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Optimal Abatement and Emission Permit Trading Policies in a Dynamic Transboundary Pollution Game

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Abstract We obtain optimal emission levels and abatement expenditures in a finite-horizon transboundary pollution game with emission trading between two regions. We show that emission trading has significant impact on the optimal strategies and profits of the two regions. We find that cooperation between the regions leads to increased abatement and lower emissions, resulting in a lower pollution stock. We also provide a stochastic extension in which the pollution stock and the emission trading price are diffusion processes and solve it numerically.

Keywords Transboundary pollution · Differential game · Pollution abatement strategies · Emission permits trading · Hamilton–Jacobi–Bellman equation

1 Introduction

In recent years, politicians and scholars have been focusing on the negative impact of rapid technological advancement and economic growth, as different types of pollution and environmental degradation have reached alarming levels. At the same time, the transboundary or the

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Suresh Sethi taught an optimal control course as a visiting professor at the Technical University in Vienna in the summer of 1981. Englelbert Dockner was a student in this course. Over the years, Suresh Sethi had many interactions with Engelbert and his family. We are honored to dedicate to the memory of Engelbert Dockner our paper on a topic to which he contributed significantly.

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cross-border pollution has also received special attention in the literature (see, for example, Benchekroun and Chaudhuri [3], Hall [14], Youssef [28], and references therein). Just as the name implies, transboundary pollution is the pollution that generates in one region and can affect the surrounding regions via water and air.

To understand the relationship between the accumulation of pollution and the region's strategies, we model the underlying problem as a differential game. The goals of the players or the neighboring regions with symmetric information in the transboundary pollution game are to maximize their respective profits by choosing their optimal allocations of emission permits and the amounts of pollution abatement. Dockner and Long [11] examine the optimal behavior of two neighboring regions in a game setting and conclude that when the two regions use only linear strategies, both regions suffer a loss caused by their competitive actions. However, when nonlinear Markov-perfect strategies are used, and the discount rate is small enough, then a subgame-perfect equilibrium can be reached. Yeung [25] studies a pollution management problem in a cooperative stochastic differential game framework, in which he proposes a payment distribution mechanism that would yield a time-consistent solution. List and Mason [19] conclude that the cooperative payoff is larger than the noncooperative one in a decentralized control setting, if the individual payoffs are sufficiently heterogeneous and the initial pollution stocks are sufficiently small. Kossioris et al. [16] use a numerical method to obtain a nonlinear feedback Nash equilibrium (FBNE) in a shallow lake pollution context and show that the equilibrium trajectory of the best FBNE is, in general, worse than the open-loop and optimal management solutions. Maler and Zeeuw [21] study open-loop and feedback Nash equilibria in an acid rain differential game and find that the depositions always converge to the critical loads. Benchekroun and Martín-Herrán [4] consider both farsighted and myopic behaviors in a transboundary pollution game. Their results suggest that it is necessary to design incentive mechanisms to induce a country to decide in a farsighted way and thereby to increase the number of farsighted countries. Huang et al. [15] present a cooperative differential game of transboundary industrial pollution, which involves a Stackelberg game between the industrial firms and their local governments. They provide a payment distribution mechanism which guarantees that cooperation would result in higher total payoff.

With increasing environmental awareness, every country in the world realizes the necessity of investing in abatement technologies. Besides, the environmental policies also give polluting firms an incentive to do R&D and invest in cleaner ways of production, to reduce their compliance costs. Therefore, some research has been done to examine the abatement decision in the process of environmental degradation. For example, Lundgren [20] presents a real option framework to explain the reasons behind producers voluntarily taking abatement actions and examining the relationship between the abatement decision and regulations, competitors' strategies, uncertainty about future green goodwill, etc. Farzin and Kort [13] study the effect of a higher pollution tax rate on abatement investment both under full certainty and when the timing and/or the size of the tax increase is uncertain. Bertinelli et al. [5] propose a differential game framework for transboundary CO_2 pollution to study the strategic behaviors of the players. In their proposed framework, the reduction of CO_2 concentration can be realized by taking advantage of the carbon capture and storage techniques. They find that if there is a high enough level of CO_2 at the beginning, then the optimal feedback strategies can lead to a higher overall environmental quality.

Furthermore, it is widely known that most pollution types are caused by over-emissions of industrial waste. To reduce greenhouse gas emissions, many regions have established and developed emission permits trading markets in recent years. In the cap-and-trade scheme, each emitter is allocated a total amount of emission, i.e., an initial quota. An emitter can sell the unused quotas in the market to other permit seekers or buy emission permits from the market if his emission level exceeds his assigned quota. This scheme reduces emission possible by adjusting the emission permits among the emitters through market means. The emission permits trading schemes have been studied by researchers including Chang et al. [6], Daskalakis et al. [12] and Seifert et al. [23], and its effects on the equilibrium of differential game have also been examined in Chang et al. [7] and Li [17].

Motivated by the above discussions, this article studies the two regions dynamic optimal strategies in transboundary pollution differential games, in which the abatement policy and emission trading are involved. By solving the derived Hamilton–Jacobi–Bellman (HJB) equations satisfied by the value functions, we work out the players' optimal emission levels and optimal pollution abatement strategies in a finite-horizon game. Moreover, we solve a stochastic extension by numerical means. Our results show that emission trading has great impact on the optimal strategies and profits of the two regions. We show that cooperation between the players leads to greater abatement, fewer emissions, as well as fewer accumulated stock of pollution. Apart from the fact that cooperation is better than noncooperation, which can also be readily shown from a simple static prisoner's dilemma, the optimal trajectory of the pollution stock is obtained from the complex dynamic game framework. Compared with the similar models without abatement, due to Chang et al. [7] and Li [17], our results demonstrate that the introduction of abatement can reduce the pollution stock. Specially, owing to the presence of the terminal point besides the stable states, our results can also provide the evolutions of emissions, abatements and revenues of the players and show that a higher unit salvage cost incentivizes the players to reduce emission, increase abatement, and then further reduce costs from the emission permits trading scheme. The results leads to an increase in net revenue that is absent in Li [17]. Additionally, in the stochastic case, different from the results in Chang et al. [7], our work shows that it is always beneficial for the players to choose cooperation. For the theory of differential games, we refer to Basar and Olsder [2], Dockner et al. [10] and Sethi and Thompson [24].

The remainder of the paper is organized as follows. In Sect. 2, the basic model is developed. Noncooperative and cooperative games are formulated in Sects. 3 and 4, respectively. Some discussion is provided with a few numerical examples in Sect. 5. A stochastic extension is treated in detail in Sect. 6. Lastly, Sect. 7 concludes the paper.

2 Basic Model

Similar to Li [17], we suppose that there are two nonidentical players (nations or regions) in our game framework. Also, the relationship between production revenue R_i and emission $E_i(t)$ is as follows:

$$R_i(E_i(t)) = A_i E_i(t) - \frac{1}{2} E_i^2(t), \tag{1}$$

$i = 1, 2$, where $A_i > 0$ is a constant, $E_i(t)$ denotes the emission of region i at time t, and $E_i(t) \in [0, A_i]$. This guarantees that R_i is an increasing concave function. Following Chang et al. [7], we set $A_1 = A$ and $A_2 = \alpha A$, where α is a positive parameter that measures the gap between the two players' abilities in obtaining benefits from production.

Then, we denote $P(t)$ as the accumulated amount of pollution in the air at time t. Then, the evolutionary process of $P(t)$ can be expressed as follows:

$$\dot{P}(t) = E_1(t) + E_2(t) - \theta P(t), \quad P(0) = P_0, \quad P(t) > 0, \tag{2}$$

where $E_1(t) \geq 0$ and $E_2(t) \geq 0$ denote the two players' emission rates at time t, and θ represents the exponential decay rate of pollution. According to Chang et al. [7], we suppose that the loss suffered by player i is a linear function of $P(t)$, namely $D_i P(t)$, where D_i is a strictly positive parameter. Without loss of generality, we let $D_1 = D$ and $D_2 = \beta D$ where β is a positive constant that measures the gap between the two players' abilities in suffering damages from pollution.

In recent years, with the signing of climate documents such as Kyoto protocol and Paris agreement, more and more countries in the world are beginning to regard abatement as a necessary issue in the process of development. Similar to Bertinelli et al. [5], by employing the abatement strategy, the regions can reduce the accumulated amount of pollution, that is

$$\dot{P}(t) = E_1(t) - a_1(t) + E_2(t) - a_2(t) - \theta P(t), \quad P(0) = P_0, \quad P(t) > 0, \tag{3}$$

where $a_1(t)$ and $a_2(t)$ are the region 1's and region 2's amounts of pollution abatement, respectively, and the two regions can suffer fewer pollution damages. However, the two regions should also face the costs of abatement which reduce the flow of net revenues. As is standard in economics, we assume the abatement cost to be convex. Specially, we assume it to be $\frac{1}{2}C_i a_i^2(t), i = 1, 2$, where C_i is a positive constant and can be regarded as the abatement cost coefficient. Similarly, we set $C_1 = C$ and $C_2 = \gamma C$, where γ measures the gap between the two regions' abilities in mastering the abatement technology.

In addition, the region i's cost from the emission permits trading scheme is given by $S(E_i(t) - E_{i0}(t) - a_i(t))$, at time t, where $S \geq 0$ is a given constant permit price and $E_{i0}(t)$ is the instantaneous permits quota allocated by the emission regulatory authorities. Specifically, $E_i(t) - E_{i0}(t) - a_i(t) > 0$ means that region i buys the permits from others who have unused permits and $E_i(t) - E_{i0}(t) - a_i(t) < 0$ means that region i sells its unused permits to other permit seekers in the market. We assume that the quota assigned to region i is exponentially decreasing over time at the rate $\rho_i > 0$, i.e., $E_{i0}(t) = E_{i0}e^{-\rho_i t}$. This assumption is entirely reasonable given what is observed in many emission trading schemes. Additionally, it encourages the players to reduce emission over time. We should also mention that the assumption of the instantaneous quota has been widely used in the environmental management and operations literature; see, e.g., Dobos [8], Dobos [9], Li [17], Chang et al. [7], Li [18], Zhang et al. [29].

The assumption of the exogenous emission permit price S, on the other hand, is a strong one. However, it is appropriate in a variety of cases. In our model, the two regions are neighbors such as countries, states, or cities and the pollutant can be gas, liquid or solid. Hence, the two players of any sizes are the contributors to the local pollution dynamics. If these are, say, located in Europe, they may not have much influence on the permits price in the big European Union permits market. In the case, when the two regions are large such as continents, we can in view of the rapid economic development and increasing globalization envision a future when there would be a global emission permit trading market, with the continental polluters having little influence on the emission permits price. A particular case (suggested by a referee) is that of carbon pricing mechanisms that work a little like feebates. If a region exceeds its target, it pays a fixed unit tax per unit exceeding the target. If it emits less than its target, it gets a fixed per unit subsidy per unit below the target. In our model, this target for region i can be thought of as its quota $E_{i0}(t)$, while S is the fixed per unit fee or rebate. A good instance of this situation is the "Carbon Competitiveness Incentive Regulation" in Canada's Alberta Province. Finally, we note that our way of modeling emission trading is an extension of Li [17] in that we allow both the terminal salvage cost and the abatement policy.

The goal of region i is to come up with the optimal emission level and the optimal abatement strategy that maximize the discounted stream of revenue and cost over a finite horizon T: production revenue, costs from the emission permits trading, pollution damages, and abatement costs. Additionally, according to Yeung and Petrosyan [26], we assume that at the end of the game, the region i's salvage cost of dealing with the excess pollution stock is $g_i(P(T) - \bar{P}_i)e^{-rT}$, where g_i is the unit cost, $\bar{P}_i$ denotes the datum pollution stock, and $P(T) - \bar{P}_i$ is the excess pollution stock which needs to be treated. Then, the optimal control problem of region i is given by

$$\max_{E_i(t)\geq 0, a_i(t)} \int_0^{\mathrm{T}} e^{-rt}\bigg[(A_i - S)E_i(t) - \frac{1}{2}E_i^2(t) + SE_{i0}e^{-\rho_i t} + Sa_i(t) - D_i P(t) - \frac{1}{2}C_i a_i^2(t)\bigg]\mathrm{d}t - g_i(P(T) - \bar{P}_i)e^{-rT},$$

subject to

$$\dot{P}(t) = E_1(t) - a_1(t) + E_2(t) - a_2(t) - \theta P(t), \qquad P(0) = P_0,$$
$$P(t) \geq 0, t \in [0, T],$$

where $r > 0$ is the risk-free discount rate.

In the next two sections, we will use the optimal control theory and HJB equations to find the optimal emission levels and optimal abatement strategies, such that the two players' discounted flows of profits are maximized under cooperative and the noncooperative games, respectively.

3 Noncooperative Game

Each player in a noncooperative game makes his own decision to maximize his own profits. Thus, the players seek a Nash equilibrium to obtain their optimal emission levels and optimal abatement strategies. That is, the problem of region 1 is described by

$$\max_{E_{N1}(t)\geq 0, a_{N1}(t)} \int_0^{\mathrm{T}} e^{-rt}\bigg[(A - S)E_{N1}(t) - \frac{1}{2}E_{N1}^2(t) + SE_{10}e^{-\rho_1 t} + Sa_{N1}(t) - DP(t) - \frac{1}{2}Ca_{N1}^2(t)\bigg]\mathrm{d}t - g_1(P(T) - \bar{P}_1)e^{-rT},$$

subject to $\dot{P}(t) = E_{N1}(t) - a_{N1}(t) + E_{N2}(t) - a_{N2}(t) - \theta P(t), \quad P(0) = P_0, \quad P(t) \geq 0,$ (4)

and that of region 2 is governed by

$$\max_{E_{N2}(t)\geq 0, a_{N2}(t)} \int_0^{\mathrm{T}} e^{-rt}\bigg[(\alpha A - S)E_{N2}(t) - \frac{1}{2}E_{N2}^2(t) + SE_{20}e^{-\rho_2 t} + Sa_{N2}(t) - \beta DP(t) - \frac{1}{2}\gamma Ca_{N2}^2(t)\bigg]\mathrm{d}t - g_2(P(T) - \bar{P}_2)e^{-rT},$$

subject to $\dot{P}(t) = E_{N1}(t) - a_{N1}(t) + E_{N2}(t) - a_{N2}(t) - \theta P(t), \quad P(0) = P_0, \quad P(t) \geq 0,$

where $E_{N1}(t)$ and $E_{N2}(t)$ denote the emission levels of regions 1 and 2, and $a_{N1}(t)$ and $a_{N2}(t)$ denote the abatement levels of regions 1 and 2 in the noncooperative game, respectively.

By using the differential games theory, we can obtain the system of HJB equations satisfied by the value functions V_{N1} and V_{N2} for regions 1 and 2 as follows:

$$\begin{cases} \max\limits_{E_{N1}(t)\geq 0, a_{N1}(t)} \left\{ \frac{\partial V_{N1}}{\partial t} + \Big(E_{N1}(t) - a_{N1}(t) + E_{N2}(t) - a_{N2}(t) - \theta P(t) \Big) \frac{\partial V_{N1}}{\partial P} - rV_{N1} + F_{N1} \right\} = 0, \\ \max\limits_{E_{N2}(t)\geq 0, a_{N2}(t)} \left\{ \frac{\partial V_{N2}}{\partial t} + \Big(E_{N1}(t) - a_{N1}(t) + E_{N2}(t) - a_{N2}(t) - \theta P(t) \Big) \frac{\partial V_{N2}}{\partial P} - rV_{N2} + F_{N2} \right\} = 0, \end{cases} \tag{5}$$

with the terminal conditions

$$V_{N1}(P,T) = -g_1\left(P - \bar{P}_1\right) \quad \text{and} \quad V_{N2}(P,T) = -g_2\left(P - \bar{P}_2\right), \tag{5a}$$

where

$$F_{N1} = (A-S)E_{N1}(t) - \frac{1}{2}E_{N1}^2(t) + SE_{10}e^{-\rho_1 t} + Sa_{N1}(t) - DP(t) - \frac{1}{2}Ca_{N1}^2(t),$$

and

$$F_{N2} = (\alpha A-S)E_{N2}(t) - \frac{1}{2}E_{N2}^2(t) + SE_{20}e^{-\rho_2 t} + Sa_{N2}(t) - \beta DP(t) - \frac{1}{2}\gamma Ca_{N2}^2(t).$$

According to the first-order optimality condition, we know that the optimal emission levels E_{N1}^* and E_{N2}^*, and the optimal abatement levels a_{N1}^* and a_{N2}^* can be given by the following equations:

$$E_{N1}^* = A - S + \frac{\partial V_{N1}}{\partial P}, a_{N1}^* = \frac{1}{C}\Big(S - \frac{\partial V_{N1}}{\partial P}\Big), \tag{6a}$$

$$E_{N2}^* = \alpha A - S + \frac{\partial V_{N2}}{\partial P}, a_{N2}^* = \frac{1}{\gamma C}\Big(S - \frac{\partial V_{N2}}{\partial P}\Big). \tag{6b}$$

Proposition 1 *In the noncooperative game,* $\{V_{\mathrm{Ni}}(P,t), E_{\mathrm{Ni}}^*(t), a_{\mathrm{Ni}}^*(t)\}$, $i = 1, 2$, *denote the Nash equilibrium solutions of the value functions and the control variables. Then, we have*

$$V_{N1}(P,t) = l_1(t)P + k_1(t), \tag{7a}$$

$$E_{N1}^*(t) = \begin{cases} A - S + l_1(t) & S < A + \left(\frac{D}{r+\theta} - g_1\right)e^{-(r+\theta)(T-t)} - \frac{D}{r+\theta}, \\ 0 & S \geq A + \left(\frac{D}{r+\theta} - g_1\right)e^{-(r+\theta)(T-t)} - \frac{D}{r+\theta}, \end{cases} \tag{7b}$$

$$a_{N1}^*(t) = \frac{1}{C}\left(S - l_1(t)\right), \tag{7c}$$

$$V_{N2}(P,t) = l_2(t)P + k_2(t), \tag{7d}$$

$$E_{N2}^*(t) = \begin{cases} \alpha A - S + l_2(t) & S < \alpha A + \left(\frac{\beta D}{r+\theta} - g_2\right)e^{-(r+\theta)(T-t)} - \frac{\beta D}{r+\theta}, \\ 0 & S \geq \alpha A + \left(\frac{\beta D}{r+\theta} - g_2\right)e^{-(r+\theta)(T-t)} - \frac{\beta D}{r+\theta}, \end{cases} \tag{7e}$$

$$a_{N2}^*(t) = \frac{1}{\gamma C}\left(S - l_2(t)\right), \tag{7f}$$

where $l_1(t)$, $l_2(t)$, $k_1(t)$, $k_2(t)$ *are given in "Appendix A" section.*

Proof See "Appendix A" section.

We can clearly see that the value functions are linear functions with respect to the state variable, namely the pollution stock P.

If we let

$$\eta = \min\left\{A + \left(\frac{D}{r+\theta} - g_1\right)e^{-(r+\theta)(T-t)} - \frac{D}{r+\theta},\right.$$

$$\alpha A+\left(\frac{\beta D}{r+\theta}-g_2\right)e^{-(r+\theta)(T-t)}-\frac{\beta D}{r+\theta}\Bigg\},$$

$$\phi=\frac{1-e^{-\theta(T-t)}}{1-e^{-r(T-t)}}\frac{r(2\gamma C+\gamma+1)}{\gamma\theta(C+1)}\left(\frac{D}{r+\theta}-g_1\right)e^{-r(T-t)}$$
$$+\frac{AC}{C+1}-\frac{D(2\gamma C+\gamma+1)}{\gamma(C+1)(r+\theta)},$$

$$\psi=\frac{1-e^{-\theta(T-t)}}{1-e^{-r(T-t)}}\frac{r(2\gamma C+\gamma+1)}{\theta(\gamma C+1)}\left(\frac{\beta D}{r+\theta}-g_2\right)e^{-r(T-t)}$$
$$+\frac{A\gamma C}{\gamma C+1}-\frac{\beta D(2\gamma C+\gamma+1)}{(\gamma C+1)(r+\theta)},$$

$$\tau_1(t)=\frac{rCe^{-\rho_1 T}\left(e^{\rho_1(T-t)}-e^{-r(T-t)}\right)}{(C+1)(1+\rho_1)(1-e^{-r(T-t)})},$$

and

$$\tau_2(t)=\frac{r\gamma Ce^{-\rho_2 T}\left(e^{\rho_2(T-t)}-e^{-r(T-t)}\right)}{(\gamma C+1)(1+\rho_2)(1-e^{-r(T-t)})},$$

then, some useful results are presented in the following corollaries. □

Corollary 1 (i) *When* $A-S+\left(\frac{D}{r+\theta}-g_1\right)e^{-(r+\theta)(T-t)}-\frac{D}{r+\theta}>E_{10}e^{-\rho_1 t}$, *region 1 should buy emission permits at time* t*; when* $A-S+\left(\frac{D}{r+\theta}-g_1\right)e^{-(r+\theta)(T-t)}-\frac{D}{r+\theta}<E_{10}e^{-\rho_1 t}$, *region 1 should sell emission permits at time* t*; and when* $A-S+\left(\frac{D}{r+\theta}-g_1\right)e^{-(r+\theta)(T-t)}-\frac{D}{r+\theta}=E_{10}e^{-\rho_1 t}$, *region 1 neither buys nor sells emission permits at time* t*. Similarly, when* $\alpha A-S+\left(\frac{\beta D}{r+\theta}-g_2\right)e^{-(r+\theta)(T-t)}-\frac{\beta D}{r+\theta}>E_{20}e^{-\rho_2 t}$, *region 2 should buy emission permits at time* t*; when* $\alpha A-S+\left(\frac{\beta D}{r+\theta}-g_2\right)e^{-(r+\theta)(T-t)}-\frac{\beta D}{r+\theta}<E_{20}e^{-\rho_2 t}$, *region 2 should sell emission permits at time* t*; and when* $\alpha A-S+\left(\frac{\beta D}{r+\theta}-g_2\right)e^{-(r+\theta)(T-t)}-\frac{\beta D}{r+\theta}=E_{20}e^{-\rho_2 t}$, *region 2 neither buys or sells emission permits at time* t.

(ii) *The profits of the two regions will increase with the decrease in pollution stock* P *and with increase in their instantaneous emission permits quotas at any time before* T.

(iii) *If* $\tau_1(t)E_{10}\geq\phi$, *region 1's profit should increase monotonically with increase in the permit price at time* t*; if* $\tau_1(t)E_{10}\leq\phi-\eta$, *region 1's profit should decrease monotonically with increase in the permit price at time* t*; if* $\phi-\eta\leq\tau_1(t)E_{10}\leq\phi$, *region 1's profit should increase monotonically with increase in permit price at time* t *when* $0<S<\phi-\tau_1(t)E_{10}$ *and decrease monotonically with increase in the permit price at time* t *when* $\phi-\tau_1(t)E_{10}<S<\eta$*. Similarly, if* $\tau_2(t)E_{20}\geq\psi$, *region 2's profit should increase monotonically with increase in the permit price at time* t*; if* $\tau_2(t)E_{20}\leq\psi-\eta$, *region 2's profit should decrease monotonically with increase in the permit price at time* t*; if* $\psi-\eta\leq\tau_2(t)E_{20}\leq\psi$, *region 2's profit should decrease monotonically with increase in the permit price at time* t *when* $0<S<\psi-\tau_2(t)E_{20}$, *and increase monotonically with increase in the permit price at time* t *when* $\psi-\tau_2(t)E_{20}<S<\eta$.

Proof See "Appendix B" section.

From the results in (iii) in Corollary 1, we can see that the instantaneous quotas play an essential role in examining the effect of emission permits price on the two regions' profits. If the instantaneous quotas are very large, then the emission levels will not exceed them, so the players in the game can sell their excess emission permits in the market and they benefit more from increases in the price of emission permits. If the two regions receive a fewer amounts of instantaneous quotas, then the emission levels can easily exceed them. In this case, the two regions have to spend money to buy emission permits and their profits will decrease with increases in the price of permits. If the amount of instantaneous quotas is adequate, the two regions' revenues will first decrease with increases in the price of permits and then increase with increases in the price of permits.

The following corollary shows the trajectories of the pollution stock along with an optimal emission path. Moreover, the evolution of the pollution stock is demonstrated to be very highly related to the initial pollution stock. □

Corollary 2 *Let*

$$X_N = \frac{1}{\theta}\left((1+\alpha)A - \frac{2\gamma C+\gamma+1}{\gamma C}S - \frac{C+1}{C}\frac{D}{r+\theta} - \frac{\gamma C+1}{\gamma C}\frac{\beta D}{r+\theta}\right)$$

and

$$Y_N = \frac{1}{r+2\theta}\left(\frac{C+1}{C}\left(\frac{D}{r+\theta}-g_1\right) + \frac{\gamma C+1}{\gamma C}\left(\frac{\beta D}{r+\theta}-g_2\right)\right).$$

Then we can obtain the trajectory of the pollution stock under the noncooperative game as follow:

$$P_N(t) = X_N + Y_N e^{-(r+\theta)(T-t)} + \left(P_0 - X_N - Y_N e^{-(r+\theta)T}\right)e^{-\theta t}. \tag{8}$$

In addition, when $P_0 < X_N + Y_N e^{-(r+\theta)T}(1+\frac{1}{\theta}(r+\theta)e^{(r+2\theta)t})$, the evolution of the pollution stock is an accumulative process; when $P_0 > X_N + Y_N e^{-(r+\theta)T}(1+\frac{1}{\theta}(r+\theta)e^{(r+2\theta)t})$, the evolution of the pollution stock is an dissipative process; when $P_0 = X_N + Y_N e^{-(r+\theta)T}(1+\frac{1}{\theta}(r+\theta)e^{(r+2\theta)t})$, the pollution stock is a constant.

Proof See "Appendix C" section. □

4 Cooperative Game

A cooperative game means that the players come to a binding agreement to make the coalition reach the optimum. In our cooperative game, the two regions find the optimal emission levels and optimal abatement strategies to maximize their joint net profit. The resulting problem can be written as follows:

$$\max_{\substack{E_{C1}(t)\geq 0, a_{C1}(t),\\ E_{C2}(t)\geq 0, a_{C2}(t)}} \int_0^{\mathrm{T}} e^{-rt}\Bigg[(A-S)E_{C1}(t) + (\alpha A - S)E_{C2}(t) + S(a_{C1}(t)+a_{C2}(t))$$
$$- \frac{E_{C1}^2(t)+E_{C2}^2(t)}{2} + (E_{10}e^{-\rho_1 t} + E_{20}e^{-\rho_2 t})S - (1+\beta)DP(t)$$
$$- \frac{1}{2}Ca_{C1}^2(t) - \frac{1}{2}\gamma C a_{C2}^2(t)\Bigg]\mathrm{d}t - \sum_{i=1,2} g_i(P(T)-\bar{P}_i)e^{-rT},$$

subject to $\dot{P}(t) = E_{C1}(t) - a_{C1}(t) + E_{C2}(t) - a_{C2}(t) - \theta P(t), \quad P(0) = P_0, \quad P(t) \geq 0,$

where $E_{C1}(t)$ and $E_{C2}(t)$ denote the emission levels of regions 1 and 2, and $a_{C1}(t)$ and $a_{C2}(t)$ denote the abatement levels of regions 1 and 2 in the cooperative game, respectively.

The corresponding HJB equation for the joint value function V_C can be written as follows:

$$\max_{\substack{E_{C1}(t)\geq 0, a_{C1}(t),\\ E_{C2}(t)\geq 0, a_{C2}(t)}} \left\{ \frac{\partial V_C}{\partial t} + (E_{C1}(t) - a_{C1}(t) + E_{C2}(t) - a_{C2}(t) - \theta P(t)) \frac{\partial V_C}{\partial P} - rV_C + F_C \right\} = 0, \tag{9}$$

with the terminal condition

$$V_C(P,T) = -g_1\left(P - \bar{P}_1\right) - g_2\left(P - \bar{P}_2\right), \tag{9a}$$

where

$$F_C = (A-S)E_{C1}(t) + (\alpha A - S)E_{C2}(t) - \frac{E_{C1}^2(t) + E_{C2}^2(t)}{2} + \left(E_{10}e^{-\rho_1 t} + E_{20}e^{-\rho_2 t}\right)S + S\left(a_{C1}(t) + a_{C2}(t)\right) - (1+\beta)DP(t) - \frac{1}{2}Ca_{C1}^2(t) - \frac{1}{2}\gamma Ca_{C2}^2(t).$$

From the first-order optimality condition, we know that the two regions' optimal emission levels E_{C1}^* and E_{C2}^*, and the optimal abatement levels a_{C1}^* and a_{C2}^* can be given by the following equations:

$$E_{C1}^* = A - S + \frac{\partial V_C}{\partial P}, \quad a_{C1}^* = \frac{1}{C}\left(S - \frac{\partial V_C}{\partial P}\right), \tag{10a}$$

$$E_{C2}^* = \alpha A - S + \frac{\partial V_C}{\partial P}, \quad a_{C2}^* = \frac{1}{\gamma C}\left(S - \frac{\partial V_C}{\partial P}\right). \tag{10b}$$

Proposition 2 *In the cooperative game,* $\{V_C(P,t), E_{Ci}^*(t), a_{Ci}^*(t)\}$, $i = 1, 2$, *denote the solutions of the value function and the control variables. Then,*

$$V_C(P,t) = l(t)P + k(t), \tag{11a}$$

$$E_{C1}^*(t) = \begin{cases} A - S + l(t) & S < A + \left(\frac{(1+\beta)D}{r+\theta} - g_1 - g_2\right)e^{-(r+\theta)(T-t)} - \frac{(1+\beta)D}{r+\theta}, \\ 0 & S \geq A + \left(\frac{(1+\beta)D}{r+\theta} - g_1 - g_2\right)e^{-(r+\theta)(T-t)} - \frac{(1+\beta)D}{r+\theta}, \end{cases} \tag{11b}$$

$$a_{C1}^*(t) = \frac{1}{C}\left(S - l(t)\right), \tag{11c}$$

$$E_{C2}^*(t) = \begin{cases} \alpha A - S + l(t) & S < \alpha A + \left(\frac{(1+\beta)D}{r+\theta} - g_1 - g_2\right)e^{-(r+\theta)(T-t)} - \frac{(1+\beta)D}{r+\theta}, \\ 0 & S \geq \alpha A + \left(\frac{(1+\beta)D}{r+\theta} - g_1 - g_2\right)e^{-(r+\theta)(T-t)} - \frac{(1+\beta)D}{r+\theta}, \end{cases} \tag{11d}$$

$$a_{C2}^*(t) = \frac{1}{\gamma C}\left(S - l(t)\right), \tag{11e}$$

where $l(t)$ *and* $k(t)$ *are given in "Appendix C" section.*

Proof See "Appendix D" section.

Letting

$$\bar{\eta} = \min\left\{A + \left(\frac{(1+\beta)D}{r+\theta} - g_1 - g_2\right)e^{-(r+\theta)(T-t)} - \frac{(1+\beta)D}{r+\theta},\right.$$
$$\left.\alpha A + \left(\frac{(1+\beta)D}{r+\theta} - g_1 - g_2\right)e^{-(r+\theta)(T-t)} - \frac{(1+\beta)D}{r+\theta}\right\},$$
$$\bar{\phi} = \frac{1-e^{-\theta(T-t)}}{1-e^{-r(T-t)}}\frac{r}{\theta}\left(\frac{(1+\beta)D}{r+\theta} - g_1 - g_2\right)e^{-r(T-t)}$$
$$+\frac{2\gamma C}{2\gamma C+\gamma+1}(1+\alpha)A - \frac{(1+\beta)D}{r+\theta},$$
$$\tau_{C1}(t) = \frac{2\gamma C r e^{-\rho_1 T}\left(e^{\rho_1(T-t)} - e^{-r(T-t)}\right)}{(r+\rho_1)(1-e^{-r(T-t)})(2\gamma C+\gamma+1)},$$

and

$$\tau_{C2}(t) = \frac{2\gamma C r e^{-\rho_2 T}\left(e^{\rho_2(T-t)} - e^{-r(T-t)}\right)}{(r+\rho_2)\left(1-e^{-r(T-t)}\right)(2\gamma C+\gamma+1)},$$

we have the following corollary. □

Corollary 3 (i) *When* $A - S + \left(\frac{(1+\beta)D}{r+\theta} - g_1 - g_2\right)e^{-(r+\theta)(T-t)} - \frac{(1+\beta)D}{r+\theta} > E_{10}e^{-\rho_1 t}$, *region 1 buys emission permits at time t; when* $A-S+\left(\frac{(1+\beta)D}{r+\theta} - g_1 - g_2\right)e^{-(r+\theta)(T-t)} - \frac{(1+\beta)D}{r+\theta} < E_{10}e^{-\rho_1 t}$, *region 1 sells emission permits at time t; and when* $A - S + \left(\frac{(1+\beta)D}{r+\theta} - g_1 - g_2\right)e^{-(r+\theta)(T-t)} - \frac{(1+\beta)D}{r+\theta} = E_{10}e^{-\rho_1 t}$, *region 1 neither buys nor sells the emission permits at time t. Similarly, when* $\alpha A - S + \left(\frac{(1+\beta)D}{r+\theta} - g_1 - g_2\right)e^{-(r+\theta)(T-t)} - \frac{(1+\beta)D}{r+\theta} > E_{20}e^{-\rho_2 t}$, *region 2 buys emission permits at time t; when* $\alpha A-S+\left(\frac{(1+\beta)D}{r+\theta} - g_1 - g_2\right)e^{-(r+\theta)(T-t)} - \frac{(1+\beta)D}{r+\theta} < E_{20}e^{-\rho_2 t}$, *region 2 sells emission permits at time t; and when* $\alpha A-S+\left(\frac{(1+\beta)D}{r+\theta} - g_1 - g_2\right)e^{-(r+\theta)(T-t)} - \frac{(1+\beta)D}{r+\theta} = E_{20}e^{-\rho_2 t}$, *region 2 neither buys nor sells emission permits at time t.*

(ii) *The joint profit of the two regions increases when the pollution stock P decreases or when their instantaneous emission permits quotas increase at any time before T.*

(iii) *If* $\tau_{C1}(t)E_{10}+\tau_{C2}(t)E_{20} \geq \bar{\phi}$, *then the two regions' total profit increases monotonically with the permit price at time t; if* $\tau_{C1}(t)E_{10}+\tau_{C2}(t)E_{20} \leq \bar{\phi}-2\bar{\eta}$, *the two regions' total profit decreases monotonically with the permit price at time t; if* $\bar{\phi} - 2\bar{\eta} < \tau_{C1}(t)E_{10} + \tau_{C2}(t)E_{20} < \bar{\phi}$, *the two regions' total profit decreases monotonically with the permit price at time t when* $0 < S < \frac{1}{2}(\bar{\phi}-\tau_{C1}(t)E_{10}-\tau_{C2}(t)E_{20})$ *and increases monotonically with the permit price at time t when* $\frac{1}{2}(\bar{\phi} - \tau_{C1}(t)E_{10} - \tau_{C2}(t)E_{20}) < S < \eta$.

Proof See "Appendix E" section.

In the cooperative game, if the sum of the instantaneous quotas is large, their joint profit increases with the permits prices, with fewer instantaneous quotas, their joint profit decreases with the permits price. If the sum of the instantaneous quotas is adequate, their joint profit first decreases and then increases with the permits price. □

Corollary 4 *Let*

$$X_C = \frac{1}{\theta}\left((1+\alpha)A - \frac{2\gamma C+\gamma+1}{\gamma C}S - \frac{2\gamma C+\gamma+1}{\gamma C}\frac{(1+\beta)D}{r+\theta}\right)$$

and

$$Y_C = \frac{1}{r+2\theta}\frac{2\gamma C+\gamma+1}{\gamma C}\left(\frac{(1+\beta)D}{r+\theta} - g_1 - g_2\right).$$

Then, the trajectory of the pollution stock in the cooperative game is:

$$P_C(t) = X_C + Y_C e^{-(r+\theta)(T-t)} + \left(P_0 - X_C - Y_C e^{-(r+\theta)T}\right)e^{-\theta t}. \tag{12}$$

In addition, when $P_0 < X_C + Y_C e^{-(r+\theta)T}\left(1+\frac{1}{\theta}(r+\theta)e^{(r+2\theta)t}\right)$, *the evolution of the pollution stock is an accumulative process; when* $P_0 > X_C + Y_C e^{-(r+\theta)T}\left(1+\frac{1}{\theta}(r+\theta)e^{(r+2\theta)t}\right)$, *the evolution of pollution stock is a dissipative process; when* $P_0 = X_C + Y_C e^{-(r+\theta)T}\left(1+\frac{1}{\theta}(r+\theta)e^{(r+2\theta)t}\right)$, *the pollution stock is a constant.*

Proof See "Appendix F" section. □

5 Discussion

In this section, we compare the optimal emission paths, the optimal abatement strategies and the value functions obtained in the two games.

5.1 Difference in the Optimal Strategies

First, we examine the difference in the optimal emission paths and the optimal abatement strategies. From Propositions 1 and 2 , we can obtain

$$\begin{aligned}
E^*_{C1} - E^*_{N1} &= \left(\frac{\beta D}{r+\theta} - g_2\right)e^{-(r+\theta)(T-t)} - \frac{\beta D}{r+\theta} < 0,\\
a^*_{C1} - a^*_{N1} &= -\frac{1}{C}\left(\left(\frac{\beta D}{r+\theta} - g_2\right)e^{-(r+\theta)(T-t)} - \frac{\beta D}{r+\theta}\right) > 0,\\
E^*_{C2} - E^*_{N2} &= \left(\frac{D}{r+\theta} - g_1\right)e^{-(r+\theta)(T-t)} - \frac{D}{r+\theta} < 0,\\
a^*_{C2} - a^*_{N2} &= -\frac{1}{\gamma C}\left(\left(\frac{D}{r+\theta} - g_1\right)e^{-(r+\theta)(T-t)} - \frac{D}{r+\theta}\right) > 0,
\end{aligned}$$

which imply that in both regions, the optimal emission levels under the cooperative game are lower than those under the noncooperative game, and the optimal abatement levels under the cooperative game are higher than those under the noncooperative game. The obvious reason is that under the noncooperative game, the two regions make decisions to maximize their respective profits. This creates a conflicting situation in which one region may try to reduce its pollution damage by reducing emissions or implementing pollution abatement only to find that its effort is offset by the other region increasing the emission level to obtain more production profit. This would not happen in the cooperative game because of their binding contract to cooperate.

Moreover, we also know from Propositions 1 and 2 that if the salvage cost coefficients g_1 and g_2 are sufficiently large, the optimal emission paths of the two regions are both decreasing over time and the optimal abatement levels of the two regions are both increasing over time in the noncooperative as well as in the cooperative game. This implies that when the two regions have to suffer from a high salvage cost, they are more willing to reduce emissions and implement pollution abatements to lower the pollution stock level.

5.2 Difference in the Optimal Trajectories of the Pollution Stock

According to (8) and (12), we can calculate the difference in the optimal trajectories of the pollution stocks as follows:

$$\begin{aligned} P_C(t) - P_N(t) &= (X_C - X_N) + (Y_C - Y_N)e^{-(r+\theta)(T-t)} \\ &\quad - \Big((X_C - X_N) + (Y_C - Y_N)e^{-(r+\theta)T}\Big)e^{-\theta t} \\ &= (X_C - X_N)\left(1 - e^{-\theta t}\right) + (Y_C - Y_N)\, e^{-(r+\theta)T} e^{rt}, \end{aligned}$$

where

$$X_C - X_N = -\frac{1}{C}\frac{1}{\theta}\frac{D}{r+\theta}\left(C + \frac{1}{\gamma} + \beta C + \beta\right) < 0$$

and

$$Y_C - Y_N = \frac{1}{r+2\theta}\left(\left(\frac{\gamma C + 1}{\gamma C} + \frac{\beta(\gamma C + \gamma)}{\gamma C}\right)\frac{D}{r+\theta} - \frac{\gamma C + 1}{\gamma C} g_1 - \frac{\gamma C + \gamma}{\gamma C} g_2\right).$$

Then, we can find a sufficiently large pair (g_1^*, g_2^*) to make $P_C(t) - P_N(t) < 0$ at each time t, since $P_C - P_N$ is a decreasing function of g_1 and g_2. This means that the pollution stock under the cooperative game will be lower than that under the noncooperative game. It naturally follows that any reduction in emission and increase in abatement can diminish the accumulation of the pollution stock.

5.3 Difference in the Optimal Net Revenues

From (7a), (7d) and (11a), we can calculate the difference in the optimal profits as follows:

$$\begin{aligned} V_C - V_{N1} - V_{N2} &= (G_C - G_{N1} - G_{N2}) + (H_C - H_{N1} - H_{N2})e^{-(r+\theta)(T-t)} \\ &\quad + (I_C - I_{N1} - I_{N2})e^{-2(r+\theta)(T-t)} - \Big((G_C - G_{N1} - G_{N2}) \\ &\quad + (H_C - H_{N1} - H_{N2}) + (I_C - I_{N1} - I_{N2})\Big)e^{-r(T-t)} \\ &= (G_C - G_{N1} - G_{N2})(1 - e^{-r(T-t)}) \\ &\quad + (H_C - H_{N1} - H_{N2})e^{-r(T-t)}\left(e^{-\theta(T-t)} - 1\right) \\ &\quad + (I_C - I_{N1} - I_{N2})e^{-r(T-t)}\left(e^{-(r+2\theta)(T-t)} - 1\right), \end{aligned}$$

where

$$\begin{aligned} G_C - G_{N1} - G_{N2} &= \frac{1}{2\gamma C}\frac{1}{r}\left(\frac{D}{r+\theta}\right)^2\left(\beta^2\gamma(1+C) + \gamma C + 1\right) > 0, \\ H_C - H_{N1} - H_{N2} &= \frac{1}{\theta}\frac{D}{r+\theta}\left(\frac{\beta^2(2\gamma C + \gamma + 1) + (1-\beta)(\gamma C + 1)}{\gamma C}\frac{D}{r+\theta}\right. \\ &\quad \left. - \frac{\gamma C + 1}{\gamma C} g_1 - \frac{\beta(\gamma C + \gamma)}{\gamma C} g_2\right), \end{aligned}$$

and

$$I_C - I_{N1} - I_{N2} = -\frac{1}{r+2\theta}\left(\frac{\gamma C + 1}{2\gamma C}\left(\frac{D}{r+\theta} - g_1\right)^2 + \frac{\gamma C + \gamma}{2\gamma C}\left(\frac{\beta D}{r+\theta} - g_2\right)^2\right) < 0.$$

Note that the coefficients $e^{-\theta(T-t)}-1$ and $e^{-(r+2\theta)(T-t)}-1$ are negative, which make $V_C - V_{N1} - V_{N2}$ to be an increasing function of g_1 and g_2 when $g_1 > \frac{D}{r+\theta}$ and $g_2 > \frac{\beta D}{r+\theta}$. Thus, we can also find a sufficiently large pair (g_1^{**}, g_2^{**}) to make $V_C - V_{N1} - V_{N2} > 0$. Therefore, we can say that when the salvage costs are large enough, it is better for the two regions to cooperate.

5.4 Numerical Example

In this subsection, we illustrate our results by some numerical examples. The parameters are: $T = 10$, $A = 20$, $\alpha = 0.9$, $E_{10} = 5$, $E_{20} = 6$, $\rho_1 = 0.05$, $\rho_2 = 0.04$, $\theta = 0.6$, $P_0 = 200$, $S = 1$, $D = 3$, $\beta = 1.2$, $C = 2$, $\gamma = 1.5$, $r = 0.08$, $g_1 = 8$, $g_2 = 9$, $\bar{P}_1 = 1100$, and $\bar{P}_2 = 1200$ (Chang et al. [7]).

5.4.1 Basic Results

The results of Propositions 1 and 2 are sketched in Fig. 1. The two regions' optimal emission levels and optimal abatement levels under the noncooperative and the cooperative games are shown in Fig. 1a, b, respectively. We can see that the emission levels and the abatement levels are relatively stable at the start, while the former shows an increasing trend and the latter shows a decreasing trend closer to the terminal time point. From Propositions 1 and 2, we know that the observed emission and abatement paths evolve in the opposite directions. This implies that in a dynamic transboundary pollution game with a terminal salvage cost, the players adopt relatively stable emission and abatement strategies most of the time and rapidly decrease emission and increase abatement closer to the expiration date to reduce the pollution stock and to avoid high salvage cost. This behavior could capture the initial revenues and reduce the loss due to pollution as much as possible.

In addition, in each region, the noncooperative emission level is higher than the cooperative emission level, and the noncooperative abatement level is lower than the cooperative abatement level. Figure 1c also demonstrates that the pollution stock under the noncooperative game is higher than that under the cooperative game. All these results imply that cooperation plays an important role in improving the environmental quality.

It is necessary to focus on the issue of how to distribute the aggregate profit to each player in the cooperative game. The payment distribution mechanism that we use in this paper is to share the aggregate profit by the proportion of their noncooperative profits. This can be expressed mathematically as

$$V_{\mathrm{Ci}} = \frac{V_{\mathrm{Ni}}}{V_{N1} + V_{N2}} V_C,$$

$i = 1, 2$, where V_{Ci} denotes the region i's value function in the cooperative game. According to Yeung [25], Yeung and Petrosyan [26] and Chang et al. [7], this payment distribution mechanism supports the subgame consistent solution, and in this case the two players should be committed to cooperation throughout the game. The two regions will cooperate when $V_C > V_{N1} + V_{N2}$ and the profit V_{Ci} in the cooperative game is higher than the profit V_{Ni} in the noncooperative game. Based on this, we can plot the two regions' value functions under the noncooperative and the cooperative games in Fig. 1d, which shows that the two regions would obtain higher profits when they cooperate. Therefore, we conclude that in both economic and environmental terms, cooperation is always better than noncooperation.

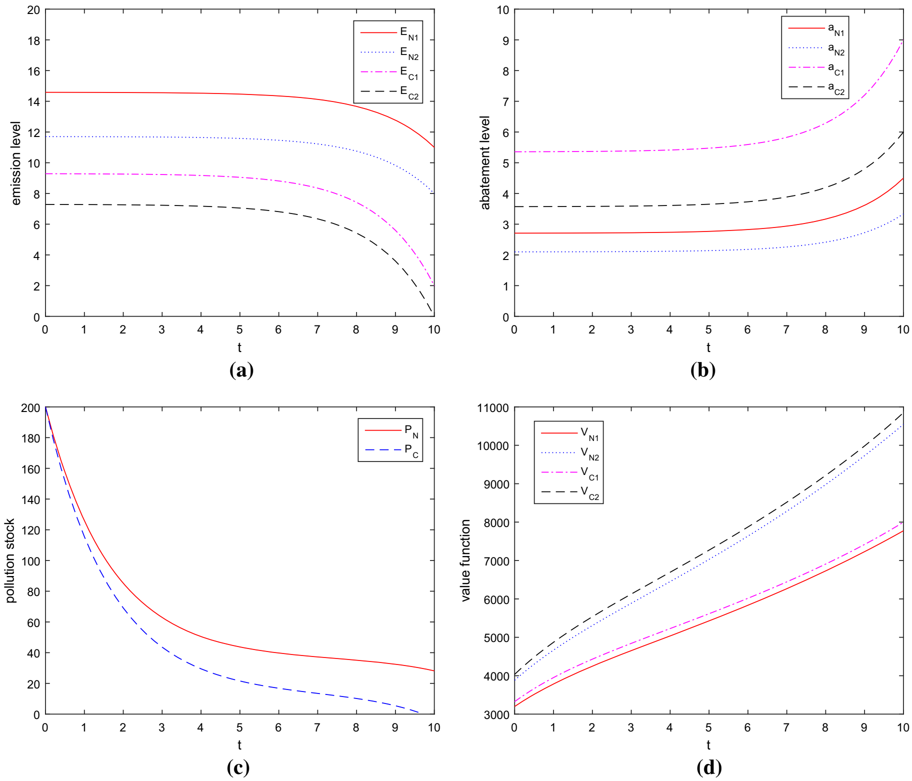

Fig. 1 Numerical results, **a** emission levels, **b** abatement levels, **c** pollution stocks, **d** value functions

5.4.2 The Effects of the Salvage Cost on the Results

It is easy to see from our theoretical results that the value of datum pollution stock $\bar{P}_i$ only influences the revenues of the players; namely, the bigger the $\bar{P}_i$ is, the more are the revenues. So, we simply examine the effects of unit salvage cost g_i. Since the results are similar for the two players under cooperative and noncooperative games, here for brevity we present the effects of g_1 on the region 1's optimal strategies for emission and abatement, optimal net revenues, and the optimal trajectories of the pollution stock under the noncooperative game only.

Figure 2, in which g_1 is set at 6, 8, and 10, shows the effects of g_1 on the results. We can see from the figure that a higher unit salvage cost results in a lower emission level, a higher abatement level and more revenues, while the optimal trajectory of the pollution stock is insensitive to g_1. This implies that a higher unit salvage cost should incentivize the players to reduce emission, increase abatement, and then further reduce costs from the emission permits trading scheme, leading to increases in net revenues.

In Fig. 3, we plot the emission levels and abatement levels for a longer horizon $T = 100$. Here we see that the results are primarily driven by the salvage value function. Specifically, the emission and abatement levels are at a steady state until close to the terminal time when the levels deviate from the steady state because of the influence of the salvage cost. Such

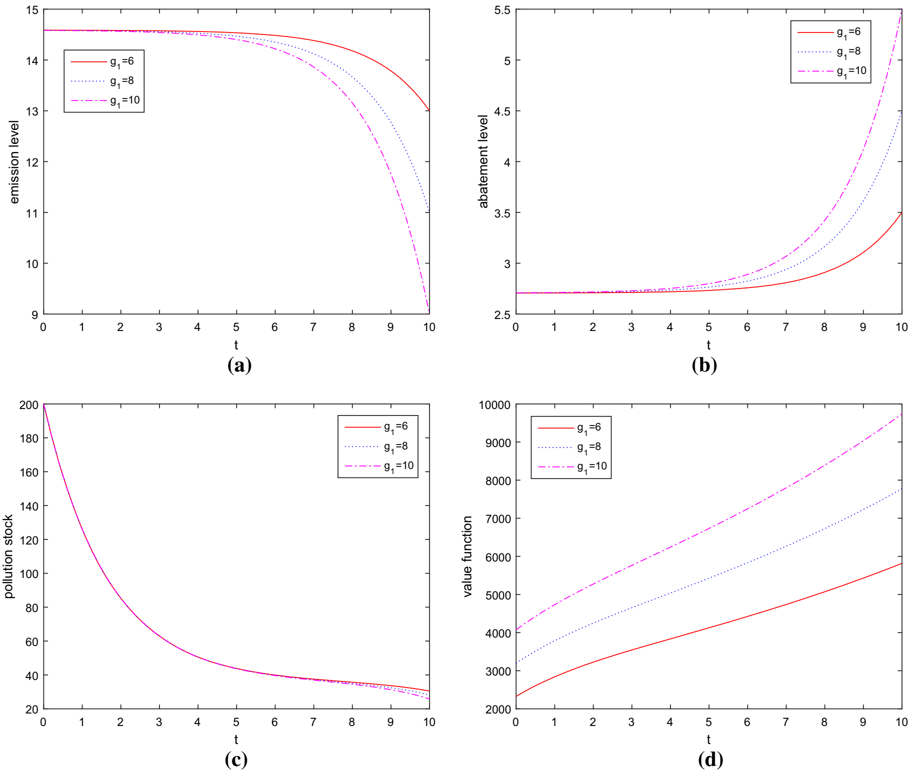

Fig. 2 The effects of g_1 on the results, **a** emission levels, **b** abatement levels, **c** pollution stocks, **d** value functions

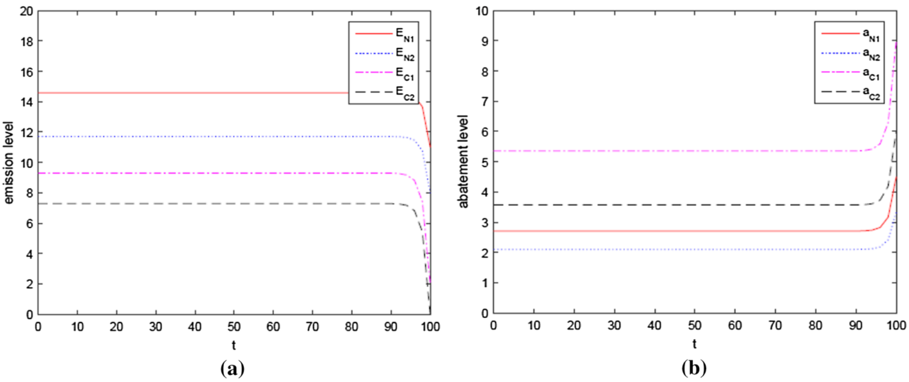

Fig. 3 The effects of time horizon on the results, **a** emission levels, **c** abatement levels

results are common in the economics literature; see, e.g., Sects. 7.2 and 13.3 in Sethi and Thompson [24].

6 Stochastic Extension

6.1 Game Framework

Our model can be extended to a stochastic multi-dimensional versions by considering stochastic emission permits prices and a stochastic evolution of the pollution stock. Several studies, such as Chang et al. [6] and Daskalakis et al. [12], have demonstrated that the emission permit price should be stochastic, as it is caused by the scarcity of emission permits and market discipline. Moreover, the dynamic process of the pollution stock may be affected by extreme weather events such as gales, nature's ability to refresh the air, and other stochastic disturbances. In recent years, several stochastic optimal control models have been proposed to deal with the uncertainty in pollution stock, such as Athanassoglou and Xepapadeas [1], Masoudi et al. [22], and Yi et al. [27]. Based on the above, we consider the following dynamics of the emission permits price $S(t)$ and the pollution stock $P(t)$:

$$\mathrm{d}S(t) = \mu_S S(t)\mathrm{d}t + \sigma_S S(t)\mathrm{d}W_S, \tag{14}$$

and

$$\mathrm{d}P(t) = (E_1(t) - a_1(t) + E_2(t) - a_2(t) - \theta P(t))\mathrm{d}t + \sigma_P P(t)\mathrm{d}W_P, \tag{15}$$

where μ_S is the drift rate of the emission permit price, σ_S and σ_P are the volatilities of the emission permit price and the pollution stock, respectively, and W_S and W_P are two correlated Brownian motions with the correlation coefficient $\rho > 0$, whereas $\rho = 0$ means they are independent of each other.

For the cooperative game with $\mathbb{E}$ denoting the expectation, the objective function and the corresponding HJB equation are

$$\begin{aligned} &\max_{\substack{E_{C1}\geq 0, a_{C1},\\ E_{C2}\geq 0, a_{C2}}} \mathbb{E}\left\{ \int_0^{\mathrm{T}} e^{-rt}\left[(A - S(t))E_{C1} + (\alpha A - S(t))E_{C2} - \frac{E_{C1}^2 + E_{C2}^2}{2} \right.\right.\\ &\quad + S(t)(a_{C1} + a_{C2}) + S(t)(E_{10}e^{-\rho_1 t} + E_{20}e^{-\rho_2 t}) - (1+\beta)DP(t) - \frac{1}{2}Ca_{C1}^2 \\ &\quad \left.\left. - \frac{1}{2}\gamma C a_{C2}^2 \right]\mathrm{d}t \right\} - \sum_{i=1,2} g_i(P(T) - \bar{P}_i)e^{-rT}, \\ &\text{subject to} \begin{cases} \mathrm{d}S(t) = \mu_S S(t)\mathrm{d}t + \sigma_S S(t)\mathrm{d}W_S,\ S(0) = S_0, \\ \mathrm{d}P(t) = (E_{C1}(t) - a_{C1}(t) + E_{C2}(t) - a_{C2}(t) - \theta P(t))\mathrm{d}t \\ \qquad\qquad + \sigma_P P(t)\mathrm{d}W_P,\ P(0) = P_0, \end{cases} \end{aligned}$$

and

$$\begin{aligned} &\max_{\substack{E_{C1}(t)\geq 0, a_{C1}(t),\\ E_{C2}(t)\geq 0, a_{C2}(t)}} \left\{ -\frac{\partial V_C}{\partial t} - \Big(E_{C1}(t) - a_{C1}(t) + E_{C2}(t) - a_{C2}(t) - \theta P(t)\Big)\frac{\partial V_C}{\partial P} \right. \\ &\left. -\frac{1}{2}\sigma_P^2 P^2 \frac{\partial^2 V_C}{\partial P^2} - \mu_S S \frac{\partial V_C}{\partial S} - \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 V_C}{\partial S^2} - \rho\sigma_P\sigma_S P S \frac{\partial^2 V_C}{\partial P \partial S} + rV_C - F_C \right\} = 0, \end{aligned} \tag{16}$$

with the terminal condition

$$V_C(P, T) = -g_1\left(P - \bar{P}_1\right) - g_2\left(P - \bar{P}_2\right),$$

where

$$F_C = (A-S)E_{C1}(t) + (\alpha A - S)E_{C2}(t) - \frac{1}{2}\left(E_{C1}^2(t) + E_{C2}^2(t)\right) + S\left(E_{10}e^{-\rho_1 t} + E_{20}e^{-\rho_2 t}\right)$$
$$+ S(t)(a_{C1}(t) + a_{C2}(t)) - (1+\beta)DP(t) - \frac{1}{2}Ca_{C1}^2(t) - \frac{1}{2}\gamma Ca_{C2}^2(t).$$

For the noncooperative game, the objective functions for regions 1 and 2, and the corresponding HJB system are, respectively,

$$\max_{\substack{E_{N1}(t)\geq 0,\\ a_{N1}(t)}} \mathbb{E}\int_0^{\mathrm{T}} e^{-rt}\Bigg[(A - S(t))E_{N1}(t) - \frac{E_{N1}^2(t)}{2}$$
$$+\ S(t)E_{10}e^{-\rho_1 t + S(t)a_{N1}(t)} - DP(t) - \frac{1}{2}Ca_{N1}^2(t)\Bigg]\mathrm{d}t - g_1(P(T) - \bar{P}_1)e^{-rT}, \tag{17}$$

$$\text{subject to} \quad \begin{cases} \mathrm{d}S(t) = \mu_S S(t)\mathrm{d}t + \sigma_S S(t)\mathrm{d}W_S,\ S(0) = S_0,\\ \mathrm{d}P(t) = (E_{N1}(t) - a_{N1}(t) + E_{N2}(t) - a_{N2}(t) - \theta P(t))\mathrm{d}t\\ \qquad\qquad + \sigma_P P(t)\mathrm{d}W_P,\ P(0) = P_0, \end{cases}$$

and for region 2,

$$\max_{\substack{E_{N2}(t)\geq 0,\\ a_{N2}(t)}} \mathbb{E}\int_0^{\mathrm{T}} e^{-rt}\Bigg[(\alpha A - S(t))E_{C2}(t) - \frac{E_{C2}^2(t)}{2} + S(t)E_{20}e^{-\rho_2 t} + S(t)a_{N2}(t)$$
$$-\ \beta DP(t) - \frac{1}{2}\gamma Ca_{C2}^2(t)\Bigg]\mathrm{d}t - g_2(P(T) - \bar{P}_1)e^{-rT}, \tag{18}$$

$$\text{subject to} \quad \begin{cases} \mathrm{d}S(t) = \mu_S S(t)\mathrm{d}t + \sigma_S S(t)\mathrm{d}W_S,\ S(0) = S_0,\\ \mathrm{d}P(t) = (E_{N1}(t) - a_{N1}(t) + E_{N2}(t) - a_{N2}(t) - \theta P(t))\mathrm{d}t\\ \qquad\qquad + \sigma_P P(t)\mathrm{d}W_P,\ P(0) = P_0, \end{cases}$$

and

$$\begin{cases} \max\limits_{E_{N1}(t)\geq 0, a_{N1}(t)} \Bigg\{ -\frac{\partial V_{N1}}{\partial t} - \Big(E_{N1}(t) - a_{N1}(t) + E_{N2}(t) - a_{N2}(t) - \theta P(t)\Big)\frac{\partial V_{N1}}{\partial P} - \frac{1}{2}\sigma_P^2 P^2 \frac{\partial^2 V_{N1}}{\partial P^2} \\ \qquad - \mu_S S\frac{\partial V_{N1}}{\partial S} - \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 V_{N1}}{\partial S^2} - \rho\sigma_P\sigma_S PS\frac{\partial^2 V_{N1}}{\partial P\partial S} + rV_{N1} - F_{N1}\Bigg\} = 0, \\ \max\limits_{E_{N2}(t)\geq 0, a_{N2}(t)} \Bigg\{ -\frac{\partial V_{N1}}{\partial t} - \Big(E_{N1}(t) - a_{N1}(t) + E_{N2}(t) - a_{N2}(t) - \theta P(t)\Big)\frac{\partial V_{N2}}{\partial P} - \frac{1}{2}\sigma_P^2 P^2 \frac{\partial^2 V_{N2}}{\partial P^2} \\ \qquad - \mu_S S\frac{\partial V_{N2}}{\partial S} - \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 V_{N2}}{\partial S^2} - \rho\sigma_P\sigma_S PS\frac{\partial^2 V_{N2}}{\partial P\partial S} + rV_{N2} - F_{N2}\Bigg\} = 0, \end{cases} \tag{19}$$

with the terminal conditions

$$V_{N1}(P,T) = -g_1\left(P - \bar{P}_1\right) \quad \text{and} \quad V_{N2}(P,T) = -g_2\left(P - \bar{P}_2\right), \tag{19a}$$

where

$$F_{N1} = (A-S)E_{N1}(t) - \frac{1}{2}E_{N1}^2(t) + SE_{10}e^{-\rho_1 t} + Sa_{N1}(t) - DP(t) - \frac{1}{2}Ca_{N1}^2(t),$$

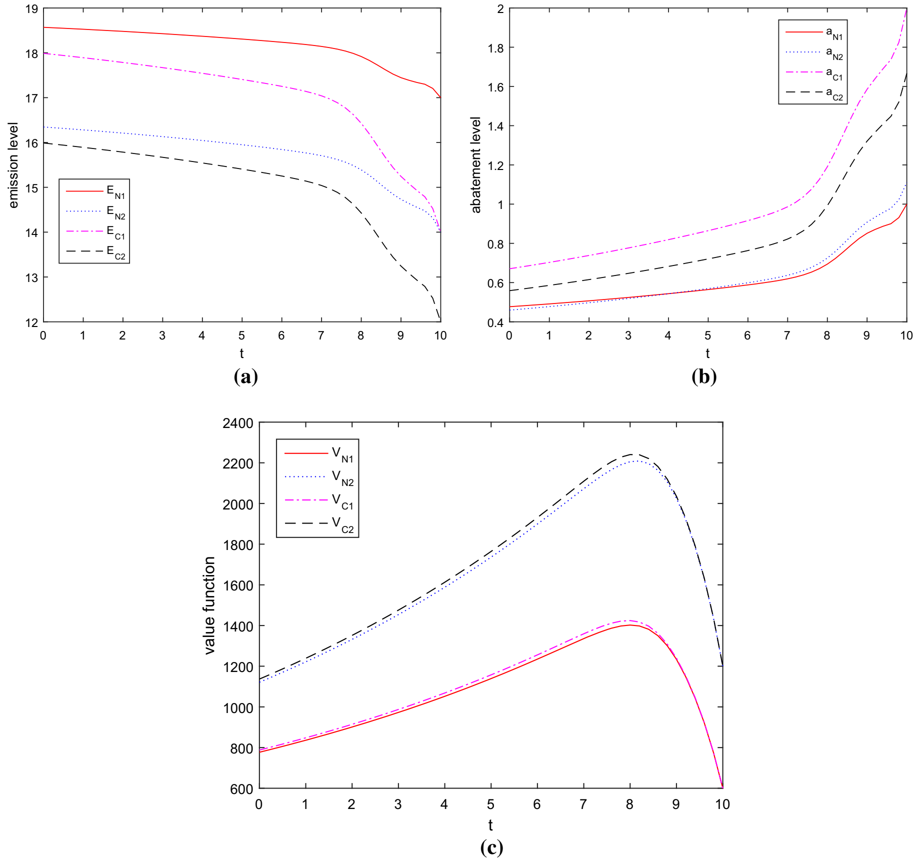

Fig. 4 Numerical results for the stochastic model, **a** emission levels, **b** abatement levels, **c** value functions

and

$$F_{N2} = (\alpha A - S)E_{N2}(t) - \frac{1}{2}E_{N2}^2(t) + SE_{20}e^{-\rho_2 t} + Sa_{N2}(t) - \beta DP(t) - \frac{1}{2}\gamma C a_{N2}^2(t).$$

Since the HJB Eqs. (19) and (19a) cannot be solved analytically, we propose a so-called fitted finite volume method to discretize the equations. This numerical scheme is presented in “Appendix G” section. The theoretical results about the method will be presented in another work. The values of parameters used in this subsection are the same as those in Chang et al. [7]: $T = 10$, $A_1 = 20$, $\alpha = 0.9$, $P_{\min} = 200$, $P_{\max} = 1000$, $S_{\min} = 0$, $S_{\max} = 2$, $E_{i0} = 5$, $E_{j0} = 6$, $\rho_1 = 0.05$, $\rho_2 = 0.04$, $\theta = 0.6$, $\sigma_P = 0.3$, $\sigma_S = 0.3$, $\mu_S = 0.2$, $\rho = 0.5$, $D_1 = 0.1$, $\beta = 1.2$, $r = 0.08$, $g_i = 2$, $g_j = 3$, $\bar{P}_i = 1100$, $\bar{P}_j = 1200$, where $[P_{\min}, P_{\max}] \times [S_{\min}, S_{\max}]$ is the computational region of our problems.

6.2 Numerical Results

In Fig. 4, we plot the numerical results of the stochastic model over time by fixing $P = 800$ and $S = 1$. Once again we see that both regions will benefit from cooperation. Further, cooperation also leads to lower emission levels and higher abatement levels for each region.

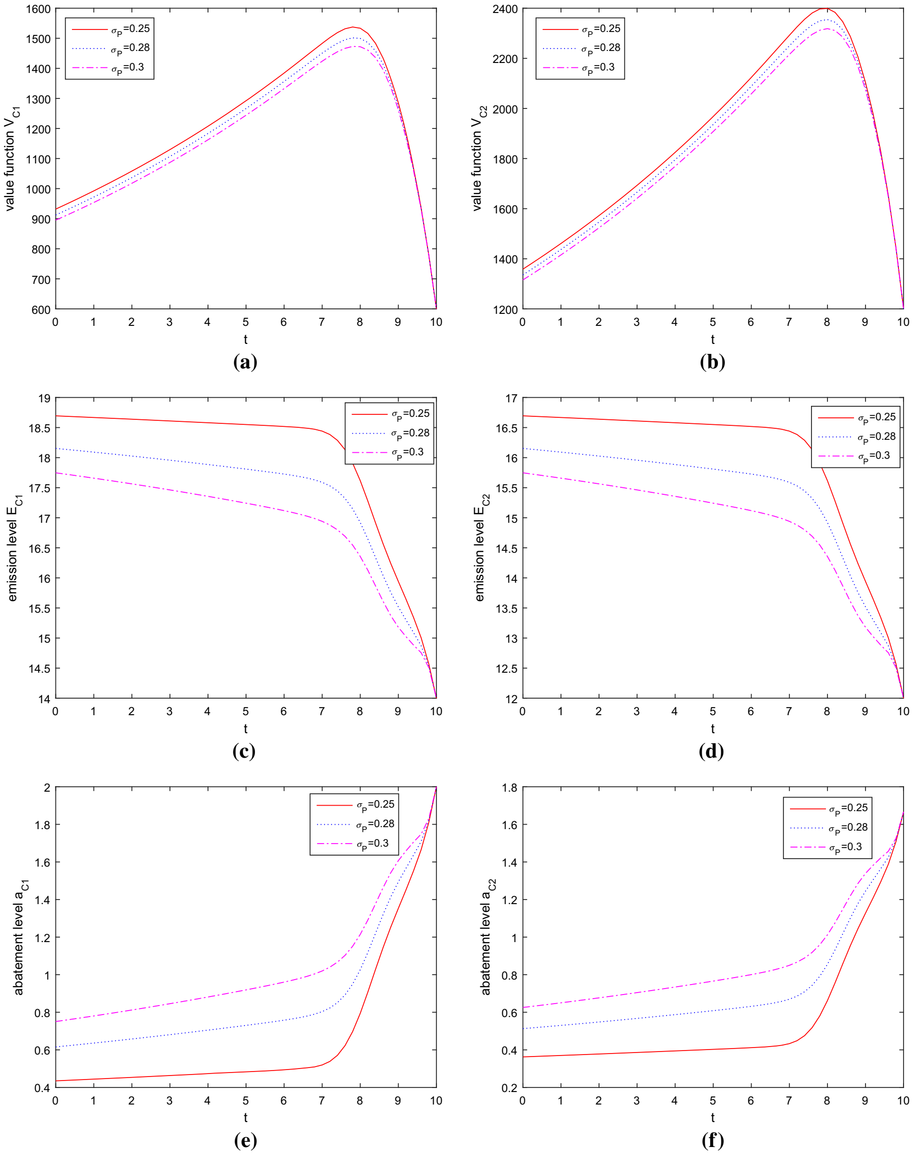

Fig. 5 The effects of σ_P in the cooperative game, **a** V_{C1}, **b** V_{C2}, **c** E_{C1}, **d** E_{C2}, **e** a_{C1}, **f** a_{C2}

It is interesting to note that the emission levels are decreasing over time and the abatement levels are increasing over time, which is similar to the deterministic case, while the profits first increase with time and then dramatically decrease until the terminal time, which is different from the deterministic case. We can conclude that this difference is caused by the randomness in the pollution stock and the emission permit price. The risk due to randomness is accumulated gradually as time progresses, and when a threshold value is reached by the risk, the players in the game suffer losses, denoting the decreasing phase in Fig. 4c.

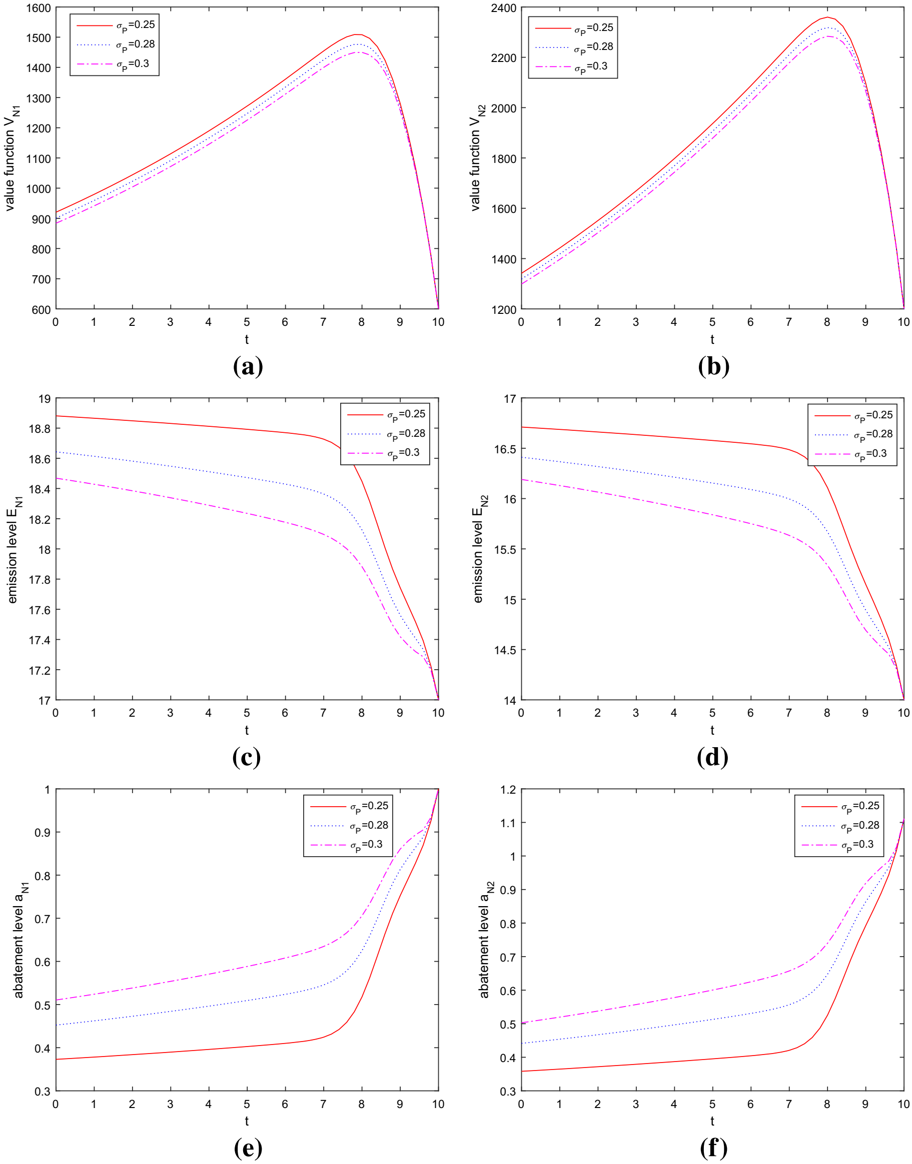

Fig. 6 The effects of σ_P in the noncooperative game, **a** V_{N1}, **b** V_{N2}, **c** E_{N1}, **d** E_{N2}, **e** a_{N1}, **f** a_{N2}

6.3 Sensitivity Analysis

In this subsection, we present the sensitivity analysis of the numerical results concerning the parameters of the Brownian motions. The effects of the other parameters on the game can be obtained from the above analytical results. Since the parameters μ_S and σ_S have little influence on the results, we will only examine the effects of σ_P on the optimal profit, the optimal emission paths as well as the optimal abatement strategies under the cooperative and noncooperative games, respectively. In Figs. 3 and 4, we fix $P = 675$, $S = 1$, and set σ_P to be 0.25, 0.3 and 0.35, respectively.

Figures 5 and 6 show the effects of the parameter σ_P on the cooperative game and the noncooperative game, respectively. On the one hand, we can see from 5a, b and 6a, b that the optimal profits of both regions decrease with σ_P. A higher volatility σ_P implies that the two regions will bear a higher risk of the pollution stock. To control this risk, more efforts, such as investments in strategic portfolios and infrastructure construction, should be made. This implies that higher volatility of the pollution stock will reduce the two regions' revenues in the game.

On the other hand, the more the volatility of the pollution stock is, the lower emission levels and the higher abatement levels are for the two regions under both types of games. This can be illustrated as follows. The regions may suffer from more pollution damage if the volatility σ_P increases, so they should do their best to reduce the pollution stock to avoid the climate risk as far as possible. Reducing emission and increasing abatement should be reasonable choices in the game for both players. Moreover, another advantage of emission reduction is that the players could save more unused emission permits and sell them in the market to receive benefits, thus offsetting the reduction in the productive revenues.

7 Concluding Remarks

In this paper, we investigate the regions' dynamic optimal strategies in finite-horizon differential games of transboundary industrial pollution. In particular, the emission trading scheme and the abatement policy are involved in our differential game models. Through solving the HJB equations satisfied by the value functions, we can obtain the two regions' optimal emission paths, optimal abatement levels, as well as the optimal trajectories of the pollution stocks under the noncooperative and the cooperative games. Moreover, a stochastic extension is also discussed.

Our results show that the two regions' instantaneous quotas affect their profits to a great extent. Additionally, cooperation leads to increased amounts of abatement, fewer emissions, and a lower pollution stock. This emphasizes the fact that cooperation is a better choice in both economic and environmental terms.

Appendix

A. Proof of Proposition 1

Proof For $i = 1, 2$, we guess the form of the value functions $V_{\mathrm{Ni}}(P, t)$ to be linear in P, that is,

$$V_{\mathrm{Ni}}(P, t) = l_i(t)P + k_i(t), \tag{20}$$

where $l_i(t)$ and $k_i(t)$ are functions of t. Then, we obtain these functions to verify that our guess is correct. By substituting the first-order conditions (6) and (20) into the system of HJB Eq. (5), we have

$$\begin{cases} \left(l_1'(t) - (r+\theta)l_1(t) - D\right) P + k_1'(t) - rk_1(t) + f_1(t) = 0, \\ \left(l_2'(t) - (r+\theta)l_2(t) - \beta D\right) P + k_2'(t) - rk_2(t) + f_2(t) = 0, \end{cases} \tag{21}$$

where

$$f_1(t) = SE_{10} + \frac{1}{2}\Big(A - S + l_1(t)\Big)^2 + l_1(t)\Big(\alpha A - S + \frac{\gamma C + 1}{\gamma C} l_2(t) + \frac{1}{2C} l_1(t)\Big),$$

and

$$f_2(t) = SE_{20} + \frac{1}{2}\Big(\alpha A - S + l_2(t)\Big)^2 + l_2(t)\Big(A - S + \frac{C + 1}{\gamma C} l_1(t) + \frac{1}{2\gamma C} l_2(t)\Big).$$

Noticing that the system (21) should be satisfied for all $P \geq 0$, we can determine $l_1(t)$, $l_2(t)$, $k_1(t)$, and $k_2(t)$ by solving the following system of ordinary differential equations:

$$\begin{cases} l_1^{'}(t) - (r + \theta) l_1(t) - D = 0, \\ l_2^{'}(t) - (r + \theta) l_2(t) - \beta D = 0, \\ k_1^{'}(t) - r k_1(t) + f_1(t) = 0, \\ k_2^{'}(t) - r k_2(t) + f_2(t) = 0. \end{cases} \tag{22}$$

Solving the above ODE system, we can obtain

$$\begin{aligned} l_1(t) &= \Big(\frac{D}{r + \theta} - g_1\Big) e^{-(r+\theta)(T-t)} - \frac{D}{r + \theta}, \\ l_2(t) &= \Big(\frac{\beta D}{r + \theta} - g_2\Big) e^{-(r+\theta)(T-t)} - \frac{\beta D}{r + \theta}, \\ k_1(t) &= G_{N1} + H_{N1} e^{-(r+\theta)(T-t)} + I_{N1} e^{-2(r+\theta)(T-t)} + \frac{SE_{10} e^{-\rho_1 t}}{r + \rho_1} \\ &\quad + \Big(g_1 \bar{P}_1 - I_{N1} - H_{N1} - G_{N1} - \frac{SE_{10} e^{-\rho_1 T}}{r + \rho_1}\Big) e^{-r(T-t)}, \\ k_2(t) &= G_{N2} + H_{N2} e^{-(r+\theta)(T-t)} + I_{N2} e^{-2(r+\theta)(T-t)} + \frac{SE_{20} e^{-\rho_2 t}}{r + \rho_2} \\ &\quad + \Big(g_2 \bar{P}_2 - I_{N2} - H_{N2} - G_{N2} - \frac{SE_{20} e^{-\rho_2 T}}{r + \rho_2}\Big) e^{-r(T-t)}, \end{aligned}$$

where

$$\begin{aligned} G_{N1} &= \frac{1}{r}\Big(\frac{S^2}{2C} + \frac{1}{2}(A - S)^2 - \frac{D}{r + \theta}\Big((1 + \alpha)A - \frac{2\gamma C + \gamma + 1}{\gamma C} S\Big) \\ &\quad + \frac{1}{C}\Big(\frac{C + 1}{2} + \beta \frac{\gamma C + 1}{\gamma}\Big)\Big(\frac{D}{r + \theta}\Big)^2\Big), \\ H_{N1} &= \frac{1}{\theta}\Big(-\Big(\frac{D}{r + \theta} - g_1\Big)\Big((1 + \alpha)A - \frac{2\gamma C + \gamma + 1}{\gamma C} S\Big) + \frac{C + 1}{C} \frac{D}{r + \theta}\Big(\frac{D}{r + \theta} - g_1\Big) \\ &\quad + \frac{\gamma C + 1}{\gamma C} \frac{D}{r + \theta}\Big(\frac{2\beta D}{r + \theta} - \beta g_1 - g_2\Big)\Big), \\ I_{N1} &= -\frac{1}{r + 2\theta}\Big(\frac{D}{r + \theta} - g_1\Big)\Big(\frac{C + 1}{2C}\Big(\frac{D}{r + \theta} - g_1\Big) + \frac{\gamma C + 1}{\gamma C}\Big(\frac{\beta D}{r + \theta} - g_2\Big)\Big), \\ G_{N2} &= \frac{1}{r}\Big(\frac{S^2}{2\gamma C} + \frac{1}{2}(\alpha A - S)^2 - \frac{\beta D}{r + \theta}\Big((1 + \alpha)A - \frac{2\gamma C + \gamma + 1}{\gamma C} S\Big) \end{aligned}$$

$$+\frac{\beta}{C}\Big(C+1+\beta\frac{\gamma C+1}{2\gamma}\Big)\frac{D^2}{(r+\theta)^2}\Big),$$

$$H_{N2}=\frac{1}{\theta}\Big(-\Big(\frac{\beta D}{r+\theta}-g_2\Big)\Big((1+\alpha)A-\frac{2\gamma C+\gamma+1}{\gamma C}S\Big)+\frac{\gamma C+1}{\gamma C}\frac{\beta D}{r+\theta}\Big(\frac{\beta D}{r+\theta}-g_2\Big)$$
$$+\frac{C+1}{C}\frac{D}{r+\theta}\Big(\frac{2\beta D}{r+\theta}-\beta g_1-g_2\Big)\Big),$$

$$I_{N2}=-\frac{1}{r+2\theta}\Big(\frac{\beta D}{r+\theta}-g_2\Big)\Big(\frac{\gamma C+1}{2\gamma C}\Big(\frac{\beta D}{r+\theta}-g_2\Big)+\frac{C+1}{C}\Big(\frac{D}{r+\theta}-g_1\Big)\Big).$$

Then, the proof can be completed by substituting the above results into (6). □

B. Proof of Corollary 1

Proof (i) Comparing (7b) and (7e) with the two regions' instantaneous quotas E_{10} and E_{20}, we can obtain the above mentioned conclusions.

(ii) From Eqs. (7a) and (7d) we know

$$\frac{\partial V_{N1}}{\partial P}=\Big(\frac{D}{r+\theta}-g_1\Big)e^{-(r+\theta)(T-t)}-\frac{D}{r+\theta}<0,$$
$$\frac{\partial V_{N1}}{\partial E_{10}}=\Big(e^{\rho_1(T-t)}-e^{-r(T-t)}\Big)\frac{e^{-\rho_1 T}S}{r+\rho_1}>0,$$
$$\frac{\partial V_{N2}}{\partial P}=\Big(\frac{\beta D}{r+\theta}-g_2\Big)e^{-(r+\theta)(T-t)}-\frac{\beta D}{r+\theta}<0,$$
$$\frac{\partial V_{N2}}{\partial E_{20}}=\Big(e^{\rho_2(T-t)}-e^{-r(T-t)}\Big)\frac{e^{-\rho_2 T}S}{r+\rho_2}>0,$$

which completes the proof of results in (ii).

(iii) Taking derivative on (7a) and (7d) with respect to S, respectively, we have

$$\frac{\partial V_{N1}}{\partial S}=\frac{1}{r}\Big(\frac{C+1}{C}S-A+\frac{2\gamma C+\gamma+1}{\gamma C}\frac{D}{r+\theta}\Big)\Big(1-e^{-r(T-t)}\Big)$$
$$+\frac{E_{10}e^{-\rho_1 T}}{1+\rho_1}\Big(e^{\rho_1(T-t)}-e^{-r(T-t)}\Big)$$
$$+\frac{1}{\theta}\frac{2\gamma C+\gamma+1}{\gamma C}e^{-r(T-t)}\Big(\frac{D}{r+\theta}-g_1\Big)\Big(e^{-\theta(T-t)}-1\Big),$$

$$\frac{\partial V_{N2}}{\partial S}=\frac{1}{r}\Big(\frac{\gamma C+1}{\gamma C}S-A+\frac{2\gamma C+\gamma+1}{\gamma C}\frac{\beta D}{r+\theta}\Big)\Big(1-e^{-r(T-t)}\Big)$$
$$+\frac{E_{20}e^{-\rho_2 T}}{1+\rho_2}\Big(e^{\rho_2(T-t)}-e^{-r(T-t)}\Big)$$
$$+\frac{1}{\theta}\frac{2\gamma C+\gamma+1}{\gamma C}e^{-r(T-t)}\Big(\frac{\beta D}{r+\theta}-g_2\Big)\Big(e^{-\theta(T-t)}-1\Big),$$

which imply that V_{N1} and V_{N2} are two quadratic functions concerning S on the real axis at any time before T, and the minimum points should be $\phi-\tau_1(t)E_{10}$ and $\psi-\tau_2(t)E_{20}$, respectively. Also, combining the condition $0<S<\eta$, which must be satisfied to keep the emissions to be positive, and the condition $S > 0$, we can conclude the results in (iii). □

C. Proof of Corollary 2

Proof Substituting (7b), (7c), (7e) and (7f) into (3), we gain the following ordinary differential equation satisfied by the dynamic process of the pollution stock:

$$\dot{P_N} + \theta P_N = (1+\alpha)A - \frac{2\gamma C + \gamma + 1}{\gamma C} S + \frac{C+1}{C} l_1(t) + \frac{\gamma C + 1}{\gamma C} l_2(t).$$

By solving the above equation, we can obtain the trajectory of the pollution stock (8).

Then, taking the derivative of (8) with respect to t, we have

$$P_N'(t) = e^{-\theta t}\left(Y_N(r+\theta)e^{-(r+\theta)(T-t)+\theta t} - \theta\left(P_0 - X_N - Y_N e^{-(r+\theta)T}\right)\right),$$

from which the conclusions follow. □

D. Proof of Proposition 2

Proof We first guess that the joint value function V_C of the two regions is of the form

$$V_C(P,t) = l_C(t)P + k_C(t), \tag{24}$$

where $l_C(t)$ and $k_C(t)$ are the functions of t. Then, by substituting (10) and (24) into the HJB Eq. (16) satisfied by the joint value function, we obtain

$$\left(l_C^{'}(t) - (r+\theta)l_C(t) - (1+\beta)DP\right)P + k_C^{'}(t) - rk_C(t) + f_C(t) = 0, \tag{25}$$

where

$$f_C(t) = S(E_{10}+E_{20}) + \frac{1}{2}\left(A - S + l_C(t)\right)^2 + \frac{1}{2}\left(\alpha A - S + l_C(t)\right)^2 + \frac{1}{2}\frac{\gamma C+1}{\gamma C} l_C^2(t).$$

Since Eq. (25) holds for every $P \geq 0$, we have the following system of ordinary differential equations

$$\begin{cases} l_C^{'}(t) - (r+\theta)l_C(t) - (1+\beta)D = 0, \\ k_C^{'}(t) - rk_C(t) + f_C(t) = 0. \end{cases} \tag{26}$$

Solving this ODE system, we obtain

$$l(t) = \left(\frac{(1+\beta)D}{r+\theta} - g_1 - g_2\right)e^{-(r+\theta)(T-t)} - \frac{(1+\beta)D}{r+\theta},$$

$$k(t) = G_C + H_C e^{-(r+\theta)(T-t)} + I_C e^{-2(r+\theta)(T-t)} + \frac{SE_{10}e^{-\rho_1 t}}{r+\rho_1} + \frac{SE_{20}e^{-\rho_2 t}}{r+\rho_2}$$
$$+ \left(g_1\bar{P}_1 + g_2\bar{P}_2 - I_C - H_C - G_C - \frac{SE_{10}e^{-\rho_1 T}}{r+\rho_1} - \frac{SE_{20}e^{-\rho_2 T}}{r+\rho_2}\right)e^{-r(T-t)},$$

where

$$G_C = \frac{1}{r}\left(\frac{1}{2}(A-S)^2 + \frac{1}{2}(\alpha A - S)^2 - \frac{(1+\beta)D}{r+\theta}\left((1+\alpha)A - \frac{2\gamma C + \gamma + 1}{\gamma C}S\right)\right.$$

$$+\left(1+\frac{\gamma+1}{2\gamma C}\right)\left(\frac{(1+\beta)D}{r+\theta}\right)^2+\frac{\gamma+1}{2\gamma C}S^2\Big),$$

$$H_C=\frac{1}{\theta}\left(\frac{(1+\beta)D}{r+\theta}-g_1-g_2\right)\left(\frac{2\gamma C+\gamma+1}{\gamma C}S-(1+\alpha)A+2\left(1+\frac{\gamma+1}{2\gamma C}\right)\frac{(1+\beta)D}{r+\theta}\right),$$

$$I_C=-\frac{1}{r+2\theta}\left(1+\frac{\gamma+1}{2\gamma C}\right)\left(\frac{(1+\beta)D}{r+\theta}-g_1-g_2\right)^2.$$

□

E. Proof of Corollary 3

Proof (i) Comparing (11b) and (11d) with the two regions' instantaneous quotas E_{10} and E_{20}, we can obtain the above conclusions.

(ii) From Eq. (11a), we have

$$\frac{\partial V_C}{\partial P}=\left(\frac{(1+\beta)D}{r+\theta}-g_1-g_2\right)e^{-(r+\theta)(T-t)}-\frac{(1+\beta)D}{r+\theta}<0,$$

$$\frac{\partial V_C}{\partial E_{10}}=\left(e^{\rho_1(T-t)}-e^{-r(T-t)}\right)\frac{Se^{-\rho_1 T}}{r+\rho_1}>0,$$

$$\frac{\partial V_C}{\partial E_{20}}=\left(e^{\rho_2(T-t)}-e^{-r(T-t)}\right)\frac{Se^{-\rho_2 T}}{r+\rho_2}>0,$$

which completes the proof of (ii).

(iii) Taking the derivative of (11a) with respect to S, we have

$$\begin{aligned}\frac{\partial V_C}{\partial S}=&\frac{1}{r}\Big(\frac{2\gamma C+\gamma+1}{2\gamma C}S-(1+\alpha)A\\&+\frac{(1+\beta)D}{r+\theta}\frac{2\gamma C+\gamma+1}{2\gamma C}\Big)(1-e^{-r(T-t)})\\&+\frac{1}{\theta}\frac{2\gamma C+\gamma+1}{2\gamma C}e^{-r(T-t)}\left(\frac{(1+\beta)D}{r+\theta}-g_1-g_2\right)\left(e^{-\theta(T-t)}-1\right)\\&+\left(e^{\rho_1(T-t)}-e^{-r(T-t)}\right)\frac{E_{10}e^{-\rho_1 T}}{r+\rho_1}+\left(e^{\rho_2(T-t)}-e^{-r(T-t)}\right)\frac{E_{20}e^{-\rho_2 T}}{r+\rho_1},\end{aligned}$$

which implies that V_C is a quadratic function of S at any time before T with the minimum point $\frac{1}{2}(\bar{\phi}-\tau_{C1}(t)E_{10}-\tau_{C2}(t)E_{20})$. Also, combining the condition $0<S<\bar{\eta}$, which must be satisfied to keep the emissions nonnegative, and the condition $S>0$, we can conclude (iii). □

F. Proof of Corollary 4

Proof Substituting (11b), (11c), (11d) and (11e) into (3), we have the following ordinary differential equation for the dynamics of the pollution stock in the cooperative game:

$$\dot{P}_C+\theta P_C=(1+\alpha)A-\frac{2\gamma C+\gamma+1}{\gamma C}S+\frac{2\gamma C+\gamma+1}{\gamma C}l_C(t).$$

Its solution gives the trajectory of the pollution stock (12). Moreover, taking its derivative with respect to t gives

$$P_C'(t) = e^{-\theta t}\left(Y_C(r+\theta)e^{-(r+\theta)(T-t)+\theta t} - \theta\left(P_0 - X_C - Y_C e^{-(r+\theta)T}\right)\right),$$

from which the results follow directly. □

G. Numerical Method for (16)

Since the structure of HJB equations arising from the noncooperative case is similar to that of the cooperative one, here we only discuss the latter to save space.

From the first-order optimality condition, we know that Eq. (16) can split into the following coupled equations:

$$\frac{\partial V_C}{\partial t} + \left(E_{C1}^*(t) + E_{C2}^*(t) - a_{C1}^*(t) - a_{C2}^*(t) - \theta_P P\right)\frac{\partial V_C}{\partial P} + \frac{1}{2}\sigma_P^2 P^2 \frac{\partial^2 V_C}{\partial P^2} + \mu_S S \frac{\partial V_C}{\partial S} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 V_C}{\partial S^2} + \rho\sigma_P\sigma_S P S \frac{\partial^2 V_C}{\partial P \partial S} - r V_C + F_C\left(P, S, E_{C1}^*(t), E_{C2}^*(t), t\right) = 0, \tag{28a}$$

$$E_{C1}^* = A - S + \frac{\partial V_C}{\partial P}, \quad a_{C1}^* = \frac{1}{C}\left(S - \frac{\partial V_C}{\partial P}\right),$$
$$E_{C2}^* = \alpha A - S + \frac{\partial V_C}{\partial P}, \quad a_{C2}^* = \frac{1}{\gamma C}\left(S - \frac{\partial V_C}{\partial P}\right). \tag{28b}$$

The Fitted Finite Volume Method for Spatial Discretization

A defined mesh for $(P_{\min}, P_{\max}) \times (S_{\min}, S_{\max})$ is significant in the process of discretization. So, we first divide the intervals I_P and I_S into N_P and N_S subintervals, respectively:

$$I_{P_i} := (P_i, P_{i+1}), \quad I_{S_j} := (S_j, S_{j+1}), \quad i = 0, 1, \ldots, N_P - 1, \quad j = 0, 1, \ldots, N_S - 1,$$

in which

$$P_{\min} = P_0 < P_1 < \cdots < P_{N_P} = P_{\max} \quad \text{and} \quad S_{\min} = S_0 < S_1 < \cdots < S_{N_S} = S_{\max}.$$

Thus, a mesh on $I_P \times I_S$, whose all mesh lines are perpendicular to the axes, is defined. Next we define another partition of $I_P \times I_S$ by letting

$$P_{i-\frac{1}{2}} = \frac{P_{i-1} + P_i}{2}, \quad P_{i+\frac{1}{2}} = \frac{P_i + P_{i+1}}{2}, \quad S_{j-\frac{1}{2}} = \frac{S_{j-1} + S_j}{2}, \quad S_{j+\frac{1}{2}} = \frac{S_j + S_{j+1}}{2}$$

for any $i = 1, 2, \ldots, N_P - 1$ and $j = 1, 2, \ldots, N_S - 1$. To keep completeness, we also define $P_{-\frac{1}{2}} = P_{\min}$, $P_{N_P+\frac{1}{2}} = P_{\max}$, $S_{-\frac{1}{2}} = S_{\min}$, and $S_{N_S+\frac{1}{2}} = S_{\max}$. The step sizes are defined by $h_{P_i} = P_{i+\frac{1}{2}} - P_{i-\frac{1}{2}}$ and $h_{S_j} = S_{j+\frac{1}{2}} - S_{j-\frac{1}{2}}$ for each $i = 0, 1, \ldots, N_P$ and $j = 0, 1, \ldots, N_S$.

Then, for the purpose of formulating finite volume scheme, we write Eq. (16) in the following divergence form:

$$-\frac{\partial V_C}{\partial t} - \nabla \cdot (A \nabla V_C + \underline{b} V_C) + c V_C = F_C, \tag{29}$$

where

$$
\begin{aligned}
A &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\sigma_P^2 P^2 & \frac{1}{2}\rho\sigma_P\sigma_S PS \\ \frac{1}{2}\rho\sigma_P\sigma_S PS & \frac{1}{2}\sigma_S^2 S^2 \end{pmatrix}, \\
\underline{b} &= \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} E_{C1}^* + E_{C2}^* - a_{C1}^* - a_{C2}^* - \theta_P P - \sigma_P^2 P - \frac{1}{2}\rho\sigma_P\sigma_S P \\ \mu_S S - \sigma_S^2 S - \frac{1}{2}\rho\sigma_P\sigma_S S \end{pmatrix}, \qquad (30) \\
c &= r + \mu_S + 2\frac{\partial^2 V_C}{\partial P^2} - \theta_P - \sigma_P^2 - \sigma_S^2 - \rho\sigma_P\sigma_S.
\end{aligned}
$$

It follows from integrating Eq. (29) over $\mathcal{R}_{i,j} = [S_{i-\frac{1}{2}}, S_{i+\frac{1}{2}}] \times [\delta_{j-\frac{1}{2}}, \delta_{j+\frac{1}{2}}]$ and applying the midpoint quadrature rule to the resulting equation that

$$
-\frac{\partial V_{C_{i,j}}}{\partial t} R_{i,j} - \int_{\mathcal{R}_{i,j}} \nabla \cdot (A\nabla V_C + \underline{b} V_C) \mathrm{d}P\mathrm{d}S + c_{i,j} V_{C_{i,j}} R_{i,j} = F_{C_{i,j}} R_{i,j} \qquad (31)
$$

for $i = 1, 2, \ldots, N_P - 1$, $j = 1, 2, \ldots, N_S - 1$, where $R_{i,j} = (P_{i+\frac{1}{2}} - P_{i-\frac{1}{2}}) \times (S_{j+\frac{1}{2}} - S_{j-\frac{1}{2}})$, $c_{i,j} = c(P_i, S_j, t)$, $V_{C_{i,j}} = V_C(P_i, S_j, t)$, and $F_{C_{i,j}} = F_C(P_i, S_j, E_{C1}^*, E_{C2}^*, a_{C1}^*, a_{C2}^*, t)$.

The approximation of the second term in Eq. (31) is the key and difficult point of this numerical method. According to the definition of flux $A\nabla V_C + \underline{b} V_C$ and integrating by parts, we have

$$
\begin{aligned}
\int_{\mathcal{R}_{i,j}} \nabla \cdot \left(A\nabla V_C + \underline{b} V_C\right) dS d\delta &= \int_{\partial\mathcal{R}_{i,j}} \left(A\nabla V_C + \underline{b} V_C\right) \cdot l \mathrm{d}s \\
&= \int_{\left(P_{i+\frac{1}{2}}, S_{j-\frac{1}{2}}\right)}^{\left(P_{i+\frac{1}{2}}, S_{j+\frac{1}{2}}\right)} \left(a_{11}\frac{\partial V_C}{\partial P} + a_{12}\frac{\partial V_C}{\partial S} + b_1 V_C\right) \mathrm{d}S \\
&\quad - \int_{\left(P_{i-\frac{1}{2}}, S_{j-\frac{1}{2}}\right)}^{\left(P_{i-\frac{1}{2}}, S_{j+\frac{1}{2}}\right)} \left(a_{11}\frac{\partial V_C}{\partial P} + a_{12}\frac{\partial V_C}{\partial S} + b_1 V_C\right) \mathrm{d}S \\
&\quad + \int_{\left(P_{i-\frac{1}{2}}, S_{j+\frac{1}{2}}\right)}^{\left(P_{i+\frac{1}{2}}, S_{j+\frac{1}{2}}\right)} \left(a_{21}\frac{\partial V_C}{\partial P} + a_{22}\frac{\partial V_C}{\partial S} + b_2 V_C\right) \mathrm{d}P \\
&\quad - \int_{\left(P_{i+\frac{1}{2}}, S_{j-\frac{1}{2}}\right)}^{\left(P_{i-\frac{1}{2}}, S_{j-\frac{1}{2}}\right)} \left(a_{21}\frac{\partial V_C}{\partial P} + a_{22}\frac{\partial V_C}{\partial S} + b_2 V_C\right) \mathrm{d}P,
\end{aligned}
\qquad (32)
$$

where l denotes the unit vector outward-normal to $\partial\mathcal{R}_{i,j}$. We approximate the first integral of Eq. (32) by a constant:

$$
\begin{aligned}
&\int_{\left(P_{i+\frac{1}{2}}, S_{j-\frac{1}{2}}\right)}^{\left(P_{i+\frac{1}{2}}, S_{j+\frac{1}{2}}\right)} \left(a_{11}\frac{\partial V_C}{\partial P} + a_{12}\frac{\partial V_C}{\partial S} + b_1 V_C\right) \mathrm{d}S \\
&\approx \left(a_{11}\frac{\partial V_C}{\partial P} + a_{12}\frac{\partial V_C}{\partial S} + b_1 V_C\right)\Bigg|_{\left(P_{i+\frac{1}{2}}, S_j\right)} \cdot h_{S_j}.
\end{aligned}
$$

Now, we are in the position to derive the approximations to $(a_{11}\frac{\partial V_C}{\partial P}+a_{12}\frac{\partial V_C}{\partial S}+b_1V_C)$ at the midpoint, $(P_{i+\frac{1}{2}},S_j)$, of the interval I_{P_i} for any $i=0,1,\ldots,N_P-1$. To begin with, the term $a_{11}\frac{\partial V_C}{\partial P}+b_1V_C$ can be approximated by a constant, which means that its derivative equals zero, that is,

$$\left(\frac{1}{2}\sigma_P^2P^2\frac{\partial V_C}{\partial P}+(E^*_{C1}+E^*_{C2}-a^*_{C1}-a^*_{C2}-\theta_PP-\sigma_P^2P-\frac{1}{2}\rho\sigma_P\sigma_SP)V_C\right)'$$

$$\equiv\left(aP^2\frac{\partial V_C}{\partial P}+b_1^{i+\frac{1}{2},j}V_C\right)'=0, \tag{33a}$$

$$V_C(P_i,S_j)=V_{C_{i,j}},\quad V_C(P_{i+1},S_j)=V_{C_{i+1,j}}, \tag{33b}$$

where $a=\frac{1}{2}\sigma_P^2$ and $b_1^{i+\frac{1}{2},j}=b_1(P_{i+\frac{1}{2}},S_j)$, $V_{C_{i,j}}$ and $V_{C_{i+1,j}}$ denote the values of V_C at (P_i,S_j) and (P_{i+1},C_j), respectively. The above first-order ordinary differential equation can be solved by integrating both sides of Eq. (33a):

$$aP^2\frac{\partial V_C}{\partial P}+b_1^{i+\frac{1}{2},j}V_C=C_1,$$

where C_1 is an arbitrary constant and can be determined by the boundary conditions Eq. (33b) as follows:

$$C_1=b_1^{i+\frac{1}{2},j}\frac{V_{C_{i+1,j}}e^{-\frac{\alpha_{i,j}}{P_{i+1}}}-V_{C_{i,j}}e^{-\frac{\alpha_{i,j}}{P_i}}}{e^{-\frac{\alpha_{i,j}}{P_{i+1}}}-e^{-\frac{\alpha_{i,j}}{P_i}}},$$

where $\alpha_{i,j}=\frac{b_1^{i+\frac{1}{2},j}}{a}$. Additionally, the derivative $\frac{\partial V_C}{\partial S}$ can be approximated by a forward difference

$$\frac{V_{C_{i,j+1}}-V_{C_{i,j}}}{h_{S_j}}.$$

As a result, we have

$$\left(a_{11}\frac{\partial V_C}{\partial P}+a_{12}\frac{\partial V_C}{\partial S}+b_1V_C\right)|(P_{i+\frac{1}{2}},S_j)\cdot h_{S_j}$$

$$\approx\left(b_1^{i+\frac{1}{2},j}\frac{V_{C_{i+1,j}}e^{-\frac{\alpha_{i,j}}{P_{i+1}}}-V_{C_{i,j}}e^{-\frac{\alpha_{i,j}}{P_i}}}{e^{-\frac{\alpha_{i,j}}{P_{i+1}}}-e^{-\frac{\alpha_{i,j}}{P_i}}}+d_{i,j}\frac{V_{C_{i,j+1}}-V_{C_{i,j}}}{h_{S_j}}\right)\cdot h_{S_j}, \tag{34}$$

where $d=\frac{1}{2}\rho\sigma_P\sigma_SPS$ and $d_{i,j}=d(P_i,S_j)$. Applying the similar method to the other three terms of Eq. (32), we get the following results:

$$\left(a_{11}\frac{\partial V_C}{\partial P}+a_{12}\frac{\partial V_C}{\partial S}+b_1V_C\right)|_{(P_{i-\frac{1}{2}},S_j)}\cdot h_{S_j}$$

$$\approx\left(b_1^{i-\frac{1}{2},j}\frac{V_{C_{i,j}}e^{-\frac{\alpha_{i-1,j}}{P_i}}-V_{C_{i-1,j}}e^{-\frac{\alpha_{i-1,j}}{P_{i-1}}}}{e^{-\frac{\alpha_{i-1,j}}{P_i}}-e^{-\frac{\alpha_{i-1,j}}{P_{i-1}}}}+d_{i,j}\frac{V_{C_{i,j+1}}-V_{C_{i,j}}}{h_{S_j}}\right)\cdot h_{S_j}, \tag{35}$$

$$\left(a_{21}\frac{\partial V_C}{\partial P}+a_{22}\frac{\partial V_C}{\partial S}+b_2V_C\right)|_{(P_i,S_{j+\frac{1}{2}})}\cdot h_{P_i}$$
$$\approx S_{j+\frac{1}{2}}\left(\bar{b}_{i,j+\frac{1}{2}}\frac{S_{j+1}^{\bar{\alpha}_{i,j}}V_{C_{i+1,j}}-S_j^{\bar{\alpha}_{i,j}}V_{C_{i,j}}}{S_{j+1}^{\bar{\alpha}_{i,j}}-S_j^{\bar{\alpha}_{i,j}}}+\bar{d}_{i,j}\frac{V_{C_{i,j+1}}-V_{C_{i,j}}}{h_{P_i}}\right)\cdot h_{P_i}, \tag{36}$$

and

$$\left(a_{21}\frac{\partial V_C}{\partial P}+a_{22}\frac{\partial V_C}{\partial S}+b_2V_C\right)|_{(P_i,S_{j-\frac{1}{2}})}\cdot h_{P_i}$$
$$\approx S_{j-\frac{1}{2}}\left(\bar{b}_{i,j-\frac{1}{2}}\frac{S_j^{\bar{\alpha}_{i,j-1}}V_{C_{i,j}}-S_{j-1}^{\bar{\alpha}_{i,j-1}}V_{C_{i,j-1}}}{S_j^{\bar{\alpha}_{i,j-1}}-S_{j-1}^{\bar{\alpha}_{i,j-1}}}+\bar{d}_{i,j}\frac{V_{C_{i,j+1}}-V_{C_{i,j}}}{h_{P_i}}\right)\cdot h_{P_i}, \tag{37}$$

where $\bar{\alpha}_{i,j}=\frac{\bar{b}_{i,j+\frac{1}{2}}}{\bar{a}_j}$, $\bar{a}=\frac{1}{2}\sigma_S^2$, $\bar{b}=\mu-\sigma_S^2-\frac{1}{2}\rho\sigma_P\sigma_S$, and $\bar{d}_{i,j}=\frac{1}{2}\rho\sigma_P\sigma_S P_i$. Hence, we obtain the following equations by combining Eqs. (31), (32), and (34)–(37):

$$-\frac{\partial V_{C_{i,j}}}{\partial t}R_{i,j}+e_{i-1,j}^{i,j}V_{C_{i-1,j}}+e_{i,j-1}^{i,j}V_{C_{i,j-1}}+e_{i,j}^{i,j}V_{C_{i,j}}+e_{i,j+1}^{i,j}V_{C_{i,j+1}}$$
$$+e_{i+1,j}^{i,j}V_{C_{i+1,j}}=F_{C_{i,j}}R_{i,j}, \tag{38}$$

where

$$e_{i-1,j}^{i,j}=-b_1^{i-\frac{1}{2},j}\frac{e^{-\frac{\alpha_{i-1,j}}{P_{i-1}}}h_{S_j}}{e^{-\frac{\alpha_{i-1,j}}{P_i}}-e^{-\frac{\alpha_{i-1,j}}{P_{i-1}}}},$$
$$e_{i,j-1}^{i,j}=-S_{j-\frac{1}{2}}\bar{b}_{i,j-\frac{1}{2}}\frac{S_{j-1}^{\bar{\alpha}_{i,j-1}}h_{P_i}}{S_j^{\bar{\alpha}_{i,j-1}}-S_{j-1}^{\bar{\alpha}_{i,j-1}}}, \tag{39}$$
$$e_{i,j}^{i,j}=h_{S_j}\left(\frac{b_1^{i+\frac{1}{2},j}e^{-\frac{\alpha_{i,j}}{P_i}}}{e^{-\frac{\alpha_{i,j}}{P_{i+1}}}-e^{-\frac{\alpha_{i,j}}{P_i}}}+\frac{b_1^{i-\frac{1}{2},j}e^{-\frac{\alpha_{i-1,j}}{P_i}}}{e^{-\frac{\alpha_{i-1,j}}{P_i}}-e^{-\frac{\alpha_{i-1,j}}{P_{i-1}}}}+\bar{d}_{i,j}\right)$$
$$+h_{P_i}\left(S_{j+\frac{1}{2}}\frac{\bar{b}_{i,j+\frac{1}{2}}S_j^{\bar{\alpha}_{i,j}}}{S_{j+1}^{\bar{\alpha}_{i,j}}-S_j^{\bar{\alpha}_{i,j}}}+S_{j-\frac{1}{2}}\frac{\bar{b}_{i,j-\frac{1}{2}}S_j^{\bar{\alpha}_{i,j-1}}}{S_j^{\alpha_{i,j-1}}-S_{j-1}^{\bar{\alpha}_{i,j-1}}}\right)+c_{i,j}R_{i,j}, \tag{40}$$
$$e_{i,j+1}^{i,j}=-S_{j+\frac{1}{2}}\bar{b}_{i,j+\frac{1}{2}}\frac{S_{j+1}^{\bar{\alpha}_{i,j}}h_{P_i}}{S_{j+1}^{\bar{\alpha}_{i,j}}-S_j^{\bar{\alpha}_{i,j}}},$$
$$e_{i+1,j}^{i,j}=-b_1^{i+\frac{1}{2},j}\frac{e^{-\frac{\alpha_{i,j}}{P_{i+1}}}h_{S_j}}{e^{-\frac{\alpha_{i,j}}{P_{i+1}}}-e^{-\frac{\alpha_{i,j}}{P_i}}}-h_{S_j}\bar{d}_{i,j}, \tag{41}$$

for $i=1,2,\ldots,N_P-1$, $j=1,2,\ldots,N_S-1$. The other elements $e_{m,n}^{i,j}$ equal zeros when $m\neq i-1,i,i+1$ and $n\neq j-1,j,j+1$. We can see that system (38) is an $(N_P-1)^2\times(N_S-1)^2$ linear system of equations for

$$V_C=(V_{C_{1,1}},\ldots,V_{C_{1,N_S-1}},V_{C_{2,1}},\ldots,V_{C_{2,N_S-1}},\ldots,V_{C_{N_P-1,1}},V_{C_{N_P-1,2}},\ldots,V_{C_{N_P-1,N_S-1}})^\top.$$

Note that $V_{C_{0,j}}(t)$, $V_{C_{i,0}}(t)$, $V_{C_{0,N_S}}(t)$, and $V_{C_{N_P,0}}(t)$ for $i = 1, 2, \ldots, N_P$ and $j = 1, 2, \ldots, N_S$ equal to the given boundary conditions. Obviously, the coefficient matrix of system (38) is pentadiagonal.

The Implicit Difference Method for Time Discretization

Next we embark on the time discretization of system (38). To this purpose, we first rewrite Eq. (38) as

$$-\frac{\partial V_{C_{i,j}}}{\partial t} R_{i,j} + D_{i,j} V_C = F_{C_{i,j}} R_{i,j}, \tag{42}$$

where

$$D_{i,j} = (0, \ldots, 0, e^{i,j}_{i-1,j}, 0, \ldots, 0, e^{i,j}_{i,j-1}, e^{i,j}_{i,j}, e^{i,j}_{i,j+1}, 0, \ldots, 0, e^{i,j}_{i+1,j}, 0, \ldots, 0)$$

for $i = 1, 2, \ldots, N_P - 1$ and $j = 1, 2, \ldots, N_S - 1$. We select $M - 1$ points numbered from t^1 to t^{M-1} between 0 and T, and let $T = t^0$, $t^M = 0$ to form a partition of time $T = t^0 > t^1 > \cdots > t^M = 0$. Then, the full discrete form of Eq. (42) can be obtained by applying the two-level implicit time-stepping method with a splitting parameter $\theta \in [\frac{1}{2}, 1]$ to it:

$$\begin{aligned}
&\left(\theta D\left(P, S, E^*_{C1}\left(t^{m+1}\right), E^*_{C2}\left(t^{m+1}\right), a^*_{C1}\left(t^{m+1}\right), a^*_{C2}\left(t^{m+1}\right), t^{m+1}\right) + G^m\right) V_C^{m+1} \\
&\quad = \theta F_C\left(P, S, E^*_{C1}\left(t^{m+1}\right), E^*_{C2}\left(t^{m+1}\right), a^*_{C1}\left(t^{m+1}\right), a^*_{C2}\left(t^{m+1}\right), t^{m+1}\right) \\
&\qquad + (1-\theta) F_C\left(P, S, E^*_{C1}\left(t^{m}\right), E^*_{C2}\left(t^{m}\right), a^*_{C1}\left(t^{m}\right), a^*_{C2}\left(t^{m}\right), t^{m}\right) \\
&\qquad + \left(G^m - (1-\theta) D\left(P, S, E^*_{C1}\left(t^{m}\right), E^*_{C2}\left(t^{m}\right), a^*_{C1}\left(t^{m}\right), a^*_{C2}\left(t^{m}\right), t^{m}\right)\right) V_C^m,
\end{aligned} \tag{43}$$

where

$$\begin{aligned}
V_C^m &= \left(V^m_{C_{1,1}}, \ldots, V^m_{C_{1,N_S-1}}, V^m_{C_{2,1}}, \ldots, V^m_{C_{2,N_S-1}}, \ldots, V^m_{C_{N_P-1,1}}, \ldots, V^m_{C_{N_P-1,N_S-1}}\right)^\top, \\
G^m &= \operatorname{diag}\left(-R_{1,1}/\Delta t^m, \ldots, -R_{N_P-1,N_S-1}/\Delta t^m\right)^\top,
\end{aligned} \tag{44}$$

for $m = 0, 1, \ldots, M-1$. Note that $\Delta t^m = t^{m+1} - t^m < 0$, and V_C^m denotes the approximation of V_C at $t = t^m$. Particularly, when we set $\theta = \frac{1}{2}$, the scheme (43) becomes the famous Crank–Nicolson scheme and is second-order accuracy; when we set $\theta = 1$, the scheme (43) becomes the backward Euler scheme and is first-order accuracy.

Decoupling of the System

In the above discussion, we have assumed that the control variables E^*_{C1}, a^*_{C1}, E^*_{C2}, and a^*_{C2} are known. However, we can see from (43) that they are coupled with V_C when $\theta \neq 0$. To deal with this dilemma, we replace $E^*_{C1}(t^{m+1})$, $a^*_{C1}(t^{m+1})$, $E^*_{C2}(t^{m+1})$, and $a^*_{C2}(t^{m+1})$ by $E^*_{C1}(t^m)$, $a^*_{C1}(t^m)$, $E^*_{C2}(t^m)$, and $a^*_{C2}(t^m)$, respectively. This method should be reasonable because the control variables are just replaced by their values in the previous time step. The error is small if Δt^m is sufficiently small. The resulting system corresponding to (43) is as follows:

$$\begin{aligned}
&\left(\theta D\left(P, S, E^*_{C1}\left(t^{m}\right), E^*_{C2}\left(t^{m}\right), a^*_{C1}\left(t^{m}\right), a^*_{C2}\left(t^{m}\right), t^{m+1}\right) + G^m\right) V_C^{m+1} \\
&\quad = \theta F\left(P, S, E^*_{C1}\left(t^{m}\right), E^*_{C2}\left(t^{m}\right), a^*_{C1}\left(t^{m}\right), a^*_{C2}\left(t^{m}\right), t^{m+1}\right) \\
&\qquad + (1-\theta) F\left(P, S, E^*_{C1}\left(t^{m}\right), E^*_{C2}\left(t^{m}\right), a^*_{C1}\left(t^{m}\right), a^*_{C2}\left(t^{m}\right), t^{m}\right) \\
&\qquad + \left(G^m - (1-\theta) D\left(P, S, E^*_{C1}\left(t^{m}\right), E^*_{C2}\left(t^{m}\right), a^*_{C1}\left(t^{m}\right), a^*_{C2}\left(t^{m}\right), t^{m}\right)\right) V_C^m.
\end{aligned} \tag{45}$$

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