Dynamic Games and Applications (2019) 9:700–723
https://doi.org/10.1007/s13235-018-0279-1

On a Solution of a Guarantee Optimization Problem Under the Functional Constraints on the Disturbance

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Published online: 4 August 2018

Abstract
The paper deals with a control problem for a dynamical system under disturbances. A motion of the system is considered on a finite interval of time and described by a nonlinear ordinary differential equation. The control is aimed at minimization of a given quality index. In addition to geometric constraints on the control and disturbance, it is supposed that the disturbance satisfies a compact functional constraint. Namely, all disturbance realizations that can happen in the system belong to some unknown set that is compact in the space L_1. Within the game-theoretical approach, the problem of optimizing the guaranteed result of the control is studied. For solving this problem, we propose a new construction of the optimal control strategy. In the linear-convex case, this strategy can be numerically realized on the basis of the upper convex hulls method. Examples are considered. Results of numerical simulations are given.

Keywords Control problem · Disturbances · Functional constraint · Optimal guaranteed result · Optimal strategy · Reconstruction · Numerical method

1 Introduction

The paper deals with a control problem for a dynamical system under disturbances. A motion of the system is considered on a finite interval of time and described by a nonlinear ordinary differential equation. The admissible values of the control and disturbance are subject to geometric constraints. The control is aimed at minimization of a given quality index. Within

This work was supported by the Integrated Program for Fundamental Research of the Ural Branch of the Russian Academy of Sciences (Project No. 18-1-1-10).

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the game-theoretical approach [5–7,18], we study the problem of optimizing the guaranteed result of the control.

In addition to the standard formulation of the guarantee optimization problem, we suppose that the disturbance satisfies a compact functional constraint. According to [8] (see also [2,16,17]), it means that all disturbance realizations that can happen in the system belong to some unknown set that is compact in the space L_1. This notion of a functional constraint is quite general and can be used in order to formalize an additional available information about the properties of the possible disturbance realizations as functions of time. However, it should be noted that this type of functional constraints substantially differs from the situation when the specific set of the possible disturbance realizations is given. The guarantee optimization problem in the latter case seems to be more complicated problem than the one studied in the paper.

The considered control problem under the functional constraint on the disturbance is formulated in the class of control strategies with full memory (see, e.g., [5,7,8]). The corresponding value of the optimal guaranteed result is introduced. The main result of the paper is a new construction of the optimal control strategy. This strategy can be considered as a control procedure with a guide. The proximity between the motions of the original system and guide is provided by the technique of dynamic reconstruction of the disturbance based on the ideas from [10]. The quality of the control process is attained due to the use of the optimal counter-strategy with full memory in the guide. Furthermore, we consider also a particular case of the problem when the right-hand side of the dynamic equation satisfies a certain additional condition [2,13,17], which allows to simplify the construction of the optimal strategy.

The proposed new construction of the optimal control strategy makes it possible to develop numerical methods for solving the guarantee optimization problems under consideration. The main difficulty here is to construct the optimal counter-strategy used in the guide. However, in the so-called linear-convex case, this can be done efficiently, for example, by applying the upper convex hulls method [1,4] (see also [9]). We consider some examples, which are close to pursuit–evasion games, and present the results of numerical simulations.

The paper is organized as follows. In Sect. 2, we give the informal statement of the guarantee optimization problem under the functional constraint on the disturbance. In Sect. 3, to emphasize the differences that arise in the mathematical statement of the problem because of the presence of this additional constraint, we consider the standard statements of guarantee optimization problem without functional constraints. The mathematical statement of the problem with the functional constraint on the disturbance is given in Sect. 4. In Sect. 5, we propose a new construction of the optimal control strategy with full memory. The proof of the corresponding result is given in Sect. 6. Section 7 is devoted to the particular case when the construction of the optimal control strategy can be simplified. Examples are considered in Sect. 8.

2 Statement of the Problem

In this section, we give the informal statement of a guarantee optimization problem under a functional constraint on the disturbance. A dynamical system and a quality index under consideration are described. The notion of a functional constraint on the disturbance is introduced. The strict mathematical statement of the problem is given in Sect. 4.

2.1 Dynamical System and Quality Index

We consider a dynamical system which motion is described by the following differential equation:

$$\begin{aligned} \frac{\mathrm{d}x(t)}{\mathrm{d}t} &= f(t, x(t), u(t), v(t)), \quad t \in T := [t_0, \vartheta], \\ x(t) \in \mathbb{R}^n, &\quad u(t) \in P \subset \mathbb{R}^p, \quad v(t) \in Q \subset \mathbb{R}^q, \end{aligned} \tag{1}$$

with the initial condition

$$x(t_0) = x_0, \quad x_0 \in B(R_0) := \big\{x \in \mathbb{R}^n : \|x\| \le R_0\big\}. \tag{2}$$

Here t is the time, x is the state vector, u is the control vector, v is the disturbance vector; t_0 and ϑ are the initial and terminal times; P and Q are known compact sets; x_0 is the initial state of the system; $R_0 > 0$ is a fixed number; the symbol $\|\cdot\|$ denotes the Euclidian norm of a vector.

It is assumed that the function $f : T \times \mathbb{R}^n \times P \times Q \to \mathbb{R}^n$ has the following properties: f is continuous; for any compact set $D \subset \mathbb{R}^n$, there exists a number $L > 0$ such that

$$\|f(t, x, u, v) - f(t, x', u, v)\| \le L\|x - x'\|, \quad t \in T, \quad x, x' \in D, \quad u \in P, \quad v \in Q;$$

and, moreover, there exists a number $a > 0$ such that

$$\|f(t, x, u, v)\| \le a(1 + \|x\|), \quad t \in T, \quad x \in \mathbb{R}^n, \quad u \in P, \quad v \in Q.$$

We assume that the segment $T = [t_0, \vartheta]$ is equipped with the Lebesgue measure. By admissible realizations $u(\cdot)$ of the control and $v(\cdot)$ of the disturbance, we mean measurable functions $u : T \to P$ and $v : T \to Q$. The sets of all such realizations are denoted by $\mathcal{U}$ and $\mathcal{V}$, respectively. One can show that, due to the properties of the function f, for any initial state $x_0 \in B(R_0)$ and any admissible realizations $u(\cdot) \in \mathcal{U}$ and $v(\cdot) \in \mathcal{V}$, there exists a unique motion $x(\cdot) = x(\cdot; x_0, u(\cdot), v(\cdot))$ of system (1) that is an absolutely continuous function $x : T \to \mathbb{R}^n$ that satisfies initial condition (2) and, together with the realizations $u(\cdot)$ and $v(\cdot)$, satisfies Eq. (1) for almost all $t \in T$. Moreover, one can choose (see, e.g., [7, pp. 8,14,15]) a number $R > 0$ such that, for any motion $x(\cdot) = x(\cdot; x_0, u(\cdot), v(\cdot))$ of system (1), generated from any initial state $x_0 \in B(R_0)$ by any realization $u(\cdot) \in \mathcal{U}$ and $v(\cdot) \in \mathcal{V}$, the following inclusions are valid:

$$x(t) \in B(R) := \big\{x \in \mathbb{R}^n : \|x\| \le R\big\}, \quad t \in T. \tag{3}$$

Let quality of a motion $x(\cdot)$ of system (1) be evaluated by the index

$$\gamma = \sigma(x(\cdot)), \tag{4}$$

where the function $\sigma : C(T, \mathbb{R}^n) \to \mathbb{R}$ is continuous.

The goal of the control is to minimize the value γ of quality index (4). Since there are unknown disturbances acting in system (1), when we pose and solve this control problem, in accordance with the guaranteed result principle (see, e.g., [3,5–7,18]), we should take into account that, in the worst case, the disturbances may be aimed at maximization of γ.

2.2 Functional Constraint on the Disturbance

According to (1), for any time $t \in T$, the value of the disturbance $v(t)$ satisfies the inclusion $v(t) \in Q$. Constraints of this kind are called geometric (or instantaneous). In the present paper, the case is considered when the disturbance satisfies an additional functional constraint imposed not on the instantaneous values $v(t)$, but on the realization $v(\cdot)$ as a whole.

By a functional constraint on the disturbance, we mean a family of subsets $\mathbb{V} \subset 2^{\mathcal{V}}$ such that

$$\mathcal{V} = \bigcup_{V \in \mathbb{V}} V. \tag{5}$$

We say that the disturbance satisfies the functional constraint $\mathbb{V}$ if there exists a set $V \in \mathbb{V}$ such that every disturbance realization $v(\cdot)$ that can happen in system (1) satisfies the inclusion $v(\cdot) \in V$. Thus, it is assumed that, when forming control actions, we know only the constraint $\mathbb{V}$, but the specific set $V \in \mathbb{V}$ is not given. This notion of a functional constraint is quite general and can be used in order to formalize an additional information about the structure and properties of the possible disturbance realizations. A functional constraint $\mathbb{V}$ is called compact if it consists of compact in $L_1(T, \mathbb{R}^q)$ subsets $V \subset \mathcal{V}$. Let us give some typical examples when an additional information about the disturbance can be formalized with the help of such a functional constraint:

1. It is known that every disturbance realization $v(\cdot)$ is a piecewise constant function with a fixed number $l \in \mathbb{N} \cup \{0\}$ of possible discontinuity points; however, this number l is unknown.
2. It is known that every disturbance realization $v(\cdot)$ is a continuous function with a fixed modulus of continuity ω; however, this modulus ω is unknown.
3. It is known that every realization $v(\cdot)$ is generated by a Carathéodory function $W : T \times \mathbb{R}^n \to Q$ such that $v(t) = W(t, x(t))$, $t \in T$; however, this function W is unknown (see, e.g., [14]).

In the paper, we consider a guarantee optimization problem for system (1), initial condition (2) and quality index (4) in the case when the disturbance satisfies a compact functional constraint $\mathbb{V}$. In order to emphasize the differences that arise in the mathematical statement of the problem due to the presence of this additional constraint, in the next section we give the standard statements of guarantee optimization problem (1), (2) and (4) without functional constraints.

3 Guarantee Optimization Without Functional Constraints

The mathematical statement of guarantee optimization problem (1), (2) and (4) depends on the way of forming the control actions. In this section, we consider three types of control strategies: quasi-strategies, counter-strategies and strategies with full memory. For each of these types, the corresponding value of the optimal guaranteed result is introduced. The comparison between these values is given.

3.1 Quasi-Strategies

The notion of a quasi-strategy, originating from works [11,12], formalizes one of the most general ways of forming the control actions in real time without using information about future. In the paper, by a quasi-strategy, we mean a function $\alpha : \mathcal{V} \to \mathcal{U}$ with the following property of nonanticipation: if, for any time $t \in T$ and any realizations $v(\cdot), v'(\cdot) \in \mathcal{V}$, the equality $v(\tau) = v'(\tau)$ is valid for almost all $\tau \in [t_0, t]$, then the corresponding images

$u(\cdot)=\alpha(v(\cdot))$ and $u'(\cdot)=\alpha(v'(\cdot))$ satisfy the equality $u(\tau)=u'(\tau)$ for almost all $\tau\in[t_0,t]$. The set of all quasi-strategies is denoted by **QS**.

For any initial state $x_0\in B(R_0)$, the value of the optimal guaranteed result in the class of quasi-strategies is defined as follows:

$$\Gamma^0_{\mathbf{QS}}(x_0):=\inf_{\alpha\in\mathbf{QS}}\ \sup_{v(\cdot)\in\mathcal{V}}\ \sigma\Big(x\big(\cdot;x_0,\alpha(v(\cdot)),v(\cdot)\big)\Big). \tag{6}$$

Note that any control procedure that forms the current value $u(t)$ on the basis of the information about the initial state x_0 and the history of the disturbance actions $v(\cdot)|_{[t_0,t]}$ (including the current value $v(t)$) can be considered as a quasi-strategy. Therefore, any such control procedure cannot guarantee the value of quality index (4) less than $\Gamma^0_{\mathbf{QS}}(x_0)$. It is known that the quasi-strategies are a convenient tool in theoretical constructions, but they are impractical in real control problems.

3.2 Counter-Strategies with Full Memory

In the paper, we use the following definition of a counter-strategy (with full memory), which goes back to the constructions from [6,8]. Let Δ be a partition of the time segment $T=[t_0,\vartheta]$ by times τ_i, $i\in 0...n_\Delta$, i.e.,

$$\Delta=\big\{\tau_i:\tau_0=t_0,\ \tau_{i-1}<\tau_i,\ i\in 1...n_\Delta,\ \tau_{n_\Delta}=\vartheta\big\}.$$

The set of all such partitions is denoted by Δ_T. By a counter-control (with full memory) on the partition Δ, we mean a family $\overline{\mathbf{U}}^\Delta=(\overline{\mathbf{U}}_i^\Delta)_{i\in 0...(n_\Delta-1)}$ of mappings

$$\overline{\mathbf{U}}_i^\Delta: C([t_0,\tau_i],\mathbb{R}^n)\to\mathbb{B}(Q,P),\quad i\in 0...(n_\Delta-1),$$

where $\mathbb{B}(Q,P)$ is the set of all Borel measurable functions from Q to P. Respectively, a counter-strategy is a family $\overline{\mathbf{U}}=(\overline{\mathbf{U}}^\Delta)_{\Delta\in\Delta_T}$ of counter-controls defined for every partition $\Delta\in\Delta_T$. The set of all counter-strategies is denoted by **CS**.

Let $x_0\in B(R_0)$ and $\overline{\mathbf{U}}=(\overline{\mathbf{U}}^\Delta)_{\Delta\in\Delta_T}\in\mathbf{CS}$. For any partition $\Delta\in\Delta_T$, the corresponding counter-control $\overline{\mathbf{U}}^\Delta=(\overline{\mathbf{U}}_i^\Delta)_{i\in 0...(n_\Delta-1)}$ in a pair with a disturbance realization $v(\cdot)\in\mathcal{V}$ forms in system (1) a control realization $u(\cdot)$ by the following step-by-step feedback rule:

$$u(t)=\overline{\mathbf{U}}_i^\Delta(x(\cdot)|_{[t_0,\tau_i]})\big(v(t)\big),\quad t\in[\tau_i,\tau_{i+1}),\quad i\in 0...(n_\Delta-1), \tag{7}$$

where $x(\cdot)|_{[t_0,\tau_i]}$ is the motion history realized up to the time τ_i. Note that, since the function $\overline{\mathbf{U}}_i^\Delta(x(\cdot)|_{[t_0,\tau_i]})$ is Borel measurable, the obtained control realization is admissible, i.e., $u(\cdot)\in\mathcal{U}$. Thus, from the initial state x_0, the counter-control $\overline{\mathbf{U}}^\Delta$ in a pair with the disturbance realization $v(\cdot)$ uniquely generates the system motion, denoted by $x(\cdot)=x(\cdot;x_0,\overline{\mathbf{U}}^\Delta,v(\cdot))$. For the counter-strategy $\overline{\mathbf{U}}$, the value of the guaranteed result is defined as follows:

$$\Gamma_{\mathbf{CS}}(x_0;\overline{\mathbf{U}}):=\lim_{\delta\downarrow 0}\ \sup_{\Delta\in\Delta_T:\mathbf{D}(\Delta)\le\delta}\ \sup_{v(\cdot)\in\mathcal{V}}\ \sigma\Big(x\big(\cdot;x_0,\overline{\mathbf{U}}^\Delta,v(\cdot)\big)\Big). \tag{8}$$

Here and below, we denote by $\mathbf{D}(\Delta):=\max_{i\in 1...n_\Delta}(\tau_i-\tau_{i-1})$ the diameter of the partition Δ. Respectively, the optimal guaranteed result in the class of counter-strategies is the following value:

$$\Gamma^0_{\mathbf{CS}}(x_0):=\inf_{\overline{\mathbf{U}}\in\mathbf{CS}}\ \Gamma_{\mathbf{CS}}(x_0;\overline{\mathbf{U}}). \tag{9}$$

Note that, according to the results of [6,7], for any number $\zeta > 0$, there exists a ζ-optimal counter-strategy $\overline{\mathbf{U}}_* \in \mathbf{CS}$ such that, for any initial state $x_0 \in B(R_0)$, we have

$$\Gamma_{\mathbf{CS}}(x_0; \overline{\mathbf{U}}_*) \leq \Gamma^0_{\mathbf{CS}}(x_0) + \zeta. \tag{10}$$

Due to the necessity of the direct measurement of the current value of the disturbance $v(t)$, the use of the counter-strategies is also quite complicated in practice.

3.3 Strategies with Full Memory

In accordance with [8], by analogy with the introduced above class of counter-strategies, we define the class of strategies (with full memory) in the following way. By a control (with full memory) on a partition $\Delta = (\tau_i)_{i \in 0 \ldots n_\Delta} \in \Delta_T$, we mean a family $\mathbf{U}^\Delta := (\mathbf{U}^\Delta_i)_{i \in 0 \ldots (n_\Delta - 1)}$ of mappings

$$\mathbf{U}^\Delta_i : C([t_0, \tau_i], \mathbb{R}^n) \to \mathcal{U}|_{[\tau_i, \tau_{i+1})}, \quad i \in 0 \ldots (n_\Delta - 1),$$

where the set $\mathcal{U}|_{[\tau_i, \tau_{i+1})}$ consists of the restrictions on $[\tau_i, \tau_{i+1})$ of all the functions $u(\cdot) \in \mathcal{U}$. A strategy is a family $\mathbf{U} = (\mathbf{U}^\Delta)_{\Delta \in \Delta_T}$ of controls defined for every partition $\Delta \in \Delta_T$. The set of all strategies is denoted by $\mathbf{S}$.

Let $x_0 \in B(R_0)$ and $\mathbf{U} = (\mathbf{U}^\Delta)_{\Delta \in \Delta_T} \in \mathbf{S}$. For any partition $\Delta \in \Delta_T$, the corresponding control $\mathbf{U}^\Delta = (\mathbf{U}^\Delta_i)_{i \in 0 \ldots (n_\Delta - 1)}$ in a pair with a disturbance realization $v(\cdot) \in \mathcal{V}$ forms in system (1) a control realization $u(\cdot)$ by the following step-by-step feedback rule:

$$u(t) = \mathbf{U}^\Delta_i (x(\cdot)|_{[t_0, \tau_i]})(t), \quad t \in [\tau_i, \tau_{i+1}), \quad i \in 0 \ldots (n_\Delta - 1).$$

Thus, from the initial state x_0, the control $\mathbf{U}^\Delta$ in a pair with the disturbance realization $v(\cdot)$ uniquely generates the system motion, denoted by $x(\cdot) = x(\cdot; x_0, \mathbf{U}^\Delta, v(\cdot))$. The value of the guaranteed result of the strategy $\mathbf{U}$ and the value of the optimal guaranteed result in the class of strategies are defined as follows:

$$\Gamma_{\mathbf{S}}(x_0; \mathbf{U}) := \lim_{\delta \downarrow 0} \sup_{\Delta \in \Delta_T : \mathbf{D}(\Delta) \leq \delta} \sup_{v(\cdot) \in \mathcal{V}} \sigma\Big(x\big(\cdot; x_0, \mathbf{U}^\Delta, v(\cdot)\big)\Big), \tag{11}$$

$$\Gamma^0_{\mathbf{S}}(x_0) := \inf_{\mathbf{U} \in \mathbf{S}} \Gamma_{\mathbf{S}}(x_0; \mathbf{U}). \tag{12}$$

Note that, when using strategies, there is no need in any information about the disturbance. It makes this way of forming the control actions more preferable in comparison with the quasi-strategies and counter-strategies.

3.4 Comparison of Optimal Guaranteed Results

The following relations between the values of optimal guaranteed results (6), (9) and (12) are valid:

$$\Gamma^0_{\mathbf{QS}}(x_0) = \Gamma^0_{\mathbf{CS}}(x_0), \quad \Gamma^0_{\mathbf{QS}}(x_0) \leq \Gamma^0_{\mathbf{S}}(x_0), \quad x_0 \in B(R_0). \tag{13}$$

The equality in (13) is derived from the results of [6, §§28, 29] [see also [5, §9]]. The inequality in (13) is a straightforward consequence of the given definitions. Note that this inequality can be strict, and a sufficient condition for the equality is the equilibrium condition in a small game (see, e.g., [7, p. 8]) or, in another terminology, the Isaacs' condition [3]:

$$\begin{gathered} \min_{u \in P} \max_{v \in Q} \langle s, f(t, x, u, v) \rangle = \max_{v \in Q} \min_{u \in P} \langle s, f(t, x, u, v) \rangle, \\ t \in T, \quad x \in \mathbb{R}^n, \quad s \in \mathbb{R}^n, \end{gathered} \tag{14}$$

where the symbol $\langle\cdot,\cdot\rangle$ denotes the inner product of vectors. Thus, the result $\varGamma^0_{\mathbf{QS}}(x_0)$ can always be guaranteed with the help of the counter-strategies, but, in general, without condition (14), cannot be guaranteed with the help of the strategies.

Another situation when the optimal guaranteed results in the classes **QS** and **S** coincide regardless of condition (14) is described in the paper and related to compact functional constraints on the disturbance.

4 Guarantee Optimization Under a Functional Constraint on the Disturbance

In this section, for system (1), initial condition (2) and quality index (4), we define the value of the optimal guaranteed result in the class of strategies **S** in the case when the disturbance satisfies a compact functional constraint $\mathbb{V}$. In accordance with the informal statement of the problem (see Sect. 2.2), the presence of the functional constraint $\mathbb{V}$ leads to the fact that, in definition (11) of the guaranteed result, we split the operation of taking the upper bound over all disturbance realizations $v(\cdot)\in\mathcal{V}$ into two parts. Firstly, inside, the upper bound is taken only over disturbance realizations $v(\cdot)$ from a set $V\in\mathbb{V}$, and, after that, outside, the upper bound is taken over the sets $V\in\mathbb{V}$.

Let us note that, for any initial state $x_0\in B(R_0)$, we can define the value of the optimal guaranteed result in the class of quasi-strategies **QS** under the functional constraint $\mathbb{V}$ as follows:

$$\varGamma^0_{\mathbf{QS}}(x_0\mid\mathbb{V}):=\inf_{\alpha\in\mathbf{QS}}\sup_{V\in\mathbb{V}}\sup_{v(\cdot)\in V}\sigma\Big(x\big(\cdot;x_0,\alpha(v(\cdot)),v(\cdot)\big)\Big). \tag{15}$$

However, due to (5), (6) and (15), we have

$$\varGamma^0_{\mathbf{QS}}(x_0\mid\mathbb{V})=\varGamma^0_{\mathbf{QS}}(x_0). \tag{16}$$

Therefore, the optimal guaranteed result in the class of quasi-strategies does not depend on the presence of the functional constraints.

Further, let us define the value of the guaranteed result of a strategy $\mathbf{U}=(\mathbf{U}^{\varDelta})_{\varDelta\in\varDelta_T}\in\mathbf{S}$ under the functional constraint $\mathbb{V}$ and the corresponding value of the optimal guaranteed result in the class of strategies **S** under the functional constraint $\mathbb{V}$ in the following way:

$$\begin{aligned}\varGamma_{\mathbf{S}}(x_0;\mathbf{U}\mid\mathbb{V})&:=\sup_{V\in\mathbb{V}}\lim_{\delta\downarrow 0}\sup_{\varDelta\in\varDelta_T:\mathbf{D}(\Delta)\le\delta}\sup_{v(\cdot)\in V}\sigma\Big(x\big(\cdot;x_0,\mathbf{U}^{\varDelta},v(\cdot)\big)\Big),\\ \varGamma^0_{\mathbf{S}}(x_0\mid\mathbb{V})&:=\inf_{\mathbf{U}\in\mathbf{S}}\varGamma_{\mathbf{S}}(x_0;\mathbf{U}\mid\mathbb{V}).\end{aligned} \tag{17}$$

According to [17], for any compact functional constraint $\mathbb{V}$, the following equality holds:

$$\varGamma^0_{\mathbf{S}}(x_0\mid\mathbb{V})=\varGamma^0_{\mathbf{QS}}(x_0\mid\mathbb{V}),\quad x_0\in B(R_0).$$

Hence, due to (16), when the disturbance satisfies a compact functional constraint $\mathbb{V}$, the optimal guaranteed result $\varGamma^0_{\mathbf{QS}}(x_0)$ in the class of quasi-strategies **QS** can be guaranteed with the help of the strategies with full memory. Thus, the considered in the paper guarantee optimization problem in the class of strategies **S** under the functional constraint $\mathbb{V}$ can be formulated as follows. For any number $\zeta>0$, we should find a strategy $\mathbf{U}_*\in\mathbf{S}$ such that, for any initial state $x_0\in B(R_0)$, the following inequality is valid:

$$\varGamma_{\mathbf{S}}(x_0;\mathbf{U}_*\mid\mathbb{V})\le\varGamma^0_{\mathbf{QS}}(x_0)+\zeta. \tag{18}$$

The main contribution of the paper is a new construction of this ζ-optimal strategy $\mathbf{U}_*$.

5 Construction of Optimal Strategy

Let $\varepsilon \in (0, 1)$ be an accuracy parameter. Let us define a strategy $\mathbf{U}_\varepsilon = (\mathbf{U}_\varepsilon^\Delta)_{\Delta \in \Delta_T} \in \mathbf{S}$ such that it satisfies inequality (18) for any sufficiently small values of ε. Let us introduce the necessary notations and constructions.

According to (10) and (13), let us fix an ε-optimal counter-strategy $\overline{\mathbf{U}}_\varepsilon = (\overline{\mathbf{U}}_\varepsilon^\Delta)_{\Delta \in \Delta_T} \in \mathbf{CS}$ such that

$$\Gamma_{\mathbf{CS}}(x_0; \overline{\mathbf{U}}_\varepsilon) \leq \Gamma^0_{\mathbf{QS}}(x_0) + \varepsilon, \quad x_0 \in B(R_0). \tag{19}$$

For the compact set P, which determines the geometric constraint on the control (see (1)), let us choose an ε–net $(u_j^\varepsilon)_{j \in 1 \dots n_\varepsilon} \subset P$:

$$\max_{u \in P} \min_{j \in 1 \dots n_\varepsilon} \|u - u_j^\varepsilon\| \leq \varepsilon. \tag{20}$$

At first, it is convenient to define the control $\mathbf{U}_\varepsilon^\Delta = (\mathbf{U}_{\varepsilon i}^\Delta)_{i \in 0 \dots (n_\Delta - 1)}$ only for partitions $\Delta \in \Delta_T$ that satisfy the following condition:

$$\mathbf{D}(\Delta) \leq 3\mathbf{d}(\Delta), \tag{21}$$

where $\mathbf{d}(\Delta) = \min_{i \in 1 \dots n_\Delta}(\tau_i - \tau_{i-1})$ is the inner diameter of the partition Δ. Let us consider the auxiliary times

$$\tau_i' := \tau_i - \varepsilon \mathbf{d}(\Delta), \quad \tau_{ij}' := \tau_i' + \frac{j(\tau_i - \tau_i')}{n_\varepsilon}, \quad j \in 0 \dots n_\varepsilon, \quad i \in 1 \dots (n_\Delta - 1). \tag{22}$$

Note that, since $\varepsilon \in (0, 1)$, the following inclusions are valid:

$$\tau_{ij}' \in (\tau_{i-1}, \tau_i], \quad i \in 1 \dots (n_\Delta - 1), \quad j \in 0 \dots n_\varepsilon.$$

For any $i \in 0 \dots (n_\Delta - 1)$ and any function $x^{(i)}(\cdot) \in C([t_0, \tau_i], \mathbb{R}^n)$, let us choose a vector $\nu_i(x^{(i)}(\cdot))$ such that

$$\nu_i(x^{(i)}(\cdot)) \in \begin{cases} Q, & i = 0, \\ \operatorname*{argmin}_{v \in Q} \max_{j \in 1 \dots n_\varepsilon} \left\| d_{ij}(x^{(i)}(\cdot)) - f(\tau_i, x^{(i)}(\tau_i), u_j^\varepsilon, v) \right\|, & i \in 1 \dots (n_\Delta - 1), \end{cases} \tag{23}$$

where, for $i \in 1 \dots (n_\Delta - 1)$ and $j \in 1 \dots n_\varepsilon$, we denote by $d_{ij}(x^{(i)}(\cdot))$ the divided difference

$$d_{ij}(x^{(i)}(\cdot)) := \frac{x^{(i)}(\tau_{ij}') - x^{(i)}(\tau_{i(j-1)}')}{\tau_{ij}' - \tau_{i(j-1)}'}. \tag{24}$$

Before proceeding to the formal definition of the control $\mathbf{U}_\varepsilon^\Delta$ on the partition Δ, let us describe it as a control procedure with a guide.

5.1 Optimal Control with a Guide

The control $\mathbf{U}_\varepsilon^\Delta$ on the partition Δ can be treated as a control procedure with a guide (see, e.g., [7, §8.2]). A motion of the guide is considered as an auxiliary motion $y(\cdot)$ of system (1). We suppose that this motion $y(\cdot)$ satisfies the same initial condition $y(t_0) = x_0$ as the motion $x(\cdot)$ of the original system, and we denote by $\overline{u}(\cdot) \in \mathcal{U}$ and $\overline{v}(\cdot) \in \mathcal{V}$ the corresponding control and "disturbance" realizations that determine this motion $y(\cdot)$. Thus, according to the introduced notations, we have $y(\cdot) = x(\cdot; x_0, \overline{u}(\cdot), \overline{v}(\cdot))$. Note that, according to choice

(3) of the number R, for any such auxiliary motion $y(\cdot)$, the inclusions $y(t) \in B(R)$, $t \in T$, are valid.

Let us describe this control procedure. In order to choose the "disturbance" $\overline{v}(\cdot)$ in the guide, when forming the control $u(\cdot)$ in the original system, we use the series of the test control actions u_j^ε, $j \in 1...n_\varepsilon$, on the small part $[\tau'_{i+1}, \tau_{i+1})$ of every step $[\tau_i, \tau_{i+1})$ of the partition Δ. By the observations of the corresponding reactions of the original system to these test controls, as in the theory of inverse problem of dynamics (see, e.g., [10]), we choose "on the fly" the "disturbance" $\overline{v}(\cdot)$ in the guide that in a some sense approximates the disturbance $v(\cdot)$ acting in the original system. After that, by the found approximation $\overline{v}(\cdot)$, we choose the control $\overline{u}(\cdot)$ in the guide according to the fixed ε-optimal counter-control $\overline{\mathbf{U}}_\varepsilon^\Delta$. Finally, the constructed control $\overline{u}(\cdot)$ is used in the original system on the current step of the partition except for the "test" part $[\tau'_{i+1}, \tau_{i+1})$. Under a suitable choice of the parameters (see Lemma 1), the obtained motion $x(\cdot)$ of the original system is close to the constructed ε-optimal motion $y(\cdot)$ of the guide.

Thus, we consider the following step-by-step procedure of forming a control realization $u(\cdot) \in \mathcal{U}$ in the original system and piecewise constant realizations $\overline{u}(\cdot)$ and $\overline{v}(\cdot)$ of the form

$$\overline{u}(t) = \overline{u}_i \in P, \quad \overline{v}(t) = \overline{v}_i \in Q, \quad t \in [\tau_i, \tau_{i+1}), \quad i \in 0...(n_\Delta - 1), \tag{25}$$

in the guide. For any $i \in 0...(n_\Delta - 1)$, let $x(\cdot)|_{[t_0,\tau_i]}$ and $y(\cdot)|_{[t_0,\tau_i]}$ be, respectively, the histories of the motions of the original system and guide realized up to the time τ_i. "Reconstructing" the disturbances acting in the original system on the interval $[\tau_{i-1}, \tau_i)$, we set

$$\overline{v}_i = \nu_i\big(x(\cdot)|_{[t_0,\tau_i]}\big). \tag{26}$$

Using the fixed counter-control $\overline{\mathbf{U}}_\varepsilon^\Delta = (\overline{\mathbf{U}}_{\varepsilon i}^\Delta)_{i \in 0...(n_\Delta - 1)}$ in the guide, according to (7), we put

$$\overline{u}_i = \overline{\mathbf{U}}_{\varepsilon i}^\Delta\big(y(\cdot)|_{[t_0,\tau_i]}\big)(\overline{v}_i). \tag{27}$$

After that, we define

$$u(t) = \begin{cases} \overline{u}_i, & t \in [\tau_i, \tau'_{i+1}), \\ u_j^\varepsilon, & t \in [\tau'_{(i+1)(j-1)}, \tau'_{(i+1)j}), \quad j \in 1...n_\varepsilon, \end{cases} \tag{28}$$

where u_j^ε are the elements of the chosen ε-net.

An illustration to the described control procedure with the guide is given in Fig. 1.

5.2 Optimal Control Strategy with Full Memory

The control procedure with guide (26)–(28) is formalized as the control with full memory $\mathbf{U}_\varepsilon^\Delta = (\mathbf{U}_{\varepsilon i}^\Delta)_{i \in 0...(n_\Delta - 1)}$ on the partition Δ as follows. Let $i \in 0...(n_\Delta - 1)$ and $x^{(i)}(\cdot) \in C([t_0, \tau_i], \mathbb{R}^n)$. Set

$$\overline{v}(t) = \nu_k\big(x^{(i)}(\cdot)|_{[t_0,\tau_k]}\big), \quad t \in [\tau_k, \tau_{k+1}), \quad k \in 0...i. \tag{29}$$

Consider the auxiliary motion $y^{(i)}(t) := x(t; x_0, \overline{\mathbf{U}}_\varepsilon^\Delta, \overline{v}(\cdot))$, $t \in [t_0, \tau_i]$, of system (1) and put

$$\mathbf{U}_{\varepsilon i}^\Delta\big(x^{(i)}(\cdot)\big)(t) := \begin{cases} \overline{\mathbf{U}}_{\varepsilon i}^\Delta\big(y^{(i)}(\cdot)\big)(\overline{v}(\tau_i)), & t \in [\tau_i, \tau'_{i+1}), \\ u_j^\varepsilon, & t \in [\tau'_{(i+1)(j-1)}, \tau'_{(i+1)j}), \quad j \in 1...n_\varepsilon. \end{cases} \tag{30}$$

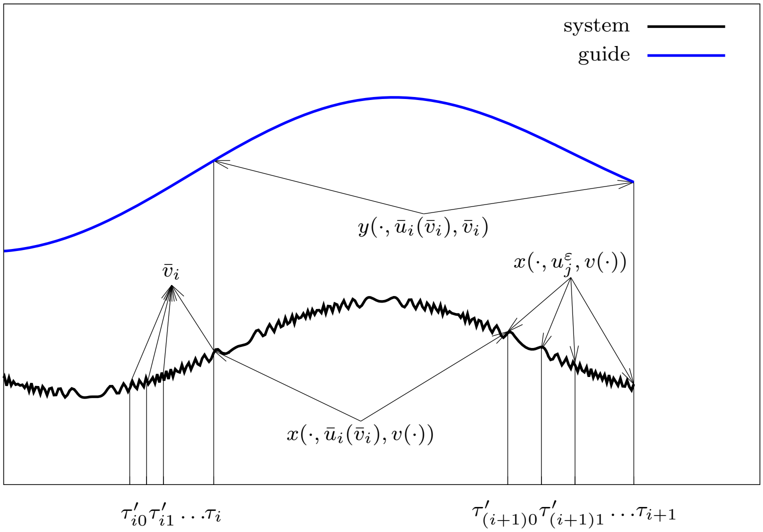

Fig. 1 The scheme of the strategy $\mathbf{U}_\varepsilon$

Thus, the control with full memory $\mathbf{U}_\varepsilon^\Delta$ is defined for any partition Δ that satisfies condition (21). Let a partition $\Delta = (\tau_i)_{i\in 0\ldots n_\Delta}$ do not satisfy this condition. Then, firstly, we "thin" this partition to a partition $\Delta^* \in \Delta_T$, $\Delta^* \subset \Delta$, satisfying (21). This can always be done, for instance, as follows:

$$\Delta^* := \Big\{\tau_0^* := t_0,\ \tau_{n_{\Delta^*}}^* := \vartheta,$$
$$\tau_i^* := \operatorname{argmin}\{\tau \in \Delta:\ \tau \geq i2\mathbf{D}(\Delta)\},\ i \in \mathbb{N},\ i \leq (\vartheta - t_0)/(2\mathbf{D}(\Delta))\Big\}.$$

Note that this "thinned" partition satisfies the condition

$$\mathbf{D}(\Delta^*) \leq 3\mathbf{D}(\Delta). \tag{31}$$

After that, the control $\mathbf{U}_\varepsilon^\Delta$ is defined with the help of the control $\mathbf{U}_\varepsilon^{\Delta^*}$ for the corresponding "thinned" partition by ignoring the times $\tau_i \in \Delta \setminus \Delta^*$. Note that, according to this definition, for any initial state $x_0 \in B(R_0)$ and any disturbance realization $v(\cdot) \in \mathcal{V}$, we have $x(\cdot; x_0, \mathbf{U}_\varepsilon^\Delta, v(\cdot)) = x(\cdot; x_0, \mathbf{U}_\varepsilon^{\Delta^*}, v(\cdot))$.

Hence, the strategy $\mathbf{U}_\varepsilon = (\mathbf{U}_\varepsilon^\Delta)_{\Delta\in\Delta_T}$ is completely defined.

Theorem 1 *Let $\mathbb{V}$ be a compact functional constraint on the disturbance. Then, for any number $\zeta > 0$, there exists a number $\varepsilon^* \in (0, 1)$ such that, for any initial state $x_0 \in B(R_0)$ and any number $\varepsilon \in (0, \varepsilon^*]$, the strategy with full memory $\mathbf{U}_\varepsilon \in \mathbf{S}$, defined by relations (29), (30), satisfies the inequality*

$$\Gamma_{\mathbf{S}}(x_0; \mathbf{U}_\varepsilon \mid \mathbb{V}) \leq \Gamma_{\mathbf{QS}}^0(x_0) + \zeta. \tag{32}$$

The proof of the theorem is given in the next section.

6 Proof of Theorem 1

The basis of the proof of Theorem 1 constitutes the following lemma, which is also of independent interest.

Lemma 1 *For any number $\xi > 0$, there exists a number $\varepsilon_* \in (0, 1)$ such that, for any number $\varepsilon \in (0, \varepsilon_*]$ and any set $V \subset \mathcal{V}$ compact in $L_1(T, \mathbb{R}^q)$, we can specify a number $\delta_* > 0$ such that, for any initial state $x_0 \in B(R_0)$ and any satisfying (21) partition $\Delta \in \Delta_T$ with the diameter $\mathbf{D}(\Delta) \leq \delta_*$, the following statement holds. Let the motions $x(\cdot)$ and $y(\cdot)$ of system (1) be generated from the initial state x_0 by realizations $u(\cdot)$, $v(\cdot)$ and $\overline{u}(\cdot)$, $\overline{v}(\cdot)$, respectively. Let the inclusion $v(\cdot) \in V$ be valid and these realizations satisfy relations (25) and (26), (28) for $i \in 0...(n_\Delta - 1)$. Then the following inequality holds:*

$$\|x(t) - y(t)\| \leq \xi, \quad t \in T. \tag{33}$$

Before proving the lemma, let us introduce the necessary notations. Due to the properties of the function f from the right-hand side of Eq. (1) and compactness of the sets T, $B(R)$, P and Q, we choose numbers $\varkappa > 0$ and $L > 0$ such that, for any $t \in T$, $x \in B(R)$, $u \in P$ and $v \in Q$, we have

$$\|f(t, x, u, v)\| \leq \varkappa, \quad \|f(t, x, u, v) - f(t, x', u, v)\| \leq L\|x - x'\|. \tag{34}$$

Let us denote

$$\begin{aligned}
\mu_t(\delta) &:= \max\Big\{\|f(t, x, u, v) - f(t', x, u, v)\| : \\
&\qquad\qquad t, t' \in T,\ x \in B(R),\ u \in P,\ v \in Q,\ |t - t'| \leq \delta\Big\},\\
\mu_u(\delta) &:= \max\Big\{\|f(t, x, u, v) - f(t, x, u', v)\| : \\
&\qquad\qquad t \in T,\ x \in B(R),\ u, u' \in P,\ v \in Q,\ \|u - u'\| \leq \delta\Big\},\\
\mu_v(\delta) &:= \max\Big\{\|f(t, x, u, v) - f(t, x, u, v')\| : \\
&\qquad\qquad t \in T,\ x \in B(R),\ u \in P,\ v, v' \in Q,\ \|v - v'\| \leq \delta\Big\},\\
\psi(\delta) &:= \mu_t(\delta) + L\varkappa\delta, \quad \delta > 0.
\end{aligned}$$

Note that these functions $\mu_t(\delta)$, $\mu_u(\delta)$, $\mu_v(\delta)$ and $\psi(\delta)$ are nondecreasing and tend to zero when $\delta \downarrow 0$. Note also that, for any motion $x(\cdot)$ of system (1) generated from an initial state $x_0 \in B(R_0)$ by realizations $u(\cdot) \in \mathcal{U}$ and $v(\cdot) \in \mathcal{V}$, the inequality

$$\big\|f(t, x(t), u, v) - f(t', x(t'), u, v')\big\| \leq \psi(|t - t'|) + \mu_v(\|v - v'\|) \tag{35}$$

holds for any $t, t' \in T$, $u \in P$ and $v, v' \in Q$.

Proof of Lemma 1 Fix a number $\xi > 0$ and choose a number $\xi_* > 0$ from the condition

$$\xi_* \exp(L(\vartheta - t_0)) \leq \xi. \tag{36}$$

Let a number $\varepsilon_* \in (0, 1)$ be such that

$$2(\vartheta - t_0)(\varepsilon_*\varkappa + \mu_u(\varepsilon_*)) \leq \xi_*/3. \tag{37}$$

Fix a number $\varepsilon \in (0, \varepsilon_*]$ and a set $V \subset \mathcal{V}$ compact in $L_1(T, \mathbb{R}^q)$. Taking into account [2, Assertion 3], one can specify a number $\delta_1 > 0$ such that, for any $\delta \in (0, \delta_1]$ and any function $v(\cdot) \in V$, the following inequality holds:

$$\int_T \frac{1}{4\delta} \int_{s-2\delta}^{s+2\delta} \mu_v\big(\|v(s) - v(\tau)\|\big)\,\mathrm{d}\tau\,\mathrm{d}s \leq \frac{\varepsilon\xi_*}{72n_\varepsilon}. \tag{38}$$

Here and below, it is assumed that $v(t) = 0$ for $t \notin T$. Further, choose a number $\delta_2 > 0$ such that

$$2\varkappa\delta_2 + 2(\vartheta - t_0)\big(2\mu_t(2\delta_2) + 3L\varkappa\delta_2\big) \leq \xi_*/3. \tag{39}$$

Put $\delta_* = \min\{\delta_1, \delta_2\} > 0$. Let us show that the assertion of the lemma holds for the chosen parameters.

In accordance with the statement of the lemma, let us assume that an initial state $x_0 \in B(R_0)$, a partition $\Delta \in \Delta_T$, realizations $u(\cdot)$, $v(\cdot)$ and $\overline{u}(\cdot)$, $\overline{v}(\cdot)$, and the corresponding motions $x(\cdot)$ and $y(\cdot)$ of system (1) are fixed. Let $\delta := \mathbf{D}(\Delta) \leq \delta_*$. Let us estimate the value $\|x(t) - y(t)\|$ for $t \in T$. Since the motions $x(\cdot)$ and $y(\cdot)$ are generated from the same initial state, we have

$$\|x(t) - y(t)\| = \left\| \int_{t_0}^{t} \big(f(s, x(s), u(s), v(s)) - f(s, y(s), \overline{u}(s), \overline{v}(s))\big)\,\mathrm{d}s \right\|.$$

In the right-hand side, we add and subtract under the integral sign the quantities $f(s, x(s), \overline{u}(s), v(s))$ and $f(s, x(s), \overline{u}(s), \overline{v}(s))$ (we continue the estimate):

$$\begin{aligned}
&\leq \left\| \int_{t_0}^{t} \big(f(s, x(s), u(s), v(s)) - f(s, x(s), \overline{u}(s), v(s))\big)\,\mathrm{d}s \right\| \\
&\quad + \left\| \int_{t_0}^{t} \big(f(s, x(s), \overline{u}(s), v(s)) - f(s, x(s), \overline{u}(s), \overline{v}(s))\big)\,\mathrm{d}s \right\| \\
&\quad + \left\| \int_{t_0}^{t} \big(f(s, x(s), \overline{u}(s), \overline{v}(s)) - f(s, y(s), \overline{u}(s), \overline{v}(s))\big)\,\mathrm{d}s \right\| \\
&:= I_1 + I_2 + I_3.
\end{aligned} \tag{40}$$

Let us estimate the integral I_1 in (40). Due to (22), (25) and (28), the measure of the set E that consists of all points $t \in T$ such that $u(t) \neq \overline{u}(t)$ does not exceed $\varepsilon(\vartheta - t_0)$. Therefore, applying (34), we obtain

$$I_1 \leq \int_E \|f(s, x(s), u(s), v(s)) - f(s, x(s), \overline{u}(s), v(s))\|\,\mathrm{d}s \leq 2\varkappa\varepsilon(\vartheta - t_0). \tag{41}$$

Let us estimate I_2 in (40). By the definition of ε-net (see (20)), we derive

$$\begin{aligned}
&\|f(s, x(s), \overline{u}(s), v(s)) - f(s, x(s), \overline{u}(s), \overline{v}(s))\| \\
&\quad \leq \max_{j \in 1\ldots n_\varepsilon} \left\| f(s, x(s), u_j^\varepsilon, v(s)) - f(s, x(s), u_j^\varepsilon, \overline{v}(s)) \right\| + 2\mu_u(\varepsilon).
\end{aligned} \tag{42}$$

Let $i \in 1\ldots(n_\Delta - 1)$, $j \in 1\ldots n_\varepsilon$ and $s \in [\tau_i, \tau_{i+1})$. We have

$$\begin{aligned}
&\left\| f(s, x(s), u_j^\varepsilon, v(s)) - f(s, x(s), u_j^\varepsilon, \overline{v}(s)) \right\| \\
&\quad \leq \left\| f(s, x(s), u_j^\varepsilon, v(s)) - d_{ij}(x(\cdot)) \right\| + \left\| d_{ij}(x(\cdot)) - f(s, x(s), u_j^\varepsilon, \overline{v}(s)) \right\|.
\end{aligned} \tag{43}$$

Let us estimate the first term from the right-hand side of this inequality. Since, due to (24) and (30), the equality $u(\tau) = u_j^\varepsilon$ holds for $\tau \in [\tau'_{i(j-1)}, \tau'_{ij})$, we get

$$\begin{aligned}
&\left\| f(s, x(s), u_j^\varepsilon, v(s)) - \frac{x(\tau'_{ij}) - x(\tau'_{i(j-1)})}{\tau'_{ij} - \tau'_{i(j-1)}} \right\| \\
&\quad = \left\| \int_{\tau'_{i(j-1)}}^{\tau'_{ij}} \frac{f(s, x(s), u_j^\varepsilon, v(s)) - f(\tau, x(\tau), u_j^\varepsilon, v(\tau))}{\tau'_{ij} - \tau'_{i(j-1)}}\,\mathrm{d}\tau \right\|
\end{aligned}$$

$$\leq \int_{\tau'_{i(j-1)}}^{\tau'_{ij}} \frac{\psi(s-\tau) + \mu_v(\|v(s) - v(\tau)\|)}{\tau'_{ij} - \tau'_{i(j-1)}} \, \mathrm{d}\tau$$
$$\leq \psi(2\delta) + \frac{n_\varepsilon}{\varepsilon \mathbf{d}(\Delta)} \int_{s-2\delta}^{s+2\delta} \mu_v(\|v(s) - v(\tau)\|) \, \mathrm{d}\tau. \tag{44}$$

For the second term from the right-hand side of (43), taking into account (23), (26) and the inclusion $s \in [\tau_i, \tau_{i+1})$, we derive

$$\max_{j \in 1 \dots n_\varepsilon} \|d_{ij}(x(\cdot)) - f(s, x(s), u_j^\varepsilon, \overline{v}(s))\|$$
$$= \max_{j \in 1 \dots n_\varepsilon} \|d_{ij}(x(\cdot)) - f(s, x(s), u_j^\varepsilon, \overline{v}_i(\tau_i))\|$$
$$\leq \max_{j \in 1 \dots n_\varepsilon} \|d_{ij}(x(\cdot)) - f(\tau_i, x(\tau_i), u_j^\varepsilon, \overline{v}_i(\tau_i))\| + \psi(\delta)$$
$$\leq \max_{j \in 1 \dots n_\varepsilon} \|d_{ij}(x(\cdot)) - f(\tau_i, x(\tau_i), u_j^\varepsilon, v(s))\| + \psi(\delta)$$
$$\leq \max_{j \in 1 \dots n_\varepsilon} \|d_{ij}(x(\cdot)) - f(s, x(s), u_j^\varepsilon, v(s))\| + 2\psi(\delta)$$
$$\leq \psi(2\delta) + 2\psi(\delta) + \frac{n_\varepsilon}{\varepsilon \mathbf{d}(\Delta)} \int_{s-2\delta}^{s+2\delta} \mu_v(\|v(s) - v(\tau)\|) \, \mathrm{d}\tau. \tag{45}$$

From (42)–(45), adding the estimate of difference (43) on the interval $[\tau_0, \tau_1]$, we obtain

$$I_2 \leq 2\varkappa\delta + 2(\vartheta - t_0)\big(\psi(2\delta) + \psi(\delta) + \mu_u(\varepsilon)\big)$$
$$+ \frac{2n_\varepsilon}{\varepsilon \mathbf{d}(\Delta)} \int_T \int_{s-2\delta}^{s+2\delta} \mu_v(\|v(s) - v(\tau)\|) \, \mathrm{d}\tau \, \mathrm{d}s. \tag{46}$$

For the integral I_3 in (40), according to (34), we have

$$I_3 \leq \int_{t_0}^{t} L\|x(s) - y(s)\| \, \mathrm{d}s. \tag{47}$$

Thus, from estimates (41), (46) and (47), for any $t \in T$, we derive

$$\|x(t) - y(t)\|$$
$$\leq \int_{t_0}^{t} L\|x(s) - y(s)\| \, \mathrm{d}s + 2\varkappa\delta + 2(\vartheta - t_0)\big(\varkappa\varepsilon + \psi(2\delta) + \psi(\delta) + \mu_u(\varepsilon)\big)$$
$$+ \frac{2n_\varepsilon}{\varepsilon \mathbf{d}(\Delta)} \int_T \int_{s-2\delta}^{s+2\delta} \mu_v(\|v(s) - v(\tau)\|) \, \mathrm{d}\tau \, \mathrm{d}s$$
$$\leq \int_{t_0}^{t} L\|x(s) - y(s)\| \, \mathrm{d}s + 2\varkappa\delta + 2(\vartheta - t_0)\big(\varkappa\varepsilon + \mu_u(\varepsilon) + 3L\varkappa\delta + 2\mu_t(2\delta)\big)$$
$$+ \sup_{v'(\cdot) \in V} \left\{ \frac{6n_\varepsilon}{\varepsilon\delta} \int_T \int_{s-2\delta}^{s+2\delta} \mu_v(\|v'(s) - v'(\tau)\|) \, \mathrm{d}\tau \, \mathrm{d}s \right\}$$
$$:= \int_{t_0}^{t} L\|x(s) - y(s)\| \, \mathrm{d}s + \Psi(\varepsilon, \delta).$$

Due to choice (37)–(39) of ε and Δ, we have $\Psi(\varepsilon, \delta) \leq \xi_*$. Then, applying the Bellman–Gronwall lemma, we deduce $\|x(t) - y(t)\| \leq \xi_* \exp(L(\vartheta - t_0))$, $t \in T$. Therefore, according to (36), inequality (33) holds. Lemma 1 is proved. □

Proof of Theorem 1 Fix a compact functional constraint on the disturbance $\mathbb{V}$ and a number $\zeta > 0$. Let us consider the set $\widetilde{D} \subset C(T, \mathbb{R}^n)$ that consists of all functions $x(\cdot)$ such that

$$\|x(t)\| \leq R, \quad \|x(t) - x(t')\| \leq \varkappa |t - t'|, \quad t, t' \in T,$$

where R and $\varkappa$ are taken from (3) and (34). Note that the set $\widetilde{D}$ is compact. Hence, since the function σ from quality index (4) is continuous, there exists a number $\xi > 0$ such that, for any functions $x(\cdot), y(\cdot) \in \widetilde{D}$, if $\|x(t) - y(t)\| \leq \xi$, $t \in T$, then

$$|\sigma(x(\cdot)) - \sigma(y(\cdot))| \leq \zeta/3.$$

By this number ξ, let us choose a number $\varepsilon_* > 0$ according to Lemma 1, and put $\varepsilon^* = \min\{\varepsilon_*, \zeta/3\}$. Let us prove that the assertion of the theorem holds for this value ε^*.

Fix an initial state $x_0 \in B(R_0)$ and a number $\varepsilon \in (0, \varepsilon^*]$. According to (8) and (19), there exists a number $\widetilde{\delta} > 0$ such that, for any partition $\Delta \in \Delta_T$, $\mathbf{D}(\Delta) \leq \widetilde{\delta}$, and any disturbance realization $v(\cdot) \in \mathcal{V}$, we have

$$\sigma\Big(x\big(\cdot; x_0, \overline{\mathbf{U}}_\varepsilon^\Delta, v(\cdot)\big)\Big) \leq \Gamma_{\mathbf{CS}}(x_0; \overline{\mathbf{U}}_\varepsilon) + \zeta/3 \leq \Gamma_{\mathbf{QS}}^0(x_0) + 2\zeta/3. \tag{48}$$

Fix a set $V \in \mathbb{V}$. Let δ_* be chosen by ε and V according to Lemma 1. Put $\delta^* = \min\{\delta_*, \widetilde{\delta}\}$. Let a partition $\Delta \in \Delta_T$ be such that $\mathbf{D}(\Delta) \leq \delta^*/3$. We assume that Δ satisfies condition (21). Otherwise, we replace Δ by the "thinned" partition Δ^* (see Sect. 5.2). In any case, we have a partition that satisfies condition (21) and has the diameter not exceeding the value of δ^* (see (31)).

Let $v(\cdot) \in V$ and $x(\cdot) = x(\cdot; x_0, \mathbf{U}_\varepsilon^\Delta, v(\cdot))$. Let $y(\cdot)$ be the corresponding motion of the guide. According to (27), we have $y(\cdot) = x(\cdot; x_0, \overline{\mathbf{U}}_\varepsilon^\Delta, \overline{v}(\cdot))$ for some $\overline{v}(\cdot) \in \mathcal{V}$, and, therefore, due to (48), we obtain

$$\sigma(y(\cdot)) \leq \Gamma_{\mathbf{QS}}^0(x_0) + 2\zeta/3.$$

Furthermore, by the choice of δ_*, we have $\|x(t) - y(t)\| \leq \xi$, $t \in T$. Hence, due to the choice of ξ, since $x(\cdot), y(\cdot) \in \widetilde{D}$, we deduce

$$\sigma(x(\cdot)) \leq \sigma(y(\cdot)) + \zeta/3.$$

Thus, we have shown that, for any set $V \in \mathbb{V}$, there exists a number $\delta^* > 0$ such that, for any partition $\Delta \in \Delta_T$, $\mathbf{D}(\Delta) \leq \delta^*$, and any disturbance realization $v(\cdot) \in V$, the following inequality holds:

$$\sigma\Big(x\big(\cdot; x_0, \mathbf{U}_\varepsilon^\Delta, v(\cdot)\big)\Big) \leq \Gamma_{\mathbf{QS}}^0(x_0) + \zeta.$$

From this fact, taking into account definition (17), we conclude the validity of inequality (32). Theorem 1 is proved. □

Let us give some remarks concerning Theorem 1 and Lemma 1.

1. For the guarantee optimization problem under a functional constraint on the disturbance, Lemma 1 can be considered as an analog of the estimates from [7, §2.3], which play a key role in establishing the properties of the extremal shift strategies.
2. Although the construction of the strategy $\mathbf{U}_\varepsilon$ and estimate (32) of its optimality are independent on a set $V \in \mathbb{V}$ (and even on a constraint $\mathbb{V}$), according to Lemma 1, to provide inequality (33) for a given number $\xi > 0$, a partition Δ should be chosen on the basis of the specific set V.

3. Coefficient “3” in condition (21) can be replaced by any other number from $[1, \infty)$. This coefficient can only affect the rate of convergence of the guaranteed result of the corresponding strategy to the optimal guaranteed result when the parameters ε and $\mathbf{D}(\Delta)$ are decreasing to zero.

7 Reduction of the Disturbance Reconstruction Problem

In numerical realization of the optimal strategy $\mathbf{U}_\varepsilon$, the rapid growth when $\varepsilon \downarrow 0$ of the dimension of disturbance reconstruction problem (23), (26) can cause difficulties. However, it is known that when the function f from (1) satisfies Property 1 (see [2]), to reconstruct the disturbance, it is sufficient to use any single value of the control instead of the series of “test” controls as in the general case. Therefore, we can simply use the previous step control value. The rest construction of the optimal control strategy remains the same.

Property 1 For any $t \in T$, $x \in B(R)$ and $v, v' \in Q$, if the equality

$$f(t, x, u, v) = f(t, x, u, v')$$

holds for some value $u = u' \in P$, then this equality holds for any value $u \in P$.

Note that Property 1 is valid for any function f that is injective with respect to $v \in Q$ for any fixed $t \in T$, $x \in B(R)$ and $u \in P$. Another example is given by the following particular case of system (1):

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \bar{f}(t, x(t), u(t)) + \bar{g}(t, x(t), u(t))\bar{h}(t, x(t), v(t)), \tag{49}$$

where $\bar{f} : T \times \mathbb{R}^n \times P \to \mathbb{R}^n$, $\bar{h} : T \times \mathbb{R}^n \times Q \to \mathbb{R}^m$, and $\bar{g}$ maps $T \times \mathbb{R}^n \times P$ into the space of $(n \times m)$-matrices. Property 1 holds for system (49) if the kernel of the linear operator $\bar{g}(t, x, u) : \mathbb{R}^m \to \mathbb{R}^n$ does not depend on $u \in P$ for any $t \in T$, $x \in B(R)$. Note also that Property 1, formulated in different terms, is considered in [13,15].

Let us define a strategy with full memory $\widehat{\mathbf{U}}_\varepsilon^\Delta := (\widehat{\mathbf{U}}_{\varepsilon i}^\Delta)_{i \in 0...(n_\Delta - 1)}$ for any value of the accuracy parameter $\varepsilon \in (0, 1)$. Let a partition $\Delta \in \Delta_T$ satisfy condition (21). Since, instead of the series of “test” controls, we now use only the previous step control, in accordance with (22), we put

$$n_\varepsilon = 1, \quad \tau'_{i0} = \tau'_i = \tau_{i-1}, \quad \tau'_{i1} = \tau_i, \quad i \in 1...(n_\Delta - 1),$$

and, therefore, due to (23) and (24), we define

$$\widehat{\nu}_i(x^{(i)}(\cdot)) \in \begin{cases} Q, & i = 0, \\ \operatorname*{argmin}_{v \in Q} \left\| d_{i1}(x^{(i)}(\cdot)) - f(\tau_i, x^{(i)}(\tau_i), \bar{u}_{i-1}, v) \right\|, & i \in 1...(n_\Delta - 1), \end{cases} \tag{50}$$

where

$$d_{i1}(x^{(i)}(\cdot)) = \frac{x^{(i)}(\tau_i) - x^{(i)}(\tau_{i-1})}{\tau_i - \tau_{i-1}}.$$

By analogy with Sect. 5.1, we consider the following control procedure with the guide. We define a piecewise constant control realization

$$u(t) = u_i \in P, \quad t \in [\tau_i, \tau_{i+1}), \quad i \in 0...(n_\Delta - 1), \tag{51}$$

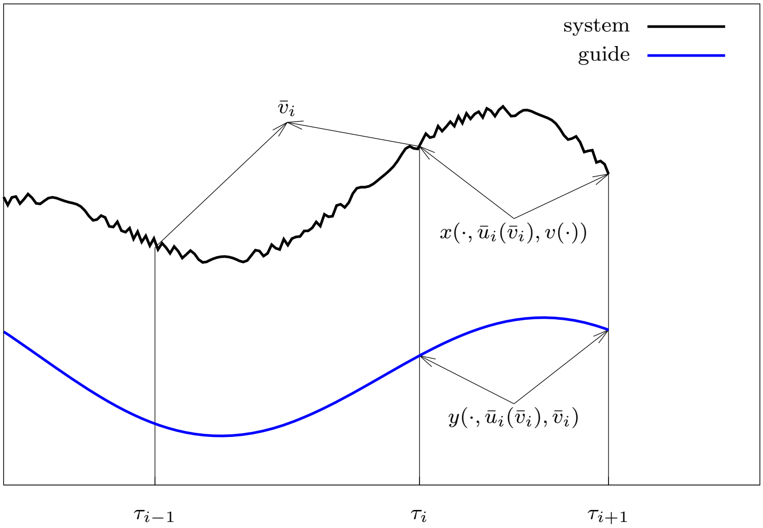

Fig. 2 The scheme of the strategy $\widehat{\mathbf{U}}_\varepsilon$

in the original system and realizations $\overline{u}(\cdot)$ and $\overline{v}(\cdot)$ of form (25) in the guide according to the following rule:

$$\overline{v}_i := \widehat{\nu}_i\big(x(\cdot)|_{[t_0,\tau_i]}\big), \tag{52}$$

$$\overline{u}_i = \overline{\mathbf{U}}^{\Delta}_{\varepsilon i}\big(y(\cdot)|_{[t_0,\tau_i]}\big)(\overline{v}_i), \tag{53}$$

$$u_i = \overline{u}_i, \tag{54}$$

where $\overline{\mathbf{U}}^{\Delta}_{\varepsilon} = (\overline{\mathbf{U}}^{\Delta}_{\varepsilon i})_{i\in 0\ldots(n_\Delta-1)}$ is fixed ε-optimal counter-control (19), and $y(\cdot)$ is the corresponding motion of the guide.

Thus, by analogy with Sect. 5.2, the control $\widehat{\mathbf{U}}^{\Delta} = (\widehat{\mathbf{U}}^{\Delta}_{\varepsilon i})_{i\in 0\ldots(n_\Delta-1)}$ on the partition Δ is defined by

$$\widehat{\mathbf{U}}^{\Delta}_{\varepsilon i}\big(x^{(i)}(\cdot)\big)(t) := \overline{\mathbf{U}}^{\Delta}_{\varepsilon i}\big(y^{(i)}(\cdot)\big)(\overline{v}(\tau_i)), \quad t \in [\tau_i, \tau_{i+1}), \quad i \in 0\ldots(n_\Delta - 1), \tag{55}$$

where

$$\begin{aligned} \overline{v}(t) &= \widehat{\nu}_k\big(x^{(i)}(\cdot)|_{[t_0,\tau_k]}\big), \quad t \in [\tau_k, \tau_{k+1}), \quad k \in 0\ldots i, \\ y^{(i)}(t) &= x\big(t; x_0, \overline{\mathbf{U}}^{\Delta}_{\varepsilon}, \overline{v}(\cdot)\big), \quad t \in [t_0, \tau_i]. \end{aligned} \tag{56}$$

As usual, if a partition $\Delta \in \Delta_T$ does not satisfy condition (21), then we should use the control $\widehat{\mathbf{U}}^{\Delta^*}_{\varepsilon}$ on the corresponding "thinned" partition Δ^*.

An illustration to the described in this section control procedure with the guide is given in Fig. 2.

Note also that the strategy $\widehat{\mathbf{U}}_\varepsilon$ differs from the one proposed in [2,13]. This new construction follows naturally from the general case (see Sect. 5) and has better convergence estimates.

Theorem 2 *Let system* (1) *satisfy Property* 1. *Let* $\mathbb{V}$ *be a compact functional constraint on the disturbance. Then, for any number* $\zeta > 0$, *there exists a number* $\varepsilon^* \in (0, 1)$ *such that, for any initial state* $x_0 \in B(R_0)$ *and any number* $\varepsilon \in (0, \varepsilon^*]$, *the strategy with full memory* $\widehat{\mathbf{U}}_\varepsilon \in \mathbf{S}$, *defined by relations* (56), (55), *satisfies the inequality*

$$\Gamma_{\mathbf{S}}(x_0; \widehat{\mathbf{U}}_\varepsilon \mid \mathbb{V}) \le \Gamma^0_{\mathbf{QS}}(x_0) + \zeta.$$

Theorem 2 is proved by the same scheme as Theorem 1, but the following lemma is used instead of Lemma 1. This lemma establishes a suitable estimate of the closeness between the motions of the original system and guide when Property 1 is satisfied and control procedure (52)–(54) is used.

Lemma 2 *Let system (1) satisfy Property 1. Then, for any number $\xi > 0$ and any set $V \subset \mathcal{V}$ compact in $L_1(T, \mathbb{R}^q)$, we can specify a number $\delta_* > 0$ such that, for any initial state $x_0 \in B(R_0)$ and any satisfying (21) partition $\Delta \in \Delta_T$ with the diameter $\mathbf{D}(\Delta) \le \delta_*$, the following statement holds. Let the motions $x(\cdot)$ and $y(\cdot)$ of system (1) be generated from the initial state x_0 by realizations $u(\cdot)$, $v(\cdot)$ and $\overline{u}(\cdot)$, $\overline{v}(\cdot)$, respectively. Let the inclusion $v(\cdot) \in V$ be valid and these realizations satisfy relations (25), (51) and (52), (54) for $i \in 0\ldots(n_\Delta - 1)$. Then the following inequality holds:*

$$\|x(t) - y(t)\| \le \xi, \quad t \in T. \tag{57}$$

In the proof of the lemma, we use the notation:

$$\mu_{uv}(\delta) := \max \Big\{ \|f(t,x,u,v) - f(t,x,u,v')\| : \\ t \in T,\ x \in B(R),\ u, u' \in P,\ v, v' \in Q,\ \|f(t,x,u',v) - f(t,x,u',v')\| \le \delta \Big\}.$$

Note that, the inequality

$$\|f(t,x,u,v) - f(t,x,u,v')\| \le \mu_{uv}\big(\|f(t,x,u',v) - f(t,x,u',v')\|\big) \tag{58}$$

holds for any $t \in T$, $x \in B(R)$, $u, u' \in P$ and $v, v' \in Q$. Furthermore, if Property 1 is satisfied, then, according to [2, Assertion 1], we have

$$\lim_{\delta \downarrow 0} \mu_{uv}(\delta) = 0. \tag{59}$$

Proof of Lemma 2 Fix a number $\xi > 0$ and a set $V \subset \mathcal{V}$ compact in $L_1(T, \mathbb{R}^q)$. Choose a number $\xi_* > 0$ from condition (36). Taking into account [2, Assertions 2, 3] and (59), one can specify a number $\delta_* > 0$ such that, for any number $\delta \in (0, \delta_*]$ and any function $v(\cdot) \in V$, the following inequality holds:

$$2\varkappa\delta + \int_T \mu_{uv}\bigg(4\psi(2\delta) + \frac{6}{\delta}\int_{s-2\delta}^{s+2\delta} \mu_v(\|v(s) - v(\tau)\|)\,\mathrm{d}\tau\bigg)\,\mathrm{d}s \le \xi_*. \tag{60}$$

Let us show that the assertion of the lemma holds for this value δ_*.

In accordance with the statement of the lemma, let us assume that an initial state $x_0 \in B(R_0)$, a partition $\Delta \in \Delta_T$, realizations $u(\cdot)$, $v(\cdot)$ and $\overline{u}(\cdot)$, $\overline{v}(\cdot)$, and the corresponding motions $x(\cdot)$ and $y(\cdot)$ of system (1) are fixed. Let $\delta := \mathbf{D}(\Delta) \le \delta_*$. Let us estimate the value $\|x(t) - y(t)\|$ for $t \in T$. In view of (25), (51) and (54), we have

$$\|x(t) - y(t)\| = \bigg\| \int_{t_0}^t \big(f(s, x(s), \overline{u}(s), v(s)) - f(s, y(s), \overline{u}(s), \overline{v}(s))\big)\,\mathrm{d}s \bigg\|.$$

In the right-hand side, we add and subtract under the integral sign the quantity $f(s, x(s), \overline{u}(s), \overline{v}(s))$ (we continue the estimate):

$$\le \bigg\| \int_{t_0}^t \big(f(s, x(s), \overline{u}(s), v(s)) - f(s, x(s), \overline{u}(s), \overline{v}(s))\big)\,\mathrm{d}s \bigg\| \\ + \bigg\| \int_{t_0}^t \big(f(s, x(s), \overline{u}(s), \overline{v}(s)) - f(s, y(s), \overline{u}(s), \overline{v}(s))\big)\,\mathrm{d}s \bigg\| := I_1 + I_2. \tag{61}$$

Let us estimate I_1 in (61). Let $i \in 1...(n_\Delta - 1)$ and $s \in [\tau_i, \tau_{i+1})$. We have

$$\begin{aligned} &\|f(s, x(s), \overline{u}_{i-1}, v(s)) - f(s, x(s), \overline{u}_{i-1}, \overline{v}(s))\| \\ &\quad \le \|f(s, x(s), \overline{u}_{i-1}, v(s)) - d_{i1}(x(\cdot))\| + \|d_{i1}(x(\cdot)) - f(s, x(s), \overline{u}_{i-1}, \overline{v}(s))\|. \end{aligned} \tag{62}$$

Let us estimate the first term in the right-hand side of this inequality. Due to (35), we obtain

$$\begin{aligned} &\left\| f(s, x(s), \overline{u}_{i-1}, v(s)) - \frac{x(\tau_i) - x(\tau_{i-1})}{\tau_i - \tau_{i-1}} \right\| \\ &\quad = \left\| \int_{\tau_{i-1}}^{\tau_i} \frac{f(s, x(s), \overline{u}_{i-1}, v(s)) - f(\tau, x(\tau), \overline{u}_{i-1}, v(\tau))}{\tau_i - \tau_{i-1}} \, \mathrm{d}\tau \right\| \\ &\quad \le \int_{\tau_{i-1}}^{\tau_i} \frac{\psi(s - \tau) + \mu_v(\|v(s) - v(\tau)\|)}{\tau_i - \tau_{i-1}} \, \mathrm{d}\tau \\ &\quad \le \psi(2\delta) + \frac{1}{\mathbf{d}(\Delta)} \int_{s-2\delta}^{s+2\delta} \mu_v(\|v(s) - v(\tau)\|) \, \mathrm{d}\tau \\ &\quad \le \psi(2\delta) + \frac{3}{\delta} \int_{s-2\delta}^{s+2\delta} \mu_v(\|v(s) - v(\tau)\|) \, \mathrm{d}\tau. \end{aligned} \tag{63}$$

For the second term in the right-hand side of (62), taking into account (50), (52) and the inclusion $s \in [\tau_i, \tau_{i+1})$, we derive

$$\begin{aligned} &\|d_{i1}(x(\cdot)) - f(s, x(s), \overline{u}_{i-1}, \overline{v}(s))\| \\ &\quad = \|d_{i1}(x(\cdot)) - f(s, x(s), \overline{u}_{i-1}, \overline{v}(\tau_i))\| \\ &\quad \le \|d_{i1}(x(\cdot)) - f(\tau_i, x(\tau_i), \overline{u}_{i-1}, \overline{v}(\tau_i))\| + \psi(\delta) \\ &\quad \le \|d_{i1}(x(\cdot)) - f(\tau_i, x(\tau_i), \overline{u}_{i-1}, v(s))\| + \psi(\delta) \\ &\quad \le \|d_{i1}(x(\cdot)) - f(s, x(s), \overline{u}_{i-1}, v(s))\| + 2\psi(\delta) \\ &\quad \le 3\psi(2\delta) + \frac{3}{\delta} \int_{s-2\delta}^{s+2\delta} \mu_v(\|v(s) - v(\tau)\|) \, \mathrm{d}\tau. \end{aligned} \tag{64}$$

From (62)–(64), we obtain

$$\begin{aligned} &\|f(s, x(s), \overline{u}_{i-1}, v(s)) - f(s, x(s), \overline{u}_{i-1}, \overline{v}(s))\| \\ &\quad \le 4\psi(2\delta) + \frac{6}{\delta} \int_{s-2\delta}^{s+2\delta} \mu_v(\|v(s) - v(\tau)\|) \, \mathrm{d}\tau. \end{aligned}$$

Therefore, due to (34) and (58), we deduce

$$I_1 \le 2\varkappa\delta + \int_T \mu_{uv} \left(4\psi(2\delta) + \frac{6}{\delta} \int_{s-2\delta}^{s+2\delta} \mu_v(\|v(s) - v(\tau)\|) \, \mathrm{d}\tau \right) \mathrm{d}s. \tag{65}$$

For the integral I_2 in (61), using (34), we obtain

$$I_2 \le \int_{t_0}^{t} L\|x(s) - y(s)\| \, \mathrm{d}s. \tag{66}$$

According to (61), (65) and (66), we have

$$\|x(t)-y(t)\| \leq \int_{t_0}^{t} L\|x(s)-y(s)\|\,\mathrm{d}s$$
$$+2\varkappa\delta + \sup_{v'(\cdot)\in V}\left\{\int_T \mu_{uv}\left(4\psi(2\delta)+\frac{6}{\delta}\int_{s-2\delta}^{s+2\delta}\mu_v(\|v'(s)-v'(\tau)\|)\,\mathrm{d}\tau\right)\mathrm{d}s\right\}$$
$$:= \int_{t_0}^{t} L\|x(s)-y(s)\|\,\mathrm{d}s + \Phi(\delta).$$

Due to choice (60) of Δ, we get $\Phi(\delta) \leq \xi_*$. Then, applying the Bellman–Gronwall lemma, we deduce $\|x(t)-y(t)\| \leq \xi_* \exp\left(L(\vartheta - t_0)\right)$, $t \in T$. Therefore, according to (36), inequality (57) holds. Lemma 2 is proved. □

8 Examples

In this section, we give examples illustrating the availability for numerical realization of the proposed in the paper (see Sects. 5 and 7) solution of the guarantee optimization problem for system (1), initial condition (2) and quality index (4) under a compact functional constraint on the disturbance $\mathbb{V}$. The most difficult part in this solution is to construct the ε-optimal counter-strategy with full memory $\overline{\mathbf{U}}_\varepsilon$. With few exceptions, optimal strategies in guarantee optimization problems (differential games) are hard to calculate. However, there are some classes of the problems for which effective procedures are known for calculating the value of the optimal guaranteed result (the game value) and, as a consequence, for constructing the corresponding optimal strategies. For example, in the so-called linear-convex case, we can apply the upper convex hulls method [1,4] (see also [9]). We use this method in Examples 2 and 3.

Example 1 The first example shows that Property 1 is essential in Lemma 2 and Theorem 2. Let a motion of a dynamical system be described by the equation

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = u(t)v(t), \quad t \in [0,1], \quad x(t) \in \mathbb{R}, \quad u(t) \in \{0,1\}, \quad v(t) \in \{-1,1\}, \tag{67}$$

with the initial condition $x(0) = 0$, and let $\gamma = x(1)$ be a quality index. Note that system (67) does not satisfy Property 1. In this problem, the optimal guaranteed result in the class of quasi-strategies (6) is $\Gamma^0_{\mathbf{QS}}(0) = 0$, and the counter-strategy $\overline{\mathbf{U}}_0(v) = 0$ for $v = 1$ and $\overline{\mathbf{U}}_0(v) = 1$ for $v = -1$ is optimal. Suppose that a set V from a compact functional constraint $\mathbb{V}$ consists of the only one function $v(t) = 1$, $t \in [0,1]$. We consider a partition Δ of the time interval $[0,1]$ with the constant step $\delta = \mathbf{D}(\Delta)$ and define piecewise constant realizations $u(\cdot)$, $\overline{u}(\cdot)$ and $\overline{v}(\cdot)$ by the following rule:

$$u_i = \overline{u}_i = \begin{cases} 1, & \text{if } i \text{ is even}, \\ 0, & \text{otherwise}, \end{cases} \qquad \overline{v}_i = \begin{cases} -1, & \text{if } i \text{ is even}, \\ 1, & \text{otherwise}, \end{cases} \qquad i \in 0...(n_\Delta - 1).$$

One can verify that such $u(\cdot)$, $\overline{u}(\cdot)$ and $\overline{v}(\cdot)$ satisfy relations (52)–(54). Substituting the realizations $u(\cdot)$ and $v(\cdot)$ into system (67), we obtain

$$\gamma = x(1) \geq 1/2 - \delta/2.$$

Thus, in this problem, the control procedure with guide (52)–(54), and, therefore, strategy $\widehat{\mathbf{U}}_\varepsilon$ (55), (56), does not guarantee for the quality index γ the value $\Gamma^0_{\mathbf{QS}}(0)$. So, the assertion of Theorem 2 does not hold in this example. The analysis of the corresponding motion of the guide shows that the assertion of Lemma 2 does not hold here either.

Example 2 Let a motion of a dynamical system be described by the equations

$$\begin{cases} \dfrac{\mathrm{d}x_1(t)}{\mathrm{d}t} = u_1(t)(v_1(t) + v_2(t)), & t \in [0, 2], \\ \dfrac{\mathrm{d}x_2(t)}{\mathrm{d}t} = u_2(t)v_1(t)v_2(t), & x(t) = \big(x_1(t), x_2(t)\big) \in \mathbb{R}^2, \end{cases} \tag{68}$$

and the initial condition $x(0) = (0, 0)$. Let the geometric constraints on the control and disturbance have the form

$$P := \big\{(u_1, u_2) \in \mathbb{R}^2 : \ 0.5 \le |u_i| \le 1.5, \ i = 1, 2\big\},$$
$$Q := \big\{(v_1, v_2) \in \mathbb{R}^2 : \ 1 \le v_1^2 + v_2^2 \le 4\big\}.$$

Table 1 The results of the numerical simulation in Example 2

δ	0.05	0.01	0.002	0.0004
γ	4.1289	**1.7502**	1.4791	**1.4694**
$\|x(\cdot) - y(\cdot)\|$	3.1970	**0.9695**	0.1657	**0.0361**

In bold, we mark the results illustrated in Figs. 3 and 4

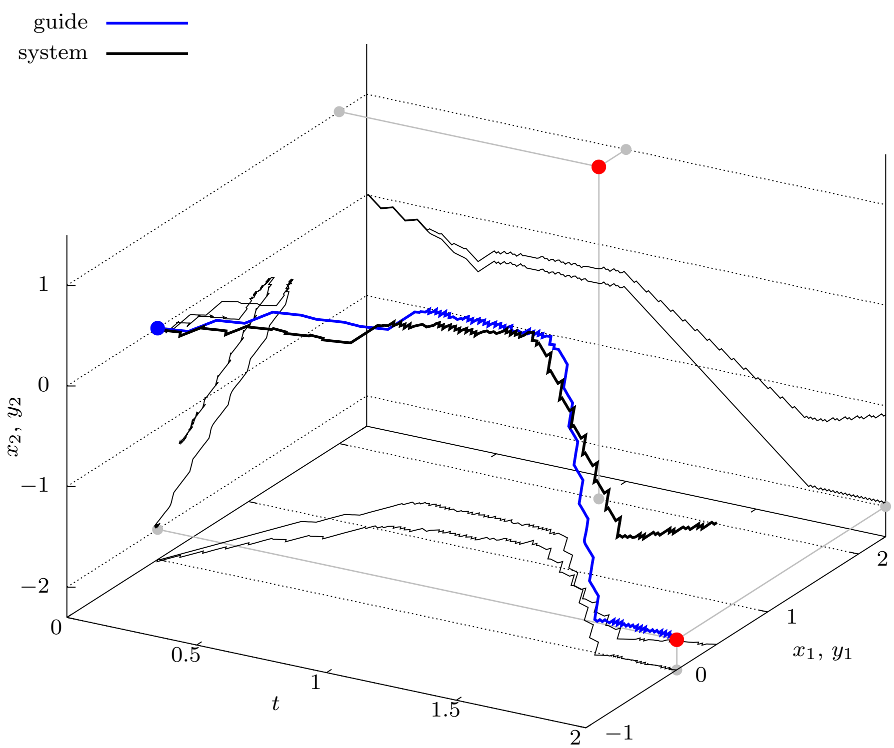

Fig. 3 Example 2. The motions of the system and guide: $\delta = 0.01$

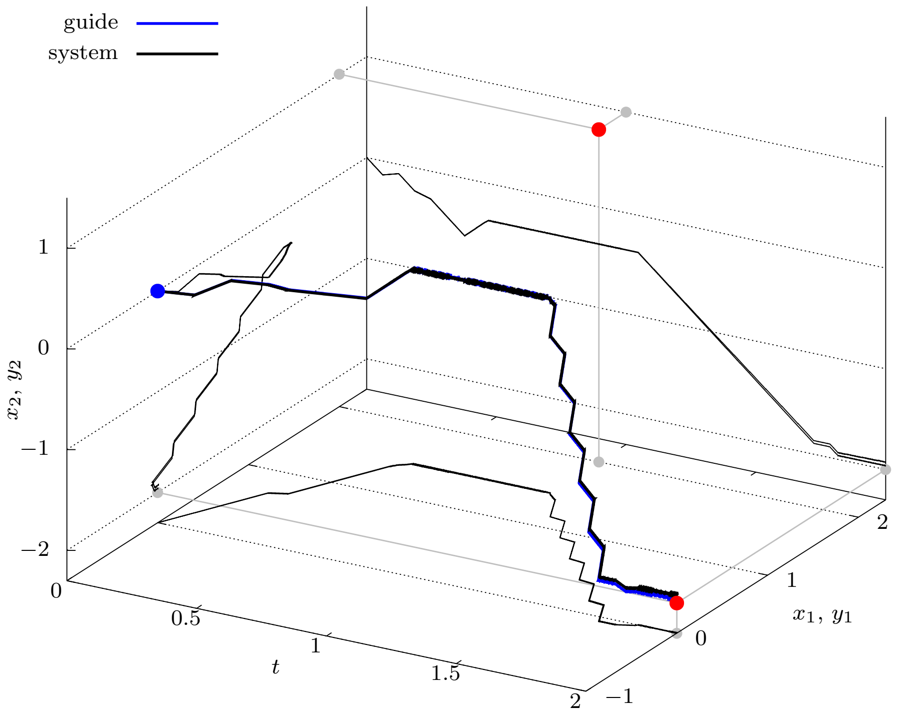

Fig. 4 Example 2. The motions of the system and guide: $\delta = 0.0004$

Let us consider the quality index

$$\gamma := \sqrt{(x_1(1) - 2)^2 + (x_2(1) - 1)^2 + x_1^2(2) + (x_2(2) + 2)^2}. \tag{69}$$

Note that the right-hand side of system (68) is not injective with respect to $v = (v_1, v_2)$. But since the system is of form (49), and the corresponding kernel is constant and equal to $\{(0, 0)\}$, system (68) satisfies Property 1. Moreover, one can show that system (68) does not satisfy condition (14).

Let a set V from a compact functional constraint on the disturbance $\mathbb{V}$ consist of all functions from $[0, 2]$ to Q that are piecewise constant on the partition of $[0, 2]$ with the constant step 0.05. So, the number of possible switchings of the disturbance is not greater than 40. Note that, for the chosen Q, the set V is compact in $L_1([0, 2], \mathbb{R}^2)$.

In simulation below we use the strategy with full memory $\widehat{\mathbf{U}}_\varepsilon$ described in Sect. 7. We construct the corresponding ε-optimal counter-strategy $\overline{\mathbf{U}}_\varepsilon$ and the value of the optimal guaranteed result $\Gamma^0_{\mathbf{QS}}(0, 0)$ on the basis of the upper convex hulls method. Furthermore, we simulate disturbance realizations on the basis of the optimal counter-strategy of the disturbance (the second player), which is also constructed by the upper convex hulls method. The step δ of the partition Δ used in the corresponding control with full memory $\widehat{\mathbf{U}}^\Delta_\varepsilon$ we vary within the set $\{0.05, 0.01, 0.002, 0.0004\}$.

For the value of optimal guaranteed result in the class of strategies with full memory under the compact functional constraint $\mathbb{V}$, we obtain

$$\Gamma^0_{\mathbf{S}}(0, 0 \mid \mathbb{V}) = \Gamma^0_{\mathbf{QS}}(0, 0) \approx 2.8760.$$

Table 2 The results of the numerical simulation in Example 3

δ	0.05	0.01	0.002	0.0004
γ	5.7575	**1.8410**	1.7109	**1.7389**
$\|x(\cdot) - y(\cdot)\|$	4.4871	**0.5631**	0.1739	**0.0486**

In bold, we mark the results illustrated in Figs. 5 and 6

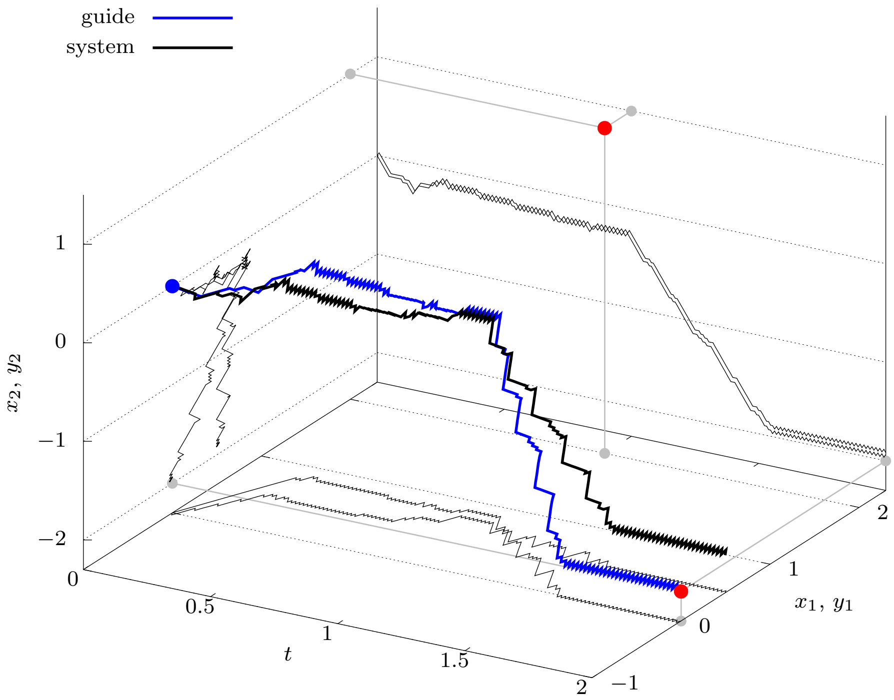

Fig. 5 Example 3. The motions of the system and guide: $\delta = 0.01$

The results of the numerical simulation are presented in Table 1, where δ is the step of the partition Δ; $\|x(\cdot) - y(\cdot)\|$ is the maximal distance between the motions of the system $x(\cdot)$ and the guide $y(\cdot)$; γ is the realized value of quality index (69). The motions of the system and guide for $\delta = 0.01$ and $\delta = 0.0004$ are shown in Figs. 3 and 4, respectively.

Example 3 Let us consider the same guarantee optimization problem as in Example 2 but with the geometrical constraints

$$P := \{u \in \mathbb{R}^2 : u \in \{(-1.1), (-1, 0), (1, 0), (1, -1)\}\},$$
$$Q := \{(v_1, v_2) \in \mathbb{R}^2 : |v_1|, |v_2| \in \{0.5, 2\}\}.$$

One can show that Property 1 is not fulfilled in this case. Therefore, in the numerical simulations, we use the strategy with full memory $\mathbf{U}_\varepsilon$ described in Sect. 5. In (20), we put $n_\varepsilon = 4$ and choose the whole set P as its ε-net. In (22), we choose $\varepsilon = 0.01$.

For the corresponding value of the optimal guaranteed result, we obtain

$$\Gamma^0_{\mathbf{S}}(0, 0 \mid \mathbb{V}) = \Gamma^0_{\mathbf{QS}}(0, 0) \approx 2.8359.$$

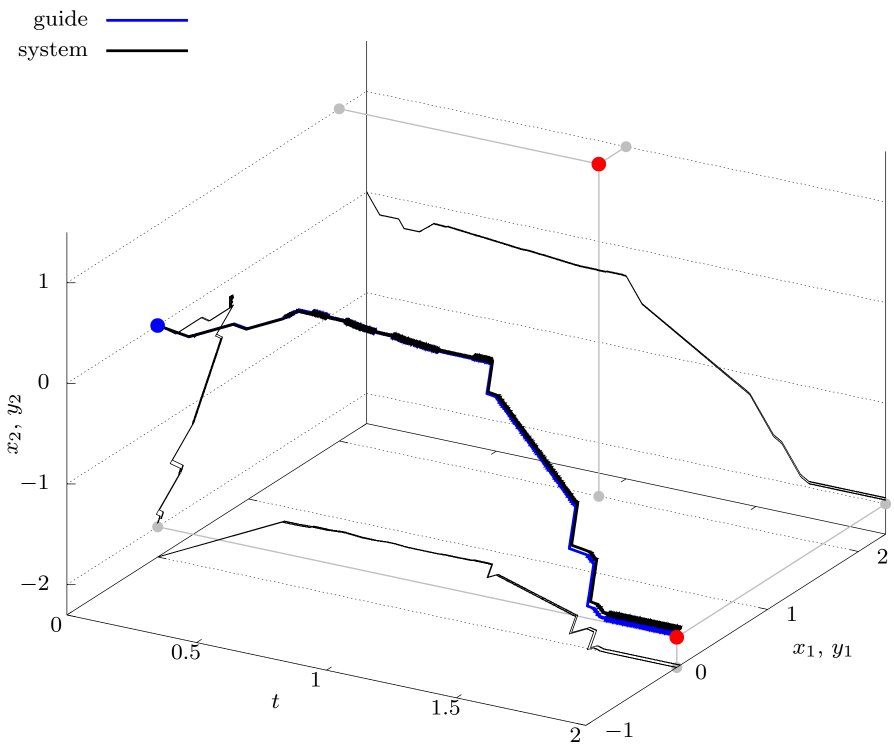

Fig. 6 Example 3. The motions of the system and guide: $\delta = 0.0004$

The results of the numerical simulation are presented in Table 2. The motions of the system and guide for $\delta = 0.01$ and $\delta = 0.0004$ are shown in Figs. 5 and 6, respectively.

Acknowledgements We thank the referees for their careful reading and their remarks that allowed us to improve the paper.

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