 1 and that of evader is $\sigma > 1$. This problem of evasion was studied in [9]. In the present paper, we propose a new evasion strategy and prove that the evader can avoid from pursuers moving in any ε-vicinity of any straight line passing through the initial state of evader. Also, we show that the number of approach times doesn't exceed $m(m+1)/2$, whereas this number was estimated by $2^m - 1$ in [9]. Another an important point to note is the fact that the approach times of each pursuer x_i to the evader may occur only on the time interval $[\tau_i, \tau_i')$ associated with the first approach time τ_i of the pursuer x_i to the evader.

It should be noted that the results of the current paper can be applied to multiple objective adversarial games such as Reach-avoid (RA) [12, 31] and Capture-the-flag [13] games. For example, for the RA games, there are not only the attacker (evader) and the defender (the team of pursuers), but there is also a target area. The attacker's goal is to reach the target area without being captured, while the defender team attempts to delay or prevent the attacker from entering the target area by capturing attacker. The results of the current paper allow us to conclude that a high-speed attacker can reach the target area without being captured.

2 Statement of Problem

Consider a simple motion differential game of m pursuers x_1, ..., x_m and one evader y in $\mathbb{R}^d$, $d \geq 2$, whose dynamics are described by the following equations

$$\begin{aligned} \dot{x}_i &= u_i, \quad x(0) = x_{i0}, \quad ||u_i|| \leq 1, \\ \dot{y} &= v, \qquad y(0) = 0, \quad ||v|| \leq \sigma, \end{aligned} \tag{1}$$

where $x_i, x_{i0}, y, y_0, u_i, v \in \mathbb{R}^n$, $x_{i0} \neq y_0$, $i = 1, 2, \ldots, m$, and $\sigma, \sigma > 1$, is a given number, u_i is control parameter of pursuer x_i, and v is that of evader y.

Definition 1 Measurable functions $u_i(t)$, $||u_i(t)|| \leq 1$, and $v(t)$, $||v(t)|| \leq \sigma$, $t \geq 0$, are called controls of the pursuer x_i and the evader y, respectively.

Definition 2 The strategy of evader is defined as a function $V : \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{(2m+1)d} \to \mathbb{R}^d$,

$$(t, \theta_1, \theta_2, y, x_1, \ldots, x_m, u_1, \ldots u_m) \to V(t, \theta_1, \theta_2, y, x_1, \ldots, x_m, u_1, \ldots u_m)$$

for which initial value problem (1) with $v = V(t, \theta_1, \theta_2, y, x_1, \ldots, x_m, u_1, \ldots u_m)$ has a unique solution $(y(t), x_1(t), \ldots, x_m(t))$ for $\theta_1 = \theta_1(t)$, $\theta_2 = \theta_2(t)$, and arbitrary controls $u_1 = u_1(t)$,..., $u_m = u_m(t)$ of pursuers, where $\theta_1(t)$ and $\theta_2(t)$, $t \geq 0$, are given functions.

The behaviors of the pursuers are arbitrary, that is, the pursuers apply any controls and the evader applies a strategy.

Definition 3 We say that evasion is possible in the game (1) if there exists a strategy V of evader such that, for any controls of pursuers, $x_i(t) \neq y(t)$ for all $t \geq 0$ and $i = 1, ..., m$.

Problem 1 *Construct a strategy V for the evader, for which evasion is possible in game (1).*

It is sufficient to consider the case where $d = 2$. We therefore study the evasion of one evader from many slow pursuers in the plane.

3 Evasion from One Pursuer

In this section, we consider an evasion differential game of one pursuer and one evader in $\mathbb{R}^2$. We use the temporary notation $x = (x_1, x_2)$ for the pursuer and $u = (u_1, u_2)$ for its control parameter only in this section. The dynamics of pursuer x and evader y are described by the equations

$$\begin{aligned} \dot{x} &= u, \ x(0) = x_0, \\ \dot{y} &= v, \ y(0) = y_0, \end{aligned} \tag{2}$$

where $x_0 \neq y_0$, $y = (y_1, y_2)$, and $v = (v_1, v_2)$ is control parameter of evader. The controls of players satisfy the inequalities

$$||u(t)|| \leq 1 \quad \text{and} \quad ||v(t)|| \leq \sigma, \quad t \geq 0. \tag{3}$$

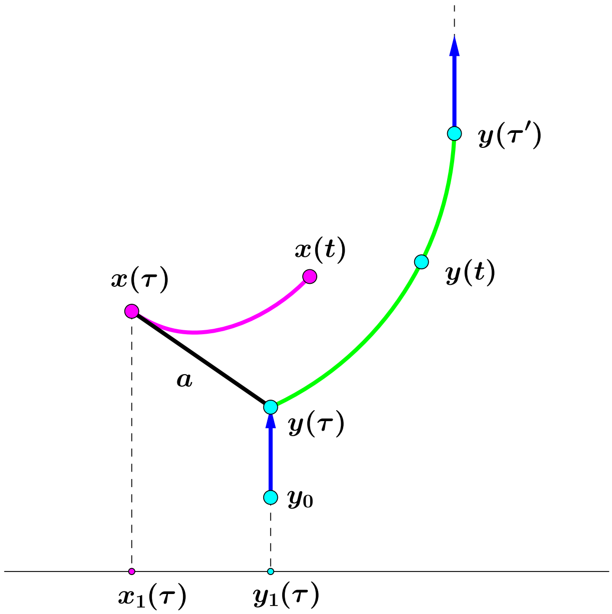


Fig. 1 Trajectory of the evader when $x_1(\tau) \le y_1(\tau)$

We fix the numbers α and a

$$0 < \alpha < \frac{1}{2}(\sigma - 1), \quad 0 < a < ||x_0 - y_0||. \tag{4}$$

The pursuer x applies an arbitrary control $u(t) = (u_1(t), u_2(t))$, $t \ge 0$, and let $x(t) = (x_1(t), x_2(t))$ be the corresponding trajectory of the pursuer.

We now construct a strategy for the evader. Fist, the evader starting from the initial time $t = 0$ moves with the velocity

$$V_0 = (0, \sigma), \quad t \in [0, \tau), \tag{5}$$

i.e., $v_1(t) = 0$, $v_2(t) = \sigma$, parallel to the Oy-axis, where τ is the first time when $||x(t) - y(t)|| = a$. We call τ the a-approach time of pursuer to the evader. The segment between the points y_0 and $y(\tau)$ in Fig. 1 is the trajectory of the evader corresponding to (5).

Note that time τ may not occur. In this case, we have $||x(t) - y(t)|| > a$ for all $t \ge 0$ and, clearly, $x(t) \ne y(t)$ for all $t \ge 0$. Therefore, we assume that the time τ occurs. Also, we define $\tau' = \tau + \frac{4a}{\sigma - 1}$.

Also, we use temporarily the notation $V(t) = (V_1(t), V_2(t))$ only in section, where

$$V_1(t) = \begin{cases} |u_1(t)| + \alpha, & x_1(\tau) \le y_1(\tau) \\ -(|u_1(t)| + \alpha), & x_1(\tau) > y_1(\tau) \end{cases}, \quad V_2(t) = \sqrt{\sigma^2 - V_1^2(t)}. \tag{6}$$

Clearly, $|V_1(t)| \le |u_1(t)| + \frac{1}{2}(\sigma - 1) \le \sigma$, and so $V_2(t)$ is defined. The evader applies the following strategy on $[\tau, \tau')$:

$$V(t) = (V_1(t), V_2(t)), \quad t \in [\tau, \tau'). \tag{7}$$

We call $V(t)$ defined by (7) a maneuver of the evader y against the pursuer x. For the final part of evader's strategy, we let

$$V_0 = (0, \sigma), \quad t \geq \tau'. \tag{8}$$

The main result of this section is the following statement which will be used to prove the main result of the paper in Sect. 4.

Lemma 1 *Let the evader use the strategy* (5), (7), *and* (8), *where τ is the a-approach time of the pursuer x to the evader y. Then,*

$$||y(t) - x(t)|| \geq a, \quad 0 \leq t \leq \tau,$$
$$||y(t) - x(t)|| > \frac{\alpha a}{2\sigma}, \quad \tau \leq t \leq \tau', \tag{9}$$
$$y_2(t) - x_2(t) > a, \quad t \geq \tau'. \tag{10}$$

Proof We will prove this lemma by considering the three parts of evader's strategy defined by formulas (5), (7), (8), respectively. First, the evader moves with the velocity $v(t) = (0, \sigma)$, $0 \leq t < \tau$, along the vertical line. The corresponding trajectory of the evader is a segment with the endpoints y_0 and $y(\tau)$ (see Fig. 1). By definition of τ, we have $||x(t) - y(t)|| \geq a$ for $0 \leq t \leq \tau$.

To prove (9), we consider the case $x_1(\tau) \leq y_1(\tau)$, hence, by (6) $V_1(t) = |u_1(s)| + \alpha$. The argument when $x_1(\tau) > y_1(\tau)$ is completely analogous. The curve between the points $y(\tau)$ and $y(\tau')$ in Fig. 1 is the trajectory of the evader corresponding to the maneuver (7). We have, for $\tau \leq t \leq \tau'$,

$$\begin{aligned} ||y(t) - x(t)|| \geq y_1(t) - x_1(t) &= y_1(\tau) - x_1(\tau) + \int_\tau^t V_1(s)ds - \int_\tau^t u_1(s)ds \\ &= y_1(\tau) - x_1(\tau) + \int_\tau^t (|u_1(s)| + \alpha)ds - \int_\tau^t u_1(s)ds \\ &\geq \alpha(t - \tau). \end{aligned}$$

On the other hand,

$$\begin{aligned} ||y(t) - x(t)|| &\geq ||y(\tau) - x(\tau)|| - \left\| \int_\tau^t V(s)ds \right\| - \left\| \int_\tau^t u(s)ds \right\| \\ &\geq a - \int_\tau^t ||V(s)||ds - \int_\tau^t ||u(s)||ds \\ &\geq a - (\sigma + 1)(t - \tau). \end{aligned}$$

Hence,

$$||y(t) - x(t)|| \geq h(t) = \max\{\alpha(t - \tau), a - (\sigma + 1)(t - \tau)\}.$$

Since the function $h_1(t) = \alpha(t - \tau)$, $t \geq \tau$, is increasing, and the function $h_2(t) = a - (\sigma + 1)(t - \tau)$, $t \geq \tau$, is decreasing, therefore the function $h(t)$, $t \geq \tau$, achieves its minimum at $t = t^*$ where $h_1(t) = h_2(t)$ (see Fig. 2). We can see that $t^* = \tau + \frac{a}{\alpha+\sigma+1} \in [\tau, \tau']$. Hence,

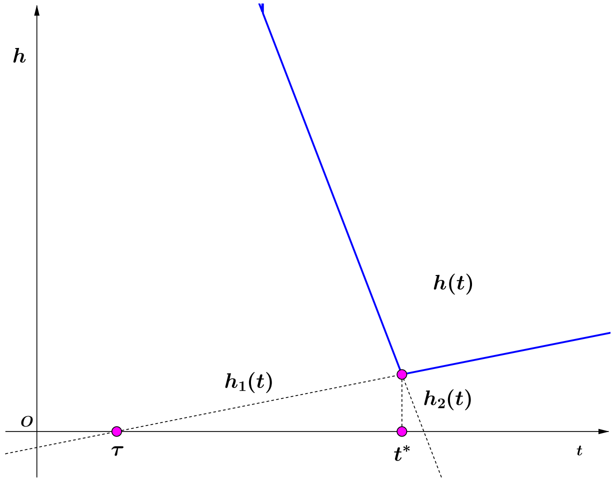


Fig. 2 Graph of the function $h(t)$

for any $t \in [\tau, \tau']$, by (4) we have

$$||y(t) - x(t)|| \geq h(t^*) = \alpha(t^* - \tau) = \frac{\alpha a}{\sigma + 1 + \alpha} > \frac{\alpha a}{2\sigma},$$

which proves (9).

Next, to prove (10), first we show that

$$y_2(\tau') - x_2(\tau') > a. \tag{11}$$

Indeed, for $\tau \leq t \leq \tau'$, due to the obvious inequality $y_2(\tau) - x_2(\tau) \geq -||y(\tau) - x(\tau)|| = -a$ we have

$$\begin{aligned} y_2(t) - x_2(t) &= y_2(\tau) - x_2(\tau) + \int_\tau^t V_2(s)ds - \int_\tau^t u_2(s)ds \\ &\geq -a + \int_\tau^t \left(\sqrt{\sigma^2 - (|u_1(s)| + \alpha)^2} - \sqrt{1 - u_1^2(s)}\right) ds. \end{aligned} \tag{12}$$

Noting that the function

$$f(\xi) = \sqrt{\sigma^2 - (\xi + \alpha)^2} - \sqrt{1 - \xi^2}, \quad \xi \in [0, 1],$$

achieves its minimum at $\xi_0 = \frac{\alpha}{\sigma - 1}$, it follows from (12) that

$$\begin{aligned} y_2(t) - x_2(t) &\geq -a + (t - \tau)\left(\sqrt{\sigma^2 - (\xi_0 + \alpha)^2} - \sqrt{1 - \xi_0^2}\right) \\ &= -a + (t - \tau)\sqrt{(\sigma - 1)^2 - \alpha^2}. \end{aligned}$$

In particular, for $t = \tau'$ we obtain

$$y_2(\tau') - x_2(\tau') \geq -a + \frac{4a}{\sigma - 1}\sqrt{(\sigma - 1)^2 - \alpha^2} > a. \tag{13}$$

Hence, (8) implies that, for $t \geq \tau'$,

$$y_2(t) - x_2(t) = y_2(\tau') - x_2(\tau') + \int_{\tau'}^{t} v_2(s)ds - \int_{\tau}^{t} u_2(s)ds$$
$$> a + (t - \tau')(\sigma - 1) > a.$$

The proof of the lemma is complete. □

In particular, Lemma 1 implies that even though the pursuer is on the vertical line and above the evader, the evader can avoid from capturing using the maneuver.

4 Evasion from Many Pursuers

We prove the following statement.

Theorem 4.1 *For any initial positions of players, evasion is possible in game* (1).

We have divided the proof into subsections.

4.1 Definitions of Parameters

Let α, a_1 and β be any fixed numbers that satisfy the following relations

$$0 < \alpha < \min\left\{1, \frac{1}{2}(\sigma - 1)\right\}, \quad 0 < a_1 < \min_{i=1,\dots,m} ||y_0 - x_{i0}||, \quad \beta = \frac{(\sigma - 1)\alpha}{64\sigma^2}.$$

We define a decreasing geometric sequence $\{a_k\}_{k=1}^{\infty}$ by the equation $a_{k+1} = \beta a_k, k = 1, 2, \dots$.

Next, we assume that the evader is moving under some strategy. We say that $t = \tau_1 > 0$ is the a_1-approach time of a pursuer x_{i_0} to the evader if $||x_{i_0}(\tau_1) - y(\tau_1)|| = a_1$ and $||x_i(t) - y(t)|| > a_1$ for all $0 \leq t < \tau_1$ and $i = 1, 2, \dots, m$. In general, if τ_{k-1}, $k \geq 2$, is the a_{k-1}-approach time, then we define the time $t = \tau_k > \tau_{k-1}$ to be the a_k-approach time if for a pursuer x_{i_1} $||x_{i_1}(\tau_k) - y(\tau_k)|| = a_k$ and $||x_i(t) - y(t)|| > a_k$ for all $0 \leq t < \tau_k$ and $i = 1, 2, \dots, m$.

Thus, we have defined a monotone increasing sequence $\tau_1 < \tau_2 < \dots$ of the approach times. Notice that the same time τ_k can be the a_k-approach time of several pursuers to the evader. For example, in Fig. 3 τ_k is an a_k-approach time of the pursuers x_i, x_j and x_k to the evader. If there are more than one pursuers, for which τ_k is the a_k-approach time, we choose any of these pursuers and, without restriction of generality, label it by x_k. Hence, by the definition of τ_k we have

$$||y(t) - x_i(t)|| > a_k, \;\; i = 1, 2, \dots, m, \;\; 0 \leq t < \tau_k, \;\; ||y(\tau_k) - x_k(\tau_k)|| = a_k. \tag{14}$$

It should be noted that the same pursuer x_k can have several other approach times $\tau_{k'}$, $\tau_{k''}$,...., beyond the approach time τ_k.

Let

$$\tau_k' = \tau_k + \frac{4a_k}{\sigma - 1}, \;\; k = 1, 2, \dots, \;\; \tau_0 = 0, \;\; \tau_0' = \infty.$$

Notice that the sequence $\tau_1', \; \tau_2', \; \tau_3', \dots$ is not necessarily monotone.

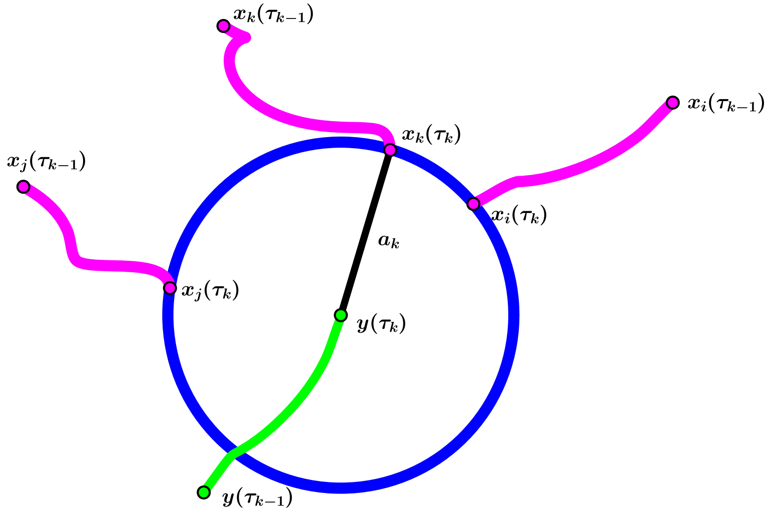


Fig. 3 τ_k is the a_k-approach time of the pursuers x_i, x_j and x_k to the evader y

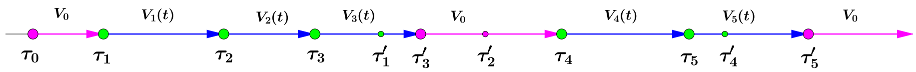


Fig. 4 Group attacks of x_1, x_2, x_3 on $[\tau_1, \tau_3')$ and x_4, x_5 on $[\tau_4, \tau_5')$

4.2 Strategy for the Evader

Without loss of generality, we assume that $y_0 = (0, 0)$, that is, the evader is at the origin at the initial time. For $k = 1, 2, \dots,$ we define the maneuvers $V_k(t) = (V_{k1}(t), V_{k2}(t))$, as follows

$$V_{k1}(t) = \begin{cases} |u_{k1}(t)| + \alpha, & x_{k1}(\tau_k) \le y_1(\tau_k), \\ -(|u_{k1}(t)| + \alpha), & x_{k1}(\tau_k) > y_1(\tau_k), \end{cases} \quad V_{k2}(t) = \sqrt{\sigma^2 - V_{k1}^2(t)}. \tag{15}$$

First, the evader moves starting from the time $\tau_0 = 0$ along the y-axis with the velocity $v(t) = V_0 = (0, \sigma)$. If the a_1-approach time $\tau_1 > 0$ doesn't occur, that is, $||y(t) - x_i(t)|| > a_1$ for all $i = 1, 2, ..., m$ and $t \ge 0$, then, clearly, $x_i(t) \ne y(t)$, $t \ge 0$, $i = 1, 2, ..., m$, and so evasion is possible in the game.

Let the a_1-approach time $\tau_1 > 0$ occur. In general, the evader constructs its strategy as follows. Let the time τ_k, $k \ge 1$, occur.

(i) if the time τ_{k+1} occurs in the interval $[\tau_k, \tau_k')$, then $v(t) = V_k(t)$ on $[\tau_k, \tau_{k+1})$.

(ii) if the time τ_{k+1} doesn't occur in $[\tau_k, \tau_k')$, then $v(t) = V_k(t)$ on $[\tau_k, \tau_k')$.

(iii) if the time τ_{k+1} occurs in $[\tau_k', \infty)$, then $v(t) = V_0$ on $[\tau_k', \tau_{k+1})$.

(iv) if the time τ_{k+1} never occurs, then $v(t) = V_0$ on $[\tau_k', \infty)$.

For the times in Fig. 4, τ_2 occurs in $[\tau_1, \tau_1')$; therefore, by item (i) $v(t) = V_1(t)$ on $[\tau_1, \tau_2)$. Similarly, $\tau_3 < \tau_2'$; therefore, $v(t) = V_2(t)$ on $[\tau_2, \tau_3)$. However, τ_4 is not in $[\tau_3, \tau_3')$; therefore, by item (ii) $v(t) = V_3(t)$ on $[\tau_3, \tau_3')$, and by item (iii) $v(t) = V_0$ on $[\tau_3', \tau_4)$. Finally, since there is no an approach time on $[\tau_5', \infty)$, therefore by item (iv) $v(t) = V_0$ on this interval.

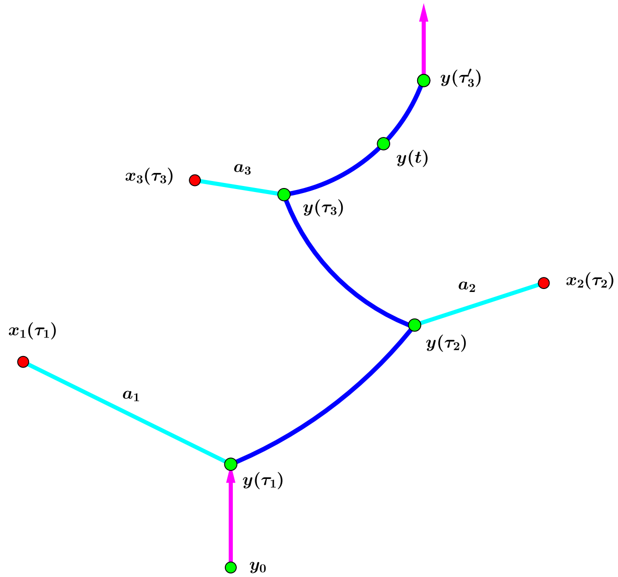


Fig. 5 Trajectory of the evader

The functions $\theta_1(t)$ and $\theta_2(t)$ in Definition 2 are defined as follows. Let $\theta_1(t) = \max_{\tau_k \le t} \tau_k = \tau_K$ for some $K \ge 1$. In other words, τ_K is the greatest of the values τ_k defined by the current time t. Then, we define $\theta_2(t) = \tau'_K$. For example, if $\tau_0 \le t < \tau_1$, then $K = 0$; if $\tau_1 \le t < \tau_2$, then $K = 1$. The strategy of the evader described by steps (i)-(iv) means that at the current time t (recall $t \ge \tau_K$) $v(t) = V_K(t)$ if $t < \tau'_K$, and $v(t) = V_0(t)$ if $t \ge \tau'_K$.

According to the description of evader's strategies (i)-(iv), there are two possible cases.

Case A. The finite approach times $\tau_1, \tau_2, ..., \tau_{k_1}$ with $\tau_1 < \tau_2 < ... < \tau_{k_1}$, occur so that $\tau_2 < \tau'_1, \tau_3 < \tau'_2, ..., \tau_{k_1} < \tau'_{k_1-1}$, and there is no an approach time in $[\tau_{k_1}, \tau'_{k_1})$ for some $k_1 \ge 1$. Then, we say that the evader is under a group attack of the pursuers $x_1, x_2, ..., x_{k_1}$ on the time interval $[\tau_1, \tau'_{k_1})$. Thus, the first group attack of pursuers ends at τ'_{k_1}.

By items (i)-(iv), the strategy of the evader on the interval $[\tau_0, \tau'_{k_1})$ can be written as follows:

$$v(t) = \begin{cases} V_0, & \tau_0 \le t < \tau_1, \\ V_k(t), & \tau_k \le t < \tau_{k+1}, \quad k = 1, 2, ..., k_1 - 1, \\ V_{k_1}(t), & \tau_{k_1} \le t < \tau'_{k_1}. \end{cases} \tag{16}$$

By item (iii) starting τ'_{k_1}, the evader starts to apply $v(t) = V_0$ and after some time the evader may undergo another group attack of pursuers.

Figure 4 illustrates two group attacks of pursuers. The evader is under a group attack of the pursuers x_1, x_2, x_3 on $[\tau_1, \tau'_3)$, and it is under a group attack of the pursuers x_4, x_5 on $[\tau_4, \tau'_5)$.

Figure 5 illustrates the three sections of the evader's trajectory between the points $y(\tau_1)$, $y(\tau_2)$, $y(\tau_3)$, and $y(\tau'_3)$ corresponding to some maneuvers $v(t) = V_1(t)$, $v(t) = V_2(t)$, and $v(t) = V_3(t)$ where $\tau_2 < \tau'_1, \tau_3 < \tau'_2$ and there is no an approach time in $[\tau_3, \tau'_3)$.

Since by the definition of approach times we have $\tau_k < \tau_{k+1}$, $k \geq 1$, therefore, in view of the conditions $\tau_{k+1} < \tau'_k$, $k = 1, ..., k_1 - 1$, we get in Case A the following inclusion

$$[\tau_1, \tau'_{k_1}) = \cup_{k=1}^{k_1-1}[\tau_k, \tau_{k+1}) \cup [\tau_{k_1}, \tau'_{k_1}) \subset \cup_{k=1}^{k_1}[\tau_k, \tau'_k), \tag{17}$$

For example, for the interval $[\tau_1, \tau'_3)$ in Fig. 4, we have $[\tau_1, \tau'_3) \subset \cup_{k=1}^{3}[\tau_k, \tau'_k)$.

It is natural to ask questions: Can the same pursuer participate several times in the same group attack? Can the number of approach times in a group attack be finite? Description (i)-(iv) doesn't exclude the case where the number of pursuers in a group attack is infinite.

Case B. Let infinitely many successive approach times τ_1, τ_2,... of pursuers x_1, x_2,... to the evader occur satisfying the conditions $\tau_k < \tau_{k+1} < \tau'_k$ for all $k = 1, 2, ...$ Then, $[\tau_k, \tau_{k+1}) \subset [\tau_k, \tau'_k)$ and so, for any $n \geq 1$, we have

$$[\tau_1, \tau_n) = \cup_{k=1}^{n-1}[\tau_k, \tau_{k+1}) \subset \cup_{k=1}^{n-1}[\tau_k, \tau'_k), \tag{18}$$

Clearly,

$$\tau_n - \tau_1 = \sum_{k=1}^{n-1}(\tau_{k+1} - \tau_k) < \sum_{k=1}^{n-1}(\tau'_k - \tau_k) < \sum_{k=1}^{\infty}\frac{4a_k}{\sigma - 1} = \frac{4a_1}{(\sigma - 1)(1 - \beta)} < \infty.$$

This means that the increasing sequence τ_n is bounded. Then, the limit $\tau_\infty = \lim_{n\to\infty} \tau_n$ exists. Note that in Case B inclusion (18) holds for any $n \geq 1$, and passing to limit as $n \to \infty$ in (18) we obtain

$$[\tau_1, \tau_\infty) \subset \cup_{k=1}^{\infty}[\tau_k, \tau'_k). \tag{19}$$

By items (i)-(iv), the evader's strategy on the interval $[\tau_1, \tau_\infty)$ is

$$v(t) = V_k(t), \quad t \in [\tau_k, \tau_{k+1}), \quad \tau_k < \tau'_{k-1}, \quad k = 1, 2, \ldots. \tag{20}$$

From now on, we use $\bar{\tau}$ to denote τ'_{k_1} in Case A, and to denote τ_∞ in Case B. We'll discuss in detail the first group attack, which starts at the time τ_1 and ends at $\bar{\tau}$. Another group attack may occur after the time $\bar{\tau}$ as well, which can be studied in a similar fashion.

The results of Sect. 4.4 show that Case B will not happen. Also, in the following subsections, we'll answer the questions: Can the same pursuer participate in several group attacks as well? Is the number of group attacks finite?

4.3 Estimation of Distance Between Evader and FE

Take any a_p-approach time τ_p of the pursuer x_p to the evader y, where $p \in \{1, 2, \ldots, k_1\}$ in Case A, and p is any positive integer in Case B, and we will estimate the distance between $x_p(t)$ and $y(t)$ for $t \geq \tau_p$. In order to obtain the desired estimate, we introduce for $t \in [\tau_p, \tau'_p]$ a fictitious evader (FE) z_p whose motion is described by the equation

$$\dot{z}_p = w_p, \ z_p(\tau_p) = y(\tau_p),$$

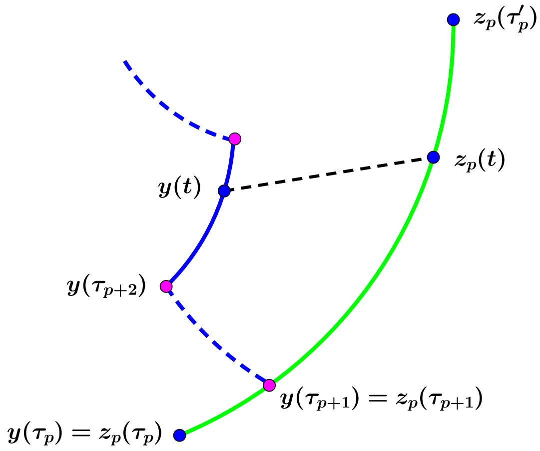
where w_p is control parameter of FE z_p. We let

$$w_p(t) = V_p(t) = (V_{p1}(t), V_{p2}(t)), \quad t \in [\tau_p, \tau'_p). \tag{21}$$

Then, by (9) we have

$$||z_p(t) - x_p(t)|| > \frac{\alpha a_p}{2\sigma} \quad \text{for} \quad \tau_p \leq t \leq \tau'_p. \tag{22}$$

Fig. 6 Evader y and fictitious evader z_p



Moreover, by (10)

$$z_{p2}(\tau_p') - x_{p2}(\tau_p') > a_p. \tag{23}$$

Note that FE z_p moves only on the time interval $[\tau_p, \tau_p')$ and its initial state $z(\tau_p)$ coincides with the initial state $y(\tau_p)$ of the real evader.

In Case A, by (16) the strategy of evader on the interval $[\tau_p, \tau_{k_1}')$, $k_1 \geq p$, is

$$v(t) = \begin{cases} V_k(t), & t \in [\tau_k, \tau_{k+1}), \quad k = p, p+1, \ldots, k_1 - 1, \\ V_{k_1}(t), & t \in [\tau_{k_1}, \tau_{k_1}'), \end{cases} \tag{24}$$

where

$$\tau_k < \tau_{k-1}', \quad k = p+1, p+2, ..., k_1; \tag{25}$$

and in Case B, by the description (i)-(iv) the evader's strategy on the interval $[\tau_p, \tau_\infty)$ is

$$v(t) = V_k(t), \quad t \in [\tau_k, \tau_{k+1}), \quad \tau_k < \tau_{k-1}', \quad k = p, p+1, \ldots. \tag{26}$$

For the distance between the points $y(t)$ and $z_p(t)$ on $[\tau_p, \tau_*]$, $\tau_* = \min\{\tau_p', \bar{\tau}\}$, where $\bar{\tau} = \tau_{k_1}'$ in Case A, and $\bar{\tau} = \tau_\infty$ in Case B, we prove the following lemma, where we use the inclusion

$$[\tau_p', t) \subset [\tau_{p+1}, t) \subset \cup_{k \geq p+1, \ \tau_k \leq t} [\tau_k, \tau_k'), \quad \tau_{p+1} \leq t \leq \bar{\tau}, \tag{27}$$

following from (17) and (19) for both Case A and Case B.

Lemma 2 *Let the evader use strategy (24) in Case A and (26) in Case B. Then,*

$$||y(t) - z_p(t)|| \leq \frac{16\sigma}{\sigma - 1} a_{p+1} \quad for \quad \tau_p \leq t \leq \tau_*. \tag{28}$$

This lemma says that if $\tau_p' \leq \bar{\tau}$, then $\tau_* = \tau_p'$ and estimate (28) is true on $\tau_p \leq t \leq \tau_p'$, and that if $\tau_p' > \bar{\tau}$, then $\tau_* = \bar{\tau}$ and estimate (28) is true on $\tau_p \leq t \leq \bar{\tau}$.

Proof Note that $y(\tau_p) = z(\tau_p)$ and there is no an approach time in the interval $[\tau_p, \tau_p')$, then by (21) and (24) $v(t) = w_p(t) = V_p(t)$, $t \in [\tau_p, \tau_p')$, and so (28) is satisfied.

We let now one or several approach times τ_{p+1}, τ_{p+2},... occur in the interval $[\tau_p, \tau_p')$. In Fig. 6, the sections of the trajectory of evader correspond to distinct maneuvers.

Since by (21) and (24) $v(t) = w_p(t) = V_p(t)$, $t \in [\tau_p, \tau_{p+1})$, therefore, $y(t) = z_p(t)$, $t \in [\tau_p, \tau_{p+1})$, and so (28) is satisfied.

For $t \in [\tau_{p+1}, \tau_*)$, we obtain

$$||y(t) - z_p(t)|| = \left\| y(\tau_p) + \int_{\tau_p}^{\tau_{p+1}} V_p(s)ds + \int_{\tau_{p+1}}^{t} v(s)ds \; -z(\tau_p) - \int_{\tau_p}^{\tau_{p+1}} V_p(s)ds - \int_{\tau_{p+1}}^{t} V_p(s)ds \right\|$$
$$= \left\| \int_{\tau_{p+1}}^{t} (v(s) - V_p(s))ds \right\| \leq \int_{\tau_{p+1}}^{t} \|v(s) - V_p(s)\|ds \leq \int_{\tau_{p+1}}^{t} 2\sigma ds, \tag{29}$$

where we used the inequality $\|v(s) - V_p(s)\| \leq \|v(s)\| + \|V_p(s)\| \leq 2\sigma$. Since due to the condition $t \leq \tau_* \leq \bar{\tau}$ we have inclusion (27), and $\beta \leq 1/2$ implies that $\sum_{k=p+1}^{\infty} a_k = \frac{a_{p+1}}{1-\beta} \leq 2a_{p+1}$, therefore we obtain from (29) that

$$\|y(t) - z_p(t)\| \leq 2\sigma(t - \tau_{p+1}) \leq 2\sigma \sum_{k \geq p+1,\ \tau_k \leq t} (\tau_k' - \tau_k) = 2\sigma \sum_{k \geq p+1,\ \tau_k \leq t} \frac{4a_k}{\sigma - 1}$$
$$< \frac{8\sigma}{\sigma - 1} \sum_{k=p+1}^{\infty} a_k = \frac{8\sigma}{\sigma - 1} \cdot \frac{a_{p+1}}{1 - \beta} \leq \frac{16\sigma}{\sigma - 1} a_{p+1}. \tag{30}$$

The proof of the lemma is complete. □

4.4 Estimation of Distance Between Evader and Pursuer

We estimate now the distance between the evader y and pursuer x_p.

Lemma 3 *Let the evader use strategy* (24) *in Case A or* (26) *in Case B. Then,*

$$\|y(t) - x_p(t)\| > a_{p+1}, \quad \tau_p \leq t \leq \tau_*, \tag{31}$$

and if for the pursuer x_p the inequality $\tau_p' \leq \bar{\tau}$ holds, then

$$y_2(t) - x_{p2}(t) > a_{p+1}, \quad t \geq \tau_p'. \tag{32}$$

Proof To prove (31), we observe that both inequalities (22) and (28) are true on $\tau_p \leq t \leq \tau_*$, and therefore,

$$\|y(t) - x_p(t)\| \geq \|x_p(t) - z_p(t)\| - \|z_p(t) - y(t)\|$$
$$> \frac{\alpha}{2\sigma} a_p - \frac{16\sigma}{\sigma - 1} a_{p+1} = \frac{\alpha}{4\sigma} a_p.$$

Consequently,

$$\|y(t) - x_p(t)\| > \frac{\alpha}{4\sigma} a_p \geq \beta a_p = a_{p+1}, \quad \tau_p \leq t \leq \tau_*,$$

and (31) is proved.

Next, to prove (32), we let $\tau_p' \leq \bar{\tau}$. Since $z_{p2}(\tau_p') - x_{p2}(\tau_p') > a_p$ by (23) and

$$\|y(\tau_p') - z_p(\tau_p')\| \leq \frac{16\sigma}{\sigma - 1} a_{p+1}$$

by (28), therefore we have

$$
\begin{aligned}
y_2(\tau_p') - x_{p2}(\tau_p') &= (z_{p2}(\tau_p') - x_{p2}(\tau_p')) + (y_2(\tau_p') - z_{p2}(\tau_p')) \\
&\geq a_p - \|y(\tau_p') - z_p(\tau_p')\| > a_p - \frac{16\sigma}{\sigma-1}a_{p+1} > \frac{1}{2}a_p.
\end{aligned} \tag{33}
$$

For $t \geq \tau_p'$, using (33) and the fact that $u_{p2}(s) \leq 1$ we have

$$
\begin{aligned}
y_2(t) - x_{p2}(t) &= y_2(\tau_p') - x_{p2}(\tau_p') + \int_{\tau_p'}^t v_2(s)ds - \int_{\tau_p'}^t u_{p2}(s)ds \\
&> \frac{1}{2}a_p + \int_{\tau_p'}^t v_2(s)ds - (t - \tau_p').
\end{aligned} \tag{34}
$$

First, we prove (32) for $\tau_p' \leq t \leq \bar{\tau}$. Indeed, since $v_2(t) > 0$ for all $t \geq 0$, therefore $\int_{\tau_p'}^t v_2(s)ds \geq 0$. Also, by (27)

$$
\begin{aligned}
t - \tau_p' &= mes([\tau_p', t)) \leq mes\left(\cup_{p+1\leq k,\ \tau_k \leq t}[\tau_k, \tau_k')\right) \leq \sum_{k\geq p+1,\ \tau_k\leq t}(\tau_k' - \tau_k) \\
&= \sum_{k\geq p+1,\ \tau_k\leq t}\frac{4a_k}{\sigma-1} \leq \frac{4}{\sigma-1}\sum_{k=p+1}^{\infty} a_k \leq \frac{8}{\sigma-1}a_{p+1}.
\end{aligned} \tag{35}
$$

Therefore, it follows from (34) that

$$
y_2(t) - x_{p2}(t) > \frac{1}{2}a_p - \frac{8\sigma}{\sigma-1}a_{p+1} > \frac{1}{2}a_p - \frac{\alpha}{8\sigma}a_p > \frac{1}{4}a_p > a_{p+1}, \quad \tau_p' \leq t \leq \bar{\tau}, \tag{36}
$$

and (32) is proved for $\tau_p' \leq t \leq \bar{\tau}$.

We conclude that, if $\tau_p' > \bar{\tau}$ for the pursuer x_p, then (31) implies that

$$
\|y(t) - x_p(t)\| > a_{p+1}, \quad \tau_p \leq t \leq \bar{\tau}, \tag{37}
$$

and if $\tau_p' \leq \bar{\tau}$, then combining the inequalities (31) and (36) we obtain (37). Hence, (37) is true for each pursuer x_p in the group attack in both Case A and Case B.

An important conclusion to draw from the inequality (37) is that, for $k \geq p+1$, there is no an a_k-approach time τ_k of the pursuer x_p to the evader on the time interval $\tau_p < t \leq \bar{\tau}$. Indeed, if there was an a_k-approach time τ_k with $\tau_p < \tau_k \leq \bar{\tau}$ and $k \geq p+1$, then we would have had $\|y(\tau_k) - x_p(\tau_k)\| = a_k$. However, this is impossible since $a_k \leq a_{p+1}$ and by (37) $\|y(\tau_k) - x_p(\tau_k)\| > a_{p+1}$.

Consequently, each pursuer x_p in the group attack has only one approach time τ_p on the time interval $\tau_p \leq t \leq \bar{\tau}$. Therefore, all the pursuers in the group attack are distinct. Moreover, the number of pursuers in the group attack $\leq m$ since there are at most m approach times in the group attack. Thus, Case B is excluded. There are only finite number of pursuers $x_1, x_2, \ldots, x_{k_1}$ with $k_1 \leq m$ in the (first) group attack. Hence, we deal with only Case A where $\bar{\tau} = \tau_{k_1}'$ and the group attack ends at τ_{k_1}'.

Next, we proceed to prove (32) for $t \geq \tau_{k_1}'$ assuming that $\tau_p' \leq \tau_{k_1}'$. Clearly, the inequality $\tau_p' \leq \tau_{k_1}'$ is satisfied at least for $p = k_1$, that is, for the pursuer x_{k_1}. We estimate now the right-hand side of (34) for $t \geq \tau_{k_1}'$. Note that if several group attacks occur on the time interval $[\tau_{k_1}', t)$, by the description of evader's strategy (i)-(iv) the evader moves with the velocity $v(t) = V_0$ between the group attacks and after the last group attack as well.

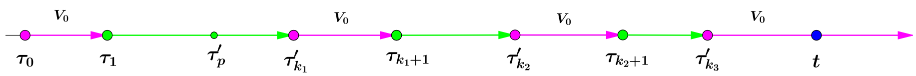


Fig. 7 Evader is under a group attack on the intervals $[\tau_1, \tau'_{k_1})$, $[\tau_{k_1+1}, \tau'_{k_2})$, $[\tau_{k_2+1}, \tau'_{k_3})$

Figure 7 illustrates three group attacks on the intervals $[\tau_{k_j+1}, \tau'_{k_{j+1}})$, $j = 0, 1, 2$, where $k_0 = 0$.

To estimate the integral $\int_{\tau'_p}^{t} v_2(s)ds$ in (34), we use the representation $[\tau'_p, t) = I \cup J$, $I \cap J = \emptyset$, where the evader undergoes a group attack of some pursuers on I, and the evader moves with the velocity $v(t) = V_0$ on J. For example, for the interval $[\tau'_p, t)$ in Fig. 7, we have

$$I = [\tau'_p, \tau'_{k_1}) \cup [\tau_{k_1+1}, \tau'_{k_2}) \cup [\tau_{k_2+1}, \tau'_{k_3}), \quad J = [\tau'_{k_1}, \tau_{k_1+1}) \cup [\tau'_{k_2}, \tau_{k_2+1}) \cup [\tau'_{k_3}, t).$$

In general, for $t \geq \tau'_{k_1}$, let $j_0 \geq 0$ be the greatest integer such that $\tau_{k_{j_0}+1} \leq t$. We then have

$$I = [\tau'_p, \tau'_{k_1}) \cup \left(\cup_{1 \leq j \leq j_0} [\tau_{k_j+1}, \tau'_{k_{j+1}}) \right) \text{ if } t \notin [\tau_{k_{j_0}+1}, \tau'_{k_{j_0+1}}),$$
$$I = [\tau'_p, \tau'_{k_1}) \cup \left(\cup_{1 \leq j \leq j_0-1} [\tau_{k_j+1}, \tau'_{k_{j+1}}) \right) \cup [\tau_{k_{j_0}+1}, t) \text{ if } t \in [\tau_{k_{j_0}+1}, \tau'_{k_{j_0+1}}).$$

Next, by (27)

$$[\tau'_p, \tau'_{k_1}) \subset [\tau_{p+1}, \tau'_{k_1}) \subset \cup_{p+1 \leq k \leq k_1} [\tau_k, \tau'_k)$$

and, for the a group attack on $[\tau_{k_j+1}, \tau'_{k_{j+1}})$, we have $\tau_{k+1} < \tau'_k$, $k = k_j + 2, ..., k_{j+1}$, and so similar to (17) we can write

$$[\tau_{k_j+1}, \tau'_{k_{j+1}}) \subset \cup_{k=k_j+1}^{k_{j+1}} [\tau_k, \tau'_k), \tag{38}$$

therefore $I \subset \cup_{k \geq p+1,\ \tau_k \leq t} [\tau_k, \tau'_k)$, and the length $|I|$ of I can be estimated as follows

$$|I| \leq \sum_{k \geq p+1,\ \tau_k \leq t} (\tau'_k - \tau_k) = \sum_{k \geq p+1,\ \tau_k \leq t} \frac{4a_k}{\sigma - 1} \leq \frac{4}{\sigma - 1} \sum_{k=p+1}^{\infty} a_k \leq \frac{8}{\sigma - 1} a_{p+1}. \tag{39}$$

Since $v_2(t) > 0$ for all $t \in I$, therefore $\int_I v_2(s)ds \geq 0$. Using this and (39) we obtain

$$\int_{\tau'_p}^{t} v_2(s)ds = \int_{I \cup J} v_2(s)ds = \int_I v_2(s)ds + \int_J \sigma ds \geq \int_J \sigma ds = \sigma |J|$$
$$= \sigma |[\tau'_p, t] \setminus I| \geq \sigma (t - \tau'_p - |I|) \geq \sigma \left(t - \tau'_p - \frac{8}{\sigma - 1} a_{p+1} \right).$$

Thus, it follows from (34) that if $\tau'_p \leq \tau'_{k_1}$, then for any $t \geq \tau'_{k_1}$,

$$y_2(t) - x_{p2}(t) > \frac{1}{2} a_p - \frac{8\sigma}{\sigma - 1} a_{p+1} + (\sigma - 1)(t - \tau'_p)$$
$$> \frac{1}{2} a_p - \frac{\alpha}{8\sigma} a_p > \frac{1}{4} a_p > a_{p+1},$$

which is the desired conclusion. The proof of the lemma is complete. □

We are now in a position to prove Theorem 4.1.

Proof Let $\tau'_p \leq \tau'_{k_1}$ for a pursuer x_p in the first group attack of pursuers. Note that this condition is satisfied at least for $p = k_1$. Combining the inequality in (14) with $k = p$, inequality (31) where $\tau_* = \tau'_p$, and inequality (32) we obtain that $\|y(t) - x_p(t)\| > a_{p+1}$ for all $t \geq 0$. Therefore, the pursuer x_p, for which $\tau'_p \leq \tau'_{k_1}$, can never reach the a_{p+1}-vicinity of the evader y. Hence, x_p will not participate in the further group attacks starting from the second one satisfying the inequality $y_2(t) - x_{p2}(t) > a_{p+1} \geq a_{k_1+1}, t \geq \tau'_p$ $(p \leq k)$.

If the time τ_{k_1+1} occurs, then the evader undergoes the second group attack of some pursuers on an interval $[\tau_{k_1+1}, \tau'_{k_2})$ for some $k_2 \geq k_1 + 1$. We can use similar arguments to obtain $\|y(t) - x_q(t)\| > a_{q+1}$ for all $t \geq 0$ and for some $q \in \{k_1 + 1, ..., k_2\}$ for which $\tau'_q \leq \tau'_{k_2}$. The pursuer x_q will not participate in further group attacks starting from the third one staying "behind" the evader satisfying the inequality $y_2(t) - x_{q2}(t) > a_{q+1} \geq a_{k_2+1}$, $t \geq \tau'_q$, and so on.

Thus, after the first group attack of pursuers $x_1, x_2, ..., x_{k_1}$ we can ignore at least one pursuer, for example, x_{k_1}, after the second group attack of pursuers $x_{k_1+1}, x_{k_1+2}, ..., x_{k_2}$ we can ignore at least one pursuer from this group of pursuers, for example, x_{k_2}, and so on. Since the total number of pursuers is m, therefore after at most m group attacks of pursuers all the pursuers remain "behind" the evader. The proof of Theorem 4.1 is complete. □

We can now estimate from above the total number of approach times. Since there are at most m approach times in the first group attack of pursuers, there are at most $m - 1$ approach times in the second group attack of pursuers and so on, therefore the total number of the approach times is at most $m + (m - 1) + ... + 1 = m(m + 1)/2$. If $\tau_{k_{j_0}}$ is the last approach time, then $k_{j_0} \leq m(m + 1)/2$ and

$$\|x_i(t) - y(t)\| > r = a_{k_{j_0}+1}, \quad t \geq 0, \quad i = 1, 2, ..., m. \tag{40}$$

This means that the evader can avoid from all pursuers moving at the distance not less than r from them.

5 Discussion

5.1 The Evader Moves in ε-Vicinity of the *Oy*-axis

For any given positive number ε, we can choose the number a_1 so that the trajectory of evader is always in the ε-vicinity of the Oy-axis (Fig. 8). Indeed, since $v(t) = V_0 = (0, \sigma)$, (hence, $v_1(t) = 0$) if $t \notin I_1 = \bigcup_{k\geq 1}[\tau_k, \tau'_k)$, therefore if $a_1 \leq \frac{\sigma-1}{8\sigma}\varepsilon$, then

$$\begin{aligned}
|y_1(t)| &= \left|\int_0^t v_1(s)ds\right| = \int_0^t |v_1(s)|ds = \int_{I_1\cap[0,t]} |v_1(s)|ds \\
&\leq \int_{I_1} |v_1(s)|ds \leq \sum_{k\geq 1} \int_{\tau_k}^{\tau'_k} |v_1(s)|ds \leq \sum_{k\geq 1} \sigma(\tau'_k - \tau_k) \\
&\leq \sum_{k=1}^{\infty} \frac{4\sigma a_k}{\sigma - 1} \leq \frac{8\sigma}{\sigma - 1} a_1 \leq \varepsilon.
\end{aligned}$$

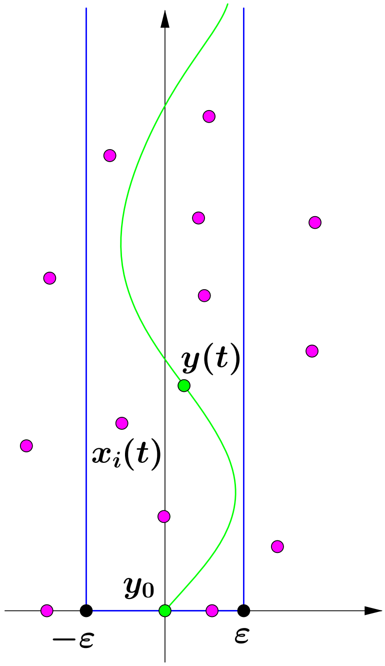


Fig. 8 Evader can avoid moving only in an ε-neighborhood of the Oy-axis

Hence, $|y_1(t)| \leq \varepsilon$ for all $t \geq 0$, meaning that the state of the evader is in the ε-vicinity of the Oy-axis for all $t \geq 0$.

For simplicity of the proof of Theorem 4.1, the evader moved around the Oy-axis. In fact, we could take any ray l with the beginning at y_0 and construct a strategy for the evader to escape from pursuers in the ε-vicinity of the ray l as well.

Next, it follows from $|I_1| \leq \frac{8}{\sigma-1}a_1 = \frac{\varepsilon}{\sigma} < \varepsilon$ that the measure of the set I_1 can be made smaller than any given positive number ε by choosing a_1. Thus, the evader moves under a group attack of some pursuers only on a subset of the set I_1 of measure less than ε, and outside the set I_1 the evader moves with velocity $v(t) = V_0 = (0, \sigma)$.

5.2 Informativeness

The estimate (40) allows the evader to weaken the condition to informativeness. We show that it suffices for the evader to use only information about $x_1(t), ..., x_m(t)$, $y(t)$, $u_1(t-\delta)$, $u_2(t-\delta)$, ..., $u_m(t-\delta)$ at the current time to avoid from capturing, where δ is a positive number.

Indeed, let

$$\hat{u}_i(t) = \begin{cases} 0, & 0 \leq t \leq \delta \\ u_i(t-\delta), & t > \delta \end{cases}, \quad i = 1, 2, ..., m. \tag{41}$$

Clearly, $\|\hat{u}_i(t)\| \leq 1$. We consider auxiliary objects (AO) $\hat{x}_i(t)$, $i = 1, 2, ..., m$, whose dynamics are given by the equations

$$\dot{\hat{x}}_i = \hat{u}_i, \quad i = 1, 2, ..., m,$$

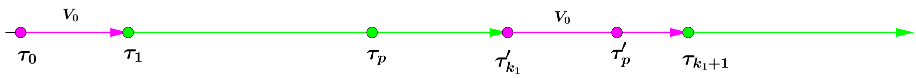


Fig. 9 For the pursuer x_p, $\tau'_{k_1} < \tau'_p$

and their trajectories are defined as follows. AO $\hat{x}_i$ moves with the trajectory $\hat{x}_i(t) = x_{i0} + \int_0^t \hat{u}_i(s)ds, t \geq 0$, if there is no an a_k-approach time of the pursuer x_i to the evader y. If the first a_i-approach time τ_i occurs, then AO $\hat{x}_i$ moves with the trajectory $\hat{x}_i(t) = x_i(\tau_k) + \int_{\tau_k}^t \hat{u}_i(s)ds$ for $t \geq \tau_i$.

In general, whenever the a_{i_l}-approach time $\tau_{i_l}, l = 1, 2, ...$, of the pursuer x_i to the evader y occurs, $\hat{x}_i$ moves with the trajectory $\hat{x}_i(t) = x_i(\tau_{i_l}) + \int_{\tau_{i_l}}^t \hat{u}_i(s)ds$ until the next approach time $\tau_{i_{l+1}}$. Clearly, the trajectory of AO $\hat{x}_i$ is, in general, discontinuous at the approach times τ_{i_l}.

For $0 \leq t \leq \delta$, by (41), we have

$$\|x_i(t) - \hat{x}_i(t)\| = \left\| x_{i0} + \int_0^t u_i(s)ds - x_{i0} - \int_0^t \hat{u}_i(s)ds \right\|$$
$$= \left\| \int_0^t u_i(s)ds \right\| \leq \int_0^t \|u_i(s)\|ds \leq t \leq \delta,$$

and, for $t > \delta$ and $\tau_{i_l} \leq t < \tau_{i_{l+1}}$, we have

$$\|x_i(t) - \hat{x}_i(t)\| = \left\| x_i(\tau_{i_l}) + \int_{\tau_{i_l}}^t u_i(s)ds - x_i(\tau_{i_l}) - \int_{\tau_{i_l}}^t u_i(s-\delta)ds \right\|$$
$$\leq \int_{\tau_{i_l}-\delta}^{\tau_{i_l}} \|u_i(s)\|ds + \int_{t-\delta}^t \|u_i(s)\|ds \leq \int_{\tau_{i_l}-\delta}^{\tau_{i_l}} 1ds + \int_{t-\delta}^t 1ds \leq 2\delta, \quad (42)$$

and so $\|x_i(t) - \hat{x}_i(t)\| \leq 2\delta$ for all $t \geq 0$.

The evader knows now information about $x_1(t), ..., x_m(t), y(t), \hat{u}_1(t), \hat{u}_2(t), ..., \hat{u}_m(t)$ at the current time t. We let the evader use strategy (16) with $u_i(t)$ replaced by $\hat{u}_i(t)$ in (15). By Theorem 4.1, we have then $\|y(t) - \hat{x}_i(t)\| > r_0$ for some $r_0 > 0$. Consequently, if we choose $\delta < r_0/4$, then by (42)

$$\|y(t) - x_i(t)\| \geq \|y(t) - \hat{x}_i(t)\| - \|\hat{x}_i(t) - x_i(t)\| > r_0 - 2\delta > r_0/2,$$

that is the evader can avoid from capturing moving from each pursuer at least $r_0/2$ distance away.

5.3 The Case Where $\tau'_{k_1} < \tau'_p$ for the Pursuer x_p

Figure 9 illustrates this case. It is important to note that if $\tau'_{k_1} < \tau'_p$ (see Fig. 4 where $\tau'_{k_1} = \tau'_3 < \tau'_2 = \tau'_p$), then the inequality $\|y(t) - x_p(t)\| > a_{p+1}$ in (31) may not be true for $\tau'_{k_1} \leq t \leq \tau'_p$. Since (31) was proved based on (28), and in its turn (28) was proved using the fact that the interval $[\tau_{p+1}, t)$ is a subset of the set $A = \cup_{k\geq p+1,\ \tau_k\leq t}\ [\tau_k, \tau_k')$. However, if $\tau_{k_1}' < \tau_p'$, then the interval $[\tau_{k_1}', t)$ is not, in general, a subset of A. For example, if $\max\limits_{p+1\leq k\leq k_1} \tau_k' < \min\{\tau_p', \tau_{k_1+1}\}$ and $\max\limits_{p+1\leq k\leq k_1} \tau_k' < t < \min\{\tau_p', \tau_{k_1+1}\}$, then, clearly, the interval $[\max\limits_{p+1\leq k\leq k_1} \tau_k', t)$ is not a subset of the set A.

Also, note that if $\tau_{k_1}' < \tau_p'$, then the pursuer x_p, in general, can approach to an a_k distance of the evader for some $k \geq p+1$ on the time interval $[\tau_{k_1}', \tau_p')$, but it is important to note that after the time τ_p' there is no an a_k-approach time to the evader for $k \geq p+1$. Indeed, by (23) we have

$$\begin{aligned} y_2(\tau_p') - x_{p2}(\tau_p') &= y_2(\tau_p') - z_{p2}(\tau_p') + z_{p2}(\tau_p') - x_{p2}(\tau_p') > y_2(\tau_p') - z_{p2}(\tau_p') + a_p \\ &= a_p + y_2(\tau_{k_1}') + \int_{\tau_{k_1}'}^{\tau_p'} v_2(s)ds - z_{p2}(\tau_{k_1}') - \int_{\tau_{k_1}'}^{\tau_p'} V_{p2}(s)ds. \end{aligned} \tag{43}$$

Since, for the set $I_2 = \bigcup\limits_{\tau_k\geq\tau_{k_1}'} [\tau_k, \tau_k') = \bigcup\limits_{k\geq k_1+1} [\tau_k, \tau_k')$, similar to (39) we have $|I_2| \leq \frac{8}{\sigma-1} a_{k_1+1} \leq \frac{8}{\sigma-1} a_{p+1}$, and by (28)

$$y_2(\tau_{k_1}') - z_{p2}(\tau_{k_1}') \geq -\|y(\tau_{k_1}') - z(\tau_{k_1}')\| \geq -\frac{16\sigma}{\sigma-1} a_{p+1},$$

therefore using the obvious inequality $V_{p2}(s) \leq \sqrt{\sigma^2 - \alpha^2}$ we obtain from (43) that

$$\begin{aligned} y_2(\tau_p') - x_{p2}(\tau_p') &\geq a_p - \frac{16\sigma}{\sigma-1} a_{p+1} + \int_{[\tau_{k_1}', \tau_p')\setminus I_2} \sigma ds - \int_{\tau_{k_1}'}^{\tau_p'} \sqrt{\sigma^2-\alpha^2} ds \\ &\geq a_p - \frac{16\sigma}{\sigma-1} a_{p+1} + \left(\tau_p' - \tau_{k_1}' - \frac{8}{\sigma-1} a_{p+1}\right)\sigma - (\tau_p' - \tau_{k_1}')\sqrt{\sigma^2-\alpha^2} \\ &= a_p - \frac{24\sigma}{\sigma-1} a_{p+1} + (\tau_p' - \tau_{k_1}')(\sigma - \sqrt{\sigma^2-\alpha^2}) \geq a_p - \frac{24\sigma}{\sigma-1} a_{p+1} \geq \frac{a_p}{2}. \end{aligned}$$

Then, for $t \geq \tau_p'$, using the fact that $v_2(s) = \sigma$, $s \in [\tau_p', t)\setminus I_2$ and $v_2(s) > 0$, $s \in I_2$, we have

$$\begin{aligned} y_2(t) - x_{p2}(t) &= y_2(\tau_p') + \int_{\tau_p'}^{t} v_2(s)ds - x_{p2}(\tau_p') - \int_{\tau_p'}^{t} u_{p2}(s)ds \\ &= y_2(\tau_p') - x_{p2}(\tau_p') + \int_{[\tau_p', t)\setminus I_2} \sigma ds + \int_{I} v_2(s)ds - \int_{\tau_p'}^{t} 1ds \\ &\geq \frac{a_p}{2} + \left(t - \tau_p' - \frac{8}{\sigma-1} a_{p+1}\right)\sigma - (t - \tau_p') \\ &= \frac{a_p}{2} - \frac{8\sigma}{\sigma-1} a_{p+1} + (\sigma-1)(t - \tau_p') > a_{p+1}. \end{aligned}$$

Hence, for the pursuer x_p, there is no an a_k, $k \geq p+1$, approach time to the evader on the interval $t \geq \tau_p'$. This fact shows that the number of pursuers participating in the group attacks decreases more faster.

6 Conclusion

We have studied a simple motion evasion differential game of many pursuers and one faster evader and proposed a new strategy for the evader.

In in the paper of Chernous'ko [9], the evader first moves along a straight line l. Then, as the a_1-approach time τ_1 occurs, the evader starting from this time moves along some spiral curves and at most 2^{m-1} approach times may occur until the evader reaches the straight line l again. The same pursuer can have several approach times during this period. As the evader reaches the straight line l, it moves again along l until the another approach time occurs. Then, the evader moves again along some spirals and at most 2^{m-2} approach times may occur until the evader reaches the straight line l again and so on. Therefore, the total number of approach times in that paper is $\leq 2^{m-1} + 2^{m-2} + \cdots + 1 = 2^m - 1$.

In the present paper, in the first group attack at most m approach times may occur and the evader, in general, doesn't reach again the Oy-axis after the first group attack, but moves parallel to the Oy-axis with the speed σ until the second group attack of pursuers occurs. Any pursuer participated in a group attack cannot have another approach time in the same group attack.

Also, we can specify at least one pursuer from each group attack, which will not participate in the following group attacks. This is namely the last pursuer joined the group attack. Therefore, the total number of approach times τ_k of m pursuers during the game doesn't exceed $m(m+1)/2$ in the present paper.

The proof of Theorem 4.1 strongly depended on the inequality that $\tau'_p \leq \tau'_{k_1}$. If $\tau'_p > \tau'_{k_1}$ for a pursuer x_p, then this pursuer can have an a_k-approach time to the evader for some $k > k_1$. In other words, this pursuer can participate in the next group attack of pursuers. However, good news is that this pursuer will not have any a_k-approach time to the evader for $k \geq p+1$ on the interval $[\tau'_p, \infty)$. Hence, this pursuer will not participate in further group attacks after the time τ'_p.

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