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Herd Behaviors in Epidemics: A Dynamics-Coupled Evolutionary Games Approach

Shutian Liu[1] (ID) **· Yuhan Zhao[1] · Quanyan Zhu[1]**

Abstract

The recent COVID-19 pandemic has led to an increasing interest in the modeling and analysis of infectious diseases. The pandemic has made a significant impact on the way we behave and interact in our daily life. The past year has witnessed a strong interplay between human behaviors and epidemic spreading. In this paper, we propose an evolutionary game-theoretic framework to study the coupled evolution of herd behaviors and epidemics. Our framework extends the classical degree-based mean-field epidemic model over complex networks by coupling it with the evolutionary game dynamics. The statistically equivalent individuals in a population choose their social activity intensities based on the fitness or the payoffs that depend on the state of the epidemics. Meanwhile, the spreading of the infectious disease over the complex network is reciprocally influenced by the players' social activities. We analyze the coupled dynamics by studying the stationary properties of the epidemic for a given herd behavior and the structural properties of the game for a given epidemic process. The decisions of the herd turn out to be strategic substitutes. We formulate an equivalent finite-player game and an equivalent network to represent the interactions among the finite populations. We develop a structure-preserving approximation technique to study time-dependent properties of the joint evolution of the behavioral and epidemic dynamics. The resemblance between the simulated coupled dynamics and the real COVID-19 statistics in the numerical experiments indicates the predictive power of our framework.

Keywords Evolutionary game · Epidemic dynamics · Complex networks · Coupled dynamics · Structure-preserving approximation

✉ Shutian Liu
   sl6803@nyu.edu

   Yuhan Zhao
   yhzhao@nyu.edu

   Quanyan Zhu
   qz494@nyu.edu

[1] Department of Electrical and Computer Engineering, Tandon School of Engineering New York University, Brooklyn, NY 11201, USA

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1 Introduction

The COVID-19 pandemic has unprecedentedly impacted our society in many ways. Companies, schools, and the government have shut down their offices. Many people work at home, shop online, and communicate over zoom. The past year has witnessed a litany of policies regarding social distancing, mask-wearing, and vaccination to prevent and mitigate the spreading of the pandemic. The pandemic has made a significant impact on the way we behave and interact in our daily life. We have observed a strong interplay between people's behaviors and the pandemic. When the pandemic transforms the pattern of social interactions, the human behaviors also change how the infectious disease spreads. When the number of COVID cases goes down, reopening policies enable more social activities to return to normal. If not done carefully, they would create second or third waves of infections, which we have witnessed recently in many countries.

The behaviors of individuals in the same fashion create a collective behavioral pattern that leads to the behavior of the population, which is also known as herd behavior. The herd behavior plays an important role in the pandemic. It is often driven by policies or individual incentives. For example, cities like New York and London have designed incentives for individuals to be vaccinated to reach targeted herd immunity. Many countries have enforced the policies of mask-wearing in public spaces to create herd behavior that reduces the risk of mass infection.

Existing works on herd behaviors have focused mainly on topics related to financial markets and economics [1, 35]. At the same time, epidemic processes are often studied as stand-alone dynamical processes without incorporating individual behaviors into the model [29]. The epidemic models alone from the literature are insufficient. There is a need for an integrated framework that gives a holistic understanding of the pandemic together with herd behaviors.

In this paper, we propose a dynamics-coupled evolutionary game-theoretic framework to model the herd behaviors that are coupled with the spreading of epidemics. Noncooperative games [2] are natural tools for the study of strategic decision-making of rational individuals in competitive environments. When populations instead of a finite number of individuals are of interest, the strategy profile of the population game captures the herd behavior. The reason lies in that the macroscopic herd behaviors of the populations result from the microscopic strategic choices of individuals without central coordination.

Evolutionary games [18, 34, 38, 40] study the strategic behaviors of the populations in which one population can mutate and choose a strategy against another population to maximize its fitness. The evolutionary game dynamics are population-level or mean-field dynamics that describe the evolution or the adaptive revision of the strategies when the populations interact with each other. The outcome of the evolutionary game and its associated dynamics is defined by the concept of evolutionarily stable strategy (ESS), which refines the concept of Nash equilibrium. The evolutionary dynamics provide a straightforward way to describe the macroscopic strategic interactions among the populations and the evolution of the herd behavior in response to the underlying changing environments, including the information received by the population and the fitness of the population affected by the epidemics.

One critical component of our evolutionary game framework is the modeling of the infectious disease. In this work, we consider a class of mean-field epidemic models over complex networks to capture the social interactions among the individuals [11, 27, 29, 30]. The individuals over the network are assumed to be statistical equivalent within the same class of

Fig. 1 Dynamics-coupled evolutionary game framework. The herd behaviors can influence epidemic spreading over the complex network, generating the epidemic state. The media reports the information containing the epidemic states, stimulating individual strategy revisions. The collective patterns of the new strategies then reform the herd behaviors
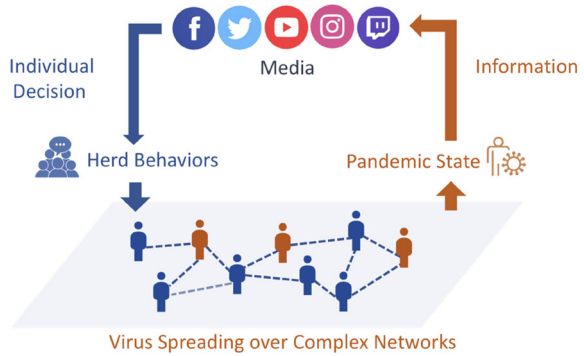
population. This assumption makes sense since we classify individuals based on their degrees, and statistically equivalent individuals share the same possibility of getting infected by the epidemic. The mean-field dynamics forthrightly describe the influences of the herd behavior on the spreading of the epidemic. We use a complex network model that is characterized by a degree distribution to represent the social interactions of the populations. Each individual in the network is associated with a degree or the number of connections that determines the probability of infection and thus the spreading of the disease.

The epidemic model is consolidated into the evolutionary game framework as illustrated in Fig. 1. The spreading of the epidemic among the populations is affected by the social activity intensities of the individuals. As the information and the policies concerning the epidemic being communicated to the population through public media, individuals can respond to them and adapt their social activities, constituting herd behavior at the population level. It is clear from Fig. 1 that the state of the epidemics and the herd behaviors are interdependent. This interdependence is one of the features of our framework which differs from previous works as will be discussed in Sect. 2.

The integrated framework in Fig. 1 can be mathematically described by a system of coupled differential equations. One set of differential equations represents the mean-field evolutionary dynamics of the game strategies. The other set of differential equations describes the epidemic process. It is critical to examine the structural properties of the coupled dynamics, including the stability and the steady state. To this end, we first discuss the stability of the epidemic dynamics under fixed herd behavior and then analyze the structural properties of the evolutionary game under the steady states. We find out that, under certain conditions, there is a unique nontrivial globally asymptotically stable steady state given the herd behaviors. The players' decisions in the game turn out to be strategic substitutes. This property makes the ESS or the Nash equilibria achievable even when the individuals revise their strategies myopically on their own. This structural property is shown to hold even when the epidemic is not at its steady state.

We formulate a unified optimization problem to compute the Nash equilibrium based on an equivalent representation of the population game as a finite-player game problem, where each population is viewed as a player. Furthermore, we develop a structure-preserving approximation technique to analyze the time-dependent evolution of the herd behaviors and the epidemics. We show that this approximation preserves the strategic substitutes property of the game.

The proposed dynamics-coupled evolutionary game provides a suitable framework to study the impact of misinformation on epidemics. We use numerical experiments to compare

the simulated infection curve with the COVID-19 statistics of the infected in New York City. The prediction of two peaks in the pandemic over a time interval of interest provides a promising analytical and policy design tool for the pandemic.

The structure of this paper is as follows. Section 2 discusses the related works. We introduce the general framework in Sect. 3. In Sect. 4, we present analytical results for the case where the epidemic evolves at a faster timescale. We characterize the steady-state behavior of the epidemic, investigate the Nash equilibrium and structural properties of the game, and study the impact of misinformation. In Sect. 5, we extend the structural properties to different timescales and develop a approximation scheme to study time-dependent behaviors. Section 6 presents the numerical experiments. We conclude the paper in Sect. 7.

2 Related Work

Epidemic models over networks The mean-field approach has been a standard tool to study the spreading of epidemics over complex networks [5, 11, 29–31]. The key components in this approach are the infection probabilities of the nodes, which bridge the degree distribution of the nodes with the contagion events. This model has been used to model spreading over networks for a diverse range of applications. For example, in [16], the authors have investigated multi-strain epidemic dynamics over complex networks to study the control policies when a single pathogen creates many strains of infections of different features. Another recent endeavor is [8], where the authors have focused on the optimal quarantining policies when multiple diseases coexist and have observed a switching phenomenon between equilibria. Our framework extends the statistical equivalence assumptions of the standard degree-based mean-field approach. The players in our framework are distinguished by their degrees of connections and the strategies they choose. The proposed coupled system of differential equations describes the flow of contagion when individuals adopt different social activity intensities.

Game modeling of networked populations The building block of our evolutionary game framework is a population game [34], where there are infinitely many players. We partition the set of players into subpopulations based on a statistical equivalence assumption. This enables us to position the population on a complex network and use the distribution of players' degrees of connectivity as the characteristics. Our method resembles the mean-field games [7, 19, 23, 41] in the setting of a large population with small players who have little influence on the whole population. The epidemic state equation in our game is also a mean-field-type differential equation. The difference between our framework and mean-field games lies in that players in our framework revise strategies myopically according to given protocols anticipating the payoffs after given time periods instead of solving optimal control problems. This type of revisions leads to evolutionary dynamics rather than Hamilton–Jacobi equations. In [37], the authors have extended mean-field games to finite communication networks. In their work, the network influences the state transitions through the adjacency matrix, which is different from our setting. Our framework is also related to network aggregative games in [15, 28], with the difference lying in the externalities. In our model, individual payoffs are influenced by all the actions of the other players in the population in an average sense, since the epidemic evolves on a complex network. However, in network aggregative games, the payoff of a player is only directly influenced by an aggregation of actions of her neighbors defined by the given finite network.

Human behavior under epidemics The connection of models of human behavior and epidemic models has been successfully established in the celebrated work [3]. The authors have characterized the vaccination decisions in populations using the concept of convergently stable Nash equilibrium. The vaccination game [3] has then been used to guide policies on social distancing [14] and analyze imitative behaviors in vaccination [12]. We are motivated by the problem settings of [3] and consider a heterogeneous population over complex networks, whose connectivity is characterized by the degree distributions of individuals. We use specific dynamic processes to model the evolution of the herd behaviors, enabling observations on the time-dependent properties of herd behaviors. This line of research has been developing in a fast pace recently. In [6], the authors have proposed the state policy coupled dynamics model for extending the current evolutionary game by introducing independent state transitions of players. They have focused on the coupling of replicator dynamics and individual player's Markov decision process. Our framework takes a different route by unifying the mean dynamic of the states of the population and the mean-field state transitions of players over the network to capture the interdependent evolution of herd behavior and epidemics over networks. In [24], the authors have considered a coupled disease–human behavior model to study the long-term impact of social distancing on disease spreading. They have focused on the replicator dynamics and the stability of the epidemic steady states. However, we emphasize the structural property of the herd behavior and include time-dependent behavior as part of our analysis. There are also recent endeavors on addressing issues caused by the COVID-19 pandemic. In [22], the authors have considered an evolutionary rate of quarantine in the compartmental epidemic model. This integration of behavioral dynamics and virus spreading has enabled them to conclude numerically the ineffectiveness of natural shield immunity in eliminating an epidemic without additional social control measures. In [44], consistency between the infection rate in the SI model and the population proportion of people not being mindful in the epidemics has been established. This discovery has led to a transmission frequency equation describing the changes in the number of infected people caused by the COVID-19 pandemic. In [39], the author has proposed a sophisticated mean-field-type game framework to capture comprehensive individual and authority behaviors under epidemic propagation. The parameters calibrating the characteristics of players and epidemic states are obtained using real COVID-19 statistics, making the model data-driven. Numerous numerical experiments have proved the practical applicability of the equilibrium of the game obtained by solving a corresponding Bellman system. In [25], the authors have used game-theoretic analysis to show that the bilateral deals between rich nations and vaccine manufacturers can be configured to improve the global supply of vaccines.

Other related works Recently in [9], the authors have studied imitation dynamics on networks. They have designed a pairwise information exchange scheme and have focused on potential games. While they have emphasized pairwise communications between nodes over a given finite network, we pay more attention to the development of the strategy profile of the populations when the required information is broadcast publicly. In [21], evolutionary games are used to analyze information diffusion over networks; evolutionary stability is also investigated for different types of networks. A random number of players have also been investigated through evolutionary Poisson games in [17], where the authors have developed global objectives for the players.

3 Problem Setting

In this section, we describe our dynamics-coupled evolutionary game framework in detail. We first introduce the general framework and then turn to the specific setting under epidemic.

3.1 The General Framework

We consider $N \in \mathbb{N}_+$ players (nodes) over a network. Each player belongs to a subset in the set $\mathcal{P} := \{1, 2, \ldots, P\}$ representing its degree of connectivity, i.e., a player in $p \in \mathcal{P}$ has degree p. Let $N^p \in \mathbb{N}_+$ denote the number of players who has degree p. We have $N = N^1 + N^2 + \cdots + N^P$. The degree distribution is then denoted by $[m^p]_{p \in \mathcal{P}} := (m^1, m^2, \ldots, m^P) \in [0, 1]^P$, where $m^p = \frac{N^p}{N}, \forall p \in \mathcal{P}$.

Let Ξ be the finite state space of all the players with $|\Xi| = L$. At a given time, a player is in a state $\xi \in \Xi$, which describes the current type of the player. In the context of epidemic spreading, elements in Ξ describe the infection status of players, such as "Susceptible," "Infected," or "Recovered" in the classic SIR model [29]. Let $\mathcal{S}^p := \{s_1^p, s_2^p, \ldots, s_{n^p}^p\} \subset [0, 1]^{n^p}$ be the finite strategy space of players with degree p with $|\mathcal{S}^p| = n^p$. A strategy in $\mathcal{S}^p$ is a social behavior that affects the spreading of the epidemic. For example, a strategy can represent the willingness to wear masks or the probability to take vaccinates.

Let $\mathcal{I}^p := \{1, 2, \ldots, n^p\}$ denote the index set of the strategies in $\mathcal{S}^p, \forall p \in \mathcal{P}$. The strategy indexed by $i \in \mathcal{I}^p$ is $s_i^p \in \mathcal{S}^p$. We assume throughout this paper that the strategies in each set $\mathcal{S}^p, \forall p \in \mathcal{P}$, are listed in an increasing order, i.e., for any $i, j \in \mathcal{I}^p$ such that $i > j$, we have that $s_i^p > s_j^p$. Let $\mathcal{S} := \prod_{p \in \mathcal{P}} \mathcal{S}^p$ with $|\mathcal{S}| = \sum_{p \in \mathcal{P}} n^p = n$. Let $w_\xi^p(t)$ denote the fraction of players with degree p who are in state ξ at time t. Let $x_i^p(t)$ denote the fraction of players with degree p playing strategy s_i^p at time t. We use $w(t) = (w^1(t), ..., w^P(t)) \in \mathcal{W} \subset \mathbb{R}_+^{PL}$ and $x(t) = (x^1(t), ..., x^P(t)) \in \mathcal{X} = \prod_{p \in \mathcal{P}} \mathcal{X}^p \subset \mathbb{R}_+^{n^1 \times \cdots \times n^P}$ to denote the concatenations of $w_\xi^p(t)$ and $x_i^p(t)$, respectively. Note that we refer to $x(t)$ as the herd behavior at time t.

Remark 1 The reason for considering only a finite number of states is that it is sufficient in describing the infection status of individuals in epidemics. Popular compartmental epidemic models use a few compartments to classify individuals based on the possibility they catch the disease, the ability they spread the disease, and their own health status. For example, three compartments are adopted in SIR model and four compartments are used in SEIR model [29]. These compartments describe comprehensive and concise patterns of virus transmission and variations on the size of each group of individuals. In [39], 17 infection states have been considered for the modeling of real COVID-19 pandemic. Their numerical experiments have shown that 17 infection states capture reported statistics of COVID-19 across many countries with various socioeconomic conditions.

Remark 2 There are two reasons for considering finite strategy spaces. First, evolutionary games emphasize the revisions of pure strategies. In our population game setting, pure NE always exists. Hence, finite strategy spaces help us elaborate the relation between revision protocols and NE. An infinite strategy space is often linked with mixed strategies. Since mixed strategies are probability distributions over the pure strategies and they form a continuum. Second, the structural property results that we will discuss in Sects. 4 and 5 are related to the externalities of players. The externalities only rely on the ordering of the strategies instead of the values of strategies or the sizes of the strategy spaces.

Suppose that players are constantly interacting physically over the network. By physical interactions, we refer to the face-to-face social interactions which can cause potential changes in players' states. This corresponds to the fact that infections can happen during our daily social lives. Interacting in online chat rooms is an example of social interaction which is not a physical interaction, since chatting online will not cause contagion. We use the degree-based mean-field approach [11, 29] to capture the coupled dynamical systems on the large network. The players are assumed to be statistically equivalent if they have the same degree and the same strategy. In other words, in the large population, the players are distinguishable only based on their degree and their strategy.

Let $Q_\xi^p : \mathcal{W} \times \mathcal{X} \times [0, 1]^P \to \mathbb{R}$ be a Lipschitz function describing the dynamical evolution of the fraction of players with degree p who are in state ξ. Functions Q_ξ^p are influenced by individual state transitions. For example, when ξ denotes the "Infected" state in an epidemic model, $Q_\xi^p, \forall p \in \mathcal{P}$ would consist of a negative term describing the natural recovery from a disease and a positive term representing infections. The coupled dynamics of players' state transitions are as follows:

$$\dot{w}_\xi^p(t) = Q_\xi^p \left(w(t), x(t), [m^p]_{p \in \mathcal{P}} \right), \quad \forall p \in \mathcal{P}, \ \forall \xi \in \Xi. \tag{1}$$

Note that in (1), the dependence on $w(t)$ emphasizes the coupling of players' state transitions, and the dependence on $[m^p]_{p \in \mathcal{P}}$ illustrates the effect of the network. We use $\bar{w}(x)$ to denote the steady-state value of $w(x)$. The game mechanism shown in Fig. 1 coordinates the information acquisition of the players. As the state evolves, public media broadcasts information relevant to the states w and the strategy profile x to all the players at times. We assume that the times between information broadcasts are independent, and they follow a rate τ exponential distribution. Each information broadcast triggers a strategic interaction where the players update their strategies based on the current information received. For all $i, j \in \mathcal{I}^p$, let $R_{ij}^p(w(t), x(t)) : \mathcal{W} \times \mathcal{X} \to [0, 1]$ denote a continuously differentiable function representing the probability of a player with degree p switching from strategy $s_i^p \in \mathcal{S}^p$ to strategy $s_j^p \in \mathcal{S}^p$. Functions R_{ij}^p resemble the revision protocols in evolutionary games, with the difference of the dependence on the state w. They capture individual perceptions of the payoffs corresponding to current state and others' strategies.

The evolution of the fraction of players with degree p playing strategy s_i^p in the strategic interactions at time t can be described as follows. Consider a small time period $(t, t + dt)$. There will be τdt expected information broadcasts during this period. The change of the number of players with degree p playing strategy s_i^p during $(t, t + dt)$ can be expressed as:

$$N \left(\sum_{j \in \mathcal{I}^p} x_j^p(t) R_{ji}^p(w(t), x(t)) - \sum_{j \in \mathcal{I}^p} x_i^p(t) R_{ij}^p(w(t), x(t)) \right) \tau dt. \tag{2}$$

By considering fractions of players in (2), we obtain the mean dynamic as follows:

$$\begin{aligned} \frac{1}{\tau} \dot{x}_i^p(t) = &\sum_{j \in \mathcal{I}^p} x_j^p(t) R_{ji}^p(w(t), x(t)) \\ &- \sum_{j \in \mathcal{I}^p} x_i^p(t) R_{ij}^p(w(t), x(t)), \quad \forall i \in \mathcal{I}^p, \forall p \in \mathcal{P}. \end{aligned} \tag{3}$$

Equations (1) and (3) constitute a system of coupled differential equations describing the joint evolution of states and strategies. This coupled system is depicted in Fig. 2. The assumption that functions R_{ij}^p are continuously differentiable with respect to w and x implies

Fig. 2 Coupled evolution of evolutionary game and epidemic dynamics. The evolutionary dynamics (3) shapes the social state x. Changes in the social states influence the spreading of the epidemic, reforming the epidemic state dynamics (1). Then, (1) outputs new state w, which in return affects the evolutionary dynamics (3)
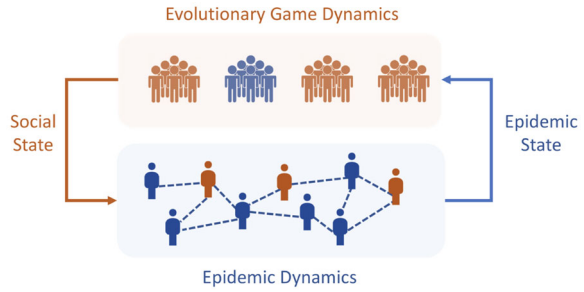
that they are locally Lipschitz with respect to w and x. Combining the assumption of functions Q_ξ^p being Lipschitz, the existence of solutions to the coupled system (1) and (3) is guaranteed [6].

Since the coupled evolution in (1) and (3) involves the subpopulations of the players instead of individual players, we naturally interpret the above setting as the interactions among populations when the total number of players goes to infinity, i.e., $N \to \infty$. In the sequel, we refer to the players with degree $p \in \mathcal{P}$ as population p.

For all $i \in \mathcal{I}^p$, let $F_i^p : \mathcal{W} \times \mathcal{X} \to \mathbb{R}$ denote the payoff function of a player with degree p who plays strategy $s_i^p \in \mathcal{S}^p$. In general, F_i^p depends on $w(t)$ and $x(t)$. The dependence on $x(t)$ characterizes the game-theoretic aspect of the framework. The dependence on $w(t)$ reveals the coupling of all players' state evolutions. We use $F = (F^1, ..., F^P) \in \mathbb{R}^{n^1 \times \cdots \times n^P}$ to denote the concatenation.

Connecting with the standard definition of evolutionary games [34], we refer to the a herd behavior x as a social state and call, with a slight abuse of terminology, $R := (R^1, ..., R^P) \in [0, 1]^{n^1(n^1-1) \times \cdots \times n^P(n^P-1)}$ a revision protocol, where $R^p := (R_{12}^p, ..., R_{1n^p}^p, ..., R_{n^p1}^p, ..., R_{n^p(n^P-1)}^p)^\top \in [0, 1]^{n^p(n^P-1)}, \forall p \in \mathcal{P}$. We refer to the game defined by the payoff function F, the evolutionary dynamics (3), and the coupled state dynamics (1) as a dynamics-coupled evolutionary game.

Definition 1 Let $\text{NE}(F)$ denote the set of Nash Equilibrium (NE) of the game defined by the payoff F and the coupled state transitions (1). A social state $x \in \mathcal{X}$ is an NE, i.e., $x \in \text{NE}(F)$, if for all $p \in \mathcal{P}$, $x^p \in m^p \text{BR}^p(x)$, where the set $\text{BR}^p(\cdot)$ denotes the set of best responses of population p, i.e., $\text{BR}^p(x) := \{z \in \mathbb{R}_+^{n^p} : \mathbf{1}^\top z = 1, z_i > 0, \text{ if } s_i^p \in \arg\max_{j \in \mathcal{I}^p} F_j^p(x, \bar{w}(x))\}$.

The difference between Definition 1 and the standard NE of population games [34] is that x is the best response to the payoffs generated by x together with the steady-state value of (1) given x. This integrates the coupling of (1) and (3) into the definition of NE of our framework.

Remark 3 In the following sections, the terms social state and herd behavior will be used interchangeably when we refer to x.

3.2 The Framework Under Epidemics

We use the susceptible–infected–susceptible (SIS) compartmental model over a degree-based network [30] with degree distribution $[m^p]_{p \in \mathcal{P}}$ to represent the state evolution of the players. The state space is $\Xi = \{\text{"Susceptible", "Infected"}\}$. Compared to the classic SIR model,

the SIS model treats individuals in the "Recovered" state as "Susceptible" again. In our framework, the SIS model is the proper choice for studying herd behaviors in epidemics, especially diseases like the COVID-19, for the following two reasons. First, our goal is not merely studying the epidemic dynamics on its own, but focusing more on the joint evolution of herd behaviors and epidemic dynamics. Hence, in our framework, we consider a fixed population interacting socially over the complex network. In other words, the size of population involved in social activities is a constant. This assumption of constant size helps us get rid of the individuals in the "Recovered" state, since if an individual is immune to the virus, it is equivalent to say that this individual is not interacting with others over the network. When it comes to herd behaviors, these individuals do not contribute. In the literature, the "Recovered" state is sometimes referred to as the "Removed" state to include the individuals who are dead. These individuals have no social impact on others either. Note that although there are many reported death cases across the world during the recent COVID-19 pandemic, the total number of deaths is still relatively small compared to the susceptible and infected portions of the population. Second, it is known that infected individuals develop no long-lasting immunity against the COVID-19 even after recovering from it. Hence, from the perspective of infectious diseases, it is proper for us to use a model which describes the spreading of diseases like influenza and common cold which can cause reinfections. The SIS model is a standard choice for infections which have no immunity upon recovery. Therefore, we adopt the SIS model to describe the spreading of infections in our framework under epidemics.

We use $I_i^p(t)$ to denote the fraction of the infected players in population p adopting strategy $s_i^p \in \mathcal{S}^p$ at time t, $\forall i \in \mathcal{I}^p$, $\forall p \in \mathcal{P}$. The strategies $s_1^p, s_2^p, \ldots, s_{n^p}^p \in [0, 1]$, $p \in \mathcal{P}$, are in the action sets of an individual with degree p. We refer to the strategy s_i^p as the choice of social inactivity intensity level i of an individual of degree p. The social inactivity intensity is quantized into n^p levels for an individual to choose from. Naturally, the chosen social activity intensity (SAI) level is given by $1 - s_i^p$, $\forall i \in \mathcal{I}^p$, $\forall p \in \mathcal{P}$, which can be viewed as the probability of a player behaving actively in the face-to-face physical interactions through all of her connections with other players. A strategy s_i^p, $i \in \mathcal{I}^p$, $p \in \mathcal{P}$, close to 1 means that the player is considerably mindful when interacting socially through her connections with others. Note that in the context of epidemics, the herd behavior describes the collective patterns in the populations' social activity intensities.

Given a herd behavior $x \in \mathcal{X}$, the evolution of $I_i^p(t)$ is also affected by a recovery rate $\gamma \in \mathbb{R}_+$ and a contagion rate $\lambda \in \mathbb{R}_+$. Under the SIS model, (1) can be written as the following dynamical system describing the time evolution of $I_i^p(t)$:

$$\dot{I}_i^p(t) = -\gamma I_i^p(t) + \lambda_i^p \left(1 - I_i^p(t)\right) p\Theta(t), \tag{4}$$

where $\lambda_i^p = \lambda(1 - s_i^p) \in \mathbb{R}_+$ denotes the activity-aware contagion rate of a player with degree p and SAI $1 - s_i^p$. The second term on the right-hand side of (4) corresponds to the growth of $I_i^p(t)$. This growth is proportional to the activity-aware contagion rate λ_i^p, the density of susceptible players $1 - I_i^p(t)$, the degree p, and the probability $\Theta(t) \in [0, 1]$ that a link is connected to an infected player. This probability can be expressed as follows:

$$\Theta(t) = \frac{\sum_{p \in \mathcal{P}} \left(\sum_{i \in \mathcal{I}^p} p x_i^p I_i^p(t)\right)}{\sum_{p \in \mathcal{P}} p m^p}. \tag{5}$$

Since we have assumed the statistical equivalence of players with the same degree and the same strategy, the numerator of the right-hand side of (5) consists of the sum of the probabilities that a link is connected to an infected player within each equivalence class. The

probability of a link connecting to an infected player with degree p choosing strategy s_i^p is proportional to $p x_i^p I_i^p(t)$. Hence, we obtain (5). The consistency of (4) and (5) follows from a similar reasoning as discussed in [30], since the effect of SAI has already been considered in the activity-aware contagion rate λ_i^p.

The probability (5) couples the dynamics in (4) corresponding to each strategy of each population. The concatenation of all n fractions of infected is $I(t) = (I^1(t), ..., I^P(t)) \in [0, 1]^{n^1 \times \cdots \times n^P}$. We use $\bar{I}_i^p$ and $\bar{\Theta}$ to denote the steady-state quantities of $I_i^p(t)$ and $\Theta(t)$. The concatenations are $\bar{I}$ and $\bar{I}^p$.

Note that the differential equation (4) corresponds to the state transition dynamics (1) in the general framework. Since the size of the population is fixed and there are only two possible states, i.e., $\Xi = \{\text{"Susceptible"}, \text{"Infected"}\}$, (4) is sufficient in describing the state transitions of the population. The mean dynamic describing the evolution of individual strategies (3) will be specified later when we discuss the stability of the game in Sect. 4.4.

Players with degree p who choose strategy $s_i^p \in \mathcal{S}^p$, $i \in \mathcal{I}^p$, $p \in \mathcal{P}$ have payoff F_i^p, which depends on the information broadcast at the time of sampling, i.e., $I_i^p(t)$, as defined in Sect. 3.1. In the context of the epidemic, it takes the following form:

$$F_i^p = s_i^p \mathcal{U}_{\text{ina}} + (1 - s_i^p)\mathcal{U}_{\text{act}}^{p,i}, \tag{6}$$

where $\mathcal{U}_{\text{act}}^{p,i} : [0, 1] \to \mathbb{R}_+$ and $\mathcal{U}_{\text{ina}} \in \mathbb{R}$. In (6), $s_i^p \mathcal{U}_{\text{ina}}$ represents the expected utility of being socially inactive; $(1 - s_i^p)\mathcal{U}_{\text{act}}^{p,i}$ represents the expected utility of being socially active.

The function $\mathcal{U}_{\text{act}}^{p,i}, \forall i \in \mathcal{I}^p, \forall p \in \mathcal{P}$ corresponds to the reward from getting infected through physical interactions on the network. Therefore, we let $\mathcal{U}_{\text{act}}^{p,i}$ be decreasing in $\eta_i^p \in [0, 1]$, which represents the probability that a player in population p playing strategy s_i^p is infected. The probability that a player is infected can be equivalently understood as the fraction of players within the same statistical equivalent class who are infected. Thus, we obtain $\eta_i^p = \mathcal{O}_i^p(I_i^p(t))$, where $\mathcal{O}_i^p : [0, 1] \to [0, 1]$ is a player's observation of the infected fraction of players at the time of an information broadcast. Note that the case of imperfect observations will be discussed in Sect. 4.5. For now, we consider the case of perfect observations, i.e., $\eta_i^p = I_i^p(t)$. Since the evolution (4) is coupled and the term (5) depends on the herd behavior x, the payoff satisfies the definition in Sect. 3.1. Note that the rate parameter τ determines the timescale of the coupled system of differential equations. The sampled epidemic status is at a steady state if $\frac{1}{\tau} \to \infty$ and is time-dependent otherwise.

In this paper, we consider $\mathcal{U}_{\text{act}}^{p,i} = -r_{\text{act}} \eta_i^p$ for all players with reward parameter $r_{\text{act}} \in \mathbb{R}_+$, for simplicity reasons. The term $\mathcal{U}_{\text{ina}}$ corresponds to isolating oneself from others. Hence, we assume that $\mathcal{U}_{\text{ina}}$ is a negative constant reward for all players, i.e., $\mathcal{U}_{\text{ina}} = r_{\text{ina}} \in \mathbb{R}_-$.

By defining $r = \frac{r_{\text{ina}}}{r_{\text{act}}} \in \mathbb{R}_-$ to be the relative reward of being socially inactive against being socially active, we obtain the payoff function suitable under (4) as follows:

$$F_i^p = s_i^p r - (1 - s_i^p)\eta_i^p. \tag{7}$$

Note that we have dropped the dependence of F_i^p and η_i^p on the epidemic state I when we analyze equilibrium behaviors, since the epidemic state is a function that only depends on the social state x, as can be observed in Definition 1.

4 Long-Term Behavior

In this section, we study the long-term behavior of our model under the assumption that $\frac{1}{\tau} \to \infty$. The epidemic dynamics (4) are assumed to reach the steady state more quickly than the herd behaviors. When an information broadcast changes the behavior, the epidemic dynamics would respond to it quickly and reach a steady state before the next information broadcast arrives.

Remark 4 The assumption that the epidemic dynamics evolve fast is for the purpose of investigating herd behaviors when the individuals anticipate the elimination of the virus in the end. This assumption also leads to results related to the relations between stability of epidemic steady states and a variant of the basic reproduction number [29]. Later in Sect. 5, we relax this assumption and extend our results to the setting where the time between information broadcasts is arbitrary.

4.1 Steady States of Epidemic Dynamics Given Social States

From (4) and (5), we obtain the steady state as

$$\bar{I}_i^p = \frac{\theta_i^p \bar{\Theta}}{\gamma + \theta_i^p \bar{\Theta}}, \tag{8}$$

where $\theta_i^p = \lambda p (1 - s_i^p)$, and

$$\bar{\Theta} = \frac{\sum_{p \in \mathcal{P}} \left(p \sum_{i \in \mathcal{I}^p} x_i^p \bar{I}_i^p \right)}{\sum_{p \in \mathcal{P}} p m^p}. \tag{9}$$

Let $\bar{p} := \sum_{p \in \mathcal{P}} p m^p$ denote the average degree of the network. By combining (8) and (9), we obtain the equation containing only $\bar{\Theta}$ as follows:

$$\bar{\Theta} = \bar{p}^{-1} \sum_{p \in \mathcal{P}} \left(p \sum_{i \in \mathcal{I}^p} \frac{x_i^p \theta_i^p \bar{\Theta}}{\gamma + \theta_i^p \bar{\Theta}} \right). \tag{10}$$

From (10), we observe that $\bar{\Theta}_0 = 0$ is always a solution. Accordingly, $\bar{I}_{i,0}^p = 0$ for all d and for all i. We refer to $(\bar{I}_0, \bar{\Theta}_0)$ as the zero steady-state pair. At this steady state, players are uninfected no matter which statistical equivalent class they belong to; and no link leads to an infectious player. The zero steady-state pair is often referred to as the disease-free state [8]. Meanwhile, there exist positive steady states, which arise from dividing $\bar{\Theta}$ from both sides of (10) when $\bar{\Theta} \neq 0$:

$$1 = \bar{p}^{-1} \sum_{p \in \mathcal{P}} \left(p \sum_{i \in \mathcal{I}^p} \frac{x_i^p \theta_i^p}{\gamma + \theta_i^p \bar{\Theta}} \right). \tag{11}$$

In a positive steady-state pair $(\bar{I}_+(x), \bar{\Theta}_+(x))$, we have $\bar{\Theta}_+(x) \in (0, 1]$. It shows that a link possesses a positive probability to connect to an infected node. In addition, $\bar{I}_{i,+}^p(x) = 0$ if and only if $s_i^p = 1$. It explains that a player can be safe from the epidemic only if she lives a totally isolated life. Note that the positive steady-state pair depends on the social state x, since (10) contains x. The next result presents the conditions on the stability of the zero steady-state pair and the positive steady-state pair.

Theorem 1 *Consider the dynamical system in (4) and (5). Given a social state $x \in \mathcal{X}$, the zero steady-state pair $(\bar{I}_0, \bar{\Theta}_0) = (0, 0) \in [0, 1]^n \times [0, 1]$ is globally asymptotically stable if $\frac{\lambda p(1-s_i^p)}{\gamma} < 1$ for all $i \in \mathcal{I}^p$ and all $p \in \mathcal{P}$; the unique positive steady-state pair $(\bar{I}_+(x), \bar{\Theta}_+(x)) \in [0, 1]^n \times [0, 1]$ is globally asymptotically stable if $\frac{\lambda p(1-s_i^p)}{\gamma} \geq 1$ for all $i \in \mathcal{I}^p$ and all $p \in \mathcal{P}$.*

Proof See "Appendix 1." □

The term $\frac{\lambda p(1-s_i^p)}{\gamma}$ can be considered as the activity-aware basic reproduction rate in the setting of complex networks. The activity-awareness, same as in the activity-aware contagion rate λ_i^p, comes from the fact that the contagion rate λ is scaled by the SAI $1 - s_i^p$. This brings individual decisions on the social behaviors into the contagion effect of a virus. The degree p reflexes the influences of the complex network. The crucial population in determining whether $(\bar{I}_+(x), \bar{\Theta}_+(x))$ is globally asymptotically stable is the one with the smallest product $p(1-s_i^p)$. In other words, if the subpopulation with the lease combination of degree and SAI has an activity-aware basic reproduction greater than 1, then the disease cannot be eliminated in the long run. On the contrary, if the subpopulation with the highest combination of degree and SAI has an activity-aware basic reproduction smaller than 1, then eventually we will reach the disease-free steady state.

We focus on the positive steady-state pair $(\bar{I}_+(x), \bar{\Theta}_+(x))$ in the following sections, since it reveals richer properties of the herd behaviors.

4.2 Numerical Computation of the Steady States

Define $M : [0, 1] \to \mathbb{R}$ as $M(z) = \bar{p}^{-1} \sum_{p \in \mathcal{P}} \left[p \sum_{i \in \mathcal{I}^p} \frac{x_i^p \theta_i^p z}{\gamma + \theta_i^p z} \right]$. The computation method to obtain a steady state relies on the next result.

Theorem 2 *The function $M(\cdot)$ is a contraction mapping on $[0, 1]$.*

Proof See "Appendix 2." □

Theorem 2 indicates that the steady state $\bar{\Theta}$ can be obtained by the fixed-point iterations using the mapping $M(\cdot)$.

4.3 Equilibrium Analysis

Before focusing on the NE, we first introduce an alternative interpretation of the population game in Sect. 3.2. Consider an equivalent P-player game where a player with degree p plays a weighted-mixed strategy x^p from the set $\mathcal{S}^p$. By weighted-mixed strategy, we refer to the restriction that $\mathbf{1}^\top x^p = m^p$. Given a social state $x = (x^p, x^{-p})$, where x^{-p} denotes the population states of populations other than population p, the expected payoff of player p playing weighted-mixed strategy x^p is $EF^p(x^p, x^{-p}) := (x^p)^\top F^p((x^p, x^{-p})) = \sum_{i \in \mathcal{I}^p} x_i^p F_i^p((x^p, x^{-p}))$. Inspired by this equivalent game, we present the next result characterizing the NE of our evolutionary game.

Theorem 3 *A social state $x^* \in \mathcal{X}$ is an NE of the game defined in Sect. 3.2 if and only if it solves the following optimization problem:*

$$
\begin{aligned}
min_{x \in \mathcal{X}, y \in \mathbb{R}^P} \quad &\sum_{p \in \mathcal{P}} -EF^p(x^p, x^{-p}) + \sum_{p \in \mathcal{P}} y^p m^p \\
s.t. \quad &-F^p(x) \geq -y^p \mathbf{1}_{n^p}, \quad \forall p \in \mathcal{P}, \\
&x^p \geq 0, \quad \mathbf{1}^T x^p = m^p, \quad \forall p \in \mathcal{P}.
\end{aligned}
\tag{12}
$$

Proof See "Appendix 3." □

The first usage of Theorem 3 is for testing whether a given social state is an NE. The details are presented in "Appendix 3." The second usage is solving for NE. A variety of algorithms can be used to numerically solve the optimization problem (12), for example, see [33]. Among them, gradient-based algorithms are popular among practitioners and have been proved to be effective in practice. At each iteration of a gradient-based algorithm, the descent direction consists of the gradient vectors $\frac{\partial F_i^p}{\partial x}(x)$ for all $i \in \mathcal{I}^p$ and for all $p \in \mathcal{P}$. We provide below the explicit expression of the gradient vector given a social state.

With a slight abuse of notation, we specify the dependence on x by writing the steady-state quantities using $\bar{\Theta}(x)$ and $\bar{I}_i^p(x)$. We express the gradient using the chain rule as $\frac{\partial F_i^p}{\partial x}(x) = -\frac{(1-s_i^p)\gamma \theta_i^p}{(\gamma + \theta_i^p \bar{\Theta}(x))^2} \cdot \frac{\partial \bar{\Theta}}{\partial x}(x)$. Next, we derive the term $\frac{\partial \bar{\Theta}}{\partial x}(x)$ leveraging (11). Define $H : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ by $H(\Theta, x) = (\bar{p})^{-1} \sum_{p \in \mathcal{P}} \left(p \sum_{i \in \mathcal{I}^p} \frac{x_i^p \theta_i^p}{\gamma + \theta_i^p \Theta} \right) - 1$. It is clear from the definition that H is continuously differentiable with respect to both arguments. Suppose, given x^*, the pair (Θ^*, x^*) solves (11), i.e., $H(\Theta^*, x^*) = 0$. The Jacobian of H with respect to the first argument at (Θ^*, x^*) is $J_\Theta(\Theta^*, x^*) := \frac{\partial H}{\partial \Theta}((\Theta^*, x^*)) \in \mathbb{R}$. From the proof of Theorem 1, we know that $\Theta^* > 0$ if x^* is a social state. Hence, $J_\Theta(\Theta^*, x^*) < 0$. Invoking the implicit function theorem, we observe that there exists a neighborhood $\mathcal{V}_\Theta$ of Θ^* and a neighborhood $\mathcal{V}_x$ of x^*, such that there is a unique continuously differentiable function $h : \mathcal{V}_x \to \mathcal{V}_\Theta$ satisfying $h(x^*) = \Theta^*$ and $H(h(x^*), x^*) = 0$. Furthermore, the derivative of $h(\cdot)$ can be expressed as

$$
\frac{\partial h}{\partial x_i^p}(x^*) = -(J_\Theta(h(x^*), x^*))^{-1} \frac{\partial H}{\partial x_i^p}(h(x^*), x^*).
\tag{13}
$$

Thus, the term $\frac{\partial \bar{\Theta}}{\partial x}(x)$ can be obtained directly using (13) at the given social state x. Therefore, the explicit gradient vectors are of the form

$$
\frac{\partial F_i^p}{\partial x}(x) = \frac{(1-s_i^p)\gamma \theta_i^p}{(\gamma + \theta_i^p \bar{\Theta}(x))^2} \left((J_\Theta(\bar{\Theta}(x), x))^{-1} \frac{\partial H}{\partial x}(\bar{\Theta}(x), x) \right),
\tag{14}
$$

where $\bar{\Theta}(x)$ is obtained from the fixed-point iterations using the mapping $M(\cdot)$ and

$$
\frac{\partial H}{\partial x}(\bar{\Theta}(x), x) = \frac{1}{\bar{p}} \left(\frac{1 \cdot \theta_1^1}{\gamma + \theta_1^1 \bar{\Theta}(x)} \quad \cdots \quad \frac{P \cdot \theta_{n_P}^P}{\gamma + \theta_{n_P}^P \bar{\Theta}(x)} \right).
\tag{15}
$$

In general, the optimization problem (12) is nonconvex. However, gradient-based algorithms are still applicable to find stationary points, i.e., points with sufficiently small gradients. Moreover, we know from the proof of Theorem 3 that the global optimal point yields a zero objective value. Therefore, we can test the stationary point obtained using gradient-based algorithms and using the objective value to determine whether it is a potential global optimal point, i.e., an NE social state.

4.4 Long-Term Property of the Game

Structural property of the game Stability studies the structural properties of the games under which sequential plays following specific revision protocols converge to an NE. In this section, we analyze players' incentives to change their strategies when the game is played sequentially.

Let $DF(x) := \frac{d}{dx} F(x) \in \mathbb{R}^{n \times n}$ denote the derivative of the payoffs with respect to the social state. From (14), we can express DF as

$$
DF(x) = \frac{\gamma}{p} (J_\Theta(\bar{\Theta}(x), x))^{-1}
\begin{pmatrix}
\frac{(1-s_1^1)\theta_1^1}{(\gamma+\theta_1^1\bar{\Theta}(x))^2} \\
\vdots \\
\frac{(1-s_{n^P}^P)\theta_{n^P}^P}{(\gamma+\theta_{n^P}^P\bar{\Theta}(x))^2}
\end{pmatrix}
\begin{pmatrix}
\frac{1\cdot\theta_1^1}{\gamma+\theta_1^1\bar{\Theta}(x)} \\
\vdots \\
\frac{P\cdot\theta_{n^P}^P}{\gamma+\theta_{n^P}^P\bar{\Theta}(x)}
\end{pmatrix}^\mathsf{T}.
\tag{16}
$$

In some classes of games, such as potential games and stable games, various evolutionary dynamics show global stability. These games require special structures of the derivative matrix $DF(x)$. Next, we investigate the structural properties of (16).

Theorem 4 *Under the assumption that every information broadcast takes place at the steady state of (4), i.e., $\frac{1}{\tau} \to \infty$, the game defined in Sect. 3.2 is a submodular game.*

Proof Let p and q be two populations in set $\mathcal{P}$. p and q can represent the same population. Let $DF_q^p(x) \in \mathbb{R}^{n^p \times n^q}$ denote the block in (16) corresponding to $\frac{dF^p(x)}{dx^q}$. We obtain the following:

$$
DF_q^p(x) = \gamma (J_\Theta(\bar{\Theta}(x), x))^{-1} \mu^p \cdot (\nu^q)^\mathsf{T},
$$

where

$$
\mu^p = \left(\frac{(1-s_1^p)\theta_1^p}{(\gamma+\theta_i^p\bar{\Theta}(x))^2} \cdots \frac{(1-s_{n^p}^p)\theta_{n^p}^p}{(\gamma+\theta_i^p\bar{\Theta}(x))^2} \right)^\mathsf{T} \in \mathbb{R}^{n^p}, \quad \nu^q = \left(\frac{q\theta_1^q}{\gamma+\theta_1^q\bar{\Theta}(x)} \cdots \frac{q\theta_{n^q}^q}{\gamma+\theta_{n^q}^q\bar{\Theta}(x)} \right)^\mathsf{T} \in \mathbb{R}^{n^q}.
$$

Define the difference matrix

$$
\Sigma^p =
\begin{pmatrix}
-1 & 1 & 0 & \cdots & 0 & 0 \\
0 & -1 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -1 & 1
\end{pmatrix}
\in \mathbb{R}^{(n^p-1) \times n^p}, \quad \forall p \in \mathcal{P}.
$$

Matrix $\Sigma^q \in \mathbb{R}^{(n^q-1) \times n^q}$ is defined in the similar fashion. To prove the submodular property of the payoff functions, we need to show that the inequality $\Sigma^p DF_q^p(x)(\Sigma^q)^\mathsf{T} \leq 0$ holds for all $p, q \in \mathcal{P}$ and for all $x \in \mathcal{X}$. Ignoring positive constant terms, we obtain the following equivalent condition for all $p, q \in \mathcal{P}$,

$$
\begin{pmatrix}
\mu_2^p - \mu_1^p \\
\vdots \\
\mu_{n^d}^p - \mu_{n^p-1}^p
\end{pmatrix}
\left(\nu_2^q - \nu_1^q \cdots \nu_{n^q}^q - \nu_{n^q-1}^q \right) \geq 0,
$$

where the dependence on x is through μ^p and ν^q. A sufficient condition is

$$
\left(\frac{(1 - s_{k+1}^p)\theta_{k+1}^p}{(\gamma + \theta_{k+1}^p\bar{\Theta}(x))^2} - \frac{(1 - s_k^p)\theta_k^p}{(\gamma + \theta_k^p\bar{\Theta}(x))^2} \right) \cdot \left(\frac{q\theta_{l+1}^q}{\gamma + \theta_{l+1}^q\bar{\Theta}(x)} - \frac{q\theta_l^q}{\gamma + \theta_l^q\bar{\Theta}(x)} \right) \geq 0 \tag{17}
$$

for all $p, q \in \mathcal{P}$ and all $k \leq n^p - 1$ and $l \leq n^q - 1$. Since $\theta_i^p = \lambda p(1 - s_i^p), \forall p \in \mathcal{P}$, after substitution, the first two terms in (17) have the form $\frac{\lambda p(1-s_i^p)^2}{(\gamma + \lambda p(1-s_i^p)\bar{\Theta}(x))^2}$, $p \in \mathcal{P}$ and the last two terms have the form $\frac{\lambda p^2(1-s_i^p)}{\gamma + \lambda p(1-s_i^p)\bar{\Theta}(x)}$, $p \in \mathcal{P}$, which are both decreasing in s_i^p on $s_i^p \in [0, 1]$. This proves (17). Therefore, we conclude the results.   □

A straightforward explanation of the above result is that the decisions in our evolutionary game are strategic substitutes; i.e., when a player with degree $p \in \mathcal{P}$ changes her strategy from s_i^p to s_j^p such that $s_i^p < s_j^p, \forall i, j \in \mathcal{I}^p$, other players, say players with degree p', are more likely to choose a strategy closer to $s_0^{p'}$ from the set $\mathcal{S}^{p'}$, and vice versa. This fact is also supported by the observations of the human behaviors under an epidemic. People tend to stay at home when the streets become crowded. There is a higher probability to get infected with a higher social interactivity. On the contrary, people tend to be outdoors if the others choose to stay at home.

Global stability of NE The counterpart to a submodular game is a supermodular game [34], where decisions of players are strategic complements. In supermodular games, increases in strategies of other players result in a relatively higher strategy of a given player. This isotone property of the payoff function makes the best response correspondences of players well-behaved and the best-response dynamics with stochastic perturbation converge to perturbed NE of the game [34]. The behavior of learning dynamics in submodular games is more involved [20]. However, following [43] and [10], we obtain guarantees on the stability of certain learning processes.

Consider the best-response dynamics of the form

$$x_{[k+1]}^p = \min\{m^p \, \mathrm{BR}^p(x_{[k]})\}, \quad \forall p \in \mathcal{P}, \tag{18}$$

where the subscript $[k]$ represents iteration k and $\min\{\cdot\}$ stands for choosing the least component. Let $x_{\min}^p = (m^p, 0, \ldots, 0)^{\mathsf{T}}$ and $x_{\max}^p = (0, \ldots, 0, m^p)^{\mathsf{T}}$ denote the minimal and maximal state of population p. Let $x_{\min} = (x_{\min}^1, \ldots, x_{\min}^P)$ and $x_{\max} = (x_{\max}^1, \ldots, x_{\max}^P)$ denote the minimal and maximal social state. The following result [43] characterizes the stability of the learning process (18).

Corollary 1 *There exists a minimal social state $x_{min}^* \in \mathrm{NE}(F)$ and a maximal social state $x_{max}^* \in \mathrm{NE}(F)$. The best-response dynamics (18) generate a monotonically increasing sequence of social states which converges to x_{min}^* when the initial social state is x_{min}; (18) generates a monotonically decreasing sequence of social states which converges to x_{max}^* when the initial social state is x_{max}.*

The results in Corollary 1 have the following interpretations. The initialization at $x_{\min}$ corresponds to the situation where players pay little attention to potential infections caused by the epidemic. In this scenario, players are at high SAIs and interact actively over the network. Through sequential revisions of strategies, players gradually become aware of the potential risks from physical interactions and they become increasingly careful about their physical interactions with others. Hence, the sequence generated by (18) starting from $x_{\min}$ is increasing. The convergence to $x_{\min}^*$ shows that by naively best responding to current payoffs, the population can eventually reach a point where no one has an incentive to further revise her strategies. On the contrary, the scenario where the starting point is $x_{\max}$ indicates cautious plays at the beginning, since players have no information about the potential consequences of the epidemic. Through sequential plays, players know more about the epidemic and they become more audacious, i.e., the subsequent social states generated by (18) after $x_{\max}$ are

decreasing. And finally, there is a point where no one is willing to take more risks (e.g., going to the supermarket without wearing a mask).

Note that the maximal and the minimal points $x^*_{\max}$ and $x^*_{\min}$ do not, in general, correspond to the equilibrium points where the payoffs of players reach the maximum and the minimum, respectively [10]. Corollary 1 enables the monotone convergence to NE of the evolutionary dynamics of the form:

$$\dot{x}^p = \min\{m^p \operatorname{BR}^p(x)\} - x^p, \quad \forall p \in \mathcal{P}, \tag{19}$$

where $\min\{\cdot\}$ selects the least element from a set. The evolutionary dynamics (19) is a result of choosing $R^p_{ij} = \mathbf{1}_{\{j=\arg\max_{l\in\mathcal{I}^p} F^p_l(x)\}}$ in (3). Note that there are other choices of R^p_{ij} which lead to different forms of the mean dynamic (3). For example, setting $R^p_{ij} = x^p_j [F^p_j(x) - F^p_i(x)]_+$ leads to the classic replicator dynamics. The reason why (19) converges to NE lies in the discretization of (19): $x^p(t + \delta) = \delta\min\{m^p \operatorname{BR}^p(x(t))\} + (1 - \delta)x^p(t)$, which has the interpretation that in a small period δ, only δ portion of the population revises their strategies to the one obtained using the best responses. The updates (18) correspond to $\delta = 1$. Suppose that t_k and t_{k+1} are two time instances corresponding to iteration $[k]$ and $[k + 1]$ in (18). Since starting from $x_{\min}$, (18) yields $x^p_{[k]} \leq x^p_{[k+1]}$, and for any $\delta \in (0, 1)$, $x^p(t_k + \delta) = \delta\min\{m^p \operatorname{BR}^p(x(t_k))\} + (1 - \delta)x^p(t_k) = \delta x^p_{[k+1]} + (1 - \delta)x^p_{[k]}$. Then, $x^p_{[k]} \leq x^p(t_k + \delta) \leq x^p_{[k+1]}$. If we pick $\delta, \delta_1, \ldots, \delta_{\mathrm{end}} \in [0, 1]$ such that $0 < \delta_1 < \delta_2 < \cdots < \delta_{\mathrm{end}} < 1$, the same relation follows: $x^p_{[k]} \leq x^p(t_k + \delta_1) \leq \cdots \leq x^p(t_k + \delta_{\mathrm{end}}) \leq x^p_{[k+1]}$. Therefore, the discretization of (19) is monotone between t_k and t_{k+1} for arbitrary choices of an increasing sequence of δ. This shows the monotonicity of (19) and its convergence to the NE from $x_{\min}$, when we let $\delta \to 0$. The scenario where the starting point is $x_{\max}$ follows the similar reasoning.

Local stability of NE The concept of ESS is often considered as a refinement of NE in the context of evolutionary games. ESS captures the local stability of a given equilibrium herd behavior when it is invaded by different aggregate behaviors played by a small group of players. Since we consider multiple populations, the definition of Taylor ESS [34] is one of the suitable extensions of ESS originally considered in [36]. Consider an equilibrium herd behavior $x^* \in \operatorname{NE}(F)$. Then x^* is Taylor ESS if and only if there exists a neighborhood $\mathcal{N}$ of x^* such that for all $x \in \mathcal{N} - \{x^*\}$, $(x - x^*)^T F(x^*) = 0$ implies that $(x - x^*)^T F(x) < 0$. This definition directly leads to the following corollary.

Corollary 2 *A social state $x^* \in \operatorname{NE}(F)$ is Taylor ESS on its neighborhood $\mathcal{N}$ if and only if the following condition holds:*

$$max_{x\in\operatorname{NE}(F)\cap(\mathcal{N}-\{x^*\})}(x - x^*)^T F(x) < 0. \tag{20}$$

The condition $x \in \operatorname{NE}(F)$ is equivalent to $(x - x^*)^T F(x^*) = 0$ and can be checked using (12). In Corollary 2, we restrict the definition of ESS to a given neighborhood $\mathcal{N}$. The reason lies in that the equilibrium points in $\mathcal{N}$ can be found by solving (12) restricted on $\mathcal{N} \subset \mathcal{X}$ and condition (20) can be check at these points. If we seek the existence of the neighborhood $\mathcal{N}$, we can check the condition $(x - x^*)^T F(x) < 0$ for all x which is greater than the minimal equilibrium point $x^*_{\min}$ and smaller than the maximal equilibrium point $x^*_{\max}$.

4.5 Misinformation Broadcasting

Information plays an important role in shaping human behaviors. In the information broadcast, the media can control the strategies of players through the design of the information.

In this subsection, we investigate manipulations on players' observations of the status of the epidemic. For simplicity reasons, we assume that the information broadcast only contains $\bar{\Theta}$, which represents the average probability that a link on the network connects to an infected node, at a given social state. This assumption can be understood as the total infected number of people reported by the news. In addition, we assume that the strategy sets $\mathcal{S}^p$ are identical for all $p \in \mathcal{P}$ with the minimal element denoted by $s_{\min}$ and the maximal element denoted by $s_{\max}$.

Define $\mathcal{F}^p : [s_{\min}, s_{\max}] \to \mathbb{R}$ by $\mathcal{F}^p(s) = sr - (1-s)\eta_i^p = sr - (1-s)\mathcal{O}_i^p(\bar{I}_i^p)$ for all $p \in \mathcal{P}$. The function $\mathcal{F}^p(s)$ extends the payoff of players with degree $p \in \mathcal{P}$ to a continuous function defined on the continuum $[s_{\min}, s_{\max}]$. This extension helps analyze the properties of the payoff when the strategies are perturbed.

Combining (7) and (8), we obtain

$$\mathcal{F}^p(s) = sr - (1-s)\frac{\lambda p(1-s)\bar{\Theta}}{\gamma + \lambda p(1-s)\bar{\Theta}}.$$

The derivative of $\mathcal{F}^p$ is

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{F}^p(s) = r + \frac{2\bar{\Theta}\frac{\lambda p(1-s)}{\gamma} + \bar{\Theta}^2(\frac{\lambda p(1-s)}{\gamma})^2}{(1 + \bar{\Theta}\frac{\lambda p(1-s)}{\gamma})^2}. \tag{21}$$

The existence of dominant strategies depends on the sign of $\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{F}^p(s)$. We observe that $\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{F}^p(s)$ is a strictly increasing function of $\bar{\Theta}$ under the condition that $\frac{\lambda p(1-s_i^p)}{\gamma} \geq 1$ for all $i \in \mathcal{I}^p$ and all $p \in \mathcal{P}$. Therefore, the smallest value of $\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{F}^p(s)$ appears when $\bar{\Theta}$ approaches 0 and the largest appears when $\bar{\Theta}$ approaches 1. The following result presents the conditions on the value of the relative reward r for achieving dominant strategies.

Theorem 5 *Under the assumption that $\frac{\lambda p(1-s_i^p)}{\gamma} \geq 1$ for all $i \in \mathcal{I}^p$ and all $p \in \mathcal{P}$, $s_{\min}$ is dominant for all $p \in \mathcal{P}$ if the following inequality holds:*

$$|r| \geq 1 - \frac{1}{\left(1 + \frac{\lambda p(1-s_{\min})}{\gamma}\right)^2}, \quad \forall p \in \mathcal{P}. \tag{22}$$

Proof From (21), we know that the minimal value of the second term on the right-hand side is 0 when $\bar{\Theta} = 0$. In this case, $\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{F}^p(s) \leq 0$ since r is negative. Therefore, dominant strategies for all players can only appear if $\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{F}^p(s) \leq 0$ when the second term on the right-hand side of (21) takes the maximal value. Hence, by requiring (21) to be negative when $\bar{\Theta} = 1$, we arrive at (22). $\qquad \square$

According to the above result, when the relative reward r satisfies (22), a player chooses $s_{\min}$ no matter what she observes from the information broadcast. We regard the relative reward satisfying the equality in (22) as the critical relative reward and denote it by r_{crit}. Condition (22) is relatively demanding, since it requires the reward perceptions of all players in the populations to go to one extreme, i.e., the reward of socially inactive is low whatever the status of the epidemic is. However, with $s_{\min}$ being a potential dominant strategy, the information broadcaster can make the condition on r for enabling dominant strategies less restrictive by taking advantage of $\bar{\Theta}$. Specifically, by misreporting $\bar{\Theta}$ with a value $\tilde{\Theta}$ satisfying $0 \leq \tilde{\Theta} < \bar{\Theta}$ in every information broadcast, the information broadcaster makes $s_{\min}$ dominant for all players even if $|r| < |r_{\text{crit}}|$. This shows the destructive impact of misinformation. Indeed, even when a player possesses a reward perception that social inactivity

during the epidemic is acceptable, if the media consecutively underreport the epidemic, this player would underestimate possible consequences of infection and become highly socially active, i.e., playing strategy $s_{\min}$. As a consequence, few infections cause regional outbreaks of the epidemic, and the epidemic eventually becomes a pandemic.

5 Time-Dependent Behavior

In Sect. 4, we have assumed that the epidemic dynamics evolve at a faster timescale. In this section, we investigate our framework at a different timescale. We first present a result analogous to that of Sect. 4.4 to make connections with the rate-τ exponential distribution that controls the times between the information broadcasts. Then, we show that the essence of the framework is maintained when we consider the approximate time-dependent epidemic dynamics.

5.1 Time-Dependent Property of the Game

We emphasize the time dependence of the probability of infection using $\eta_i^p(\Delta t)$, where Δt denotes the time between information broadcasts. We rewrite the payoffs of players as:

$$F_i^p = s_i^p r - (1 - s_i^p)\eta_i^p(\Delta t). \tag{23}$$

The next result is the time-dependent counterpart of Theorem 4.

Theorem 6 *Under the assumption that all the players have the same probability to be infected at the beginning, i.e., $I_i^p(0) = i_0, \forall i \in \mathcal{I}^p, \forall p \in \mathcal{P}$, the game defined in Sect. 3.2 with payoff functions (23) is a submodular game for any time $\Delta t > 0$ between information broadcasts.*

Proof We first show that $I_i^p(t)$ are ordered when $I_i^p(0) = i_0, \forall i \in \mathcal{I}^p, \forall p \in \mathcal{P}$. From (4), we observe that the distinct term is $\lambda_i^p = \lambda(1 - s_i^p)$. For all $i, j \in \mathcal{I}^p$ such that $i > j$ and $s_i^p, s_j^p \in \mathcal{S}^p$, since $s_i^p > s_j^p$, $\dot{I}_i^p(t)$ is an upper bound of $\dot{I}_j^p(t)$ for all $p \in \mathcal{P}$. At an arbitrary time $t > 0$, we obtain the expression

$$I_i^p(t) = I_i^p(0) + \int_0^t -\gamma I_i^p(t) + \lambda(1 - s_i^p)(1 - I_i^p(t))p\Theta(t)\mathrm{d}t.$$

Since $I_i^p(0) = i_0, \forall i \in \mathcal{I}^p, \forall p \in \mathcal{P}$, we conclude that $I_j^p(t) > I_i^p(t)$ for all $t > 0$ and $p \in \mathcal{P}$ if $i > j$ using arguments in [4]. Note that the social state x only appears in the expression of $\Theta(t)$ and $\Theta(t)$ stays the same for all $s_i^p \in \mathcal{S}^p$ and $p \in \mathcal{P}$. Therefore, the relation $I_j^p(t) > I_i^p(t)$ holds when the social state x evolves.

Next, let p and q be two populations in set $\mathcal{P}$. We derive the structural properties of the matrix $DF_q^p(x) := \frac{\partial F_i^p}{\partial x_j^q}$ with the payoff (23). We focus on the time period $[t, t + \Delta t]$. The evolution of the epidemic dynamics for population p is

$$I_i^p(t + \Delta t) = I_i^p(t) + \int_t^{t+\Delta t} -\gamma I_i^p(t) + \lambda(1 - s_i^p)(1 - I_i^p(t))p\Theta(t)\mathrm{d}t. \tag{24}$$

The only term that depends on the social state in (24) is $\Theta(t)$. Hence, the derivative of the payoff (23) with respect to the social state is

$$\frac{\partial F_i^p}{\partial x_j^q} = -\frac{\lambda}{\bar{p}}pq\int_t^{t+\Delta t}[(1 - s_i^p)^2(1 - I_i^p(t))][I_j^q(t)]\mathrm{d}t. \tag{25}$$

The matrix $DF_q^p(x)$ takes (25) as the element on the row corresponding to strategy $s_i^p \in \mathcal{S}^p$ of population p and the column corresponding to strategy $s_j^q \in \mathcal{S}^q$ of population q. We use the same difference matrices Σ^p and Σ^q in the proof of Theorem 4. Our objective is to show that $\Sigma^p DF_q^p(x)(\Sigma^q)^\mathsf{T} \leq 0$ for all $p, q \in \mathcal{P}$. We arrive at the expression of the element in $\Sigma^p DF_q^p(x)(\Sigma^q)^\mathsf{T}$ on row i column j as follows:

$$[\Sigma^p DF_q^p(x)(\Sigma^q)^\mathsf{T}]_{i,j}^{p,q} = -\frac{\lambda pq}{\bar{p}}(\Upsilon_{i+1,j+1}^{p,q} - \Upsilon_{i,j+1}^{p,q} - \Upsilon_{i+1,j}^{p,q} + \Upsilon_{i,j}^{p,q}), \tag{26}$$

where $\Upsilon_{i,j}^{p,q} = \int_t^{t+\Delta t}[(1-s_i^p)^2(1-I_i^p(t))][I_j^q(t)]\mathrm{d}t$. After simplification, we obtain

$$\begin{aligned}
&\Upsilon_{i+1,j+1}^{p,q} - \Upsilon_{i,j+1}^{p,q} - \Upsilon_{i+1,j}^{p,q} + \Upsilon_{i,j}^{p,q} \\
&= \int_t^{t+\Delta t}[I_{j+1}^q(t) - I_j^q(t)][(1-s_{i+1}^p)^2(1-I_{i+1}^p(t)) - (1-s_i^p)^2(1-I_i^p(t))]\mathrm{d}t.
\end{aligned} \tag{27}$$

Then, it suffices to prove that (27) is nonnegative.

A sufficient condition for (27) to be nonnegative is that the integrand in (27) is nonnegative on the interval $[t, t+\Delta t]$. We have shown that $I_j^p(t) > I_i^p(t)$ when $i > j$ for all $p \in \mathcal{P}$. Hence, it suffices to show that $[(1-s_{i+1}^p)^2(1-I_{i+1}^p(t)) - (1-s_i^p)^2(1-I_i^p(t))]$ is negative. Note that by rearranging the differential equation (4), for any population $p \in \mathcal{P}$, we arrive at

$$\frac{\dot{I}_i^p(t) + \gamma I_i^p(t)}{\lambda p\Theta(t)} = (1-s_i^p)(1-I_i^p(t)). \tag{28}$$

Multiplying $1 - s_i^p$ on both sides of (28), we obtain

$$\frac{\dot{I}_i^p(t) + \gamma I_i^p(t)}{\lambda p\Theta(t)}(1-s_i^p) = (1-s_i^p)^2(1-I_i^p(t)). \tag{29}$$

By (29), it suffices to prove that the left-hand side of (29) is decreasing with respect to i. We have shown that $\dot{I}_j^p(t)$ upper bounds $\dot{I}_i^p(t)$ and $I_j^p(t) > I_i^p(t)$ if $i > j$. In addition, the strategies in the set $\mathcal{S}^p$ follow an increasing order. Therefore, we conclude that the left-hand side of (29) decreases as i increases. This completes the proof. $\qquad\square$

We remark that in the proof of Theorem 6, the time Δt is arbitrary. This suggests the possibility of different time intervals between two information broadcasts. Hence, we can assume that the times between the information broadcasts are independent and follow a rate τ exponential distribution, which is consistent with the settings described in Sect. 3.

With Theorem 6, the convergence result in Corollary 1 can be extended to the setting where the epidemic evolves for an arbitrary time between any two information broadcasts. Indeed, after the learning procedure defined in (18) does not result in new social states given additional information broadcasts, the epidemic will gradually converge to the unique positive steady state associated with the current social state as $t \to \infty$. This leads to Nash equilibria defined in Definition 1.

5.2 Equivalent Networks and Approximations

In general, the time-dependent behaviors depend on the solution of a system of nonlinear differential equations. Here, we study it using approximations.

Combining (4) and (5), we obtain

$$
\begin{aligned}
\dot{I}_i^p(t) &= -\gamma I_i^p(t) + \lambda p(1 - s_i^p)\left[1 - I_i^p(t)\right] \frac{\sum_{p \in \mathcal{P}} \sum_{i \in \mathcal{I}^p} p x_i^p I_i^p(t)}{\sum_{p \in \mathcal{P}} p m^p} \\
&= -\gamma I_i^p(t) + \lambda \left[1 - I_i^p(t)\right] \frac{p(1 - s_i^p) \sum_{p \in \mathcal{P}} \sum_{i \in \mathcal{I}^p} p x_i^p I_i^p(t)}{\bar{p}} \\
&= -\gamma I_i^p(t) + \lambda \left[1 - I_i^p(t)\right] \sum_{p' \in \mathcal{P}, i' \in \mathcal{I}^{p'}} \tilde{A}_{(p,i),(p',i')} I_{i'}^{p'}(t),
\end{aligned}
\tag{30}
$$

where $\tilde{A}_{(p,i),(p',i')} = \frac{p(1-s_i^p)p' x_{i'}^{p'}}{\bar{p}}$ denotes an entry in the matrix $\tilde{A} \in \mathbb{R}^{n \times n}$, with (p, i) representing the row index and (p', i') representing the column index. We use a pair (p, i) to represent an index to emphasize that this index is associated with population $p \in \mathcal{P}$ and strategy $s_i^p \in \mathcal{S}^p$, $i \in \mathcal{I}^p$. The matrix $\tilde{A}$ acts as an equivalent adjacency matrix if we regard our epidemic dynamics as a dynamical system on a small network with n nodes. The entry $\tilde{A}_{(p,i),(p',i')}$ stands for the weight on the link from node (p, i) to node (p', i'). In general, matrix $\tilde{A}$ is asymmetric, indicating that the equivalent network is directed. Since we have interpreted the population game as a P-player game in Sect. 4.3, the interactions among the P players can be captured by the small network.

To analyze the time-dependent behavior of the epidemics, we ignore the quadratic terms in the equivalent dynamics and arrive at:

$$
\dot{I}_i^p(t) \simeq -\gamma I_i^p(t) + \lambda \sum_{p' \in \mathcal{P}, i' \in \mathcal{I}^{p'}} \tilde{A}_{(p,i),(p',i')} I_{i'}^{p'}(t).
$$

Combining the two terms and rewriting it in matrix form, we obtain

$$
\dot{I}(t) \simeq \lambda A I(t),
\tag{31}
$$

where $A = \tilde{A} - \frac{\gamma}{\lambda} \Lambda \in \mathbb{R}^{n \times n}$ and Λ represents the identity matrix. Let v_k and κ_k denote the kth eigenvector and eigenvalue of matrix A. Then, we can express $I(t)$ as a linear combination of the eigenvectors as

$$
I(t) \simeq \sum_{k=1}^{n} \alpha_k(t) v_k,
$$

where $\alpha_k(t)$ is a time-dependent parameter associated with v_k. To solve for $\alpha_k(t)$, we use the following equation:

$$
\sum_{k=1}^{n} \dot{\alpha}_k(t) v_k = \dot{I}(t) = \lambda A I(t) = \lambda A \sum_{k=1}^{n} \alpha_k(t) v_k = \lambda \sum_{k=1}^{n} \kappa_k \alpha_k(t) v_k.
$$

Hence,

$$
\dot{\alpha}_k(t) = \lambda \kappa_k \alpha_k(t).
\tag{32}
$$

The differential equation (32) has solutions $\alpha_k(t) = \alpha_k(0) e^{\lambda \kappa_k t}$. Therefore, we obtain the time-dependent behavior of the epidemic:

$$
I(t) \simeq \sum_{k=1}^{n} \alpha_k(0) e^{\lambda \kappa_k t} v_k.
\tag{33}
$$

The exponential term determines the growth speed of $I(t)$. Hence, the largest eigenvalue κ_1 corresponds to the fastest evolution component.

In general, (33) can be approximated using $I(t) \simeq \alpha_1(0)e^{\lambda\kappa_1 t}v_1$ by assuming that the largest eigenvalue is significantly greater than the second largest eigenvalue. In our case, we can leverage the structural properties of matrix $\tilde{A}$ to justify the approximation given by

$$I(t) \simeq \alpha_1(0)e^{\lambda\kappa_1 t}v_1. \tag{34}$$

From the definition of $\tilde{A}_{(p,i),(p',i')}$, we obtain

$$\tilde{A} = \frac{1}{\bar{p}}\begin{pmatrix} 1 \cdot (1-s_1^1) \\ \vdots \\ D(1-s_{n^P}^P) \end{pmatrix} \cdot \begin{pmatrix} 1 \cdot x_1^1 & \cdots & Px_{n^P}^P \end{pmatrix}.$$

This shows that matrix $\tilde{A}$ has rank 1. Furthermore, $\tilde{A}$ has a nonnegative eigenvalue $\tilde{\kappa}_1 = \bar{p}^{-1}\sum_{p\in\mathcal{P}}\sum_{i\in\mathcal{I}^p}p^2(1-s_i^p)x_i^p$ of order 1 associated with the eigenvector $\tilde{v}_1 = \begin{pmatrix} 1 \cdot (1-s_1^1) & \cdots & P(1-s_{n^P}^P) \end{pmatrix}^\mathsf{T}$, and an eigenvalue 0 of order $n-1$ associated with the eigenvectors $\tilde{v}_k \in Null(\tilde{A})$ if $k \neq 1$. Since $A = \tilde{A} - \frac{\gamma}{\lambda}\Lambda$, we arrive at $\kappa_1 = \tilde{\kappa}_1 - \frac{\gamma}{\lambda}$ associated with $v_1 = \tilde{v}_1$, and $\kappa_k = -\frac{\gamma}{\lambda}$ associated with $v_k = \tilde{v}_k$ if $k \neq 1$. Consider the conditions $\frac{\lambda p(1-s_i^p)}{\gamma} \geq 1, \forall i \in \mathcal{I}^p, \forall p \in \mathcal{P}$ which we have assumed in Theorem 1 to obtain the positive steady state. Then, we obtain $\tilde{\kappa}_1 \geq \bar{p}^{-1}\frac{\gamma}{\lambda}\sum_{p\in\mathcal{P}}\sum_{i\in\mathcal{I}^p}px_i^p = \frac{\gamma}{\lambda}$. Hence, $\kappa_1 \geq 0$. This shows that the largest eigenvalue is nonnegative when $\frac{\lambda p(1-s_i^p)}{\gamma} \geq 1$. In addition, we know that $\kappa_k = -\frac{\gamma}{\lambda} < 0$ for all $k \neq 1$. Therefore, the initial values $\alpha_k(0)$ in (33) decays exponentially in time if $k \neq 1$. This justifies the approximation (34).

Some of the properties of our framework introduced in the previous sections depends on the submodularity of the game. The approximation (34) does not break this structural feature. Consider the derivative of the payoff for two populations $p, q \in \mathcal{P}$ under the approximation (34):

$$\frac{\partial}{\partial x_j^q}F_i^p(x,t) = -\alpha_1(0)\lambda t e^{\lambda\kappa_1 t}p(1-s_i^d)^2q^2(1-s_j^c). \tag{35}$$

In (35), both $p(1-s_i^p)^2$ and $q^2(1-s_j^q)$ are decreasing in $s_i^p \in [0,1]$ and $s_j^q \in [0,1]$, respectively. Hence, we can show that the inequality $\Sigma^p DF_q^p(x)(\Sigma^q)^\mathsf{T} \leq 0$ holds when $DF_q^p(x)$ is based on (35). Therefore, after the approximation, players' decisions remain strategic substitutes. This shows the structure-preserving property of the approximation (34). As a result, simulations of the real epidemics can be done using our framework with (34) instead of the original epidemic dynamics over complex networks. In Sect. 6.1, we provide numerical results supporting the applicability of (34).

6 Numerical Experiments

In this section, we present the experiment results to corroborate the previous analysis and the effectiveness of our model. To make the presentation more clear, we set the number of populations $P = 2$ and set the number of strategies $n^1 = 3$ and $n^2 = 2$. We choose a scale-free network [30] where the degree distribution $[m^p]_{p\in\mathcal{P}}$ follows a power law. The strategy spaces for each population are set as $\mathcal{S}^1 = \{0.1, 0.3, 0.5\}$ and $\mathcal{S}^2 = \{0.3, 0.6\}$. To
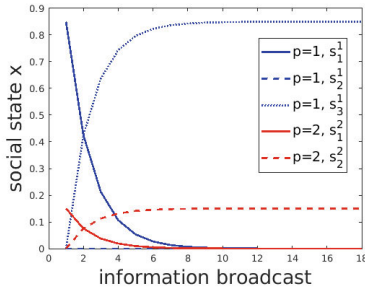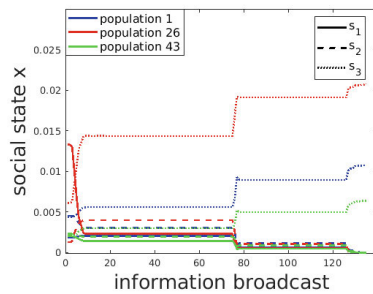
(a) Using best-response dynamics (18).
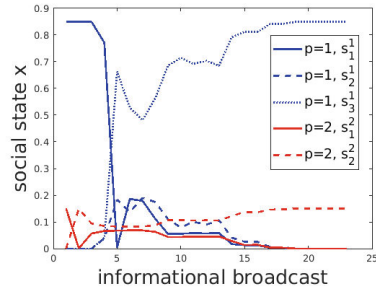
(b) Solving the optimization problem (12).
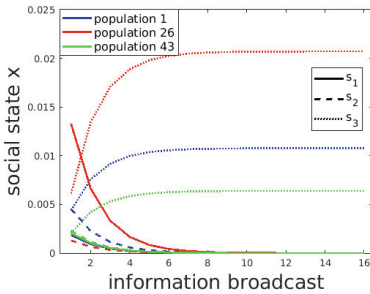
(c) Best response dynamics (18) for 50 populations.

(d) Optimization (12) for 50 populations.

Fig. 3 Convergence to the NE using different approaches

avoid trivial cases where dominant strategies and the zero steady state exist, we select the relative reward $r = -0.1$. We set $\frac{\gamma}{\lambda} = 0.4$ to simulate an epidemic of high infectiousness. We also normalize the timescale for information broadcast to 1 in the experiment. To simulate the continuous best response dynamics (19), we take $\delta = 0.5$ for discretization. Note that our framework is effective for modeling variants of infectious diseases by changing the parameter $\frac{\gamma}{\lambda}$, which is the reciprocal of the basic reproduction rate of a given virus. With the knowledge of p, s_i^p and $\frac{\gamma}{\lambda}$, we can calculate the activity-aware basic reproduction rate defined by $\frac{\lambda p(1-s_i^p)}{\gamma}$.

6.1 Convergence to Nash Equilibrium

In Fig. 3a, b, the best-response dynamics (18) and the iterations of the optimization problem (12) converge to the same NE. From the evolution of the social state, we observe that the best-response dynamics provide a smoother learning process where populations gradually learn the equilibrium strategies through sequential strategic interactions. This is due to the monotonicity of the updates of the social states in (18) under the submodular property of the game. The learning process provided by the optimization problem (12) is not as well-shaped as the one given by the best-response dynamics. However, the optimization problem provides a different learning approach containing a global objective for all players, differing from merely considering individual player's myopic reactions toward the payoff realizations.
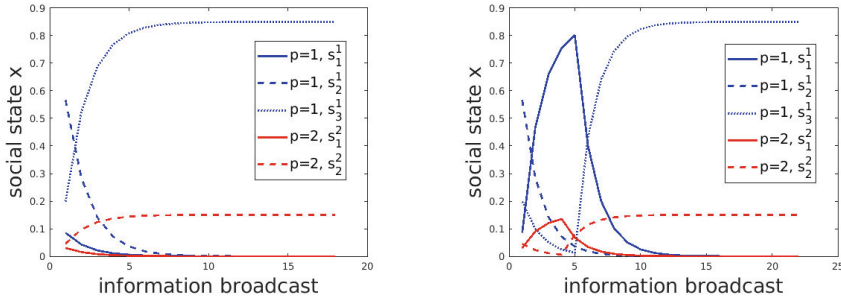
(a) Information broadcasts take place at the steady states of the epidemic dynamics.

(b) Information broadcasts take place before the epidemic dynamics reach the steady states under the linearly approximated epidemic model.

Fig. 4 Study of the convergences of best-response dynamics under different scenarios
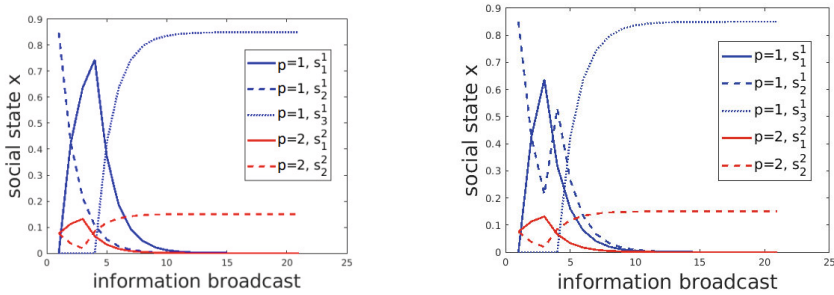
(a) Information broadcasts take place before the epidemic dynamics reach the steady states.

(b) Delaying the information broadcasts.

Fig. 5 Effect of delaying the information broadcasts

We interpret the optimization of the problem (12) as an objective-guided evolutionary process describing strategy revisions in the population, with the explicit gradients (14) acting as the incentives of the players in revising their strategies. We also experiment the best-response dynamics (18) and the iterations of the optimization problem (12) for 50 populations in total and observe the convergence. In each population, players have three strategies. We observed from Fig. 3c, d that the two approaches converge to the same NE.

Next, we demonstrate the convergence results using the linear approximation method from Sect. 5. In Fig. 4, we compare the convergence of the best-response dynamics when information broadcasts take place at the steady states of the epidemic dynamics and when information broadcasts take place before the epidemic dynamics reach the steady states. Interestingly, in Fig. 4a, b, the evolution of the social state converges to the same NE, despite the different evolution patterns at the beginning. This result corroborates the approximation methods we have discussed in Sect. 5.2.

We also study the impact of delayed information broadcast to our model. The delays are common in practice. For example, the published infectious data by the authority can be obsoleted for one day or two and cannot reflect the current real situation. Such delay
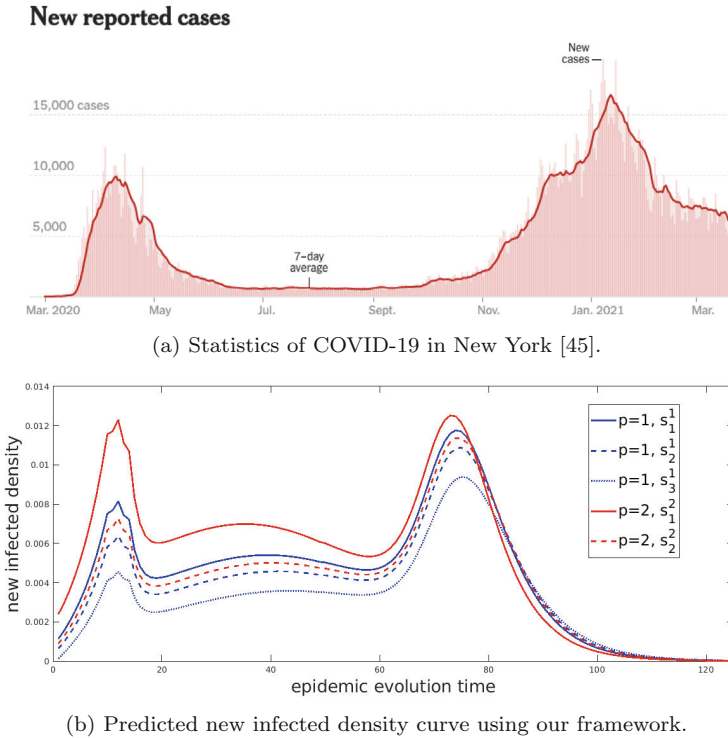
New reported cases

(a) Statistics of COVID-19 in New York [45].

(b) Predicted new infected density curve using our framework.

Fig. 6 Multiple peaks of new infections

may lead to different strategies for fighting against the epidemic. In Fig. 5, we compare the evolutionary processes with and without delays in the information broadcasts. As shown in Fig. 5, the delays in the information broadcasts significantly influence the evolution of the behaviors. The curve corresponding to $p = 1$ and s_2^1 is smoother in Fig. 5a than in b. However, both the processes in Fig. 5a, b converge to the same equilibrium point. Delays in information broadcasts often arise from either the delays in data collection and processing or purposeful deferral of the broadcast. The results suggest that even if the broadcaster has difficulty in obtaining the epidemic status in real time, the out-of-date information can still lead the herd to reach the equilibrium, as long as the reported information is chronological and precise. Therefore, our framework has the potential to guide real-world policy developing, since neither information reporting nor behavioral revision of the herd is perfectly on time.

6.2 Infection Waves Due to Strategy Changes

In Fig. 6, we compare the curve of the reported cases using the real COVID-19 data in New York and the simulated curve of the new infected density using our framework. The multi-peak curve in Fig. 6a shows the 7-day average statistics of new reported cases in New York from March 2020 to March 2021 [42]. The first outbreak corresponds to the initial spreading of the epidemic. The decrease in new cases between July 2020 and November 2020 corresponds to the period when people start to avoid close contacts and social policies are enforced to mitigate the epidemic, such as wearing masks all the time. The second infection
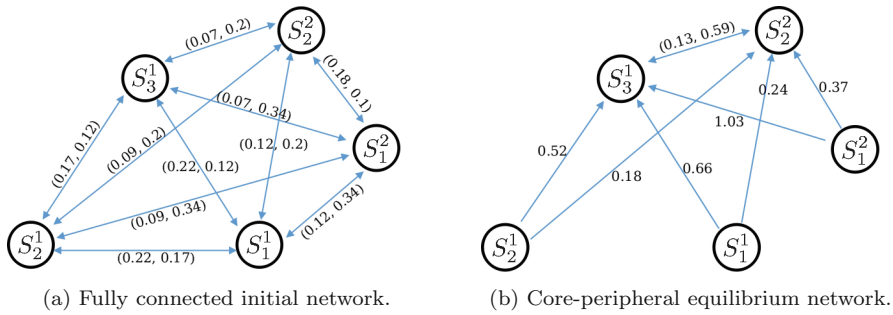
(a) Fully connected initial network.   (b) Core-peripheral equilibrium network.

Fig. 7 Equivalent small networks: the networks are directed graphs. Each link is associated with weights

peak is a consequence of a relaxed social guidance policy [32] and the violations of existing quarantine policies. In Fig. 6b, the prediction of the density of the new infected admits a similar two-peak pattern. The first peak represents the natural evolution of the epidemic when it first appears among populations. At time $t = 50$, the strategy of each population is changed, corresponding to the change of policies in Fig. 6a. This change captures the populations' overconfidence on the epidemic status as the number of new cases decreases. When the populations become less careful, i.e., more people play strategies close to s_1^p in the set $\mathcal{S}^p$, $p \in \mathcal{P}$, the new infected density curve increases and shows a second peak. This second peak is an indication of the influence of the herd behaviors on the epidemic evolution. The new infected density then goes down because the overall infected density has already reached a high level, which decreases the infectious rate. Note that the parameters chosen in the experiments guarantee that the steady state of the epidemic dynamics is positive. This means that though the new infected density in Fig. 6b approaches 0 when the evolution time approaches 120, the total infected density can be significantly large depending on the degree distribution of the network $[m^p]_{p \in \mathcal{P}}$ and the activity-aware basic reproduction rates $\frac{\lambda p (1 - s_i^p)}{\gamma}$.

The comparison in Fig. 6 shows the potential of our framework in assessing risks of real-world infectious diseases.

6.3 Equivalent Networks

Figure 7 illustrates the equivalent small networks discussed in Sect. 5.2. From the randomized initial network in Fig. 7a, we eventually arrive at the equilibrium network in Fig. 7b, which has a special structure, where a few nodes are in the center with others being peripheral nodes. This pattern is often called a core–peripheral structure [13]. In Fig. 7b, the nodes s_3^1 and s_2^2 act as the cores, since all the other nodes have a link pointing toward them but they only have links pointing toward each other. Core–peripheral networks are consequences of network formation games where the directed links represent resource flows [13]. This observation suggests that the evolution of populations' strategies can be statistically equivalent to an N-person sequential network formation game. Each player represents a population in which the individuals are indistinguishable and choose the same strategy.

7 Conclusion

In this paper, we have proposed an evolutionary game framework that couples the state dynamics of the epidemics with the evolution of strategies to study the herd behaviors of the

population over complex networks. We have designed the mechanism containing physical interactions and information broadcasts to combine coupled state transitions over a complex network and sequential strategy revisions in the populations. Taking the epidemic model as a particular case, we have found a unique nontrivial steady state when the evolution of the epidemic evolves at a faster timescale.

We have characterized the Nash equilibrium of the game at the steady state and have explicitly expressed its gradient that facilitates numerical computations and the analysis of structural properties of the game. In addition, we have shown that decisions in the game are strategic substitutes under arbitrary timescales. This observation has enabled simple learning processes to reach the equilibrium point. We have constructed an equivalent small network to represent the complex network and proposed a structure-preserving approximation method that can maintain the strategic substitutes property of the game. Having applied our framework to study the impact of misinformation on epidemics, we have shown that misreports lead to a high social activity intensity, which can exacerbate the spreading of the infectious disease. Our numerical examples have indicated the predictive power of our framework by comparing the simulated dynamics to the real COVID-19 statistics. The multi-peak pattern observed in the case study has shown that the herd behaviors have contributed to multiple outbreaks of an epidemic.

As for future work, we would like to study the stability of the coupled dynamical system in a direct way. We have leveraged the structural property of the game to aid the discussion of the strategy revisions in this work. However, the NE may also be found by directly studying the behavior of the coupled dynamical system in a similar way as in Sect. 4.1 with more advanced mathematical tools. We have adopted the SIS model as a special case in our framework. We would also extend our discussions to other compartmental models to cover infectious diseases of various types. Investigating the role of the information in those models would also be an interesting extension. In addition, the numerical experiments on the delays in the information broadcasts suggest an extension of our framework to scenarios where the populations react to the underlying dynamics with a delay.

Proof of Theorem 1

Proof Suppose that $\frac{\lambda p(1-s_i^p)}{\gamma} < 1$ for all $i \in \mathcal{I}^p$ and for all $p \in \mathcal{P}$. Since $1 - I_i^p(t) \le 1$, we obtain from (4) that: $\frac{\mathrm{d}}{\mathrm{d}t} I_i^p(t) \le -\gamma I_i^p(t) + \lambda(1 - s_i^p)p\Theta(t)$. Then, it suffices to discuss the stability of the system

$$\frac{\mathrm{d}}{\mathrm{d}t} I(t) = \begin{pmatrix} -\gamma I_1^1(t) \\ \vdots \\ -\gamma I_{n^P}^P(t) \end{pmatrix} + \lambda \begin{pmatrix} 1 \cdot (1 - s_1^1) \\ \vdots \\ P \cdot (1 - s_{n^P}^P) \end{pmatrix} \Theta(t).$$

Consider the Lyapunov function $V(t) = \sum_{p \in \mathcal{P}} \sum_{i \in \mathcal{I}^p} b_i^p I_i^p(t)$, where $b_i^p = \frac{p x_i^p}{\gamma} \ge 0$. The time derivative of the Lyapunov function is:

$$\frac{\mathrm{d}}{\mathrm{d}t} V(t) = -\sum_{p \in \mathcal{P}} \sum_{i \in \mathcal{I}^p} p x_i^p I_i^p(t) + \sum_{p \in \mathcal{P}} \sum_{i \in \mathcal{I}^p} \frac{p x_i^p \lambda(1 - s_i^p)p\Theta(t)}{\gamma}$$

$$= -\Theta(t) \sum_{p \in \mathcal{P}} p m^p + \Theta(t) \sum_{p \in \mathcal{P}} \sum_{i \in \mathcal{I}^p} \frac{p^2 x_i^p \lambda(1 - s_i^p)}{\gamma}$$

$$= \Theta(t) \sum_{p \in \mathcal{P}} \left[-pm^p + \frac{p^2 \lambda}{\gamma} \sum_{i \in \mathcal{I}^p} x_i^p (1 - s_i^p) \right].$$

Combining the assumption that $\frac{\lambda p (1 - s_i^p)}{\gamma} < 1$ and the condition that $\sum_{i \in \mathcal{I}^p} x_i^p = m^p$, we conclude that $\frac{d}{dt} V(t) < 0$ when $\Theta(t) \neq 0$. This result shows that the system $\frac{d}{dt} I_i^p(t) = -\gamma I_i^p(t) + \lambda(1 - s_i^p) p \Theta(t)$ converges to 0 as $t \to \infty$. Therefore, the system (4) is globally asymptotically stable at the zero steady state.

Suppose that the opposite condition holds, i.e., $\frac{\lambda p (1 - s_i^p)}{\gamma} \geq 1$ for all $i \in \mathcal{I}^p$ and for all $p \in \mathcal{P}$. We drop the dependence on x of the positive steady-state pair for simplicity. We first show that a solution $\bar{\Theta}_+ \in (0, 1]$ exists for (10). Define $\Psi : [0, 1] \to \mathbb{R}$ as: $\Psi(z) = \sum_{p \in \mathcal{P}} \left[p \sum_{i \in \mathcal{I}^p} \frac{x_i^p \theta_i^p}{\gamma + \theta_i^p z} \right]$. Since $\Psi(z)$ is a strictly decreasing function of z, $\Psi(0)$ achieves the maximum value and $\Psi(1)$ achieves the minimum value of $\Psi(\cdot)$. Under the condition $\frac{\lambda p (1 - s_i^p)}{\gamma} \geq 1$, we obtain the inequality

$$\frac{\theta_i^p}{\gamma} \geq 1 \geq \frac{\theta_i^p}{\gamma + \theta_i^p}.$$

Multiplying by px_i^p and taking the summation over all $i \in \mathcal{I}^p$ and all $p \in \mathcal{P}$, we arrive at

$$\sum_{p \in \mathcal{P}} \left[p \sum_{i \in \mathcal{I}^p} \frac{x_i^p \theta_i^p}{\gamma} \right] \geq \sum_{p \in \mathcal{P}} pm^p \geq \sum_{p \in \mathcal{P}} \left[p \sum_{i \in \mathcal{I}^p} \frac{x_i^p \theta_i^p}{\gamma + \theta_i^p} \right],$$

which is equivalent to $\Psi(0) \geq \bar{p} \geq \Psi(1)$. Hence, there exists $\bar{\Theta}_+ \in (0, 1]$ such that $\Psi(\bar{\Theta}_+) = \bar{p}$. Moreover, $\bar{\Theta}_+$ is unique because $\Psi(\cdot)$ is monotone. Accordingly, every element of $\bar{I}_+$ is positive. Now, we proceed to study the stability of the positive steady-state pair $(\bar{I}_+, \bar{\Theta}_+)$. Define $\phi_i^p = \frac{\lambda(1 - s_i^p)}{\gamma}$. Consider the following equivalent system of (4):

$$\frac{d}{dt} I_i^p(t) = -I_i^p(t) + \phi_i^p p(1 - I_i^p(t)) \Theta(t).$$

Let the density of the susceptible be $U_i^p(t) = 1 - I_i^p(t)$, (5) can be rewritten as

$$\frac{d}{dt} \Theta(t) = \bar{p}^{-1} \sum_{p \in \mathcal{P}} \sum_{i \in \mathcal{I}^p} px_i^p \frac{d}{dt} I_i^p(t)$$

$$= \bar{p}^{-1} \sum_{p \in \mathcal{P}} \sum_{i \in \mathcal{I}^p} px_i^p \left[-I_i^p(t) + \phi_i^p p U_i^p(t) \Theta(t) \right]$$

$$= \Theta(t) \left[\bar{p}^{-1} \sum_{p \in \mathcal{P}} \sum_{i \in \mathcal{I}^p} \phi_i^p U_i^p(t) p^2 x_i^p - 1 \right].$$

Consider the following Lyapunov function for the equivalent dynamical systems above: $V(t) = \frac{1}{2} \sum_{p \in \mathcal{P}} \sum_{i \in \mathcal{I}^p} \left[b_i^p (U_i^p(t) - \bar{U}_i^p)^2 \right] + \Theta(t) - \bar{\Theta} - \bar{\Theta} \ln \frac{\Theta(t)}{\bar{\Theta}}$, where the parameters b_i^p are defined as $b_i^p = \frac{px_i^p}{\bar{p} \bar{U}_i^p}$, and the term $\bar{U}_i^p$ denotes the steady-state quantity of $U_i^p(t)$. The time derivative of $V(t)$ is

$$\frac{\mathrm{d}}{\mathrm{d}t} V(t) = \sum_{p\in\mathcal{P}} \sum_{i\in\mathcal{I}^p} b_i^p (U_i^p(t) - \bar{U}_i^p) \frac{\mathrm{d}U_i^p(t)}{\mathrm{d}t} + \frac{\Theta(t) - \bar{\Theta}}{\Theta(t)} \cdot \frac{\mathrm{d}\Theta(t)}{\mathrm{d}t}$$

$$= \sum_{p\in\mathcal{P}} \sum_{i\in\mathcal{I}^p} b_i^p (U_i^p(t) - \bar{U}_i^p)(I_i^p(t) - \phi_i^p p U_i^p(t)\Theta(t))$$

$$+ (\Theta(t) - \bar{\Theta}) \left[\frac{\sum_{p\in\mathcal{P}} \sum_{i\in\mathcal{I}^p} \phi_i^p U_i^p(t) p^2 x_i^p}{\bar{p}} - 1 \right].$$

Since $\bar{U}_i^p = 1 - \bar{I}_i^p$ and $\bar{p}^{-1} \sum_{p\in\mathcal{P}} \sum_{i\in\mathcal{I}^p} p^2 x_i^p \phi_i^p \bar{U}_i^p = 1$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} V(t) = \sum_{p\in\mathcal{P}} \sum_{i\in\mathcal{I}^p} b_i^p (U_i^p(t) - \bar{U}_i^p) \left[(I_i^p(t) - \bar{I}_i^p) - \phi_i^p p(U_i^p(t) - \bar{U}_i^p) \right]$$

$$+ (\Theta(t) - \bar{\Theta}) \left[\bar{p}^{-1} \sum_{p\in\mathcal{P}} \sum_{i\in\mathcal{I}^p} \phi_i^p p^2 x_i^p (U_i^p(t) - \bar{U}_i^p) \right]$$

$$= \sum_{p\in\mathcal{P}} \sum_{i\in\mathcal{I}^p} b_i^p \left[(U_i^p(t) - \bar{U}_i^p)(I_i^p(t) - \bar{I}_i^p) \right]$$

$$+ \bar{p}^{-1} \sum_{p\in\mathcal{P}} \sum_{i\in\mathcal{I}^p} \frac{\phi_i^p p^2 x_i^p}{\bar{U}_i^p} (U_i^p(t) - \bar{U}_i^p) \left[U_i^p(t)\Theta(t) - \bar{U}_i^p \bar{\Theta} \right]$$

$$+ \bar{p}^{-1} \sum_{p\in\mathcal{P}} \sum_{i\in\mathcal{I}^p} \phi_i^p p^2 x_i^p \left[(\Theta(t) - \bar{\Theta})(U_i^p(t) - \bar{U}_i^p) \right]$$

$$= -\sum_{p\in\mathcal{P}} \sum_{i\in\mathcal{I}^p} b_i^p (U_i^p(t) - \bar{U}_i^p)^2$$

$$- \bar{p}^{-1} \sum_{p\in\mathcal{P}} \sum_{i\in\mathcal{I}^p} \phi_i^p p^2 x_i^p \Theta(t) \left[\frac{(U_i^p(t))^2}{\bar{U}_i^p} - 2U_i^p(t) + \bar{U}_i^p \right].$$

Since $\forall U_i^p(t) \in [0, 1]$, $\frac{(U_i^p(t))^2}{\bar{U}_i^p} - 2U_i^p(t) + \bar{U}_i^p \geq 0$, we conclude that $\frac{\mathrm{d}}{\mathrm{d}t} V(t) \leq 0$. Therefore, the positive steady-state pair $(\bar{I}_+, \bar{\Theta}_+)$ is globally asymptotically stable.   □

Proof of Theorem 2

Proof Consider the component $\frac{x_i^p \theta_i^p z}{\gamma + x_i^p \theta_i^p z}$. For arbitrary $z_1, z_2 \in [0, 1]$, the following holds:

$$\left| \frac{x_i^p \theta_i^p z_1}{\gamma + x_i^p \theta_i^p z_1} - \frac{x_i^p \theta_i^p z_2}{\gamma + x_i^p \theta_i^p z_2} \right|$$

$$= x_i^p \theta_i^p \left| \frac{\gamma(z_1 - z_2)}{(\gamma + x_i^p \theta_i^p z_1)(\gamma + x_i^p \theta_i^p z_2)} \right|$$

$$= x_i^p \theta_i^p \beta_i^p |z_1 - z_2|,$$

where $\beta_i^p = \frac{1}{(1+\gamma^{-1}x_i^p\theta_i^p z_1)(1+\gamma^{-1}x_i^p\theta_i^p z_2)} < 1$. Then, summing all components, we obtain

$$|M(z_1) - M(z_2)| = \frac{|z_1 - z_2|}{\bar{p}} \left(\sum_{p\in\mathcal{P}} \sum_{i\in\mathcal{I}^p} px_i^p\theta_i^p\beta_i^p \right).$$

Since $\bar{p} = \sum_{p\in\mathcal{P}} pm^p$, $m^p = \sum_{i\in\mathcal{I}^p} x_i^p$, $\theta_i^p \in [0, 1]$, and $\beta_i^p \in (0, 1)$, we conclude that $\bar{p}^{-1}(\sum_{p\in\mathcal{P}} \sum_{i\in\mathcal{I}^p} px_i^p\theta_i^p\beta_i^p) \in (0, 1)$. Therefore, $M(\cdot)$ is a contraction mapping on $[0, 1]$.
□

Proof of Theorem 3

Proof The constraint $-F^p(x) \geq -y^p\mathbf{1}_{n^p}$ leads to

$$-EF^p(x) = -(x^p)^\mathsf{T} F^p(x) \geq -(x^p)^\mathsf{T} y^p\mathbf{1}_{n^p} \geq -y^p m^p.$$

This implies that the objective function is nonnegative, i.e.,

$$\sum_{p\in\mathcal{P}} -EF^p(x) + \sum_{p\in\mathcal{P}} y^p m^p \geq 0.$$

Suppose that $x^* = (x^{1*}, ..., x^{P*})$ is an NE of the population game. Define $y^* = (y^{1*}, ..., y^{P*})$ by $y^{p*} = (m^p)^{-1} EF^p(x^{p*}, x^{-p*})$ for all $p \in \mathcal{P}$. We prove that the pair (x^*, y^*) is an optimal solution to problem (12) by showing that it is feasible and $\sum_{p\in\mathcal{P}} -EF^p(x^*) + \sum_{p\in\mathcal{P}} y^{p*} m^p = 0$. To prove the feasibility of (x^*, y^*), it suffices to prove $-F^p(x^*) \geq -y^{p*}\mathbf{1}_{n^p}$ for all $p \in \mathcal{P}$. Since

$$y^{p*} = (m^p)^{-1}(x^{p*})^\mathsf{T} F^p(x^{p*}, x^{-p*}),$$

we obtain

$$-y^{p*}\mathbf{1}_{n^p} = -(m^p)^{-1}(x^{p*})^\mathsf{T} F^p(x^{p*}, x^{-p*})\mathbf{1}_{n^p}.$$

Then, it suffices to prove

$$-(x^{p*})^\mathsf{T} F^p(x^{p*}, x^{-p*})\mathbf{1}_{n^p} \leq -m^p \begin{pmatrix} (\mathbf{e}_1^p)^\mathsf{T} F^p(x^{p*}, x^{-p*}) \\ \vdots \\ (\mathbf{e}_{n^p}^p)^\mathsf{T} F^p(x^{p*}, x^{-p*}) \end{pmatrix}, \qquad (36)$$

where $\mathbf{e}_i^p \in \mathbb{R}^{n^p}$ is the vector of all zeros except for a 1 at the ith entry for population $p \in \mathcal{P}$. From Definition 1, we know that if $x_i^{p*} > 0$, $s_i^p \in \arg\max_{j\in\mathcal{I}^p} F_j^p(x^*)$. This shows that for all $i \in \mathcal{I}^p$ such that $x_i^{p*} > 0$, the values of $F_i^p(x^{p*}, x^{-p*})$ are all equivalent to $\max_{j\in\mathcal{I}^p} F_j^p(x^*)$. Then, for all i such that $x_i^{p*} > 0$, since $\mathbf{1}^\mathsf{T} x^{p*} = m^p$, equality holds in the ith row of (36). For $j \in \mathcal{I}^p$ such that $x_j^{p*} = 0$, inequality holds in the jth row of (36) since $F_j^p(x^*) \leq \arg\max_{i\in\mathcal{I}^p} F_i^p(x^*)$. Hence, the pair (x^*, y^*) is feasible. From the definition of y^*, we conclude that the objective function is zero under (x^*, y^*). Therefore, (x^*, y^*) solves (12).

Suppose that $\tilde{x} = (\tilde{x}^1, ..., \tilde{x}^P)$ and $\tilde{y} = (\tilde{y}^1, ..., \tilde{y}^P)$ solve (12). Since we have found the pair (x^*, y^*) under which the objective value is zero, the objective value must be zero under

the pair $(\tilde{x}, \tilde{y})$, i.e., $\sum_{p \in \mathcal{P}} -EF^p(\tilde{x}^p, \tilde{x}^{-p}) + \sum_{p \in \mathcal{P}} \tilde{y}^p m^p = 0$. For all x such that $x \geq 0$ and $\mathbf{1}^\mathsf{T} x^p = m^p, \forall p \in \mathcal{P}, -(x^p)^\mathsf{T} F^p(\tilde{x}) \geq -\tilde{y}^p m^p$ holds. This leads to

$$\sum_{p \in \mathcal{P}} -(x^p)^\mathsf{T} F^p(\tilde{x}) \geq \sum_{p \in \mathcal{P}} -\tilde{y}^p m^p = \sum_{p \in \mathcal{P}} -EF^p(\tilde{x}^p, \tilde{x}^{-p})$$
$$= \sum_{p \in \mathcal{P}} -(\tilde{x^p})^\mathsf{T} F^p(\tilde{x}).$$

Since $\forall p \in \mathcal{P}, -(\tilde{x}^p)^\mathsf{T} F^p(\tilde{x}) \geq -\tilde{y}^p m^p$ holds. Hence, $\forall p \in \mathcal{P}, -(\tilde{x}^p)^\mathsf{T} F^p(\tilde{x}) = -\tilde{y}^p m^p$. Therefore, $\forall p \in \mathcal{P}$ and $\forall x^p$ feasible, we obtain

$$-(\tilde{x}^p)^\mathsf{T} F^p(\tilde{x}) \leq -(x^p)^\mathsf{T} F^p(\tilde{x}). \tag{37}$$

Setting $x^p = m^p \mathbf{e}_1^p$, $x^p = m^p \mathbf{e}_2^p$, up to $x^p = m^p \mathbf{e}_{n^p}^p$ in (37), we arrive at

$$(\tilde{x}^p)^\mathsf{T} F^p(\tilde{x}) \geq \left(\max_{i \in \mathcal{I}^p} F_i^p(\tilde{x}) \right) m^p. \tag{38}$$

Since $\tilde{x} \geq 0$ and $\mathbf{1}^\mathsf{T} \tilde{x}^p = m^p$, equality holds in (38). Thus, we conclude that for $i \in \mathcal{I}^p$ such that $\tilde{x}_i^p > 0$, $i \in \arg\max_{i \in \mathcal{I}^p} F_i^p(\tilde{x})$. Therefore, $\tilde{x}$ is an NE of the population game, which completes the proof. $\square$

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