Dyn Games Appl (2018) 8:352–378
https://doi.org/10.1007/s13235-017-0226-6

Simple Motion Evasion Differential Game of Many Pursuers and Evaders with Integral Constraints

Gafurjan Ibragimov[1] · Massimiliano Ferrara[2,3] · Atamurat Kuchkarov[4] · Bruno Antonio Pansera[2]

Published online: 15 July 2017

Abstract We study a simple motion evasion differential game of many pursuers and evaders. Control functions of players are subjected to integral constraints. If the state of at least one evader does not coincide with that of any pursuer forever, then evasion is said to be possible in the game. The aim of the group of evaders is to construct their strategies so that evasion can be possible in the game and the aim of the group of pursuers is opposite. The problem is to find a sufficient condition of evasion. If the total energy of pursuers is less than or equal to that of evaders, then it is proved that evasion is possible, and moreover, evasion strategies are constructed explicitly.

Keywords Differential game · Many pursuers · Many evaders · Integral constraint · Evasion · Strategy

Mathematics Subject Classification Primary 91A23; Secondary 49N75

✉ Massimiliano Ferrara
massimiliano.ferrara@unirc.it; massimiliano.ferrara@unibocconi.it

Gafurjan Ibragimov
ibragimov@upm.edu.my

Atamurat Kuchkarov
kuchkarov1@yandex.ru

Bruno Antonio Pansera
bruno.pansera@unirc.it

1 Department of Mathematics, Institute for Mathematical Research, Universiti Putra Malaysia, Serdang, Malaysia

2 Department of Law and Economics, University Mediterranea of Reggio Calabria, Reggio Calabria, Italy

3 ICRIOS, Bocconi University, Milan, Italy

4 Institute of Mathematics, National University of Uzbekistan, Tashkent, Uzbekistan

1 Introduction

Differential games have been an object of research since the 1960 (see, for example, Isaacs [18], Friedman [12], Hajek [14], Pontryagin [27], Krasovskii and Subbotin [21], Petrosyan [26]). A great number of works were devoted to simple motion pursuit and evasion differential games of many players. Mainly, controls of players are subjected to either geometric or integral constraints. In the case of geometric constraints, Croft [11] showed that in the n-dimensional Euclidean ball n lions can catch the man while the man can escape from $n-1$ lions. A similar game problem was studied by Ivanov [19] on any convex compact set and an estimate from above was obtained for guaranteed pursuit time. In the case of an unbounded region, interesting results were obtained by Alexander et al. [1].

A new evasion maneuver was proposed by Mishchenko et al. [25] in the game of many pursuers. Chernous'ko [7] studied an evasion game of one evader and several pursuers where the evader was faster than the pursuers. It was proved that evader can avoid pursuers by remaining in a neighborhood of a given ray. Later on, the result of this paper was extended by Zak to more general differential game problems (see, for example, [33]). Also, evasion from a group of pursuers were studied by Borowko et al. [6] and Chernous'ko [9].

A simple motion differential game of many pursuers and one evader, when all players have the same dynamic possibilities, was studied in $\mathbb{R}^n$ by Pshenichnii [28]. Namely, it was proved that if the initial state of the evader belongs to the convex hull of pursuers' initial states, then pursuit can be completed; otherwise, evasion is possible. Based on this work, Pshenichnii et al. [29] developed the method of resolving functions for solving linear pursuit problems with many pursuers. In the case of integral constraints, the method of resolving functions was developed by Belousov [4]. Later on, the results of the paper [28] were extended by many researchers. For example, when control sets of players are convex compact sets, Grigorenko [13] obtained the necessary and sufficient conditions of evasion of one evader from several pursuers. The papers [8,23] are also extensions of [28]. In the work [22], the game problem of many pursuers and one evader was studied on a cylinder. In the recent work of Kuchkarov et al. [24], the results of [28] were extended to differential games on manifolds with Euclidean metric.

Petrov [5] obtained necessary and sufficient condition of evasion in a simple motion differential games of a group of pursuers and a group of evaders in $\mathbb{R}^n$ where all evaders use the same control. By definition, pursuit is considered completed if the state of a pursuer coincides with the state of at least one evader. Also, the works [3,32] relate to such games.

The present paper is devoted to the evasion differential game of K pursuers and M evaders described by equations

$$\dot{x}_i = u_i, \quad x_i(0) = x_{i0}, \quad i = 1, \dots, K, \tag{1.1}$$
$$\dot{y}_j = v_j, \quad y_j(0) = y_{j0}, \quad j = 1, \dots, M, \tag{1.2}$$

with integral constraints

$$\int_0^\infty |u_i(s)|^2 \mathrm{d}s \le \rho_i^2, \quad i = 1, \dots, K, \tag{1.3}$$
$$\int_0^\infty |v_j(s)|^2 \mathrm{d}s \le \sigma_j^2, \quad j = 1, \dots, M, \tag{1.4}$$

where $x_i, y_j, u_i, v_j \in \mathbb{R}^n$, $n \geq 2$, and ρ_i, σ_j are given positive numbers, $x_{i0} \neq y_{j0}$ for all $i = 1, \dots, K$, and $j = 1, \dots, M$. Control systems with integral constraints on the control functions arise in problems, where the control resource is exhausted by consumption, such as energy, finance, food (see, e.g.,[10,15,20]).

A linear differential game of many pursuers and one evader ($M = 1$) when the control functions of players are subjected to integral constraints was first studied by Satimov et al. [30]. For the game (1.1)–(1.4), the result of that paper can be formulated as follows: if $\rho_1^2 + \dots + \rho_K^2 > \sigma_1^2$, then pursuit can be completed. In the paper of Satimov et al. [31], it was shown that if

$$\rho_1^2 + \dots + \rho_K^2 \leq \sigma_1^2, \tag{1.5}$$

then evasion is possible from some initial positions of players. In general, an evasion game of K pursuers and $M = 1$ evader was studied in the work of Ibragimov et al. [17], and it was proven that if (1.5) holds, then for any initial positions of players evasion is possible. Later on, Ibragimov and Satimov [16] studied a pursuit game problem of K pursuers and M evaders described by Eqs. (1.1)–(1.4) in a closed convex subset of $\mathbb{R}^n$. It was established that if

$$\rho_1^2 + \dots + \rho_K^2 > \sigma_1^2 + \dots + \sigma_M^2,$$

then pursuit can be completed. In the evasion game problem studied by Idham et al. [2] in ℓ_2, strategies of evaders were constructed based on the fact that the space ℓ_2 is infinite dimensional.

In the present paper, we study the game (1.1)–(1.4) in $\mathbb{R}^n$ in the case where

$$\rho_1^2 + \dots + \rho_K^2 \leq \sigma_1^2 + \dots + \sigma_M^2.$$

If this is the case, we show that evasion is possible from any initial positions of players. In addition, we construct explicit strategies for the evaders.

We use the following notations.

- $\zeta_0 = (x_{10}, \dots, x_{K0}, y_{10}, \dots, y_{M0})$ is initial position of players.
- V_j is the strategy of j-th evader,
- $p_i(t) = \rho_i^2 - \int\limits_0^t |u_i(s)|^2 ds$, $q_j(t) = \sigma_j^2 - \int\limits_0^t |v_j(s)|^2 ds$,
- $\sigma = \left(\sigma_1^2 + \dots + \sigma_M^2\right)^{1/2}$, $\rho = \left(\rho_1^2 + \dots + \rho_K^2\right)^{1/2}$, $d = \min\limits_{i=1,\dots,K,\ j=1,\dots,M} |x_{i0} - y_{j0}|$,
- $e_j = \frac{y_{j0} - z_0}{|y_{j0} - z_0|}$, and e'_j is a unit vector orthogonal to the vector e_j to be obtained from e_j by rotating it to the angle $-\frac{\pi}{2}$,
- $S_j = \{\xi \in \mathbb{R}^2 \mid |(\xi - y_{j0}, e'_j)| \leq a + b, (\xi - y_{j0}, e_j) \geq -b\}$, and $S'_j = \{\xi \in \mathbb{R}^2 \mid |(\xi - y_{j0}, e'_j)| \leq a,\ (\xi - y_{j0}, e_j) \geq -b\}$ are strips,
- $I_j(t) = \{s \in \{1, \dots, K\} \mid x_s(t) \in S_j\}$,
- $d_0 = \min\{|y_{i0} - y_{j0}| \ \mid\ y_{i0} \neq y_{j0}, i, j = 1, \dots, M\}$,
- $a_{j1}, a_{j2}, \dots$ are given numbers that satisfy $a_{j,i+1} = \kappa \cdot a_{j,i}^4$ where $\kappa \in (0, 1)$,
- τ_{ji} is the a_{ji}-approach time of the evader y_j with some pursuers; x_{ji} is the pursuer that is chosen to apply maneuver at τ_{ji}; $\tau'_{ji} = \tau_{ji} + \frac{a_{ji}}{\alpha}$ (later they are used without j),
- Continuous attack of a group of the pursuers $x_{\omega_{k-1}+1}, \dots, x_{\omega_k}$, is started at $\tau_{\omega_{k-1}+1}$ and is completed at $\tau^*_{\omega_k} = \max\{\tau'_{\omega_{k-1}+1}, \dots,\ \tau'_{\omega_k}\}$, $\omega_0 = 0$, $\tau^*_0 = 0$, $\tau'_0 = \tau'_1$,
- $J_{i+1} = \cup_{j=i+1}^{\omega_k} [\tau_j, \tau'_j)$, $\omega_{k-1} + 1 \leq i < \omega_k$; $J_{\omega_k+1} = \emptyset$,
- z_i is i-th fictitious evader, and w_i is its control parameter.

2 Statement of Problem

We consider a differential game described by Eqs. (1.1)–(1.4) where $K \geq 1$, $M \geq 1$. First, we give definitions for control functions of players and strategies of evaders.

Definition 2.1 Borel measurable functions $u_i(t)$, $t \geq 0$, and $v_j(t)$, $t \geq 0$, that satisfy the constraints (1.3) and (1.4), respectively, are called controls of the pursuer x_i, $i \in \{1, \dots, K\}$ and evader y_j, $j \in \{1, \dots, M\}$, respectively.

Definition 2.2 If $y_j(t_j) = x_i(t_j)$ at some $j \in \{1, \dots, M\}$, $i \in \{1, \dots, K\}$ and $t_j > 0$, then we say that the evader y_j is captured by the pursuer x_i at the time t_j.

To define the strategies of evaders, we introduce new scalar parameters $q_j = q_j(t)$, $t \geq 0$, by equations

$$\dot{q}_j = -|v_j(t)|^2,\ q_j(0) = \sigma_j^2, \quad j = 1, 2, \dots, M.$$

Clearly, $q_j(t) = \sigma_j^2 - \int_0^t |v_j(s)|^2 \mathrm{d}s$, and $q_j(t)$ expresses the amount of energy of evader y_j remained at the time t. Let $\zeta_0 = (x_{10}, \dots, x_{K0}, y_{10}, \dots, y_{M0})$.

Definition 2.3 A function $V_j(\zeta_0, t, y_j, q_j, x_1, \dots, x_K, u_1, \dots, u_K)$, $j \in \{1, \dots, M\}$, is called strategy of the evader y_j if

(i) for any controls of the pursuers $u_i = u_i(t)$, $i = 1, \dots, K$, the initial value problem

$$\begin{cases} \dot{x}_1 = u_1(t), & x_1(0) = x_{10}, \\ \vdots & \vdots \\ \dot{x}_K = u_K(t), & x_K(0) = x_{K0}, \\ \dot{y}_j = V_j(\zeta_0, t, y_j, q_j, x_1, \dots, x_K, u_1(t), \dots, u_K(t)), & y_j(0) = y_{j0}, \\ \dot{q}_j = -|V_j(\zeta_0, t, y_j, q_j, x_1, \dots, x_K, u_1(t), \dots, u_K(t))|^2, & q_j(0) = \sigma_j^2, \end{cases} \tag{2.1}$$

has a unique solution $(x_1(t), \dots, x_K(t), y_j(t), q_j(t))$, $t \geq 0$,

(ii) along this solution

$$\int_0^\infty |V_j(\zeta_0, t, y_j(t), q_j(t), x_1(t), \dots, x_K(t), u_1(t), \dots, u_K(t))|^2 \mathrm{d}t \leq \sigma_j^2. \tag{2.2}$$

By a solution of the initial value problem (2.1), we mean $(K+2)$-tuple $(x_1(t), \dots, x_K(t), y_j(t), q_j(t))$, $t \geq 0$, with absolutely continuous components $x_i(t)$, $i = 1, \dots, K$, $y_j(t)$, and $q_j(t)$ that satisfy initial conditions in (2.1), differential equations in (2.1) almost everywhere on $[0, \infty)$ and $q_j(t) \geq 0$.

Definition 2.4 We say that evasion is possible in the game (1.1)–(1.4) if there are strategies of evaders such that, for any controls of the pursuers, for all $t > 0$ and $i = 1, \dots, K$, there exists $j_0 \in \{1, 2, \dots, M\}$ such that $x_i(t) \neq y_{j_0}(t)$.

Thus, evaders apply strategies, whereas the pursuers use any controls. In other words, behaviors of pursuers are any. Pursuers try to capture each evader, and evaders try to avoid. If at least one evader is not captured by pursuers for all $t \geq 0$, then by Definition 2.4 evasion is possible. It should be noted that we will not study the existence and uniqueness of solution for the system of Eq. (2.1). We construct strategies V_j, $j = 1, \dots, M$, for evaders such that the initial value problem (2.1) has a unique solution and (2.2) is satisfied as well.

Now, formulate the problem.

Problem 1 *Find a condition of evasion in terms of* $\rho_1, \ldots, \rho_K$ *and* $\sigma_1, \ldots, \sigma_M$.

3 Main Result

Let $\sigma = \left(\sigma_1^2 + \cdots + \sigma_M^2\right)^{1/2}, \ \rho = \left(\rho_1^2 + \cdots + \rho_K^2\right)^{1/2}$. The main result of the paper is the following statement.

Theorem 3.1 *If* $\sigma \geq \rho$, *then for any initial position of players* ζ_0, *evasion is possible in the game* (1.1)–(1.4).

First, we give the structure of proof. In constructing the strategies for evaders the fact that $\sigma > \rho$ is crucial. Therefore, the case $\sigma = \rho$ is reduced to the case $\sigma > \rho$ (Sect. 3.1). Then disjoint strips corresponding to evaders are constructed and each evader further moves only in his own strip (Sect. 3.2). Next, strategies for evaders are constructed (Sect. 3.3).

If for a pursuer $x_i \in S$, the inequalities $(y(t), e) \geq (x_i(t), e)$ and $y(t) \neq x_i(t)$ hold at some time $t = \theta \geq 0$, then they hold for all $t \geq \theta$ while the pursuer moves in the strip S and the evader's energy is positive (Sect. 3.4). In particular, this is true for all pursuers x_i for which $x_{i0} \in S$ and $(y_0, e) \geq (x_{i0}, e)$, where $\theta = 0$. In other words, such pursuers moving in S can never capture the evader while its energy $q(t)$ is positive.

To estimate the distance between any pursuer x_p moving in S and the evader y on the time interval $[\tau_p, \tau_p']$, where τ_p is a_p-approach time, we introduce fictitious evaders (FE) z_i (Sect. 3.5) and estimate their energies. Then, we estimate the distance $|z_p(t) - x_p(t)|$ on $[\tau_p, \tau_p']$ from below (Sect. 3.6). Next, we estimate $|z_p(t) - y(t)|$ from above (Sect. 3.7). Use these estimates to estimate $|y(t) - x_p(t)|$ on $[\tau_p, \tau_p']$ from below, which allows to conclude that $x_p(t) \neq y(t)$, and moreover, a_{p+1}-approach will not occur with the pursuer x_p on $[\tau_p, \tau_p']$ (Sect. 3.8). In addition, we show that $(y(\tau_p'), e) \geq (x_p(\tau_p'), e)$ and $y(\tau_p') \neq x_p(\tau_p')$. As mentioned above, pursuer x_p moving in S cannot capture the evader on $t \geq \tau_p'$ as long as $q(t) > 0$.

Further, we estimate the evaders' energies to prove that $q_j(t) > 0$, $t \geq 0$, for some $j \in \{1, \ldots, M\}$ and then prove that evasion is possible on any interval $[0, T]$ providing that the number of approach times in $[0, T]$ is finite (Sect. 3.9). Then finiteness of the number of approach times in any interval $[0, T]$ is established (Sect. 3.10).

Finally, we choose parameters to satisfy all imposed conditions throughout the paper (Sect. 3.11), show that evader moves by remaining in the set S' whose width $2a$ can be made as small as we wish, give a method of reduction of the game in $\mathbb{R}^n$ to the game in $\mathbb{R}^2$, and then discuss the strategies of evaders (Sect. 4).

Proof The fact that evasion is possible in $\mathbb{R}^2$ implies that evasion is possible in $\mathbb{R}^n$ (see Sect. 4.2). Therefore, we prove the theorem for $\mathbb{R}^2$. □

3.1 Reduction to the Case $\sigma > \rho$

Show that the case $\sigma = \rho$ can be reduced to the case $\sigma > \rho$. Indeed, let $\sigma = \rho$. Set

$$v_j(t) = 0, \quad 0 \leq t \leq \tau^0, \quad j = 1, \ldots, M, \tag{3.1}$$

where $t = \tau^0$ is the first time at which

$$|x_i(t) - x_{i0}| = \frac{d}{4}, \text{ where } d = \min_{i=1,\ldots,K,\ j=1,\ldots,M} |x_{i0} - y_{j0}| \tag{3.2}$$

for at least one of the numbers $i \in \{1, \ldots, K\}$. Such a time τ^0 may not exist meaning that $|x_i(t) - x_{i0}| < d/4$ for all $i \in \{1, \ldots, K\}$ and $t \geq 0$. In this case, we let $\tau^0 = \infty$, and evasion is possible since

$$\begin{aligned}|x_i(t) - y_j(t)| &= |x_i(t) - x_{i0} + x_{i0} - y_{j0}| \\ &\geq |x_{i0} - y_{j0}| - |x_i(t) - x_{i0}| \\ &> d - \frac{d}{4} = \frac{3d}{4}\end{aligned}$$

for all i and j, and, hence, $x_i(t) \neq y_j(t)$, $t \geq 0$. Let τ^0 be finite. Say $|x_s(t) - x_{s0}| = d/4$ at some $t = \tau^0$ and $s \in \{1, \ldots, K\}$. Then in view of (3.1) at the time $t = \tau^0$ the total resource of the evaders is still $\sigma_1^2 + \cdots + \sigma_M^2$ since they have not moved on $[0, \tau^0]$. The total resource of the pursuers remained at the time τ^0 is

$$p(\tau^0) = \sum_{i=1}^{K} p_i(\tau^0), \quad p_i(t) = \rho_i^2 - \int_0^t |u_i(s)|^2 \mathrm{d}s.$$

We have

$$p(\tau^0) = \sum_{i=1}^{K} \rho_i^2 - \sum_{i=1}^{K} \int_0^{\tau^0} |u_i(t)|^2 \mathrm{d}t \leq \rho^2 - \int_0^{\tau^0} |u_s(t)|^2 \mathrm{d}t. \tag{3.3}$$

Since

$$\frac{d}{4} = |x_s(\tau^0) - x_{s0}| = \left| \int_0^{\tau^0} u_s(t) \mathrm{d}t \right| \leq \left(\tau^0 \int_0^{\tau^0} |u_s(t)|^2 \mathrm{d}t \right)^{1/2},$$

the pursuer x_s has spent a positive resource $\int_0^{\tau^0} |u_s(t)|^2 \mathrm{d}s \geq \frac{d^2}{16\tau^0}$, and so by (3.3) $p(\tau^0) < \rho^2$. Thus, at the time $t = \tau^0$, we have [see (3.1)]

$$\sum_{j=1}^{M} q_j(\tau^0) = \sigma^2 > \rho^2 - \frac{d^2}{16\tau^0} \geq p(\tau^0)$$

meaning that the total energy of evaders is greater than that of pursuers at the time τ^0. Moreover, $x_i(\tau^0) \neq y_j(\tau^0)$ for all i and j. Thus, without loss of generality, we can assume that $\sigma > \rho$.

Remark 3.2 According to Definition 2.3, evaders are not allowed to know information about $p_i(\tau^0)$. Hence, evaders do not know the value $p(\tau^0)$. However, evaders know the fact that $p(\tau^0) \leq \rho^2 - \frac{d^2}{16\tau^0} < \sigma^2$ at the time τ^0. Therefore, if $\sigma = \rho$, then at the time τ^0 evaders use $\bar{\rho} = \sqrt{\rho^2 - \frac{d^2}{16\tau^0}}$ instead of ρ to construct their strategies at $t \geq \tau^0$. At the time τ^0 evaders exactly know that $\sigma > \bar{\rho}$.

3.2 Disjoint Strips

To construct disjoint strips, consider two cases.

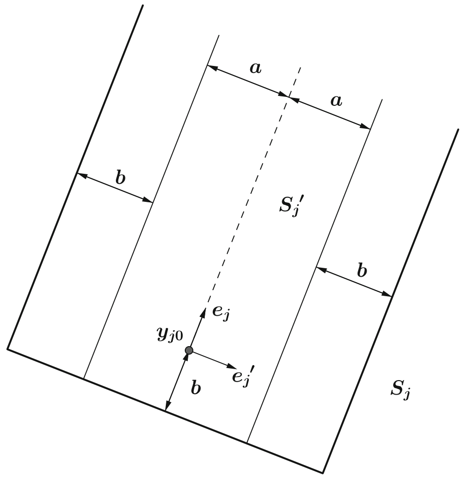


Fig. 1 Sets S_j and S'_j

Case 1 $y_{i0} \neq y_{j0}$, **for all** $i \neq j, i, j = 1, \dots, M$. Let z_0 be any point that doesn't coincide with y_{10} if $M = 1$; and any point that doesn't lie on any straight line passing through the points y_{i0} and y_{j0} for all $i, j = 1, \dots, M, i \neq j$, if $M \geq 2$. Hence, $z_0 \neq y_{j0},\ j = 1, \dots, M$.

Let

$$e_j = \frac{y_{j0} - z_0}{|y_{j0} - z_0|}, \quad j = 1, \dots, M,$$

and let e'_j be a unit vector orthogonal to the vector e_j to be obtained from e_j by rotating it to the angle $-\frac{\pi}{2}$, that is, by rotating clockwise to the angle $\frac{\pi}{2}$.

We define a strip S_j associated with each point y_{j0} as follows:

$$S_j = \{\xi \in \mathbb{R}^2 \mid |(\xi - y_{j0}, e'_j)| \leq a + b, (\xi - y_{j0}, e_j) \geq -b\},$$

where the positive numbers a and b are any if $M = 1$, and if $M \geq 2$, they are chosen so that $S_i \cap S_j = \varnothing$ if $i \neq j$, and $z_0 \notin S_j$ for all $i, j = 1, \dots, M$ (see Fig. 1). If $M \geq 2$, we can choose such numbers a and b, since the rays $\zeta_j(t) = z_0 + e_j t,\ t > 0,\ j = 1, \dots, M$, have no common point. Let

$$I_j(t) = \{s \in \{1, \dots, K\} \mid x_s(t) \in S_j\}, \quad j = 1, \dots, M,\ t \geq 0.$$

In other words, $I_j(t)$ is the set of numbers of pursuers in the strip S_j at the current time t. In the case $M \geq 2$, because of $S_i \cap S_j = \varnothing$, we have $I_i(t) \cap I_j(t) = \varnothing$ for all $i \neq j$.

Case 2 There exist initial states $y_{i0}, y_{j0}, i \neq j$ **with** $y_{i0} = y_{j0}$. Let $y'_{i0} = y_{i0} + \varepsilon v_{i0},\ i = 1, \dots, M$, where unit vectors

$$v_{i0} = \left(\cos\frac{2\pi(i-1)}{M},\ \sin\frac{2\pi(i-1)}{M}\right), \quad i = 1, \dots, M, \tag{3.4}$$

are the vertices of regular M-gon with the center at the origin, ε is a positive number that satisfies the following condition

$$\varepsilon < \min\left\{\frac{\sigma^2 - \rho^2}{M}, \frac{d^2}{64\rho^2}, \rho^2, \frac{d_0}{2}\right\}, \tag{3.5}$$

where

$$d_0 = \min\{|y_{i0} - y_{j0}| \mid y_{i0} \neq y_{j0}, \quad i, j = 1, \dots, M\},$$

and d is defined by (3.2). Since $\varepsilon < d_0/2$, we have $y'_{i0} \neq y'_{j0}$ for all $i \neq j, i, j = 1, \dots, M$. We now use the procedure of construction of strips in Case 1, with y_{i0} replaced by y'_{i0} for all $i = 1, \dots, M$.

In Case 2, first we bring all the evaders to points $y'_{i0}, \quad i = 1, \dots, M$. To this end, we let

$$v_i(t) = v_{i0}, \; j = 1, \dots, M, \quad 0 \le t \le \varepsilon.$$

Then, clearly, at the time $t = \varepsilon$, we have $y_i(\varepsilon) = y'_{i0}$ for all $i = 1, \dots, M$, and

$$\sum_{j=1}^{M} q_j(\varepsilon) = \sum_{j=1}^{M} \sigma_j^2 - M\varepsilon > \sum_{i=1}^{K} \rho_i^2 \ge \sum_{i=1}^{K} p_i(\varepsilon),$$

showing that the total energy of the evaders at the time $t = \varepsilon$ is still greater than that of the pursuers. Moreover,

$$y_j(t) \neq x_i(t), \quad 0 \le t \le \varepsilon, \quad j = 1, \dots, l, \quad i = 1, \dots, K,$$

because by (3.5)

$$\begin{aligned} |y_j(t) - x_i(t)| &\ge |y_0 - x_{i0}| - \left| \int_0^{\varepsilon} v_{j0} \mathrm{d}s \right| - \left| \int_0^{\varepsilon} u_i(s) \mathrm{d}s \right| \\ &> \frac{3d}{4} - \varepsilon - \sqrt{\varepsilon}\rho \ge \frac{3d}{4} - 2\sqrt{\varepsilon}\rho \ge \frac{d}{2} > 0. \end{aligned} \tag{3.6}$$

Thus, Case 2 can be reduced to Case 1. Therefore, we can assume at the beginning that $y_{i0} \neq y_{j0}$ for all $i \neq j, i, j = 1, \dots, M$.

3.3 The Construction of Strategies for Evaders

We construct a strategy for each evader y_j to move in the strip (Fig. 1)

$$S'_j = \{\xi \in \mathbb{R}^2 \mid \; |(\xi - y_{j0}, e'_j)| \le a, \quad (\xi - y_{j0}, e_j) \ge -b\}.$$

Let us choose a number a_{j1} that satisfies the condition

$$0 < a_{j1} < \min\left\{\frac{1}{2}, \frac{a}{2}, b, \frac{d}{2}\right\}, \tag{3.7}$$

where d is defined by (3.2), and let $a_{j,i+1} = \kappa \cdot a_{ji}^4$, $i = 1, 2, \dots$. The number $\kappa \in (0, 1)$ will be specified later.

Then, clearly, terms of the sequence $a_{j1}, a_{j2}, \dots$ satisfy the following inequalities

$$a_{j,i+1} < a_{j,i}^4, \quad i = 1, 2, \dots, \tag{3.8}$$

and, therefore, by the inequality $a_{j1} < 1/2$, we obtain

$$\begin{aligned} a_{j,i+1} + a_{j,i+2} + \cdots &< a_{j,i+1} + a_{j,i+1}^4 + a_{j,i+1}^{16} + \cdots \\ &< a_{j,i+1} + a_{j,i+1}^2 + \cdots < 2a_{j,i+1}, \quad i = 0, 1, 2, \dots, \end{aligned} \tag{3.9}$$

and

$$\sum_{i=0}^{\infty} \sqrt{a_{j,i+1}} < \sqrt{a_{j1}} + a_{j1}^2 + a_{j1}^8 + \cdots < 2\sqrt{a_{j1}}. \tag{3.10}$$

We say that a pursuer x_s is j-active at the time $t \geq 0$ if $x_s(t) \in S_j$ and $(e_j, x_s(t)) > (e_j, y_j(t))$, otherwise x_s is called j-passive at the time t. We say that $t = \tau_{ji} \geq 0$ is a_{ji}-approach time of a pursuer x_s to the evader y_j if this pursuer is j-active at τ_{ji} and the equation $|x_s(\tau_{ji}) - y_j(\tau_{ji})| = a_{ji}$ is first satisfied at the time τ_{ji}. Thus, for the specified sequence of numbers $a_{j1}, a_{j2}, \ldots$, first we define τ_{j1} as the a_{j1}-approach time, then we define τ_{j2} as the a_{j2}-approach time and so on. Therefore, $\tau_{j1} < \tau_{j2} < \cdots$.

It should be noted that the same τ_{ji} can be a_{ji}-approach time for several pursuers to the evader y_j. If there are more than one of such pursuers, then we take any of these pursuers and, for convenience, we denote this pursuer by x_{ji}. Starting from the time τ_{ji} on some time interval, the evader y_j uses a maneuver to be defined against the specified pursuer x_{ji}. While a pursuer x_s is j-passive, for x_s, neither approach time is defined nor maneuver is applied against. Note that j-passive pursuer at some t' might be able to be j-active at some $t'' \geq t'$. Therefore, one pursuer can have several approach times to the same evader.

We make the following convention: If a pursuer makes a_{ji}- and a_{jl}- approaches at times τ_{ji} and τ_{jl}, respectively, then this pursuer is labeled by x_{ji} and x_{jl} respectively at these times. Also, for convenience, we'll write $x_{ji}, y_j, y_{j0}, q_j, \tau_{ji}, a_{ji}, I_j(t), e_j, e'_j, S_j, S'_j, v_j$ without j as $x_i, y, y_0, q, \tau_i, a_i, I(t), e, e', S, S', v$.

Define numbers

$$\tau'_i = \tau_i + \frac{a_i}{\alpha}, \quad i = 1, 2, \ldots,$$

where α is a positive number, which will be specified later. Note that, in general, the sequence $\tau'_1, \tau'_2, \ldots$ is neither increasing nor decreasing. Throughout the paper we impose conditions on parameters $\alpha, a_1, a_2, \ldots$ and we choose the parameters in Sect. 3.11.

Describe the idea of construction of evader's strategy. On the time interval $[0, \tau_1)$, the evader y moves parallel to the vector e. Note that τ_1, the first time of a_1-approach with a pursuer may not occur. In this case, either $|y(t) - x_i(t)| > a_1$ or $(y(t), e) \geq (x_i(t), e)$ and $y(t) \neq x_i(t)$ for all $t \geq 0$ and $i = 1, \ldots, K$. The latter case will be discussed in Sect. 3.4. In both cases, we let $\tau_1 = \infty$, and because of $y(t) \neq x_i(t),\ t \geq 0$, evasion is possible.

Let a_i-approaches occur at finite times $\tau_i, i = 1, 2, \ldots$. On the set $[\tau_i, \tau'_i) \setminus \cup_{j \geq i+1} [\tau_j, \tau'_j)$, the evader y uses a maneuver against the pursuer $x_i, i = 1, 2, \ldots$. Also we say that the evader is under the attack of pursuer x_i on this set.

Note that a_2-approach may occur before the time τ'_1, that is, $\tau_2 < \tau'_1$, and then a_3-approach may occur before the time $\max\{\tau'_1, \tau'_2\}$, that is, $\tau_3 < \max\{\tau'_1, \tau'_2\}$ and so on. If this process is broken off at some first time $\tau^*_p \doteq \max\{\tau'_1, \ldots, \tau'_p\}$ with $p \geq 1$, that is, $\tau_j < \max\{\tau'_1, \ldots, \tau'_{j-1}\}$ $(\tau'_0 \doteq \tau'_1)$ for all $j = 1, \ldots, p$ and $\tau_{p+1} > \tau^*_p$, then we say that the evader undergo the continuous attack of the pursuers $x_1, \ldots, x_p$ on the time interval $[\tau_1, \tau^*_p)$. In other words,

(i) $[\tau_1, \tau^*_p) = \cup_{i=1}^{p} [\tau_i, \tau'_i)$, that is, at each $t \in [\tau_1, \tau^*_p)$, the evader is under the attack of a pursuer x_j, $j \in \{1, \ldots, p\}$,
(ii) all pursuers $x_i,\ i = 1, \ldots, p$, participate in attack on $[\tau_1, \tau^*_p]$,
(iii) $\tau_{p+1} > \tau^*_p$ and on the interval $[\tau^*_p,\ \tau_{p+1})$ the evader is not under the attack of any pursuer.

It should be noted that one pursuer can attend several times in the sequence $x_1, \ldots, x_p$ with different labels.

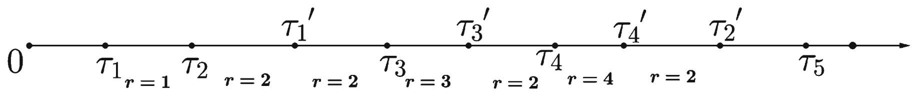


Fig. 2 Continuous attack of the pursuers $x_1,\ x_2,\ x_3$, and x_4 is completed at τ_2'

At the time τ_p^*, a continuous attack of the group of pursuers $x_1, \dots, x_p$ is stopped. The evader y moves again parallel to the vector e on $[\tau_p^*, \tau_{p+1})$. In general, on the set $\mathbb{R}\setminus \cup_{i\geq 1} [\tau_i, \tau_i')$, the evader is not under attack of any pursuer. Starting from τ_{p+1} the evader may undergo a continuous attack of another group of pursuers. It should be noted that one pursuer can participate in several continuous group attacks too.

Let $[\tau_{\omega_{k-1}+1}, \tau_{\omega_k}^*)$, $k = 1, 2, \dots, (\omega_0 = 0)$, be time intervals where the evader undergoes a continuous attack of a group of pursuers $x_{\omega_{k-1}+1}, \dots, x_{\omega_k}$, where $\tau_{\omega_k}^* = \max\{\tau'_{\omega_{k-1}+1}, \dots, \tau'_{\omega_k}\}$. Hence, on the time intervals $[\tau_{\omega_k}^*, \tau_{\omega_k+1})$, $k = 1, 2, \dots$, the evader is not under attack of any pursuer. For convenience, let $\tau_0^* = 0$. If for some ω_k there is no approach time greater than $\tau_{\omega_k}^*$, then we put $\tau_{\omega_k+1} = \infty$.

Define a natural-valued function $r = r(t)$, $\tau_{\omega_{k-1}+1} \leq t < \tau_{\omega_k}^*$, which may change its value only at points $\tau_i,\ \tau_i',\ i = \omega_{k-1}+1, \dots, \omega_k$, as follows:

$$r(t) = \max\{j \in \{\omega_{k-1}+1, \dots, \omega_k\} \mid \tau_j \leq t < \tau_j'\} \quad \text{if } t \in [\tau_{\omega_{k-1}+1}, \tau_{\omega_k}^*). \tag{3.11}$$

Here some properties of the function $r(t)$.

Property 3.3 *Let $i \in \{\omega_{k-1}+1, \dots \omega_k\}$ be any number. Then*

(i) $r(t) = i$, *if* $t \in [\tau_i, \tau_i')\setminus J_{i+1}$, *where* $J_{i+1} = \cup_{j=i+1}^{\omega_k} [\tau_j, \tau_j')$, $i < \omega_k$; $\quad J_{\omega_k+1} = \emptyset$, (3.12)

(ii) *if* $\tau_{i+1} < \tau_i'$, *then* $r(t) = i$ *on* $[\tau_i, \tau_{i+1})$,
(iii) *if* $\tau_{i+1} \geq \tau_i'$, *then* $r(t) = i$ *on* $[\tau_i, \tau_i')$.

Proof (i) Indeed, let $t \in [\tau_p, \tau_p')\setminus J_{p+1}$. Then $t \in [\tau_p, \tau_p')$, and so by (3.11) $r(t) \geq p$. However, because of $t \notin J_{p+1}$, we get $r(t) < p+1$. Hence, $r(t) = p$.
(ii) Let $\tau_{i+1} < \tau_i'$. Then $[\tau_i, \tau_{i+1}) \subset [\tau_i, \tau_i')\setminus J_{i+1}$, and therefore by (3.12) $r(t) = i$.
(iii) Let now $\tau_{i+1} \geq \tau_i'$. Then $[\tau_i, \tau_i')\setminus J_{i+1} = [\tau_i, \tau_i')$, and so therefore by (3.12) we get $r(t) = i,\ t \in [\tau_i, \tau_i')$. □

Thus, $r(t)$ is a step function.

Give an example (see Fig. 2). On the interval $[0, \tau_1)$ the evader is not under attack of any pursuer; $r(t) = 1$ if $t \in [\tau_1, \tau_2)$; $r(t) = 2$ if $t \in [\tau_2, \tau_3) \cup [\tau_3', \tau_4) \cup [\tau_4', \tau_2')$; $r(t) = 3$ if $t \in [\tau_3, \tau_3')$; $r(t) = 4$ if $t \in [\tau_4, \tau_4')$ and the evader uses (3.14). At the time $\tau_4^* = \tau_2'$, continuous attack of the pursuers x_1, x_2, x_3 and x_4 is completed. Then on the interval $[\tau_2', \tau_5)$, the evader is not under attack of any pursuer. Starting from the time τ_5 the evader undergoes an attack of another group of pursuers.

We now start constructing a strategy for the evader y. Set $v(t) = 0$ if $I(t) = \emptyset$, else

$$v(t) = \begin{cases} \left(\sum_{k\in I(t)} |u_k(t)|^2\right)^{1/2} e, & q(t) > 0, \\ 0, & q(t) = 0, \end{cases}, \quad t \in [\tau_{\omega_k}^*, \tau_{\omega_k+1}), \quad k = 0, 1, 2, \dots \tag{3.13}$$

For $k = 0$, we have $[\tau_{\omega_0}^*, \tau_{\omega_0+1}) = [0, \tau_1)$, therefore, the evader uses (3.13) on $[0, \tau_1)$ as well. Thus, by (3.13) the evader moves parallel to the vector e on the intervals $[\tau_{\omega_k}^*, \tau_{\omega_k+1})$, $k = 0, 1, 2, \dots$.

For $i = 1, 2, \ldots,$ let

$$V_{1i}(t) = \begin{cases} \alpha + |(u_i(t), e')|, & (y(\tau_i), e') \geq (x_i(\tau_i), e'), \\ -(\alpha + |(u_i(t), e')|), & (y(\tau_i), e') < (x_i(\tau_i), e'), \end{cases}$$

$$V_{2i}(t) = \alpha + \left(\sum_{k \in I(t) \setminus \{i\}} |u_k(t)|^2 + (u_i(t), e)^2 \right)^{1/2}.$$

For $t \in [\tau_{\omega_{k-1}+1}, \tau^*_{\omega_k}),\ k = 1, 2, \ldots,$ the maneuver of the evader against the pursuer x_r is defined as follows:

$$v(t) = \begin{cases} V_{1r}(t)e' + V_{2r}(t)e, & q(t) > 0, \\ 0 & q(t) = 0, \end{cases} \tag{3.14}$$

where $r = r(t)$ defined by (3.11).

3.4 Evasion from the Pursuer x_i for which $(y(\theta), e) \geq (x_i(\theta), e)$

Let $(y(\theta), e) \geq (x_i(\theta), e)$ and $y(\theta) \neq x_i(\theta)$ for a pursuer x_i at some time $\theta \geq 0$. We show that if $x_i(s) \in S, \theta \leq s \leq t$, then $x_i(t) \neq y(t)$ for any behavior of the pursuer x_i, as long as $q(t) > 0$. Let $x_i(s) \in S, \theta \leq s \leq t$.

Consider two cases 1) $\theta \in [\tau^*_{\omega_p}, \tau_{\omega_p+1})$, and 2) $\theta \in [\tau_{\omega_p+1}, \tau^*_{\omega_{p+1}})$ at some $p \in \{0, 1, \ldots\}$, where $\tau^*_{\omega_p}$ is the time when continuous attack of a group of the pursuers $x_{\omega_{p-1}+1}, \ldots, x_{\omega_p}$ is completed, and the attack of another group of pursuers $x_{\omega_p+1}, \ldots, x_{\omega_{p+1}}$ is started at the time τ_{ω_p+1} and completed at the time $\tau^*_{\omega_{p+1}}$.

Case 1 Let $\theta \in [\tau^*_{\omega_p}, \tau_{\omega_p+1}),\ p \in \{0, 1, \ldots\}$.

A. Let $\theta \leq t < \tau_{\omega_p+1}$. Since $(y(\theta) - x_i(\theta), e) \geq 0$, then by (3.13) we have $\left(y(\theta) - x_i(\theta), \int_\theta^t v(s)\mathrm{d}s\right) \geq 0$, and hence

$$\begin{aligned} |y(t) - x_i(\theta)|^2 &\geq |y(\theta) - x_i(\theta)|^2 + \left(y(\theta) - x_i(\theta), \int_\theta^t v(s)\mathrm{d}s \right) + \left| \int_\theta^t v(s)\mathrm{d}s \right|^2 \\ &\geq |y(\theta) - x_i(\theta)|^2 + \left(\int_\theta^t \left(\sum_{k \in I(t)} |u_k(s)|^2 \right)^{1/2} \mathrm{d}s \right)^2 \\ &\geq |y(\theta) - x_i(\theta)|^2 + \left(\int_\theta^t |u_i(s)|\mathrm{d}s \right)^2. \end{aligned}$$

Therefore,

$$\begin{aligned} |y(t) - x_i(t)| &\geq |y(t) - x_i(\theta)| - \left| \int_\theta^t u_i(s)\mathrm{d}s \right| \\ &\geq \left(|y(\theta) - x_i(\theta)|^2 + \left(\int_\theta^t |u_i(s)|\mathrm{d}s \right)^2 \right)^{1/2} - \int_\theta^t |u_i(s)|\mathrm{d}s. \end{aligned}$$

Since the right-hand side of this inequality is positive, and so $y(t) \neq x_i(t)$. Moreover, in the view of (3.13), $(y(t) - x_i(t), e) \geq 0,\ \theta \leq t \leq \tau_{\omega_p+1}$.

B. Let now $\tau_{\omega_p+1} \leq t < \tau^*_{\omega_{p+1}}$. We get

$$(y(t), e) - (x_i(t), e) = (y(\theta), e) - (x_i(\theta), e) + \int_\theta^t (v(s), e)\mathrm{d}s - \int_\theta^t (u_i(s), e)\mathrm{d}s$$

$$\geq \int_\theta^{\tau_{\omega_p+1}} (v(s), e)\mathrm{d}s + \int_{\tau_{\omega_p+1}}^t (v(s), e)\mathrm{d}s$$

$$- \int_\theta^{\tau_{\omega_p+1}} (u_i(s), e)\mathrm{d}s - \int_{\tau_{\omega_p+1}}^t (u_i(s), e)\mathrm{d}s.$$

In view of (3.14), we then have

$$(y(t), e) - (x_i(t), e) \geq \int_\theta^{\tau_{\omega_p+1}} ((v(s), e) - (u_i(s), e))\, \mathrm{d}s$$

$$+ \int_{\tau_{\omega_p+1}}^t \left[\alpha + \left(\sum_{k \in I(t)\setminus\{r\}} |u_k(s)|^2 + (u_r(s), e)^2 \right)^{1/2} - (u_i(s), e) \right] \mathrm{d}s > 0, \quad (3.15)$$

where $r = r(t)$. The first integral is not negative since by (3.13), $(v(s), e) \geq (u_i(s), e),\ \theta \leq t \leq \tau_{\omega_p+1}$, and the second integral is clearly positive. Hence, $y(t) \neq x_i(t),\ \tau_{\omega_p+1} \leq t < \tau^*_{\omega_{p+1}}$.

C. Let now $t \geq \tau^*_{\omega_{p+1}}$. Then it can be shown that inequalities (3.15) and $(v(t), e) \geq (u_i(t), e)$ [see (3.13) and (3.14)], imply that $(y(t), e) - (x_i(t), e) > 0$ for all $t \geq \tau^*_{\omega_{p+1}}$. Hence, $x_i(t) \neq y(t)$ for all $t \geq \theta$.

Case 2 Let $\theta \in [\tau_{\omega_p+1}, \tau^*_{\omega_{p+1}})$. Then according to (3.14)

$$(v(t), e) \geq \alpha + |u_i(t)| > (u_i(t), e), \quad \theta \leq t < \tau^*_{\omega_p},$$

and therefore for any $t > \theta$, we obtain $(y(t), e) > (x_i(t), e)$, provided $x_i(s) \in S$, $s \in [\theta, t]$.

Thus, in both cases, if $(y(\theta), e) \geq (x_i(\theta), e)$, $y(\theta) \neq x_i(\theta)$ and $x_i(s) \in S, \theta \leq s \leq t$, for a pursuer x_i and some time θ, then $(y(t), e) \geq (x_i(t), e)$ and $y(t) \neq x_i(t)$. What if this pursuer goes out of the set S and moves outside S till some time τ for which $(x_i(\tau), e) > (y(\tau), e)$ and gets again into the set S to make some a_j-approach? We will discuss this question in Sect. 3.10.

3.5 Fictitious Evaders

To estimate distances between pursuers and evaders, we define fictitious evaders (FEs) $z_i,\ i = \omega_{k-1}, \ldots, \omega_k$, by the equations

$$\dot{z}_i = w_i, \quad z_i(\tau_i) = y(\tau_i), \quad \tau_i \leq t < \tau'_i, \quad (3.16)$$

where w_i is the control parameter of the FE z_i. According to (3.16), the initial position of FE z_i at τ_i coincides with the position of the evader at the same time τ_i, and FE

z_i is defined only on the time interval $[\tau_i, \tau_i')$. We define the strategy of FE z_i as follows:

$$w_i(t) = V_{1i}(t)e' + V_{2r}(t)e, \quad \tau_i \le t < \tau_i', \tag{3.17}$$

where $V_{2r}(t)$ is the same as in (3.14). Note that $(w_i(t), e) = (v(t), e) = V_{2r}(t)$, $\tau_i \le t < \tau_i'$, that is speeds of the FE z_i and the evader y along the direction e are the same. Also combining (3.14) and (3.17), we observe that $z_r(t) = y(t)$, $\tau_r \le t \le \tau_r'$.

Next, show that

$$\int_{\tau_i}^{\tau_i'} |w_i(s)|^2 \mathrm{d}s \le 2\sigma^2. \tag{3.18}$$

Indeed, introducing the following vector functions

$$f(s) = (\alpha, \alpha) \text{ and } g(s) = \left(|(u_i(s), e')|, \left(\sum_{k \in I(t) \setminus \{r\}} |u_k(s)|^2 + (u_r(s), e)^2 \right)^{1/2} \right),$$

we obtain

$$\begin{aligned} &V_{1i}^2(s) + V_{2r}^2(s) \\ &\quad = (\alpha + |(u_i(s), e')|)^2 + \left(\alpha + \left(\sum_{k \in I(s) \setminus \{r\}} |u_k(s)|^2 + (u_r(s), e)^2 \right)^{1/2} \right)^2 \\ &\quad = |f(s) + g(s)|^2. \end{aligned}$$

Then by the Minkowskii inequality

$$\begin{aligned} \left(\int_{\tau_i}^{\tau_i'} |w_i(s)|^2 \mathrm{d}s \right)^{1/2} &= \left(\int_{\tau_i}^{\tau_i'} \left(V_{1i}^2(s) + V_{2r}^2(s) \right) \mathrm{d}s \right)^{1/2} \\ = \left(\int_{\tau_i}^{\tau_i'} |f(s) + g(s)|^2 \mathrm{d}s \right)^{1/2} &\le \left(\int_{\tau_i}^{\tau_i'} |f(s)|^2 \mathrm{d}s \right)^{1/2} + \left(\int_{\tau_i}^{\tau_i'} |g(s)|^2 \mathrm{d}s \right)^{1/2} \\ = \left(\int_{\tau_i}^{\tau_i'} 2\alpha^2 \mathrm{d}s \right)^{1/2} &+ \left(\int_{\tau_i}^{\tau_i'} \left((u_i(s), e')^2 + \sum_{k \in I(s) \setminus \{r\}} |u_k(s)|^2 + (u_r(s), e)^2 \right) \mathrm{d}s \right)^{1/2} \end{aligned} \tag{3.19}$$

Since

$$\int_{\tau_i}^{\tau_i'} (u_i(s), e')^2 \mathrm{d}s \le \int_{\tau_i}^{\tau_i'} |u_i(s)|^2 \mathrm{d}s \le \rho_i^2,$$

and

$$\int_{\tau_i}^{\tau_i'} \left(\sum_{k \in I(s) \setminus \{r\}} |u_k(s)|^2 + (u_r(s), e)^2 \right) \mathrm{d}s \leq \rho^2.$$

then requiring that

$$\alpha a_1 \leq (\sigma - \rho)^2 \tag{3.20}$$

and using definition of τ_i', we obtain from (3.19)

$$\left(\int_{\tau_i}^{\tau_i'} |w_i(s)|^2 \mathrm{d}s \right)^{1/2} \leq \sqrt{2\alpha a_i} + \sqrt{\rho_i^2 + \rho^2}$$
$$\leq \sqrt{2\alpha a_1} + \rho\sqrt{2} \leq \sigma\sqrt{2}$$

which is the desired conclusion.

3.6 Estimation of the Distance Between FE and Pursuer

Let us estimate the distance $|x_p(t) - z_p(t)|$, $t \in [\tau_p, \tau_p')$, between the FE z_p and pursuer x_p for any $p \in \{\omega_{k-1} + 1, \ldots, \omega_k\}$. To this end, estimate $|x_p(t) - z_p(t)|$ in two ways. Since $|x_p(\tau_p) - z_p(\tau_p)| = a_p$, then by the Cauchy–Schwartz inequality and inequality (3.18), we have

$$|x_p(t) - z_p(t)| \geq |x_p(\tau_p) - z_p(\tau_p)| - \left| \int_{\tau_p}^{t} w_p(s) \mathrm{d}s \right| - \left| \int_{\tau_p}^{t} u_p(s) \mathrm{d}s \right|$$
$$\geq a_p - (\sqrt{2} + 1)\sqrt{t - \tau_p} \cdot \sigma \tag{3.21}$$

provided

$$\int_{\tau_p}^{t} |u_p(s)|^2 \mathrm{d}s \leq \rho_p^2.$$

On the other hand by (3.14), signs of

$$(z_p(\tau_p), e') - (x_p(\tau_p), e') = (y(\tau_p), e') - (x_p(\tau_p), e')$$

and $\pm(\alpha + |(u_p(s), e')|)$ are the same, and therefore

$$|z_p(t) - x_p(t)| \geq |(z_p(t) - x_p(t), e')| \geq |(z_p(\tau_p), e') - (x_p(\tau_p), e')|$$
$$+ \int_{\tau_p}^{t} (\alpha + |(u_p(s), e')|) \mathrm{d}s - \int_{\tau_p}^{t} |(u_p(s), e')| \mathrm{d}s$$
$$\geq \alpha(t - \tau_p). \tag{3.22}$$

From (3.21) and (3.22), we conclude that

$$|z_p(t) - x_p(t)| \geq h(t), \quad h(t) = \max\{a_p - (\sqrt{2} + 1)\sigma\sqrt{t - \tau_p}, \quad \alpha(t - \tau_p)\}.$$

Since the function $h_1(t) = a_p - (\sqrt{2}+1)\sigma\sqrt{t-\tau_p}$ decreases on $[\tau_p, \tau'_p]$ from a_p to $a_p - (\sqrt{2}+1)\sigma\sqrt{\frac{a_p}{\alpha}}$, and $h_2(t) = \alpha(t-\tau_p)$ increases on $[\tau_p, \tau'_p]$ from 0 to a_p, and therefore the function $h(t)$ has the only minimum point $t_* \in (\tau_p, \tau'_p)$, which is the only root of the equation $h_1(t) = h_2(t)$. We can see that

$$t_* = \tau_p + \frac{4a_p^2}{\left((\sqrt{2}+1)\sigma + \sqrt{(3+2\sqrt{2})\sigma^2 + 4\alpha a_p}\right)^2}.$$

Then, $h_2(t_*) > \frac{\alpha a_p^2}{6\sigma^2}$, provided α and a_p are required to satisfy the inequalities

$$0 < \alpha < 1,\ 12\alpha a_p < \sigma^2. \tag{3.23}$$

Then we obtain the following estimate for the distance between FE z_p and pursuer x_p

$$|z_p(t) - x_p(t)| > \frac{\alpha a_p^2}{6\sigma^2}, \quad \tau_p \le t \le \tau'_p. \tag{3.24}$$

In addition,

$$\begin{aligned}(z_p(\tau'_p) - x_p(\tau'_p), e) &= (z_p(\tau_p) - x_p(\tau_p), e) + \int_{\tau_p}^{\tau'_p} ((v(s), e) - (u_p(s), e))\mathrm{d}s \\ &\ge -a_p + \int_{\tau_p}^{\tau'_p} (\alpha + |(u_p(s), e)| - (u_p(s), e))\mathrm{d}s \\ &\ge -a_p + \alpha(\tau'_p - \tau_p) = 0,\end{aligned}$$

then according to (3.16) and (3.17), we obtain

$$\begin{aligned}(y(\tau'_p), e) &= (y(\tau_p), e) + \int_{\tau_p}^{\tau'_p} (v(s), e)\mathrm{d}s \\ &= (z_p(\tau_p), e) + \int_{\tau_p}^{\tau'_p} (w_p(s), e)\mathrm{d}s \\ &= (z_p(\tau'_p), e) \ge (x_p(\tau'_p), e).\end{aligned} \tag{3.25}$$

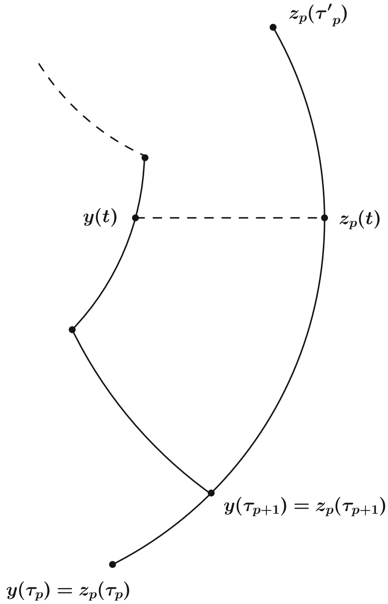
The inequality (3.25) shows that at the time τ'_p the pursuer x_p becomes passive.

3.7 Estimation of the Distance Between Evader and FE

To estimate the distance $|x_p(t) - y(t)|$ between the evader y and pursuer x_p for $t \in [\tau_p, \tau'_p)$, we first estimate the distance $|y(t) - z_p(t)|$ between the evader y and FE z_p on the time interval $[\tau_p, \tau'_p)$ assuming that $q(t) > 0$.

If a_{p+1}-approach time $\tau_{p+1} > \tau'_p$, then by Property 3.3 (iii) $r = r(t) = p$ for all $t \in [\tau_p, \tau'_p)$. Hence, comparing (3.14) with (3.17) we get $w_p(t) = v(t),\quad \tau_p \le t < \tau'_p$. Consequently $y(t) = z_p(t)$, $t \in [\tau_p, \tau'_p]$, and so $|y(t) - z_p(t)| = 0$, $t \in [\tau_p, \tau'_p]$.

Fig. 3 Evader y and FE z_p



Let now $\tau_{p+1} \in (\tau_p, \tau_p')$. Then by Property 3.3 (ii) $r = r(t) = p$ for $t \in [\tau_p, \tau_{p+1})$, and comparing (3.14) with (3.17) we get $w_p(t) = v(t),\ \tau_p \le t < \tau_{p+1}$.

Then (see Fig. 3)

$$\begin{aligned} y(\tau_{p+1}) &= y(\tau_p) + \int_{\tau_p}^{\tau_{p+1}} v(t)\mathrm{d}t \\ &= z_p(\tau_p) + \int_{\tau_p}^{\tau_{p+1}} w_p(t)\mathrm{d}t = z_p(\tau_{p+1}). \end{aligned}$$

Therefore, for $\tau_{p+1} \le t < \tau_p'$, we get

$$\begin{aligned} |y(t) - z_p(t)| &= \left| y(\tau_{p+1}) + \int_{\tau_{p+1}}^{t} v(s)\mathrm{d}s - z_p(\tau_{p+1}) - \int_{\tau_{p+1}}^{t} w_p(s)\mathrm{d}s \right| \\ &\le \int_{\tau_{p+1}}^{t} |v(s) - w_p(s)|\mathrm{d}s \end{aligned} \tag{3.26}$$

Since $w_p(t) = v(t)$ on $[\tau_{p+1}, t]\backslash J_{p+1}$, where J_{p+1} is defined in (3.12), then combining (3.14) and (3.17) with (3.26), yields

$$\begin{aligned} |y(t) - z_p(t)| &\le \int_{[\tau_{p+1}, t)\cap J_{p+1}} |v(s) - w_p(s)|\mathrm{d}s \\ &\le \int_{J_{p+1}} (2\alpha + |(u_r(s), e')|) + |(u_p(s), e')|)\mathrm{d}s, \end{aligned} \tag{3.27}$$

where $r = r(s)$. As

$$|(u_r(s), e')| \le |u_r(s)| \le \sqrt{\sum_{i=1}^{K} |u_i(s)|^2 \mathrm{d}s}$$

then from (3.27) by using the Cauchy–Schwartz inequality we get

$$\begin{aligned} |y(t) - z_p(t)| &\le 2\alpha \text{ mes}(J_{p+1}) + 2 \int_{J_{p+1}} \sqrt{\sum_{i=1}^{K} |u_i(s)|^2 \mathrm{d}s} \\ &\le 2\alpha \text{ mes}(J_{p+1}) + 2\rho\sqrt{\text{mes}(J_{p+1})}, \end{aligned} \tag{3.28}$$

where $\text{mes}(J_{p+1})$ denotes the Lebesgue measure of the set J_{p+1}. Since $J_{p+1} \subset \cup_{i \ge p+1}[\tau_p, \tau'_p)$, then (3.9) implies that

$$\text{mes}(J_{p+1}) \le \sum_{i \ge p+1} (\tau'_i - \tau_i) \le \sum_{i=p+1}^{\infty} \frac{a_i}{\alpha} \le \frac{2a_{p+1}}{\alpha},$$

Thus, the last inequality and (3.28) yield that

$$|y(t) - z_p(t)| \le 4a_{p+1} + 2\rho\sqrt{\frac{2a_{p+1}}{\alpha}} < 4\rho\sqrt{\frac{2a_{p+1}}{\alpha}} \tag{3.29}$$

provided that the parameters a_{p+1} and α satisfy the condition

$$2a_{p+1}\alpha < \rho^2. \tag{3.30}$$

Thus, we have obtained an estimate for $|y(t) - z_p(t)|$ which is given by (3.29).

3.8 Estimation of Distance Between Pursuer and Evader

Finally, estimate the distance between the pursuer x_p and evader on the time interval $[\tau_p, \tau'_p)$ when evader's energy is positive. It follows from (3.24) and (3.29) that

$$\begin{aligned} |x_p(t) - y(t)| &\ge |x_p(t) - z_p(t)| - |z_p(t) - y(t)| \\ &> \frac{\alpha a_p^2}{6\sigma^2} - 4\rho\sqrt{\frac{2a_{p+1}}{\alpha}} \ge \frac{\alpha a_p^2}{12\sigma^2} \end{aligned} \tag{3.31}$$

provided that $\frac{\alpha a_p^2}{12\sigma^2} \ge 4\rho\sqrt{\frac{2a_{p+1}}{\alpha}}$. Solve this inequality for a_{p+1} to obtain a new requirement for parameters $a_{p+1},\ a_p$, and α

$$a_{p+1} \le \frac{\alpha^3}{144 \cdot 32 \cdot \rho^2 \cdot \sigma^4} a_p^4. \tag{3.32}$$

Thus, for any $t \in [\tau_p, \tau'_p)$,

$$|x_p(t) - y(t)| > \frac{\alpha a_p^2}{12\sigma^2}. \tag{3.33}$$

Also, require that

$$\frac{\alpha a_p^2}{12\sigma^2} \ge a_{p+1}. \tag{3.34}$$

Then (3.33) implies that $|x_p(t)-y(t)| > a_{p+1}$. Therefore, for any $j \geq p+1$, a_j-approach doesn't occur on $[\tau_p, \tau'_p]$ with the pursuer x_p. According to (3.25) $(y(\tau'_p), e) \geq (x_p(\tau'_p), e)$. As it was proved earlier in Sect. 3.4, the inequalities $x_p(\tau'_p) \neq y(\tau'_p)$ [which follows from (3.33)] and $(y(\tau'_p), e) \geq (x_p(\tau'_p), e)$ imply that $(y(t), e) \geq (x_p(t), e)$ and $y(t) \neq x_p(t)$ for all $t > \tau'_p$, while $x_p(s) \in S$ for all $\tau'_p \leq s \leq t$, and $q(t) > 0$. That is this pursuer becomes passive while $x_p(t) \in S$, $t > \tau'_p$.

3.9 Estimation of Evaders Energy and Possibility of Evasion

Let T be any positive number, and $\tau^*_{\omega_l}$ $(\omega_0 = 0,\ \tau^*_0 = 0)$, $l = 0, 1, \ldots$, be the times when continuous attack of a group of pursuers is completed, and τ_{ω_l+1}, $l = 0, 1, \ldots$, be the times when continuous attack of a group of pursuers is started. The time T can be either in an interval $[\tau^*_{\omega_p}, \tau_{\omega_p+1})$ or in an interval $[\tau_{\omega_{p-1}+1}, \tau^*_{\omega_p})$ for some p. First, estimate the energy of the evader spent on the time interval $[0, T]$. Since on the intervals $[\tau^*_{\omega_l}, \tau_{\omega_l+1})$, $l = 0, 1, \ldots,$ the evader uses (3.13), and on the intervals $[\tau_{\omega_{l-1}+1}, \tau^*_{\omega_l})$, $l = 1, 2, \ldots,$ the evader is under the attack of pursuers and uses (3.14). Consequently, the evader spends the following amount of energy

$$\int_0^T |v(t)|^2 \mathrm{d}t \leq A_1 + A_2 + A_3, \tag{3.35}$$

where

$$A_1 = \sum_{l=0}^{p-1} \int_{\tau^*_{\omega_l}}^{\tau_{\omega_l+1}} \left(\sum_{i \in I(t)} |u_i(t)|^2 \right) \mathrm{d}t,$$

$$A_2 = \sum_{l=1}^{p-1} \int_{\tau_{\omega_{l-1}+1}}^{\tau^*_{\omega_l}} \left(V_{1r}^2(t) + V_{2r}^2(t) \right) \mathrm{d}t,$$

$$A_3 = \begin{cases} \int_{\tau^*_{\omega_p}}^{T} \left(\sum_{i \in I(t)} |u_i(t)|^2 \right) \mathrm{d}t, & \text{if } \ T \in [\tau^*_{\omega_p}, \tau_{\omega_p+1}), \\ \int_{\tau_{\omega_{p-1}+1}}^{T} \left(V_{1r}^2(t) + V_{2r}^2(t) \right) \mathrm{d}t, & \text{if } \ T \in [\tau_{\omega_{p-1}+1}, \tau^*_{\omega_p}), \end{cases}$$

where $r = r(t)$.

Since $(u_r(t), e')^2 + (u_r(t), e)^2 = |u_r(t)|^2$, then similar to (3.19), with i replaced by r, we obtain for any $\lambda, \mu \in [\tau_{\omega_{p-1}+1}, \tau^*_{\omega_p})$ and $\lambda \leq \mu$ that

$$\left(\int_\lambda^\mu \left(V_{1r}^2(s) + V_{2r}^2(s) \right) \mathrm{d}s \right)^{1/2}$$
$$= \left(\int_\lambda^\mu 2\alpha^2 \mathrm{d}s \right)^{1/2} + \left(\int_\lambda^\mu \left((u_r(s), e')^2 + \sum_{i \in I(s) \setminus \{r\}} |u_i(s)|^2 + (u_r(s), e)^2 \right) \mathrm{d}s \right)^{1/2}$$

$$= \sqrt{2\alpha^2(\mu-\lambda)} + \left(\int_\lambda^\mu \left(\sum_{i\in I(s)} |u_i(s)|^2\right) \mathrm{d}s\right)^{1/2}. \tag{3.36}$$

Then, by the inequality $\int_\lambda^\mu \left(\sum_{i\in I(s)} |u_i(s)|^2\right) \mathrm{d}s \le \rho^2$, we can see that

$$\int_\lambda^\mu \left(V_{1r}^2(s) + V_{2r}^2(s)\right) \mathrm{d}s$$
$$= 2\alpha^2(\mu-\lambda) + 2\sqrt{2\alpha^2(\mu-\lambda)}\rho + \int_\lambda^\mu \left(\sum_{i\in I(s)} |u_i(s)|^2\right) \mathrm{d}s.$$

Therefore, (3.35) can be rewritten as follows:

$$\int_0^T |v(t)|^2 \mathrm{d}t \le \int_0^T \sum_{i\in I(t)} |u_i(t)|^2 \mathrm{d}t + B_1 + B_1' + B_2 + B_2', \tag{3.37}$$

where

$$B_1 = \sum_{l=1}^{p-1} 2\alpha^2(\tau^*_{\omega_l} - \tau_{\omega_{l-1}+1}), \quad B_2 = \sum_{l=1}^{p-1} 2\sqrt{2\alpha^2(\tau^*_{\omega_l} - \tau_{\omega_{l-1}+1})}\rho,$$

$$B_1' = \begin{cases} 0, & T \in [\tau^*_{\omega_p}, \tau_{\omega_p+1}), \\ 2\alpha^2(T - \tau_{\omega_{p-1}+1}), & T \in [\tau_{\omega_{p-1}+1}, \tau^*_{\omega_p}), \end{cases} \tag{3.38}$$

$$B_2' = \begin{cases} 0, & T \in [\tau^*_{\omega_p}, \tau_{\omega_p+1}), \\ 2\rho\sqrt{2\alpha^2(T - \tau_{\omega_{p-1}+1})}, & T \in [\tau_{\omega_{p-1}+1}, \tau^*_{\omega_p}). \end{cases} \tag{3.39}$$

Next, estimate B_1, B_1', B_2, B_2'. Using (3.9) yields for $l = 1, \ldots, p-1$,

$$\alpha^2(\tau^*_{\omega_l} - \tau_{\omega_{l-1}+1}) \le \alpha^2 \sum_{\tau_j \in [\tau_{\omega_{l-1}+1}, \tau^*_{\omega_l})} (\tau_j' - \tau_j) = \alpha^2 \sum_{\tau_j \in [\tau_{\omega_{l-1}+1}, \tau^*_{\omega_l})} \frac{a_j}{\alpha}$$
$$\le \alpha(a_{\omega_{l-1}+1} + a_{\omega_{l-1}+2} + \ldots) \le 2\alpha a_{\omega_{l-1}+1}, \tag{3.40}$$

Similarly, for $T \in [\tau_{\omega_{p-1}+1}, \tau^*_{\omega_p})$,

$$\alpha^2(T - \tau_{\omega_{p-1}+1}) \le \alpha^2(\tau^*_{\omega_p} - \tau_{\omega_{p-1}+1}) \le 2\alpha a_{\omega_{p-1}+1}. \tag{3.41}$$

Then, combining the inequalities (3.40) and (3.41) with (3.38) and then using (3.9), we get

$$B_1 + B_1' \le 2\sum_{l=1}^{p-1} \alpha^2(\tau^*_{\omega_l} - \tau_{\omega_{l-1}+1}) + 2\alpha^2(T - \tau_{\omega_{p-1}+1})$$
$$\le 4\alpha \sum_{i=1}^{p-1} a_{\omega_{l-1}+1} + 4\alpha a_{\omega_{p-1}+1} \le 4\alpha \sum_{i=1}^{\infty} a_i \le 8\alpha a_1. \tag{3.42}$$

Combining (3.40) and (3.41) with (3.39), and then using (3.10), we obtain

$$B_2 + B_2' \le 2\rho \sum_{l=1}^{p-1} \sqrt{2\alpha^2(\tau_{\omega_l}^* - \tau_{\omega_{l-1}+1})} + 2\rho\sqrt{2\alpha^2(T - \tau_{\omega_{p-1}+1})}$$
$$\le 4\rho \sum_{i=1}^{p-1} \sqrt{\alpha a_{\omega_{l-1}+1}} + 4\rho\sqrt{\alpha a_{\omega_{p-1}+1}} \le 4\rho \sum_{i=1}^{\infty} \sqrt{\alpha a_i} \le 8\rho\sqrt{\alpha a_1}. \quad (3.43)$$

Therefore, using (3.42) and (3.43), we obtain from (3.37)

$$\int_0^T |v(t)|^2 \mathrm{d}t \le 4\alpha a_1 + 8\rho\sqrt{\alpha a_1} + \int_0^T \sum_{i \in I(t)} |u_i(t)|^2 \mathrm{d}t. \quad (3.44)$$

Let α, a_1 satisfy the following inequality

$$4\alpha a_1 + 8\rho\sqrt{\alpha a_1} < \frac{\sigma^2 - \rho^2}{2M}. \quad (3.45)$$

Then it follows from (3.44) that

$$\int_0^T |v(t)|^2 \mathrm{d}t < \frac{\sigma^2 - \rho^2}{2M} + \int_0^T \sum_{i \in I(t)} |u_i(t)|^2 \mathrm{d}t. \quad (3.46)$$

We again think of all evaders moving in the sets S_j, $j = 1, \dots, M$, and proceed to show that evasion is possible in the game (1.1)–(1.4). For the evader y_j, $j \in \{1, \dots, M\}$, the inequality (3.46) can be written as follows:

$$\int_0^T |v_j(t)|^2 \mathrm{d}t < \frac{\sigma^2 - \rho^2}{2M} + \int_0^T \sum_{i \in I_j(t)} |u_i(t)|^2 \mathrm{d}t. \quad (3.47)$$

Clearly, the evader y_j can move using strategies (3.13) and (3.14) not being captured on $[0, T]$ providing that $q_j(t) > 0$ and the number of approach times in $[0, T]$ for this evader is finite. We claim that for any $T > 0$, the inequality $q_i(t) > 0$, $0 \le t \le T$, holds for at least one $i \in \{1, \dots, M\}$. Finiteness of approach times will be shown in the next subsection. This means that at least one of the evaders is not captured on $[0, T]$. Assume the contrary, that is

$$\int_0^T |v_j(t)|^2 \mathrm{d}t \ge \sigma_j^2 \quad \text{for all } j = 1, \dots, M,$$

at some $T > 0$. Since $I_{j_1}(t) \cap I_{j_2}(t) = \emptyset$ for every $j_1 \ne j_2$, and $I_j(t) \subset 1, 2, \dots, M$, then from (3.47), we obtain

$$\sigma^2 = \sigma_1^2 + \cdots + \sigma_M^2 \le \sum_{j=1}^{M} \int_0^T |v_j(t)|^2 \mathrm{d}t$$
$$< \frac{\sigma^2 - \rho^2}{2} + \sum_{j=1}^{M} \int_0^T \sum_{i \in I_j(t)} |u_i(t)|^2 \mathrm{d}s$$

$$\leq \frac{\sigma^2 - \rho^2}{2} + \rho^2 = \frac{\sigma^2 + \rho^2}{2} < \sigma^2.$$

A contradiction. Thus, for any time $T > 0$, at least one of the evaders is not captured on $[0, T]$. Note that strategies of evaders do not depend on T.

3.10 Finiteness of the Number of Approaches

Is the number of approach times $\tau_1, \tau_2, \ldots$ in any time interval $[0, T]$ finite? or are there infinitely many approach times $\tau_1, \tau_2, \ldots$ on an interval $[0, T]$? We show that for any finite time interval $[0, T]$, the number of approach times $\tau_1, \tau_2, \ldots$ is finite. To see this, we take any pursuer x_i and show that the number of approach times of x_i in $[0, T]$ is finite.

Let τ_i be the first an a_i-approach time of the pursuer x_i with the evader. On the interval $[\tau_i, \tau_i')$, as shown in Sect. 3.9 that a new a_j-approach ($j \geq i + 1$) will not occur with this pursuer. According to (3.33), at the time τ_i' the inequality $(y(\tau_i'), e) \geq (x_i(\tau_i'), e)$ holds. It was shown in Sect. 3.9 that, if $x_i(s) \in S$ for all $\tau_i' \leq s \leq t$, then $(y(t), e) \geq (x_i(t), e)$ and $y(t) \neq x_i(t)$, and therefore, starting from τ_i', this pursuer is passive, and because of the inequality $(y(t), e) \geq (x_i(t), e)$, $t \geq \tau_i'$, further approach times are not defined while $x_i(t) \in S$.

There is only one way for the pursuer x_i to make a new a_j- approach with the evader: The pursuer x_i has to go out of the set S, and move outside S for some time to get $(x_i(t), e) > (y(t), e)$, then get into the set S again to make a new approach with the evader.

Show that at any approach time τ_k the pursuer x_i will be in S'. Indeed, by the definition of the approach time τ_k, we have $|y(\tau_k) - x_i(\tau_k)| = a_k$ and therefore

$$\begin{aligned} |(x_k(\tau_k) - y_0, e')| &\leq |(x_k(\tau_k) - y(\tau_k), e')| + |(y(\tau_k) - y_0, e')| \\ &\leq a_k + \frac{a}{2} < a, \end{aligned}$$

meaning that $x_i(\tau_k) \in S'$. Here, the inequality $|(y(\tau_k) - y_0, e')| < \frac{a}{2}$, which will be proved in Sect. 4.2, and assumption

$$a_k < \frac{a}{2} \quad \text{for all } i = 1, 2, \ldots, \tag{3.48}$$

are used.

We now estimate the number of approach times on the interval $[0, T]$. Let t' be an approach time. Then by the reasoning above we have $x_i(t') \in S'$. Let $x_i(t'') \notin S$ at some time t''. That is, at the time t' the pursuer is in S', and at the time t'' the pursuer x_i is outside S. Then

$$b \leq |x_i(t'') - x_i(t')| = \left| \int_{t'}^{t''} u_i(s)\mathrm{d}s \right| \leq \int_{t'}^{t''} |u_i(s)|\mathrm{d}s. \tag{3.49}$$

If there are N approach times in $(\tau_i, T]$, then the pursuer x_i must have gone out of S at least N times. Note that at each approach time the pursuer is in S'. Let $x_i(t_j') \in S'$ and $x_i(t_j'') \notin S$, $j = 1, \ldots, N$. According to (3.49),

$$b \leq \int_{t_j'}^{t_j''} |u_i(s)|\mathrm{d}s, \quad j = 1, \ldots, N. \tag{3.50}$$

Clearly, $t_j' < t_j'' < t_{j+1}'$, $j = 1, 2, \ldots, N$, and so intervals $[t_j', t_j'']$, $j = 1, 2, \ldots, N$, are disjoint. Then, we obtain from (3.50)

$$Nb \le \sum_{j=1}^{N} \int_{t_j'}^{t_j''} |u_i(s)|\mathrm{d}s \le \int_0^T |u_i(s)|\mathrm{d}s \le \sqrt{T} \cdot \rho_i \le \rho\sqrt{T}.$$

Hence, $N \le \frac{\rho\sqrt{T}}{b}$. Thus, the number of approach times with the pursuer x_i on $[0, T]$ is finite and does not exceed $\left[\frac{\rho\sqrt{T}}{b}\right] + 1$.

Therefore, the total number of approaches with all pursuers on $[0, T]$ does not exceed $K\left(\left[\frac{\rho\sqrt{T}}{b}\right] + 1\right)$. Thus, we can conclude that the number of approach times $\tau_1, \tau_2, \ldots$, is finite on any interval $[0, T]$ and hence they have no limit point on $[0, T]$.

3.11 Estimate of Parameters

We imposed the following conditions on the parameters κ, α, a_i, $i = 1, 2, \ldots$:

A. $0 < \kappa < 1$, $0 < a_1 < \min\left\{\frac{1}{2}, \frac{a}{2}, b, d\right\}$ [see (3.7)];
B. $0 < \alpha < 1$, $12\alpha a_i < \sigma^2$ [see (3.23)];
C. $\frac{\alpha a_i^2}{12\sigma^2} > a_{i+1}$ [see (3.34)];
D. $a_{i+1} < a_i^4$ [see (3.8)];
E. $4\alpha a_1 + 8\rho\sqrt{\alpha a_1} < \frac{\sigma^2 - \rho^2}{2M}$ [see (3.45)];
F. $a_i < \frac{a}{2}$ [see (3.48)];
G. $2\alpha a_{i+1} < \rho^2$ [see (3.30)];
H. $a_{i+1} \le \frac{\alpha^3}{144 \cdot 32 \cdot \rho^2 \sigma^4} a_i^4$ [see (3.32)];
I. $2\alpha a_1 \le (\sigma - \rho)^2$ [see (3.20)];

To satisfy all these inequalities, we choose parameters. Let

$$a_1 = \frac{a^2\alpha}{16\sigma^2},$$

where a is the number in definition of S'. Next, choose the number α, $0 < \alpha < \min\{1, \sigma^2\}$, to satisfy the inequalities

$$12\alpha a_1 < (\sigma - \rho)^2, \quad \alpha a_1 < \frac{(\sigma^2 - \rho^2)^2}{64M^2(2\rho + 1)^2}, \quad a_1 < \min\left\{\frac{1}{2}, \frac{\rho^2}{2}, \frac{a}{2}, b, d\right\}. \tag{3.51}$$

Then choose κ by the formula

$$\kappa = \frac{\alpha^3}{144 \cdot 32\sigma^6}. \tag{3.52}$$

Observe that $\kappa < 1$. After that choose numbers $a_2, a_3, \ldots$ by the formula

$$a_{i+1} = \kappa a_i^4, \quad i = 1, 2, \ldots. \tag{3.53}$$

Note that by the third inequality of (3.51) and (3.53) the inequalities $1/2 \ge a_1 > a_2 > \ldots$ hold. Then, we show that all the above inequalities A–I are satisfied.

Indeed, A follows from (3.51) and (3.52). B is satisfied since by the first inequality of (3.51)

$$12\alpha a_i \le 12\alpha a_1 < (\sigma - \rho)^2 < \sigma^2. \tag{3.54}$$

Next, show the inequality in C. Indeed, to show

$$\frac{\alpha a_i^2}{12\sigma^2} > a_{i+1} = \frac{\alpha^3 a_i^4}{144 \cdot 32\sigma^6},$$

it is sufficient to show that

$$a_i^2 \alpha^2 < 12 \cdot 32\sigma^4.$$

This inequality is valid since by (3.54)

$$(a_i\alpha)^2 \le (a_1\alpha)^2 < \sigma^4/144 < 12 \cdot 32\sigma^4.$$

Since $\kappa < 1$, then (3.53) implies D. Since $\alpha \in (0, 1)$, $0 < a_{i+1} < a_1$ and $a_1 < \rho^2/2$, then we obtain

$$2\alpha a_{i+1} < 2a_{i+1} < 2a_1 < \rho^2,$$

which gives us the validity of option G.

We now think of E. The numbers α and a_1 are less than 1, and therefore $\alpha a_1 < \sqrt{\alpha a_1}$. Then according to the second inequality of (3.51) we have

$$4\alpha a_1 + 8\rho\sqrt{\alpha a_1} < 4\sqrt{\alpha a_1} + 8\rho\sqrt{\alpha a_1} = 4(1+2\rho)\sqrt{\alpha a_1} < \frac{\sigma^2 - \rho^2}{2M},$$

and so E is satisfied. Next, F follows from the inequalities $a_1 < \min\{\frac{1}{2}, \frac{a}{2}\}$ and $a_i > a_{i+1}$, $i = 1, 2, \ldots$. Since $\sigma \ge \rho$, H follows from (3.52) and (3.53). I also follows from the first inequality in (3.51).

Now, we turn to the game with the evaders $y_1, \ldots, y_M$. We choose the parameters α_i, a_{j1}, $a_{j2}, \ldots$ as follows. Let $a_{j1} = a_1$ for all $j = 1, \ldots, M$. We choose $\alpha_i = \alpha$. We define the numbers a_{j2}, $a_{j3}, \ldots$ by formula

$$a_{j,i+1} = \kappa a_{ji}^4, \quad i = 1, 2, \ldots, \quad j = 1, \ldots, M.$$

Note that by the choice of the numbers a_{j1} we obtain $a_{1i} = \ldots = a_{Mi}$, $i = 1, 2, \ldots$. Therefore, inequalities $a_{ji} < \frac{a}{2}$ are satisfied. Proof of Theorem 3.1 is complete.

4 Discussion

4.1 The Evader Moves in a Corridor of width *a*

If parameters are chosen as in Sect. 3.11, then we show that the evader moves in a corridor of width a. More precisely, we show that $|(y(t) - y_0, e')| \le a/2$. Indeed,

$$|(y(t) - y_0, e')| = \left| \int_0^t (v(s), e')\mathrm{d}s \right| \le \int_0^t |(v(s), e')|\mathrm{d}s$$
$$\le \left(\int_E + \int_{[0,t]\setminus E} \right) |(v(s), e')|\mathrm{d}s$$

where

$$E = \bigcup_{\{i | \tau_i \in [0,t]\}} [\tau_i, \tau_i'].$$

Note that by (3.13), $(v(t), e') = 0,\ t \in [0, t]\backslash E$. Therefore,

$$|(y(t) - y_0, e')| \le \int\limits_E |(v(s), e')|\mathrm{d}s$$
$$\le \sum_{i=1}^{\infty} \int\limits_{\tau_i}^{\tau_i'} |(v(s), e')|\mathrm{d}s \le \sum_{i=1}^{\infty} \sqrt{(\tau_i' - \tau_i) \int\limits_{\tau_i}^{\tau_i'} (v(s), e')^2 \mathrm{d}s}$$
$$\le \sigma \cdot \sum_{i=1}^{\infty} \sqrt{(\tau_i' - \tau_i)} \le \sigma \cdot \sum_{i=1}^{\infty} \sqrt{\frac{a_i}{\alpha}}.$$

From this using (3.10) and the formula $a_1 = \frac{a^2\alpha}{16\sigma^2}$, we get

$$|(y(t) - y_0, e')| < 2\sigma\sqrt{\frac{a_1}{\alpha}} = \frac{a}{2}.$$

This means the evader always moves in the set

$$S' = \{\xi |\ \ |(\xi - y_0, e')| \le \frac{a}{2}, \quad (\xi - y_0, e) \ge -b\}.$$

We have shown that the evader moves in $a/2$ neighborhood of the ray $\zeta(t) = y_0 + et,\ t \ge 0$.

4.2 Reduction to the Game in $\mathbb{R}^2$

Give a procedure of reduction of the game in $\mathbb{R}^n,\ n \ge 3$, to the game in $\mathbb{R}^2$. Given $x_{i0} - y_{j0} \ne 0,\ i = 1, \dots, K;\ j = 1, \dots, M$, we choose a unit vector e_1 different from the vectors $\pm(x_{i0} - y_{j0})/|x_{i0} - y_{j0}|,\ i = 1, \dots, K;\ j = 1, \dots, M$. Let $\bar{x}$ stand for $x - (x, e_1)e_1$, the projection of x on the subspace $L_1 = \{x \in \mathbb{R}^n |\ (x, e_1) = 0\}$. One can verify that $x_{i0} \ne y_{j0}$ implies that $\bar{x}_{i0} \ne \bar{y}_{j0}$. For the projections of the pursuers on L_1, we obtain from (1.1) the following equations

$$\dot{\bar{x}}_i = \bar{u}_i,\ \bar{x}_i(0) = \bar{x}_{i0},$$

where

$$\int\limits_0^\infty |\bar{u}_i(s)|^2 \mathrm{d}s = \int\limits_0^\infty \left(|u_i(s)|^2 - (u_i(s), e_1)^2\right) \mathrm{d}s \le \int\limits_0^\infty |u_i(s)|^2 \mathrm{d}s \le \rho_i^2.$$

Consider the evader's projection $\bar{y}_j \in L_1$ whose motion is described by the equation

$$\dot{\bar{y}}_j = v_j,\ \bar{y}_j(0) = \bar{y}_{j0},\ \int\limits_0^\infty |v_j(s)|^2 \mathrm{d} \le \sigma_j^2,$$

where $v_j(t) \in L_1,\ t \ge 0$. Since $\bar{x}_i(t) \ne \bar{y}_j(t)$ implies $x_i(t) \ne y_j(t)$, therefore if evasion is possible in the game in L_1, then evasion is possible in the game (1.1)–(1.2). Note that the evader $y_j(t)$ uses the same control function $v_j(t)$ as its projection $\bar{y}_j(t)$. Thus, evasion problem in $\mathbb{R}^n$ is reduced to an evasion problem in $(n-1)$-dimensional space L_1. We can proceed analogously to reduce the problem in L_1 to evasion problem in an $(n-2)$-dimensional space and so on. After some finite steps, we obtain a two-dimensional space.

4.3 Important Points of the Solution

According to the constructed strategies of the evaders, the total energy of the evaders spent by a time t might be greater than that of pursuers. Indeed, it can be seen from (3.14) and (3.13) that if any pursuer is in one of the sets S_j, the total energy of the evaders can be estimated from below as follows. When the evader y_j is moving on the time interval $[0, \tau_{1j}]$, by (3.13)

$$\int_0^t |v_j(t)|^2 \mathrm{d}t = \int_0^t \sum_{i \in I_j(t)} |u_i(t)|^2 \mathrm{d}t,$$

meaning that the evader y_j spends energy same as all the pursuers in the strip S_j. When the evader y_j moves on a time interval $[\tau_{\omega_{l-1}+1}, \tau^*_{\omega_l}]$, by (3.14) we have

$$\int_{\tau_{\omega_{l-1}+1}}^{\tau^*_{\omega_l}} |v_j(t)|^2 \mathrm{d}t \geq \int_{\tau_{\omega_{l-1}+1}}^{\tau^*_{\omega_l}} \sum_{i \in I_j(t)} (2\alpha^2 + |u_i(t)|^2) \mathrm{d}t > \int_{\tau_{\omega_{l-1}+1}}^{\tau^*_{\omega_l}} \sum_{i \in I_j(t)} |u_i(t)|^2 \mathrm{d}t.$$

Hence, on the interval $[\tau_{\omega_{l-1}+1}, \tau^*_{\omega_l}]$, the evader y_j spends energy more than that of all pursuers in S_j. In a similar fashion, we can verify that if the evader y_j moves on the interval $[\tau^*_j, \tau_{j+1}]$ according to (3.13), then the evader y_j spends energy same as all the pursuers in the strip S_j on the time interval $[\tau^*_j, \tau_{j+1}]$, where τ^*_j is time group pursuit is completed at and at τ_{j+1} another group pursuit is started.

Therefore, we can conclude that the total energy spent by evaders at any time t greater than or equal to that of all the pursuers. By this reason, we first reduced the case $\sigma_1^2 + \cdots + \sigma_M^2 = \rho_1^2 + \ldots + \rho_K^2$ to the case $\sigma_1^2 + \cdots + \sigma_M^2 > \rho_1^2 + \cdots + \rho_K^2$. Then we constructed strategies for the evaders, which depend on some parameters. We have chosen the parameters so that the total energy of the evaders be less than $\sigma_1^2 + \cdots + \sigma_M^2$.

Next, the movement of the evaders in the disjoint strips $S_j,\ j = 1, \ldots, M$, is crucial. Because of disjointness of the strips, the control parameter of any pursuer cannot appear in the strategies of two different evaders y_i and y_j with $i \neq j$. This condition is important for admissibility of the strategies of evaders.

The fact that approach times τ_i have no finite limit point is another key point in the proof of the main result since if the approach times have a finite limit point, the evader can be captured before his energy is exhausted. In fact, one pursuer, say x_i, can make infinitely many a_{ω_1}-, a_{ω_2}-, …, -approaches at some times $\tau_{\omega_1},\ \tau_{\omega_2}, \ldots \to \infty$ respectively.

4.4 Realization of Evaders' Strategies

By Definition 2.3, to construct the strategy V_j, the evader y_j knows information about parameters $t, y_j, q_j, x_1, \ldots, x_K, u_1, \ldots, u_K$ at each time $t \geq 0$.

The initial position ζ_0, which is known to evaders, is used to find the numbers d, d_0 and ε. Also, the evaders employ only ζ_0 to construct the points y'_{j0} and vectors v_{j0}, e_j, e'_j, and, hence, strips $S_j,\ S'_j,\ j = 1, \ldots, M$.

Step 1 Check the equation $\sigma = \rho$. If it is so, using techniques of Sect. 3.1, we obtain $q(\tau^0) > p(\tau^0)$ where τ^0 is the time for which $|x_s(\tau^0) - x_{i0}| = d/4$ at some $1 \leq s \leq K$ then we go to Step 2. Here, the evader y_j needs the information about $x_i(t),\ i = 1, \ldots, K$, and time t to check the condition $|x_i(t) - x_{i0}| = d/4$. To this end, the evaders employ the following controls

$$v_j(t) = 0, \quad j = 1, \ldots, M, \quad 0 \le t \le \tau^0.$$

As mentioned in Remark 3.2 that the evaders do not know $p(\tau^0)$, but they know $\bar{\rho}$ and the fact that $p(\tau^0) \le \bar{\rho}$. From now on the evaders use $\bar{\rho}$ instead of ρ. If $\sigma > \rho$, then we put $\tau^0 = 0$ and go to Step 2.

Step 2 Let $q(\tau^0) > p(\tau^0)$. We check whether $y_{i0} = y_{j0}$ for some $i \neq j$, $i, j \in \{1, \ldots, M\}$. If so the evaders move to the points y'_{j0} by using the following controls

$$v_j(t) = v_{j0}, \quad j = 1, \ldots, M, \quad \tau^0 \le t \le \tau^0 + \varepsilon.$$

This control is constructed based on ζ_0. To choose the parameter a_{j1} [see (3.7)], we use d which depends also on ζ_0. Note that $a_{j1} < d/2$ and by (3.6) $|x_i(t) - y_j(t)| > d/2$ for all $i = 1, \ldots, K$; $j = 1, \ldots, M$, and therefore there is no a_{j1}-approach time in the time interval $\tau^0 \le t \le \tau^0 + \varepsilon$.

If $y_{i0} \neq y_{j0}$ for all $i \neq j$, $i, j \in \{1, \ldots, M\}$, then we put $\varepsilon = 0$ and go to Step 3.

Step 3 Let $q(\tau^0 + \varepsilon) > p(\tau^0 + \varepsilon)$ and $y'_{i0} \neq y'_{j0}$ for all $i \neq j$, $i, j \in \{1, \ldots, M\}$. Then evaders use strategies (3.13) and (3.14) where we have to put $\tau_0^* = \tau^0 + \varepsilon$. In this step, the strategies of evaders are constructed based on the information about t, y_j, q_j, $x_1, \ldots, x_K$, $u_1, \ldots, u_K$ at each time $t \ge 0$. While $q_j(t) > 0$ the evader y_j is not captured by any pursuer. If $q_j(t) = 0$, the energy of the evader y_j is exhausted, y_j is not able further to move, and then it can be captured by a pursuer.

Importantly, the number of summands in the definition of $V_{2i}(t)$ depends on t, therefore the measurability of $V_{2i}(t)$ needs to be explained. This follows from the fact that any pursuer x_i can stay in the strip S only on a closed time interval since $x_i(t)$ is continuous and S is closed.

5 Conclusion

In the present paper, we have solved completely the simple motion evasion differential game of many evaders from many pursuers when control functions of the players are subject to integral constraints. We have shown that if

$$\rho_1^2 + \cdots + \rho_K^2 \le \sigma_1^2 + \cdots + \sigma_M^2,$$

then evasion is possible from any initial position ζ_0 of players. In addition, we have constructed explicit strategies for the evaders. Note that this inequality is a necessary condition of evasion as well since if it does not hold, then pursuit can be completed from any initial position of players [16]. Thus, the result of the present paper fills the long standing gap in the simple motion differential game of many players with integral constraints. In future, this result can be extended to differential games described by more general linear differential equations.

Acknowledgements The present research was partially supported by the National Fundamental Research Grant Scheme FRGS of Malaysia, 01-01-13-1228FR.

References

1. Alexander S, Bishop R, Christ R (2009) Capture pursuit games on unbounded domain. L'Enseignement Mathématique 55(1/2):103–125
2. Alias IA, Ibragimov G, Rakhmanov A (2016) Evasion differential game of infinitely many evaders from infinitely many pursuers in Hilbert space. Dyn Games Appl 6(2):1–13. doi:10.1007/s13235-016-0196-0

3. Bannikov AS, Petrov NN (2010) To non-stationary group pursuit problem. Trudy Inst Math Mech UrO RAN 16(1):40–51
4. Belousov AA (2010) Method of resolving functions for differential games with integral constraints. Theory Optim Solut 9:10–16
5. Blagodatskikh AI, Petrov NN (2009) Conflict interaction between groups of controlled objects. Udmurt State University Press, Izhevsk (in Russian)
6. Borowko P, Rzymowski W, Stachura A (1988) Evasion from many pursuers in the simple motion case. J Math Anal Appl 135(1):75–80
7. Chernous'ko FL (1976) A problem of evasion of several pursuers. Prikl Mat Mekh 40(1):14–24
8. Chikrii AA, Prokopovich PV (1992) Simple pursuit of one evader by a group. Cybern Syst Anal 28(3):438–444. doi:10.1007/BF01125424
9. Chodun W (1989) Differential games of evasion with many pursuers. J Math Anal Appl 142(2):370–389
10. Conti R (1974) Problemi di Controllo e di Controllo Ottimale. UTET, Torino **(in Italian)**
11. Croft HT (1964) Lion and man: a postscript. J Lond Math Soc 39:385–390
12. Friedman A (1971) Differential games. Wiley, New York
13. Grigorenko NL (1990) Mathematical methods of control of several dynamic processes. MSU Press, Moscow **(in Russian)**
14. Hajek O (1975) Pursuit games. Academic Press, New York
15. Huseyin A, Huseyin N, Guseinov KG (2015) Approximation of the sections of the set of trajectories of the control system described by a nonlinear Volterra integral equation. Math Model Anal 20(4):502–515
16. Ibragimov G, Satimov N (2012). A multiplayer pursuit differential game on a closed convex set with integral constraints. In: Abstract and applied analysis. Article ID 460171: 12 p. doi:10.1155/2012/460171
17. Ibragimov GI, Salimi M, Amini M (2012) Evasion from many pursuers in simple motion differential game with integral constraints. Eur J Oper Res 218(2):505–511
18. Isaacs R (1965) Differential games. Wiley, New York
19. Ivanov RP (1980) Simple pursuit-evasion on a compact convex set. Doklady Akademii Nauk SSSR 254(6):1318–1321
20. Krasovskii NN (1968) Theory of control of motion: linear systems. Nauka, Moscow
21. Krasovskii NN, Subbotin AI (1988) Game-theoretical control problems. Springer, New York
22. Kuchkarov AS, Risman MH, Malik AH (2012) Differential games with many pursuers when evader moves on the surface of a cylinder. ANZIAM J 53(E):E1–E20
23. Kuang FH (1986) Sufficient conditions of capture in differential games of m pursuer and one evader. Kibernetika 6:91–97
24. Kuchkarov A, Ibragimov G, Ferrara M (2016) Simple motion pursuit and evasion differential games with many pursuers on manifolds with Euclidean metric. Discrete Dyn Nat Soc. doi:10.1155/2016/1386242
25. Mishchenko EF, Nikol'skii MS, Satimov NY (1977) Evoidance encounter problem in differential games of many persons. Trudy MIAN USSR 143:105–128
26. Petrosyan LA (1993) Differential games of pursuit. World Scientific, Singapore
27. Pontryagin LS (1988) Selected works. Nauka, Moscow
28. Pshenichnii BN (1976) Simple pursuit by several objects. Cybern Syst Anal 12(3):145–146
29. Pshenichnii BN, Chikrii AA, Rappoport JS (1981) An efficient method of solving differential games with many pursuers. Dokl Akad Nauk SSSR 256:530–535 **(in Russian)**
30. Satimov NY, Rikhsiev BB, Khamdamov AA (1983) On a pursuit problem for n person linear differential and discrete games with integral constraints. Math USSR Sbornik 46(4):456–469
31. Satimov NY, Fazilov AZ, Hamdamov AA (1984) On pursuit and evasion problems for multi-person linear differential and discrete games with integral constraints. Dif Uravneniya 20(8):1388–1396
32. Vagin D, Petrov N (2001) The problem of the pursuit of a group of rigidly coordinated evaders. J Comput Syst Sci Int 40(5):749–753
33. Zak VL (1978) On a problem of evading many pursuers. J Appl Maths Mekhs 43(3):492–501