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Dynamic Price Competition with Switching Costs

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Abstract We characterize a relatively simple Markov Perfect equilibrium in a continuous-time dynamic model of competition with switching costs. When firms cannot price-discriminate between old and new consumers, the effect of switching costs on prices critically depends on the degree of market share asymmetries: If firms' market shares are sufficiently asymmetric, an increase in switching costs leads to higher prices. However, as market shares become sufficiently symmetric, price competition turns fiercer, and in the long-run, switching costs have a pro-competitive effect. If firms can price-discriminate, an increase in switching costs make all consumers better off regardless of market structure.

Keywords Switching costs · Continuous-time model · Markov Perfect equilibrium · Differential games · Market concentration · Price discrimination

1 Introduction

Many products and technologies exhibit switching costs (i.e., costs that customers must bear when they adopt a new product or technology). For example, switching costs arise when there is limited compatibility between an old product (or technology) and a newly adopted one. In this case, the specific investments that a customer may have incurred in relation to the utilization of the old product (or technology) may fully or partially depreciate. When compatibility is highly valued by customers, they are discouraged from changing products. Other switching costs stem from contract structure (e.g., service contracts with a

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certain minimum term) and/or loyalty programs (e.g., discount coupons, frequent flyer cards). Competition in mobile telephony and residential broadband shares some of these features. As new features and services are periodically added, consumers valuations may vary over time, thus eventually rendering switching an attractive course of action despite the presence of switching costs.

Switching costs alter the nature of dynamic strategic competition as consumers face a potential lock-in effect which gives market power to the firms. As the history of past choices affects future product choice or technology adoption decisions, market share is a valuable asset. The incentives to exploit current customers (by charging high prices) and to increase market share (by offering low prices) are countervailing. A priori, it seems that the net effect of switching costs on the nature of dynamic price competition is unclear. Nonetheless, the conventional wisdom distilled from the literature seems to suggest that switching costs are anti-competitive (see [8,12] for a survey of this literature).

In this paper, we show that when switching costs are symmetric and not too high (so that switching takes place in equilibrium), this conventional wisdom does not apply—at least in steady state. Our analysis is based upon a continuous-time dynamic equilibrium model in which switching costs are independent and identically distributed across consumers and over time. In each period, consumers are randomly given the opportunity to switch according to a Poisson process. Their decisions are based upon price differences across firms, their realized valuations for the goods, and the value of the switching cost. We consider stationary Markovian strategies, with market shares being the state variable, and characterize a relatively simple Markov Perfect pricing equilibrium. We find that market shares are sticky and converge monotonically. If switching costs are equal across firms, the market structure is symmetric in steady state.

When firms cannot price-discriminate between old and new consumers, the competitive effects of switching costs depend on market structure. In particular, when firms are of equal size, the mass of consumers loyal to a firm is the same as the mass of potential consumers switching into the firm. Hence, the static incentives— exploiting loyal consumers vs. attracting new ones—cancel each other out, so that the dynamic incentives to attract *new* consumers in order to exploit them *in the future* prevail. Since the value of new customers is greater the higher the switching costs, an increase in switching costs fosters more competitive outcomes.

However, if switching costs are asymmetric, market shares do not become symmetric in the long-run and the static effects remain. In particular, the firm from which it is costlier to switch charges higher prices and still serves more than half of the market. Since an increase in switching costs allows the firm to charge higher prices, the conventional wisdom is recovered. Namely, (asymmetric) switching costs have anti-competitive effects. Nevertheless, we show that if firms are allowed to endogenously choose whether or not to invest in a given level of switching costs, in equilibrium only the symmetric switching costs configuration survives. This suggests that in steady state, the pro-competitive effect of switching costs is more likely to arise.

To assess the robustness of the above results, we also analyze the case in which firms are allowed to price-discriminate between old and new consumers. Under price discrimination, firms no longer face the trade-off between the investing and the harvesting effect as they can charge low prices to attract new consumers and high prices to exploit the old ones. As a consequence, new consumers are always better off the higher the switching costs. The effect on the old consumers results from the interplay of two effects: on the one hand, the higher the switching cost, the higher the price that a firm can charge to its current customers without losing them; on the other hand, since higher switching costs also induce the alternative supplier to price more aggressively, the firm has to reduce its own price to retain its current customers. We show that, for moderate values of discount rates, the second effect dominates and the price for old customers is also decreasing with increasing switching costs. Thus, with price discrimination, switching costs are pro-competitive regardless of market structure.

This paper is organized as follows. In Sect. 2, we review the relevant literature on the subject and relate it to our results. In Sect. 3, we describe and analyze our basic model, in which firms have symmetric switching costs and are not allowed to price-discriminate between the old and the new consumers. In Sect. 4, we explore various extensions of the basic model: price discrimination, asymmetric switching costs, and endogenous switching costs. We conclude in Sect. 5.

2 Are Switching Costs Pro- or Anti-Competitive?

Klemperer [10,11] are the first published works that aimed to analyze the impact of switching costs on the nature of price competition. In these papers, the author developed a two-period model in which, in the first period, consumers choose a product (or technology) for the first time. In the second period, consumers may choose a different product in which case they face switching costs. In equilibrium, prices follow a pattern of "bargains" followed by "rip-offs." To circumvent the potential "end of horizon" effect in these two-period models, infinite-horizon models have been analyzed [9,13,16,18].

In all of these papers, firms face a trade-off between maximizing current versus future profits. Maximizing current profits calls firms to exploit their loyal consumers ("harvesting" effect), whereas maximizing future profits calls firms to decrease current prices in order to attract new customers ("investing" effect). The previous papers concluded that the former effect dominates, so that switching costs have anti-competitive effects.

However, these analyses omitted an equally important effect: the fact that switching costs affect current market competition even in a static setting. This effect was hidden in these models by the lack of switching in equilibrium (either by assumption, or because switching costs were assumed very large). Instead, if switching takes place in equilibrium, firms want to attract new consumers, not just to exploit them in the future, but also as a source of current profits. This effect, which similarly to [14] we refer to as the "poaching" effect, partially mitigates the "harvesting" effect. However, since "harvesting" by large firms dominates over "poaching" by small firms, the overall short-run effect of an increase in switching costs is an increase in prices.

However, in a dynamic setting, the "investing" effect induces firms to reduce current prices in order to attract new customers who will become loyal in the future. Therefore, the net effect of switching costs on equilibrium prices becomes ambiguous. In this paper, we show that the overall effect—resulting from the interplay of the harvesting, poaching, and investing effects—critically depends of the degree of market share asymmetry. If firms' market shares are symmetric, the poaching and harvesting effects cancel out. As the investing effect is the only effect that remains, an increase in switching costs reduces prices. In contrast, if market shares are very asymmetric, the current switching effect barely mitigates the harvesting effect, so that an increase in switching costs leads to higher prices. It turns out that, in steady state, firms' market shares become symmetric precisely because in previous periods the dominant firm priced less aggressively than the smaller one. This implies that long-run equilibrium prices decrease with switching costs (as long as this value is small enough so as to allow for switching in equilibrium).

There is a recent string of papers showing that switching costs can be pro-competitive as they lead to lower prices [2–4,6,7,14,15,17]. By means of a numerical test bed, [7] show that depending upon the magnitude of switching costs, switching may indeed occur in equilibrium and that the net effect on prices is ambiguous. In an empirical paper, [17] finds that lower switching costs (i.e., number portability) led to lower prices for toll-free services. While most of the theoretical literature appeals to price discrimination to show that switching costs can lower prices, [2] identify another channel which is common to our model: If consumers switch in equilibrium, then a firm may lower price to partially offset the costs of consumers that are switching to the firm. [14] also arrives at similar conclusions using an overlapping generations model.

Interestingly, *all* the papers in this literature are based on discrete-time models of strategic price competition in which the time interval that separates the firms' pricing decisions is a constant of a certain specified length. Our paper differs from this literature in that we analyze a continuous-time model of dynamic pricing. In general, the set of equilibrium strategies for discrete-time approximations does not necessarily correspond to the set of equilibria for the continuous-time game. Hence, it is important to analyze the qualitative properties of continuous-time equilibrium pricing. While our model setting is not as general as those compared in ,e.g., [2,14], we are able to derive closed-form expressions for equilibrium pricing which in turn reveal the structure of equilibrium pricing dynamics *both* in the short-run *and* in steady state. Furthermore, since we derive results for the case of symmetric and asymmetric switching costs, we are able to shed light on the strategic choice of switching costs.

3 The Model

We consider a market in which two firms,[1] with marginal costs normalized to zero compete to provide a service which is demanded continuously over time. Let $x_i(t)$ denote the market share of firm $i \in \{1, 2\}$. There is a unit mass of infinitely lived consumers. We assume $x_1(t) + x_2(t) = 1$, i.e., all consumers are served.[2] Let p_i denote the price charged by firm $i \in \{1, 2\}$ to all its customers, i.e., firms are not allowed to price-discriminate. [3]

At every moment in time, some consumers are given the option to switch to the alternative supplier, in which case they have to incur a switching cost. More specifically, switching opportunities for consumers take place over time according to independent Poisson processes with unit rate,[4] i.e., in the interval $(t, t + dt)$, the expected fraction of consumers considering switching or not between firms is dt. We assume that consumers cannot anticipate infinite

[1] Our model could be extended to multiple competing firms assuming a "circular city" model of product differentiation with evenly spaced product varieties as in Salop's model. In that model, demand for a product variety is a function of the two adjacent product varieties. So in a dynamic model with switching, demand for a product will be a function of prices *and* market shares of the two adjacent product varieties. A dynamic model would therefore require a multi-dimensional state space, which is out of the scope of the current paper.

[2] In the context of a discrete choice model with product differentiation, the relaxation of the full market coverage assumption would imply that not all consumers are served continuously: Some consumers may at times drop out of the market and then sign back up for service at a future time (*without* experiencing a switching cost). Thus, relaxing the full market assumption would add an element of competitive pressure on firms equilibrium pricing strategies.

[3] In Sect. 4.1, we analyze the effects of price discrimination.

[4] The analysis is robust to arrival rates different from one. However, this would add an additional parameter in the model, with only a scaling effect.

equilibrium price trajectories and thus can only react to current prices.[5] Conditional on having the opportunity to switch, $q_{ji} \in (0, 1)$ denotes the probability with which a customer currently served by firm j switches to firm i. Accordingly, $q_{jj} = 1 - q_{ji}$ is the probability that a customer already served by firm j maintains this relationship. Firm i's net (expected) change in market share in the infinitesimal time interval $(t, t + dt]$ can be expressed as

$$x_i(t + \mathrm{d}t) - x_i(t) = q_{ji} x_j(t)\mathrm{d}t - (1 - q_{ii})x_i(t)\mathrm{d}t, \tag{1}$$

i.e., the net (expected) change in market share is equal to the expected number of customers that firm i steals from firm j, minus the customers that firm j steals from firm i. The revenue accrued in the infinitesimal time interval $(t, t+\mathrm{d}t]$ is the sum of $p_i x_i(t)\mathrm{d}t$ (i.e., revenue from current customers) and $p_i[q_{ji}x_j(t) - (1 - q_{ii})x_i(t)]\mathrm{d}t$ (i.e., revenue gain/loss from new/old customers). Hence, the rate at which revenue is accrued by firm i, say $\pi_i(t)$, can be written as

$$\pi_i(t) = p_i x_i(t) + p_i[q_{ji}x_j(t) - (1 - q_{ii})x_i(t)]. \tag{2}$$

In order to characterize the switching probabilities, we assume a discrete choice model in which the net surplus from product $i \in \{1, 2\}$ at time $t > 0$ is of the form

$$u_i(t) = v_i(t) - p_i(t)$$

wherein we make the following standing assumption:[6]

Assumption 1 The collection $\{v_i(t) : t > 0\}$ is *i.i.d.* and $v_i(t) - v_j(t)$ is uniformly distributed in $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

In what follows, we analyze two cases. First, we consider the case in which switching costs are symmetric. Second, we analyze the case in which only one firm is protected by switching costs, while the rival firm is not. This will allow us to shed light on the endogenous choice of switching costs.

We first assume that switching costs are symmetric. In this case, all customers who switch incur the same switching cost regardless of whether they switch from firm 1 to firm 2 or from 2 to 1. When an opportunity to switch arises for a given customer, he would opt for firm i (if currently served by firm j) provided that

$$u_i - \frac{s}{2} = v_i - p_i - \frac{s}{2} > u_j = v_j - p_j$$

where $\frac{s}{2}$ is the switching cost incurred ($s < 1$). Hence, the probability that such a customer served by firm j switches to firm i, q_{ji}, is given by

$$q_{ji} = \Pr\left(v_j - v_i < -\frac{s}{2} + p_j - p_i\right) = \frac{1-s}{2} - p_i + p_j$$

where we assume $p_i - p_j \in \left[-\frac{1}{2}(1 - s), \frac{1}{2}(1 + s)\right]$. Conversely, if firm i serves the selected consumer, he will maintain this relationship if

$$u_i = v_i - p_i > u_j = v_j - p_j - \frac{s}{2}.$$

5 The main conclusions of the analysis are preserved if we allowed for more sophisticated consumers. See "Appendix 1".

6 Note that this assumption is consistent with Hotelling's model of product differentiation with product varieties at the extremes of a linear city uniformly distributed in $[0, \frac{1}{2}]$. Results are robust to allowing for more general distributions.

Hence, the probability q_{ii} that a customer already served by firm i maintains this relationship is[7]

$$q_{ii} = \Pr\left(v_j - v_i < \frac{s}{2} - p_i + p_j\right) = \frac{1+s}{2} - p_i + p_j.$$

Substituting q_{ji} and q_{ii} into (1) and taking the limit, as $\mathrm{d}t \to 0$, we obtain

$$\dot{x}_i(t) = -x_i(t)(1-s) + \frac{1-s}{2} - p_i + p_j\cdot$$

Substituting q_{ji} and q_{ii} into (2) and using the condition $x_1(t) + x_2(t) = 1$, we obtain the rate at which revenue is accrued by firms 1 and 2,

$$\pi_1(t) = p_1\left(x_1(t)s + \frac{1-s}{2} - p_1 + p_2\right)$$
$$\pi_2(t) = p_2\left(-x_1(t)s + \frac{1+s}{2} + p_1 - p_2\right)\cdot$$

Given the assumption of full market coverage, payoff relevant histories are subsumed in the state variable $x_1 \in [0, 1]$. Assume a discount rate $\rho > 0$. A stationary Markovian pricing policy is a map $p_i : [0, 1] \to [0, \bar{p}]$ where $\bar{p} > 0$ is the maximum price ensuring full market coverage. We restrict our attention to the set of continuous and bounded Markovian pricing policies, say $\mathcal{P}$. For a given strategy combination $(p_i, p_j) \in \mathcal{P} \times \mathcal{P}$ and initial condition, $x_1(\tau) \in [0, 1]$ and $\tau < \infty$, the value function is defined as

$$V_i^{(p_i,p_j)}(x_1(\tau)) = \int_\tau^\infty \mathrm{e}^{-\rho t}\pi_i(p_i(x_1(t)), p_j(x_1(t)), x_1(t))\mathrm{d}t\cdot$$

A stationary Markovian strategy combination $(p_i^*, p_j^*) \in \mathcal{P} \times \mathcal{P}$ is a Markov Perfect equilibrium (MPE) if and only if

$$V_i^{(p_i^*,p_j^*)}(x_1(\tau)) \geq V_i^{(p_i,p_j^*)}(x_1(\tau)),$$

for all $p_i \in \mathcal{P}$, $i \in \{1, 2\}$, $x_1(\tau) \in [0, 1]$ and $\tau < \infty$.

3.1 Equilibrium Pricing

As we shall show, when switching costs are not too high, the tension between the "harvesting" effect (exploit loyal customers via high prices), the "poaching" effect (attract new customers via low prices to increase *current* sales), and the "investing" effect (attract new customers via low prices to increase *future* sales) leads to more competitive prices in the long-run. However, in the short-run, the degree of market share asymmetries will determine which of these effects dominates.

Static Setting

To understand pricing incentives in the long-run, let us first characterize equilibrium pricing in the static setting. This allows to isolating the harvesting and poaching effects from the

[7] Note that $q_{ii} \geq q_{ji}$ reflects the fact that, for given prices, firm i is more likely to retain a randomly chosen current customer than to "steal" one from firm j.

investing effect, as the latter only arises in the dynamic setting. The first order conditions in the static setting are:

$$\frac{\partial \pi_1}{\partial p_1} = x_1 s + \frac{1-s}{2} - 2p_1 + p_2 = 0$$
$$\frac{\partial \pi_2}{\partial p_2} = -x_1 s + \frac{1+s}{2} + p_1 - 2p_2 = 0,$$

from which we derive the best reply functions:

$$R_1(p_2) = \frac{1}{2}\left(p_2 + s\left(x_1 - \frac{1}{2}\right) + \frac{1}{2}\right)$$
$$R_2(p_1) = \frac{1}{2}\left(p_1 - s\left(x_1 - \frac{1}{2}\right) + \frac{1}{2}\right).$$

Adding switching costs does not alter the fact that prices are strategic complements, that is, a firm optimally responds to a rival's price increase by increasing its own. Figure 1 below plots firms' best reply functions for two values of switching costs.

The large firm (e.g., firm 1) has more to gain by increasing the price and exploit its loyal consumers (more than half) than it has to lose by reducing the price to attract its rival's loyal consumers (less than half). Hence, the large firm behaves less aggressively than the small one, i.e., $R_1(p) > R_2(p)$.

Solving for equilibrium prices, we obtain

$$p_1(x_1) = \frac{s}{3}\left(x_1 - \frac{1}{2}\right) + \frac{1}{2}$$
$$p_2(x_1) = -\frac{s}{3}\left(x_1 - \frac{1}{2}\right) + \frac{1}{2}.$$

The equilibrium price of the large firm exceeds that of its smaller competitor,

$$p_1 - p_2 = \frac{2}{3}s\left(x_1 - \frac{1}{2}\right) > 0.$$

As a consequence, the large firm loses customers in favor of its smaller competitor, but it still remains large.

The following lemma summarizes the comparative statics of equilibrium outcomes in the static setting as switching costs s increase:

Lemma 1 *In a static setting:*

(i) If market shares are asymmetric, an increase in switching costs s raises the price charged by the large firm, reduces the price charged by the small firm, increases the average market price, and makes market shares even more asymmetric.
(ii) If market shares are symmetric, switching costs have no effect on equilibrium outcomes.

When firms' market shares are asymmetric, i.e., $x_1 > \frac{1}{2}$, an increase in switching costs s implies an outward shift in the large firm's best reply function and an inward shift in the small firm's best reply function (see Fig. 1). In other words, switching costs make the large firm less aggressive and the small firm more so. As already noted, this is consistent with the view that switching costs can be interpreted as a subsidy for the large firm and a tax for the small one.

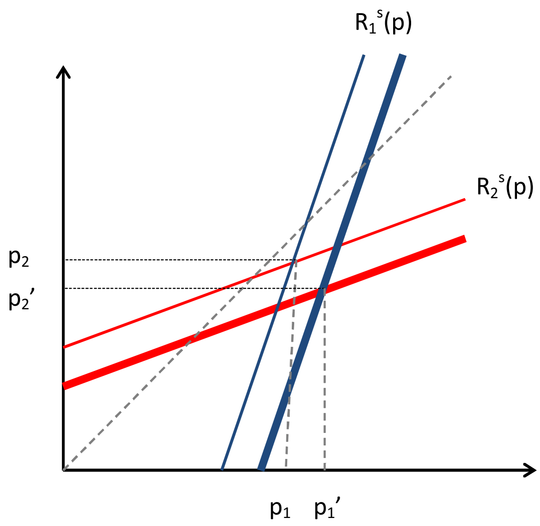

Fig. 1 Firms' best replies in the static setting for low (*thin lines*) and high (*thick lines*) swicthing costs *s*

Since, as s increases, the shifts in firms' best replies are of the same magnitude, the equilibrium point moves down to the right, i.e., the price of the large firm goes up while the price of the small firm goes down. Given that the price charged by the large firm has a stronger impact on the average market price, an increase in s implies that the average price in the market also goes up. This corresponds to the conventional wisdom according to which prices are increasing in switching costs. It also follows that an increase in s enlarges the price differential, so that market shares become even more asymmetric.

Last note that if firms were symmetric, i.e., $x_1 = \frac{1}{2}$, switching costs would have no impact on equilibrium outcomes in a static setting.

Dynamic Setting

We are now ready to characterize equilibrium pricing in the dynamic setting.

Proposition 1 *The unique Markov Perfect Equilibrium in affine pricing strategies is:*[8]

$$p_1(x_1) = \tfrac{1}{3}(s-a)\left(x_1 - \tfrac{1}{2}\right) + p^*$$
$$p_2(x_1) = -\tfrac{1}{3}(s-a)\left(x_1 - \tfrac{1}{2}\right) + p^*$$

where $a \in \left(0, \frac{s}{2}\right)$ *is the smallest root of the quadratic equation*

$$2a^2 - 3\left(2 + \rho - \frac{7}{9}s\right)a + \frac{2}{3}s^2 = 0, \quad (3)$$

[8] We believe that there would be little difference between the affine MPE presented in the paper and an equilibrium (should it exist) in nonlinear strategies. In the linear MPE, the dominant firm exploits its market share, while the firm with smaller market share offers a price discount that decreases over time. The rate at which the dominant firm loses market share is constant. Suppose firms follow nonlinear pricing strategies and the firm with lower market share offers a price discount. In this case, nonlinearity changes the speed at which the dominant firm loses market share. There may be a nonlinear equilibrium in which market dominance decreases more slowly initially (with high prices by both firms) and then speeds up (with relatively larger price discounts) as market shares equilibrate. However, there would be no qualitative difference between this equilibrium (should it exist) and the one presented in the paper. With relatively high discount rates, there can not be a nonlinear equilibrium in which the dominant firm maintains (or increases) dominance because this entails a price discount by the dominant firm which would make it worse off.

and[9]

$$p^* = \frac{1}{2} + \frac{a}{2(1+\rho)} - \frac{s(1+\frac{a}{2})}{3(1+\rho)}\cdot$$

Proof See the "Appendix 2". □

In the proof of Proposition 1, we make use of the notion of a Hamiltonian (see [5]), that is:

$$\mathcal{H}_i = \mathrm{e}^{-\rho t}[\pi_i + \lambda_i \dot{x}_1],$$

for $i \in \{1, 2\}$, where $\lambda_i = \frac{\partial V_i}{\partial x_1}$ is the co-state variable. The necessary and sufficient conditions for a Markov Perfect equilibrium are:

$$\frac{\partial \pi_i}{\partial p_i} = -\lambda_i \frac{\partial \dot{x}_1}{\partial p_i}, \tag{4}$$

which captures the inter-temporal trade-offs inherent in equilibrium pricing, i.e., marginal revenue equals the (marginal) opportunity cost (value loss) associated with market share reduction and the Hamilton–Jacobi equations

$$-\frac{\partial \mathcal{H}_i}{\partial x_1} - \frac{\partial \mathcal{H}_i}{\partial p_j}\frac{\partial p_j}{\partial x_1} = \dot{\lambda}_i - \rho\lambda_i\cdot \tag{5}$$

In the proof of Proposition 1, we show that the system of partial differential equations (4) and (5) has a closed-form solution.

Note that condition (4) gives rise to a sort of "instantaneous" best reply functions:

$$R_1(p_2) = \frac{1}{2}\left(p_2 + s\left(x_1 - \frac{1}{2}\right) + \frac{1}{2}\right) - \frac{\lambda_1}{2} \tag{6}$$

$$R_2(p_1) = \frac{1}{2}\left(p_1 - s\left(x_1 - \frac{1}{2}\right) + \frac{1}{2}\right) + \frac{\lambda_2}{2} \tag{7}$$

Therefore, as compared to the static setting (in which λ_i equals zero), firms' best reply functions in the dynamic setting shift in, thus implying that equilibrium prices, are lower. In particular, $R_i(p_j)$ now shifts in by $\frac{\lambda_i}{2}$. This is a direct consequence of the "investing effect": firms compete more aggressively to attract new customers as these will become loyal in the future. In the proof of Proposition 1 we show that $\lambda_1 = ax_1 + b > -\lambda_2 = -ax_1 + b > 0$. Hence, in the dynamic setting, it is still true that the large firm behaves less aggressively than the small firm, i.e., $R_1(p) > R_2(p)$, thus implying that the large firm's equilibrium price is higher than that of the small firm, regardless of the value of s,

$$p_1(x_1) - p_2(x_2) = \frac{2}{3}(s-a)\left(x_1 - \frac{1}{2}\right) > 0\cdot$$

As compared to the static setting, the large firm's best reply function has shifted in by a larger amount, $\frac{\lambda_1}{2}$, than that of the small one, $-\frac{\lambda_2}{2}$ (recall that $\lambda_1 > -\lambda_2$). This derives from the fact that the investing effect is stronger for the large firm than for the small one: attracting new customers today is more valuable for the large firm, given that the price it charges to its loyal consumers is higher. It follows that the price differential, while still positive, is now smaller than in the static setting.

[9] Note that $a \to 0^+$ as $s \to 0^+$, the equilibrium pricing corresponds to the "static" case (i.e., without switching costs) with $p_1 = p_2 = \frac{1}{2}\cdot$

Concerning dynamics, the fact that the large firm has the high price implies that the large firm concedes market share in favor of the smaller one. Therefore, market share asymmetries fade away over time. Note that, unlike other papers in the literature, market share convergence is monotone. In particular, the equilibrium state dynamics are described by:

$$\begin{aligned}\dot{x}_1(t) &= -x_1(t)(1-s) + \frac{1-s}{2} - p_1(x_1(t)) + p_2(x_1(t)) \\ &= -\left(x_1(t) - \frac{1}{2}\right)\left(1 - \frac{s+2a}{3}\right) < 0,\end{aligned}$$

whose solution is:

$$x_1(t) = x_1(0)e^{-\left(1-\frac{s+2a}{3}\right)t} + \frac{1}{2}\cdot$$

Furthermore, as the large firm loses market share, its incentives to price highly diminish, and competition becomes more intense. Hence, the average price in the market is *decreasing* over time. In detail, let $p(t) = p_1(x_1(t))x_1(t) + p_2(x_1(t))x_2(t)$ denote the average price charged in the market. After some algebra, it follows that:

$$\dot{p}(t) = \left[\frac{4}{3}(s-a)\left(x_1 - \frac{1}{2}\right)\right]\dot{x}_1 < 0\cdot$$

Note that in steady state, market shares become symmetric, as $\lim_{t\to\infty} x_1(t) = \frac{1}{2}$, and both firms' equilibrium prices converge to their lowest level,

$$\lim_{t\to\infty} p_i(t) = p^* = \frac{1}{2} + \frac{a}{2(1+\rho)} - \frac{s(1+\frac{a}{2})}{3(1+\rho)}\cdot$$

These results are summarized next:

Lemma 2 *In a dynamic setting:*

(i) Firms' market shares become more symmetric over time, and they become fully symmetric in steady state.
(ii) The average market price is decreasing *over time, and it is thus lowest in steady state.*

3.2 Comparative Dynamics

We end this section by performing comparative statics of equilibrium outcomes as switching costs s increase. We start by focusing on equilibrium prices charged by the two firms at a given point in time before reaching the steady state:

Lemma 3 *In the short-run:*

(i) An increase in s reduces the price charged by the small firm;
(ii) There exists $\widehat{x}_1 > 1/2$ such that an increase in s reduces the price charged by the large firm if and only if $x_1 < \widehat{x}_1$.
(iii) An increase in s enlarges the price differential.
(iv) There exists $\widetilde{x}_1 > \widehat{x}_1$ such that an increase in s reduces the average market price if and only if $x_1 < \widetilde{x}_1$.

Proof See the "Appendix 2". □

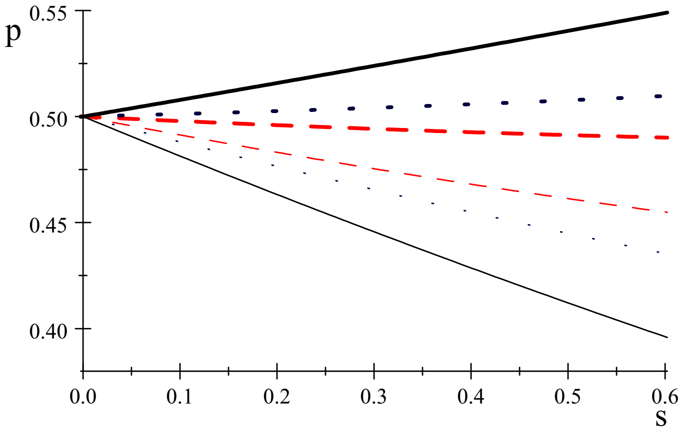

Fig. 2 Prices charged by the large firm (*thick lines*) and small firm (*thin lines*) as a function of the switching cost s, assuming $\rho = 5$ and $x_1 = 0.6$ (*dash*), $x_1 = 0.8$ (*dots*), $x_1 = 0.9$ (*solid*)

When switching costs increase, price choices reflect two countervailing incentives. An increase in s changes the harvesting and poaching effects, inducing the large firm to price less aggressively and the small firm to price more aggressively. However, in a dynamic setting, a higher s also changes the investing effect, as it implies a greater value of attracting customers in order to increase future profits.

For the small firm, all three effects point to the same direction. Accordingly, the price charged by the small firm unambiguously decreases in the switching cost parameter, s. In contrast, the large firm faces countervailing incentives as s increases. Since the incentives to charge higher prices today are greater the larger the firm's market share, there exists a critical market share $\widehat{x}_1$ below (above) which the investing (harvesting) effect dominates, so that the price charged by the large firm decreases (increases) in s.

As an illustration, Fig. 2 depicts the price charged by the two firms as a function of the switching cost parameter, s, for different values of the large firm's market share. As it can be seen, the price charged by the large firm decreases in s for low values of x_1 but increase in s for high values of x_1. In contrast, the price charged by the small firm is always decreasing in s. In all cases, the vertical distance between the prices charged by the two firms widens up as s goes up.

Let $p(t)$ denote the average price charged in the market. As s increases, the average price changes as follows:

$$\frac{\partial p\,(t)}{\partial s} = \frac{\partial\,(p_1 - p_2)}{\partial s} x_1 + (p_1 - p_2)\frac{\partial x_1}{\partial s} + \frac{\partial p_2}{\partial s}.$$

The first term is positive given that an increase in s enlarges the price differential. However, the second and third terms are negative. Hence, the sign of the effect of s on average prices is ambiguous. In particular, an increase in s leads to a reduction in the average price only when firms are sufficiently symmetric, i.e., if $x_1 < \widetilde{x}_1$. Note that $\widetilde{x}_1 > \widehat{x}_1$ as $x_1 < \widehat{x}_1$ is a sufficient condition for the average price to go down in s, as both firms' prices are decreasing in s (part (ii) of the Lemma). In sum, an increase in switching costs might be pro-competitive

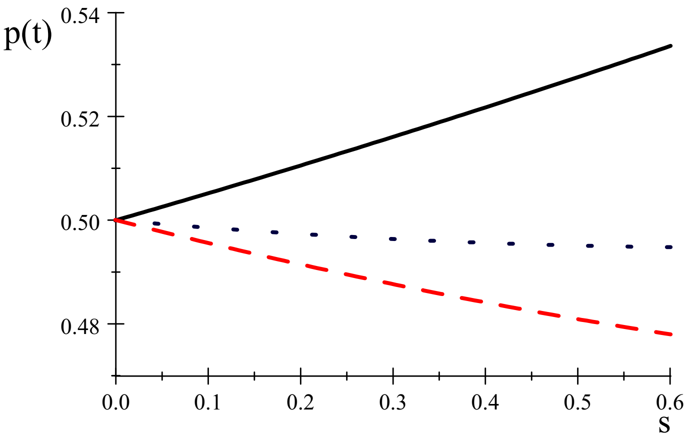

Fig. 3 Average price as a function of the switching cost s, assuming $\rho = 5$ and $x_1 = 0:6$ (*dash*), $x_1 = 0:8$ (*dots*), and $x_1 = 0.9$ (*solid*)

or anti-competitive depending on whether firms' market share are more or less symmetric. Figure 3 provides numerical support to this claim.

The fact that the price differential across firms goes up in s (part (iii) of the Lemma) implies that higher switching costs also slow down the transition to a symmetric market structure, and hence lead to a lower rate of decline in average prices. To see this more clearly, note that when $s = 0$ firms charge the static price, half of consumers switch, and therefore the large firm's market share erodes quickly. When $s > 0$, the large firm's market share also decreases over time, but not as fast. The reason is that although it looses some consumers, it keeps more than half among those who have the opportunity to switch because $\partial (p_1 - p_2) / \partial s < 1$. This can be seen as an anti-competitive effect of switching costs in the short-run, which arises regardless of the degree of market share asymmetries.

The following Lemma summarizes the effect of switching costs on the equilibrium dynamics.

Lemma 4 *An increase in switching costs s:*

(i) reduces the rate of decline of average prices and
(ii) delays the transition to the steady state.

Proof See the "Appendix 2". □

Nonetheless, in the long-run, switching costs are pro-competitive: the *higher* the switching cost, the *lower* the equilibrium price in steady state. Indeed, in steady state, once firms' market shares have become fully symmetric, only the investing effect plays a role. Hence, an increase in s, which increases the future value of current sales, makes competition fiercer and thus lowers equilibrium prices.

Lemma 5 *In steady state, an increase in switching costs reduces prices.*

Proof See the "Appendix 2". □

4 Extensions

4.1 Price Discrimination

So far, we have assumed that firms cannot price-discriminate between old and new consumers. In this section, we explore whether the possibility of price discriminating between old and new consumers alters our previous results.[10]

We keep all our previous assumptions, but allow firm i to set a price p_{ii} to its current consumers and (possibly) a different price p_{ji} to consumers previously served by firm j. Accordingly, the revenue accrued in the infinitesimal time interval $(t, t + \mathrm{d}t]$ can be written as

$$\pi_i(t) = p_{ii}x_i(t) - p_{ii}\,(1 - q_{ii})\,x_i(t) + p_{ji}q_{ji}x_j(t) \tag{8}$$

where the first two terms represent the revenue from the old customers who do not switch to the rival, and the third term represents the revenue from new customers.

The switching probabilities are analogous to those in the no-discrimination case. In particular, the probability that a customer served by firm j switches to firm i, q_{ji}, is now given by

$$q_{ji} = \frac{1-s}{2} - p_{ji} + p_{jj}$$

where we assume $p_{ji} - p_{jj} \in \left[-\frac{1}{2}(1+s), \frac{1}{2}(1-s)\right]$. Conversely, the probability q_{ii} that a customer already served by firm i maintains this relationship is now given by,

$$q_{ii} = \frac{1+s}{2} - p_{ii} + p_{ij}$$

where we assume $p_{ii} - p_{ij} \in \left[-\frac{1}{2}(1-s), \frac{1}{2}(1+s)\right]$.

Substituting q_{ji} and q_{ii} into (1) and taking the limit, as $\mathrm{d}t \to 0$ we have:[11]

$$\dot{x}_i(t) = -x_i(t)(1 - s + p_{ii} - p_{ji} + p_{jj} - p_{ij}) + \frac{1-s}{2} - p_{ji} + p_{jj}.$$

Similarly, substituting q_{ji} and q_{ii} into (8), and using condition $x_1(t) + x_2(t) = 1$, we obtain the rate at which revenue is accrued by firms 1 and 2,

$$\begin{aligned}
\pi_1(t) &= p_{21}\left(\frac{1-s}{2} - p_{21} + p_{22}\right)\\
&\quad + \left[p_{11}\left(\frac{1+s}{2} - p_{11} + p_{12}\right) - p_{21}\left(\frac{1-s}{2} - p_{21} + p_{22}\right)\right]x_1(t)\\
\pi_2(t) &= p_{22}\left(\frac{1+s}{2} - p_{22} + p_{21}\right)\\
&\quad + \left[p_{12}\left(\frac{1-s}{2} - p_{12} + p_{11}\right) - p_{22}\left(\frac{1+s}{2} - p_{22} + p_{21}\right)\right]x_1(t)
\end{aligned}$$

We are now ready to characterize equilibrium pricing under price discrimination.

10 [4] also considers the case when the seller is able to discriminate between locked-in and not locked-in consumers.

11 Note that in the no-discrimination case, $p_{ii} = p_{ji}$ and $p_{jj} = p_{ij}$. Hence, the price terms in parenthesis canceled out in that case.

Proposition 2 *The unique Markov Perfect Equilibrium in affine pricing strategies is:*[12]

$$p_{11} = p_{22} = \frac{1}{2} - \frac{s}{3}\left(\frac{1}{1+\rho-\frac{2s}{3}} - \frac{1}{2}\right)$$

$$p_{21} = p_{12} = \frac{1}{2} - \frac{s}{3}\left(\frac{1}{1+\rho-\frac{2s}{3}} + \frac{1}{2}\right).$$

Proof See the "Appendix 2". □

In the no price discrimination case, the trade-off between the "harvesting" and the "investment" effects led the large firm to charge higher prices than the small one. In contrast, under price discrimination, such a trade-off no longer exists as firms can set higher prices for the old consumers and lower prices for the new ones.[13] As a consequence, equilibrium prices do not depend on market shares, as can be seen in Proposition 2.[14]

Intuitively, one would expect bigger firms to charge higher prices to old customers than smaller firms, just as monopolists charge higher prices in bigger markets. However, there is a countervailing effect: The bigger the firm's market share, the smaller its rival's, and hence the more aggressive the latter becomes. Therefore, while a large firm has strong incentives to charge higher prices, it also faces a tougher rival. As it turns out, the two effects cancel each other, implying that prices do not depend on firms' market shares.

Therefore, in this context, market shares are inconsequential. Still, it is interesting to point out a robust effect of switching costs on market share dynamics. Regardless of whether it is possible to price-discriminate or not, symmetric switching costs imply that market shares converge to full symmetry over time. Indeed, along the equilibrium path in the price discrimination case,

$$\dot{x}_1(t) = -\left(1 - \frac{s}{3}\right)\left(x_1(t) - \frac{1}{2}\right)$$

so that $x_1(t) \to \frac{1}{2}$ as $t \to \infty$.

For the new consumers, switching costs foster more competitive outcomes. Indeed, the higher the switching cost, the lower the price for the new consumers. The reason is twofold: First, competition to attract new consumers becomes fiercer as higher switching costs make them more valuable for the future, and second, new consumers have to be compensated with lower prices as switching becomes more costly.

The impact of higher switching costs on the price charged to old consumers depends on the interplay between two countervailing effects. On the one hand, the higher the switching cost, the higher the price that the firm can charge without inducing its current consumers to switch. On the other hand, the higher the switching cost, the lower the prices these consumers can get if they switch to the rival. For $\rho < 1$, the latter effect dominates, so that an increase in switching costs leads to lower prices for the old consumers. Therefore, the overall effect of switching costs is pro-competitive regardless of market structure.

These results are summarized below:

Lemma 6 *If $\rho < 1$, under price discrimination, an increase in switching costs s reduces prices for all consumers.*

---

[12] Similarly to the no price discrimination case (Proposition 1), as $s \to 0$, we retrieve the prices of the static game for both types of consumers, $\lim_{s\to 0} p_{11}^* = \lim_{s\to 0} p_{21}^* = \frac{1}{2}$.

[13] Indeed, the price discount offered to new consumers is constant at $s/3$.

[14] Mathematically, this arises since the co-state variable $\lambda_1 = \frac{\partial V_1}{\partial x_1}$ is constant.

Proof See the "Appendix 2." □

This result relies on the assumption that switching opportunities arise according to a Poisson process with unit rate. Lower rates for switching imply the firm with dominant market share is less exposed to the possibility of losing customers. In other words, with a lower rate for switching opportunities, there are limited inter-temporal effects to current pricing decisions and (as in the static game) an increase in switching cost might not be pro-competitive.

4.2 Asymmetric Switching Costs

In the basic model, switching costs were symmetric across firms. We now allow switching costs to be asymmetric. In particular, we assume that only customers switching from firm 1 to firm 2 experience a cost, whereas customers switching from firm 1 do not face any cost. Following a similar logic as in the basic model, the model specification would now be the following:

$$q_{21} = \frac{1}{2} - (p_1 - p_2) \quad q_{11} = \frac{1}{2}(1+s) - (p_1 - p_2)$$

where $p_1 - p_2 \in \left[-\frac{1}{2}(1-s), \frac{1}{2}(1+s)\right]$ so that q_{21} and q_{11} belong to (0, 1). Following the same steps as in the case with symmetric costs, we can derive the results for the asymmetric switching costs case.

We reconstruct our differential game model so that the difference equation (1) is now

$$\frac{x_1(t+\mathrm{d}t) - x_1(t)}{\mathrm{d}t} = \frac{1}{2} - p_1 + p_2 - x_1(t)\left(1 - \frac{s}{2}\right).$$

In the limit as $\mathrm{d}t \to 0$, we obtain:

$$\dot{x}_1 = -x_1(t)\left(1 - \frac{s}{2}\right) + \frac{1}{2} - p_1 + p_2.$$

Substituting in (2), we obtain the instantaneous rate at which revenue is accrued by firms 1 and 2,

$$\pi_1(t) = p_1\left(x_1(t)\frac{s}{2} + \frac{1}{2} - p_1 + p_2\right)$$
$$\pi_2(t) = p_1\left(-x_1(t)\frac{s}{2} + \frac{1}{2} - p_2 + p_1\right).$$

In the following result, we revisit the structure of dynamic equilibrium pricing policies for the case of asymmetric switching costs:

Proposition 3 *The unique Markov Perfect Equilibrium in affine pricing strategies is:*

$$p_1(x_1) = \frac{1}{3}\left(\frac{s}{2} - a\right)x_1 + p_1^*$$
$$p_2(x_1) = -\frac{1}{3}\left(\frac{s}{2} - a\right)x_1 + p_2^*,$$

where $a \in \left(0, \frac{s}{2}\right)$ *is the smallest root of the quadratic equation*

$$2a^2 - 3\left(2 + \rho - \frac{7}{18}s\right)a + \frac{s^2}{6} = 0,$$

and

$$p_1^* = \frac{1}{2} - \frac{1}{2}\left(\frac{s}{3(1+\rho)} + \frac{a}{1+\rho-\frac{2}{3}\left(a+\frac{s}{3}\right)}\right),$$
$$p_2^* = \frac{1}{2} - \frac{1}{2}\left(\frac{s}{3(1+\rho)} - \frac{a}{1+\rho-\frac{2}{3}\left(a+\frac{s}{3}\right)}\right)$$

are a Markov Perfect Equilibrium.

Proof See the "Appendix 2". □

In the dynamic equilibrium, firm 1 prices less aggressively regardless of whether it is large or not, i.e., regardless of whether $x_1 > \frac{1}{2}$ or $x_1 < \frac{1}{2}$. In this case, the equilibrium dynamics are:

$$\begin{aligned}\dot{x}_1 &= \left(\frac{s}{2}-1\right)x_1 + \frac{1}{2} - \frac{2}{3}\left(\frac{s}{2}-a\right)x_1 - p_1^* + p_2^* \\ &= -\left(1-\frac{1}{3}\left(\frac{s}{2}+2a\right)\right)x_1 + \frac{1}{2} + \frac{a}{1+\rho-\frac{2}{3}\left(a+\frac{s}{3}\right)}.\end{aligned}$$

The solution is

$$x_1(t) = (x_1(0) - x_1(\infty))e^{-(1-\frac{1}{3}(\frac{s}{2}+2a))t} + x_1(\infty),$$

which, in steady state becomes,

$$x_1(\infty) = \lim_{t\to\infty} x_1(t) = \frac{\frac{1}{2} + \frac{a}{1+\rho-\frac{2}{3}(a+\frac{s}{3})}}{1-\frac{1}{3}\left(\frac{s}{2}+2a\right)} > \frac{1}{2}.$$

Note that

$$\lim_{t\to\infty}[p_1(x_1(t)) - p_2(x_1(t))] = \frac{2}{3}\left(\frac{s}{2}-a\right)x_1(\infty) - \frac{a}{1+\rho-\frac{2}{3}(a+\frac{s}{3})} > 0.$$

These two expressions show that, in the long-run, the asymmetric structure of switching costs allows the firm from which it is more costly to switch (i.e., firm 1) to remain dominant despite charging higher prices. To see why this is the case, recall that switching in both directions is *always* possible. Hence, once customers switch to firm 1 they are in a certain sense "locked-in," implying that firms 2 can lose market share despite charging lower prices. The degree of asymmetry in the long-run market structure is increasing in the magnitude of the switching cost, which in turn leads to higher steady-state prices.

4.3 Strategic Choice of Switching Costs

Building on the previous analysis, we now consider a situation in which firms can determine whether or not to impose a switching cost s on their costumer base prior to choosing prices. Proposition 3 suggests that asymmetric switching costs might have anti-competitive effects by creating or reinforcing firm dominance. However, for ex-ante symmetric firms and sufficiently high discount factors, we show that the unique equilibrium of the switching costs choice game involves *both* firms imposing switching costs on their customers.

This results derives from a prisoner's dilemma incentives structure, which is reminiscent of several results in industrial organization, e.g., the impact of forward contracting on spot market outcomes [1]. More specifically, starting from the no switching costs case, each firm has unilateral incentives to impose switching costs on its customers in order to be able to charge higher prices. However, this cannot constitute an equilibrium as the firm from which it is *not* costly to switch would also have incentives to impose a switching cost in order to gain market share. Therefore, even if firms are better off in the no switching costs case than in the symmetric switching costs case, the former cannot be sustained in equilibrium.

To see this formally, assume that firms are ex-ante symmetric, i.e., they evenly share the market in the first period, $x_1(0) = x_2(0) = \frac{1}{2}$ and consider the following game:

	Switching Cost	No Switching Cost
Switching Cost	$(V_1(s,s);\, V_2(s,s))$	$(V_1(s,0);\, V_2(s,0))$
No Switching Cost	$(V_1(0,s);\, V_2(0,s))$	$(V_1(0,0);\, V_2(0,0))$

The next proposition characterizes the unique equilibrium of this game:

Proposition 4 *There exists* $\bar{\rho} > 0$ *such that for all* $\rho < \bar{\rho}$, (s, s) *constitutes the unique equilibrium of the switching costs choice game with ex-ante symmetric firms.*

To prove the above result, let $p_i^{s,s}$ and $p_i^{s,0}$, respectively, denote the equilibrium pricing strategies for firm $i \in \{1, 2\}$ under symmetric switching costs (Proposition 1) and asymmetric switching costs (Proposition 2), when it is costly to switch from firm 1 but not from firm 2. Similarly, let us denote $a^{s,s}$ and $a^{s,0}$ the solutions of the quadratic equations in Propositions 1 and 2. After tedious algebra,[15] it follows that

$$\lim_{t\to\infty} [p_2^{s,s}(t)x_2^{s,s}(t) - p_2^{s,0}(t)x_2^{s,0}(t)] > 0.$$

Essentially, the effect on revenues due to the price decrease is smaller than the effect on revenues due to the market share gain. In steady state, this revenue difference implies that the difference in value functions is also positive,

$$V_2(s,s) - V_2(s,0) = \int_0^\infty \mathrm{e}^{-\rho t}[p_2^{s,s}(t)x_2^{s,s}(t) - p_2^{s,0}(t)x_2^{s,0}(t)]\mathrm{d}t > 0$$

for $\rho < \bar{\rho}$. In words, as long as the discount factor is sufficiently high so that the steady-state profits matter, firm 2' best response to firm 1 investing in switching costs s is to also invest in s. This rules out the existence of an asymmetric equilibrium. Moreover,

$$\begin{aligned} V_1(s,0) - V_1(0,0) &= \int_0^\infty \mathrm{e}^{-\rho t}\left[p_1^{s,0}(t)x_1^{s,0}(t) - p_1^{0,0}(t)x_1^{0,0}(t)\right]\mathrm{d}t \\ &= \int_0^\infty \mathrm{e}^{-\rho t}\left[p_1^{s,0}(t)x_1^{s,0}(t) - \frac{1}{4}\right]\mathrm{d}t > 0, \end{aligned}$$

since $x_1^{s,0}(t) - \frac{1}{2} > \frac{1}{2} - p_1^{s,0}(t) > 0$ for $t > 0$. Thus, imposing a switching cost is a dominant strategy. This proves the existence of a unique Nash equilibrium, the one with symmetric switching costs. Furthermore, note that this equilibrium is Pareto dominated by the case with no switching costs as,

$$V_i(0,0) - V_i(s,s) = \int_0^\infty \mathrm{e}^{-\rho t}\left[p_i^{0,0}(t) - p_i^{s,s}(t)\right]\frac{1}{2}\mathrm{d}t > 0.$$

15 Available from the authors upon request.

5 Conclusions

Many information technology products and technologies exhibit switching costs (i.e., costs that customers must bear when they adopt a new product or technology). Typically, switching costs arise when there is limited compatibility between an old product (or technology) and a newly adopted one. In this paper, we have analyzed the effect of switching costs on the nature of dynamic price competition.

We have shown that symmetric switching costs can be pro-competitive when the magnitude of switching costs is not too high and firms' market shares are not too asymmetric. In a Markov Perfect equilibrium, the dominant firm concedes market share by charging higher prices to current customers in the short-run. As the market structure becomes more symmetric, price competition becomes fiercer. The average price charged in the market is decreasing over time and in the long-run equilibrium prices are decreasing in the magnitude of switching costs.

However, in the short-run, switching costs have an ambiguous effect on market prices. When market shares are sufficiently asymmetric, an increase in switching costs implies that the large firm behaves less aggressively in order to exploit its customer base. In other words, the harvesting effect dominates, and switching costs lead to higher prices. It is only when firms' market shares become sufficiently symmetric over time that the investing effect dominates, thus leading to lower prices in markets with higher switching costs.

The analysis of the case with asymmetric switching costs suggests that, in the long-run, the asymmetric structure of switching costs allows the firm from which it is more costly to switch to remain dominant while charging higher prices. Since there is no convergence to full market share symmetry, switching costs are anti-competitive even in the steady state.

However, we have shown that for ex-ante symmetric firms, the asymmetric switching costs configuration does not constitute an equilibrium since the firm from which it is not costly to switch would optimally invest in switching costs. Indeed, the unique equilibrium of the switching costs choice game has both firms choosing symmetric costs, even when these make the market more competitive. This may explain the prevalence of certain business practices such as loyalty cards and frequent flyer programs by all firms in a given industry.

Our model also helps to explain the prevalence of other business practices such as introductory pricing, which is particularly widespread in markets with switching costs. Allowing for price discrimination between old and new consumers, we show that those consumers who decide to switch into a new firm are given a price discount that is increasing in the value of switching costs. Furthermore, for reasonable values of the discount factor, an increase in switching costs reduces prices for all consumers, regardless of market structure.

In sum, the presumption that switching costs are anti-competitive is misplaced as a general statement. There are several instances in which switching costs should indeed raise competitive concerns. For instance, this should be the case in concentrated markets in which firms cannot condition their prices on past purchasing decisions, or in markets in which one firm enjoys a competitive advantage as consumers find it relatively cheaper to switch to it. However, since there are many other instances in which switching costs foster more competitive outcomes, e.g., when price discrimination is possible, the assessment of the competitive effects of switching costs should be carried out on a case-by-case basis.

Appendix 1: Forward-Looking Consumers

In this appendix, we show that the pro-competitive effect of switching costs on steady-state prices found in the basic model of Sect. 2 is robust to allowing for more forward-looking consumers who aim to maximize their total discounted consumption surplus over the infinite horizon. Hence, they may abstain from switching when they correctly anticipate price increases in the future. In our extended model of switching, we assume that consumers are able to correctly anticipate infinite price trajectories when deciding whether or not to switch. However, we only consider the open-loop solution to the stochastic optimization problem in which a consumer has to decide whether or not to switch taking into account future opportunities to switch (at random times) subject to (random) switching costs. For simplicity, we only focus on the case of (exogenously given) symmetric switching costs.

Let us denote by $p_i^e(\tau)$ consumers' anticipation of firm i's price at time $\tau > t$. Given current prices, $p_i(t)$, consumers assume

$$p_1^e(\tau) = \alpha p_1(t)\mathrm{e}^{-\gamma(\tau-t)} + \beta \tag{9}$$

$$p_2^e(\tau) = \alpha p_2(t)\mathrm{e}^{-\gamma(\tau-t)} + \beta, \tag{10}$$

where $\alpha, \beta, \gamma > 0$. Note that consumers anticipate that price differences will gradually decrease over time. We assume all consumers use a normalized discount rate.

When an opportunity to switch arises at time $t > 0$, a consumer currently served by firm j would opt for firm i provided that

$$\int_t^\infty (v - p_i^e(\tau))\mathrm{e}^{-\tau}\mathrm{d}\tau - \tilde{s} > \int_t^\infty (v - p_j^e(\tau))\mathrm{e}^{-\tau}\mathrm{d}\tau.$$

Thus, the probability $\bar{q}_{ji}(t)$ that a customer served by firm j switches to firm i at time $t > 0$ is given by

$$\bar{q}_{ji}(t) = \Pr\left(\int_t^\infty (v - p_i^e(\tau))\mathrm{e}^{-\tau}\mathrm{d}\tau - \tilde{s} > \int_t^\infty (v - p_j^e(\tau))\mathrm{e}^{-\tau}\mathrm{d}\tau\right).$$

With some algebra,

$$\bar{q}_{ji}(t) = \frac{1}{2}(1-s) - \frac{\alpha}{1+\gamma}\left(p_i(t) - p_j(t)\right),$$

assuming that $p_i(t) - p_j(t) \in \left[-\frac{1+\gamma}{\alpha}(1-s), \frac{1+\gamma}{\alpha}(1+s)\right]$. Conversely, the probability that a customer already served by firm i maintains this relationship at time $t > 0$ is given by

$$\begin{aligned}\bar{q}_{ii}(t) &= \Pr\left(\tilde{s} > \int_t^\infty (p_i^e(\tau) - p_j^e(\tau))\mathrm{e}^{-(\tau-t)}\mathrm{d}\tau\right)\\ &= \frac{1}{2}(1+s) - \frac{\alpha}{1+\gamma}\left(p_i(t) - p_j(t)\right).\end{aligned}$$

Note that when $\frac{\alpha}{1+\gamma} < 1$, forward-looking consumers are *less* sensitive to current price differences than myopic consumers.

By comparing switching probabilities with myopic and with forward-looking consumers, it can be seen that equilibrium prices $\bar{p}_i(x_1)$ in the latter case can be obtained by a simple rescaling of the equilibrium prices in the former, i.e., $\bar{p}_i(x_1) = \frac{1+\gamma}{\alpha} p_i(x_1)\cdot$ Therefore, the price dynamics with forward-looking consumers remained unaltered as compared to the myopic consumers case. In other words, the pro-competitive effect of switching costs on steady- state prices survives the presence of more sophisticated consumers. Still, the pro-competitive effect is slightly attenuated because forward-looking consumers are less sensitive to current price differences.

For completeness, let us stress that consumers' anticipation of price differences over time is correct provided that

$$p_1^e(t) - p_2^e(t) = \frac{1+\gamma}{\alpha}\left(p_1(t) - p_2(t)\right),$$

or equivalently, using expressions (9), (10) and Proposition 1,

$$\alpha \mathrm{e}^{-\gamma t} = \frac{1+\gamma}{\alpha}\frac{2}{3}\left(s-a\right) x_1(0)\mathrm{e}^{-\left(1-\frac{s+2a}{3}\right)t}.$$

For this to be the case, α and must take the following values,

$$\gamma = 1 - \tfrac{s+2a}{3} \quad \text{and} \quad \alpha = \sqrt{2\left(1-\tfrac{s+2a}{3}\right)\left(\tfrac{s-a}{3}\right)x_1(0)}.$$

Appendix 2: Proofs of Lemmas and Propositions

Basic Model

Proof of Proposition 1

The Hamiltonians are

$$\mathcal{H}_i = \mathrm{e}^{-\rho t}[\pi_i + \lambda_i \dot{x}_1],$$

for $i = 1, 2$. The Hamiltonians are strictly concave so that first order conditions for MPE are also sufficient (see [5]),

$$\frac{\partial \mathcal{H}_i}{\partial p_i} = 0$$
$$-\frac{\partial \mathcal{H}_i}{\partial x_1} - \frac{\partial \mathcal{H}_i}{\partial p_j}\frac{\partial p_j}{\partial x_1} = \dot{\lambda}_i - \rho\lambda_i,$$

for $i = 1, 2$. These, respectively, lead to:

$$p_1 = \frac{1}{2}\left(p_2 + s\left(x_1 - \frac{1}{2}\right) + \frac{1}{2} - \lambda_1\right) \tag{11}$$

$$-sp_1 + (1-s)\lambda_1 - (p_1 + \lambda_1)\frac{\partial p_2}{\partial x_1} = \dot{\lambda}_1 - \rho\lambda_1 \tag{12}$$

$$p_2 = \frac{1}{2}\left(p_1 - s\left(x_1 - \frac{1}{2}\right) + \frac{1}{2} + \lambda_2\right) \tag{13}$$

$$sp_2 + (1-s)\lambda_2 - (p_2 - \lambda_2)\frac{\partial p_1}{\partial x_1} = \dot{\lambda}_2 - \rho\lambda_2 \tag{14}$$

Equations (11) and (13) are firms' best reply functions. Using them we can obtain equilibrium prices,

$$p_1 = \frac{s}{3}\left(x_1 - \frac{1}{2}\right) + \frac{1}{2} + \frac{1}{3}(\lambda_2 - 2\lambda_1)$$
$$p_2 = -\frac{s}{3}\left(x_1 - \frac{1}{2}\right) + \frac{1}{2} + \frac{1}{3}(2\lambda_2 - \lambda_1)$$

Thus,

$$\tfrac{\partial p_2}{\partial x_1} = -\tfrac{s}{3} \quad \tfrac{\partial p_1}{\partial x_1} = \tfrac{s}{3}$$

Substituting into (12) and (14) we obtain

$$\frac{2s}{3}p_2 + \left(1 - \frac{2s}{3}\right)\lambda_2 = \dot{\lambda}_2 - \rho\lambda_2$$
$$-\frac{2s}{3}p_1 + \left(1 - \frac{2s}{3}\right)\lambda_1 = \dot{\lambda}_1 - \rho\lambda_1$$

We solve this system of differential equations by the method of undetermined coefficients. Assume $\lambda_i = a_i(x_1 - \frac{1}{2}) + b_i$ for $i = 1, 2$. Substitution into the last equation yields

$$\begin{aligned}
&-\frac{2s}{3}\left(\frac{s}{3}\left(x_1 - \frac{1}{2}\right) + \frac{1}{2} + \frac{1}{3}\left(a_2(x_1 - \frac{1}{2}\right) + b_2 - 2a_1\left(x_1 - \frac{1}{2}\right) - 2b_1)\right)\\
&\quad + \left(1 - \frac{2s}{3}\right)\left(a_1\left(x_1 - \frac{1}{2}\right) + b_1\right) = a_1\dot{x}_1 - \rho\left(a_1\left(x_1 - \frac{1}{2}\right) + b_1\right)\\
&= a_1\left(-x_1(1-s) + \frac{1-s}{2} - p_1 + p_2\right) - \rho a_1\left(x_1 - \frac{1}{2}\right) - \rho b_1\\
&= a_1\left(-x_1(1-s) + \frac{1-s}{2} - \frac{2s}{3}(x_1 - \frac{1}{2}) + \frac{\lambda_1 + \lambda_2}{3}\right) - \rho a_1\left(x_1 - \frac{1}{2}\right) - \rho b_1\\
&= a_1\left(-(1-s)\left(x_1 - \frac{1}{2}\right) - \frac{2s}{3}\left(x_1 - \frac{1}{2}\right) + \frac{a_1\left(x_1 - \frac{1}{2}\right) + b_1 + a_2\left(x_1 - \frac{1}{2}\right) + b_2}{3}\right)\\
&\quad - \rho a_1\left(x_1 - \frac{1}{2}\right) - \rho b_1
\end{aligned}$$

This results in the following two equations:

$$-\frac{2}{9}s^2 + \frac{2}{9}s(2a_1 - a_2) + \left(1 - \frac{2s}{3}\right)a_1 = -\left(1 - \frac{s}{3}\right)a_1 + \frac{1}{3}(a_1 + a_2)a_1 - \rho a_1 \quad (15)$$
$$-\frac{s}{3} - \frac{2s}{9}(b_2 - 2b_1) + b_1\left(1 - \frac{2s}{3}\right) = \frac{1}{3}(b_1 + b_2)a_1 + \frac{a_1}{2}\left(1 - \frac{s}{3}\right) - \rho b_1 \quad (16)$$

In a similar fashion,

$$\frac{2s}{3}\left(-\frac{s}{3}\left(x_1-\frac{1}{2}\right)+\frac{1}{2}+\frac{1}{3}(2a_2\left(x_1-\frac{1}{2}\right)+2b_2-a_1\left(x_1-\frac{1}{2}\right)-b_1)\right)$$
$$+(1-\frac{2s}{3})(a_2\left(x_1-\frac{1}{2}\right)+b_2)$$
$$=a_2\dot{x}_1-\rho(a_2\left(x_1-\frac{1}{2}\right)+b_2)$$
$$=a_2\left(-x_1(1-s)+\frac{1-s}{2}-\frac{2s}{3}\left(x_1-\frac{1}{2}\right)+\frac{a_1\left(x_1-\frac{1}{2}\right)+b_1+a_2\left(x_1-\frac{1}{2}\right)+b_2}{3}\right)$$
$$-\rho a_2\left(x_1-\frac{1}{2}\right)-\rho b_2$$
$$=a_2\left(-(1-s)\left(x_1-\frac{1}{2}\right)-\frac{2s}{3}(x_1-\frac{1}{2})+\frac{a_1\left(x_1-\frac{1}{2}\right)+b_1+a_2\left(x_1-\frac{1}{2}\right)+b_2}{3}\right)$$
$$-\rho a_2\left(x_1-\frac{1}{2}\right)-\rho b_2$$

We obtain two additional equations:

$$-\frac{2}{9}s^2+\frac{2}{9}s(2a_2-a_1)+\left(1-\frac{2s}{3}\right)a_2=-\left(1-\frac{s}{3}\right)a_2+\frac{1}{3}(a_1+a_2)a_2-\rho a_2 \quad (17)$$
$$\frac{s}{3}+\frac{2s}{9}(2b_2-b_1)+b_2\left(1-\frac{2s}{3}\right)=\frac{1}{3}(b_1+b_2)a_2+\frac{a_2}{2}\left(1-\frac{s}{3}\right)-\rho b_2 \quad (18)$$

Thus, substracting (15) from (17) we get:

$$\left[1-\frac{s}{3}-\frac{1}{3}(a_1+a_2)+\frac{2}{9}s+1-\frac{2s}{3}+\rho\right](a_1-a_2)=0$$

Hence, $a_1-a_2=0$. Let $a_1=a_2=a$, we solve the quadratic equation implicit in (15):

$$2a^2-3\left(2+\rho-\frac{7}{9}s\right)a+\frac{2}{3}s^2=0$$

Then (16) and (18) imply $b_1+b_2=0$ and

$$b_1=\frac{s(1+\frac{a}{2})}{3(1+\rho)}-\frac{a}{2(1+\rho)}$$

So the equilibrium strategies can be rewritten as

$$p_1=\frac{s}{3}\left(x_1-\frac{1}{2}\right)+\frac{1}{2}+\frac{1}{3}(\lambda_2-2\lambda_1)$$
$$p_2=-\frac{s}{3}\left(x_1-\frac{1}{2}\right)+\frac{1}{2}+\frac{1}{3}(2\lambda_2-\lambda_1)$$

Or equivalently,

$$p_1=\frac{s-a}{3}\left(x_1-\frac{1}{2}\right)+\frac{1}{2}+\frac{a}{2(1+\rho)}-\frac{s(1+\frac{a}{2})}{3(1+\rho)}$$
$$p_2=-\frac{s-a}{3}\left(x_1-\frac{1}{2}\right)+\frac{1}{2}+\frac{a}{2(1+\rho)}-\frac{s(1+\frac{a}{2})}{3(1+\rho)}$$

Finally, we note that given the assumption $s < \frac{3}{5}$ the pricing policies satisfy:

$$p_1(x_1) - p_2(x_1) = \frac{2}{3}(s-a)\left(x_1 - \frac{1}{2}\right) \in \left(-\frac{1-s}{2}, \frac{1-s}{2}\right)$$

so that $q_0, q_1 \in (0, 1)$.

Proof of Lemma 3

We first note that implicit differentiation in (3) yields:

$$\frac{\partial a}{\partial s} = \frac{4s + 7a}{9(2+\rho) - 7s - 12a} \in (0, 1)$$

(i) Using this result, it is straightforward to see that $\frac{\partial p_2}{\partial s} < 0$. Taking derivatives,

$$\begin{aligned}\frac{\partial p_2}{\partial s} = &-\frac{1}{3}\left(\left(1 - \frac{\partial a}{\partial s}\right)\left(x_1 - \frac{1}{2}\right) + \frac{\partial a}{\partial s}\right) \\ &- \frac{1}{3}\frac{1}{1+\rho}\left(\frac{2}{3}\left(\frac{s}{3} + \frac{a}{2}\right) + \left(1 - \frac{s}{3}\right)\left(\frac{1}{3} + \frac{1}{2}\frac{\partial a}{\partial s}\right)\right)\end{aligned}$$

(ii) Taking derivatives,

$$\begin{aligned}\frac{\partial p_1}{\partial s} = &\frac{1}{3}\left(\left(1 - \frac{\partial a}{\partial s}\right)x_1 - \frac{1}{2}\left(1 + \frac{\partial a}{\partial s}\right)\right) \\ &- \frac{1}{3}\frac{1}{1+\rho}\left(\frac{2}{3}\left(\frac{s}{3} + \frac{a}{2}\right) + \left(1 - \frac{s}{3}\right)\left(\frac{1}{3} + \frac{1}{2}\frac{\partial a}{\partial s}\right)\right)\end{aligned}$$

The second term is negative, while the sign of the first term cannot be determined in general. Solving for x_1, expression above is positive if and only if

$$x_1 > \widehat{x}_1 = \frac{1}{\left(1 - \frac{\partial a}{\partial s}\right)}\left(\frac{1}{1+\rho}\left(\frac{2}{3}\left(\frac{s}{3} + \frac{a}{2}\right) + \left(1 - \frac{s}{3}\right)\left(\frac{1}{3} + \frac{1}{2}\frac{\partial a}{\partial s}\right)\right) + \frac{1}{2}\left(1 + \frac{\partial a}{\partial s}\right)\right)$$

The fact that $\widehat{x}_1 > \frac{1}{2}$ follows since $\frac{\partial p_1}{\partial s}$ is weakly increasing in x_1 and $\frac{\partial p_1}{\partial s} < 0$ for $x_1 = \frac{1}{2}$, as the first term becomes $-\frac{\partial a}{\partial s} < 0$.

(iii) It follows from the fact that the price differential $p_1 - p_2$ is directly proportional to $s - a$ and, as shown above, $\frac{\partial a}{\partial s} < 1$.

(iv) The proof is provided in the main text.

Proof of Lemma 4

(i) It follows from the fact that $\dot{p}(t)$ is inversely proportional to $s - a$ and, as shown above, $\frac{\partial a}{\partial s} < 1$. (ii) The transition to the steady state occurs at a rate which is inversely proportional to $\frac{s+2a}{3}$, and as shown above, $\frac{\partial a}{\partial s} > 0$.

Proof of Lemma 5

Steady-state prices are

$$\lim_{t\to\infty} p_i(t) = p^* = \frac{1}{2} + \frac{a}{2(1+\rho)} - \frac{s(1+\frac{a}{2})}{3(1+\rho)}.$$

Taking derivatives w.r.t. s,

$$\frac{\partial p^*}{\partial s} = \frac{a}{2(1+\rho)}\frac{\partial a}{\partial s} - \frac{1+\frac{a}{2}}{3(1+\rho)} - \frac{s}{6(1+\rho)}\frac{\partial a}{\partial s} < 0,$$

where the inequality follows from $\frac{\partial a}{\partial s} \in (0, 1)$ and $a < \frac{s}{2}$.

Price Discrimination

Proof of Proposition 2

The Hamiltonians are:

$$\mathcal{H}_i = \mathrm{e}^{-\rho t}[\pi_i + \lambda_i \dot{x}_1]$$

for $i = 1, 2$. A necessary condition for optimality is:

$$\frac{\partial \pi_i}{\partial p_{ii}} = -\lambda_i \frac{\partial \dot{x}_1}{\partial p_{ii}} \frac{\partial \pi_i}{\partial p_{ji}} = -\lambda_i \frac{\partial \dot{x}_1}{\partial p_{ji}}$$

These conditions imply:

$$\frac{1+s}{2} - 2p_{11} + p_{12} - \lambda_1 = 0$$
$$\frac{1-s}{2} - 2p_{12} + p_{11} + \lambda_2 = 0$$
$$\frac{1-s}{2} - 2p_{21} + p_{22} - \lambda_1 = 0$$
$$\frac{1+s}{2} - 2p_{22} + p_{21} + \lambda_2 = 0$$

Note that these "instantaneous" best reply functions do not depend on x_1. Hence,

$$p_{11} = p_{22} = \tfrac{1}{2}\left(1 + \tfrac{s}{3}\right) + \tfrac{\lambda_2 - 2\lambda_1}{3}$$
$$p_{12} = p_{21} = \tfrac{1}{2}\left(1 - \tfrac{s}{3}\right) + \tfrac{2\lambda_2 - \lambda_1}{3}$$

The remaining conditions are:

$$-\frac{\partial \mathcal{H}_i}{\partial x_1} - \frac{\partial \mathcal{H}_i}{\partial p_{ij}}\frac{\partial p_{ij}}{\partial x_1} - \frac{\partial \mathcal{H}_i}{\partial p_{jj}}\frac{\partial p_{jj}}{\partial x_1} = \dot{\lambda}_i - \rho\lambda_i.$$

Since $\frac{\partial p_{ii}}{\partial x_1} = \frac{\partial p_{ij}}{\partial x_1} = 0$ we have

$$-p_{11}\left(\frac{1+s}{2} - p_{11} + p_{12}\right) + p_{21}\left(\frac{1-s}{2} - p_{21} + p_{22}\right)$$
$$+\lambda_1(1 - s + p_{11} - p_{12} - p_{21} + p_{22}) = \dot{\lambda}_1 - \rho\lambda_1$$

and

$$-p_{12}\left(\frac{1-s}{2} - p_{12} + p_{11}\right) + p_{22}\left(\frac{1}{2}(1+s) - p_{22} + p_{21}\right)$$
$$+\lambda_2(1 - s + p_{11} - p_{12} - p_{21} + p_{22}) = \dot{\lambda}_2 - \rho\lambda_2$$

We posit a solution of the form $\lambda_i = b_i, i = 1, 2$. Solving the equations above, we respectively get:

$$-\left(\frac{1}{2}+\frac{s}{6}+\frac{b_2-2b_1}{3}\right)\left(\frac{1}{2}+\frac{s}{6}+\frac{b_2+b_1}{3}\right)$$
$$+\left(\frac{1}{2}-\frac{s}{6}+\frac{2b_2-b_1}{3}\right)\left(\frac{1}{2}-\frac{s}{6}+\frac{b_2+b_1}{3}\right)+b_1\left(1-\frac{s}{3}\right)$$
$$=-b_1\rho$$
$$-\left(\frac{1}{2}-\frac{s}{6}+\frac{2b_2-b_1}{3}\right)\left(\frac{1}{2}-\frac{s}{6}+\frac{b_2+b_1}{3}\right)$$
$$+\left(\frac{1}{2}+\frac{s}{6}+\frac{b_2-2b_1}{3}\right)\left(\frac{1}{2}+\frac{s}{6}+\frac{b_2+b_1}{3}\right)+b_2\left(1-\frac{s}{3}\right)$$
$$=-b_2\rho$$

Adding these equations we obtain:

$$(b_1+b_2)\left(1+\rho-\frac{s}{3}\right)=0.$$

Thus, $b_1 = -b_2$. Plugging this into the FOCs above,

$$b_1=\frac{\frac{s}{3}}{1+\rho-\frac{2s}{3}}.$$

Thus, prices are

$$p_{11}=p_{22}=\frac{1}{2}-\frac{s}{3}\left(\frac{1}{1+\rho-\frac{2s}{3}}-\frac{1}{2}\right)$$
$$p_{12}=p_{21}=\frac{1}{2}-\frac{s}{3}\left(\frac{1}{1+\rho-\frac{2s}{3}}+\frac{1}{2}\right).$$

Note that the price discount to new consumers is:

$$p_{ii}-p_{ij}=\frac{s}{3}.$$

Condition $s < 1$ guarantees that $p_{ii} - p_{ij} \in \left[-\frac{1}{2}(1-s), \frac{1}{2}(1+s)\right]$ and $p_{ji} - p_{jj} \in \left[-\frac{1}{2}(1+s), \frac{1}{2}(1-s)\right]$, as we had previously assumed.

Proof of Lemma 6

The comparative statics of prices with respect to s is,

$$\frac{\partial p_{ij}}{\partial s}=-3\frac{1+\rho}{(2s-3\rho-3)^2}-\frac{1}{6}<0$$
$$\frac{\partial p_{ii}}{\partial s}=\frac{\partial p_{ij}}{\partial s}+\frac{1}{3}$$
$$=-3\frac{1+\rho}{(2s-3\rho-3)^2}+\frac{1}{6}<0$$

under the assumption $\rho < 1$.

Asymmetric Switching Costs

Proof of Proposition 3

As in the proof of Proposition 1, the Hamiltonians are

$$\mathcal{H}_i = \mathrm{e}^{-\rho t}[\pi_i + \lambda_i \dot{x}_1]$$

for $i = 1, 2$. First order conditions (which in this case due to concavity are also sufficient) are:

$$\frac{\partial \mathcal{H}_i}{\partial p_i} = 0$$
$$-\frac{\partial \mathcal{H}_i}{\partial x_1} - \frac{\partial \mathcal{H}_i}{\partial p_j}\frac{\partial p_j}{\partial x_1} = \dot{\lambda}_i - \rho\lambda_i$$

The first order conditions lead to:

$$p_1 = \frac{1}{2}\left(p_2 + \frac{s}{2}x_1 + \frac{1}{2} - \lambda_1\right) \tag{19}$$
$$-\frac{s}{2}p_1 + \left(1 - \frac{s}{2}\right)\lambda_1 - (p_1 + \lambda_1)\frac{\partial p_2}{\partial x_1} = \dot{\lambda}_1 - \rho\lambda_1 \tag{20}$$
$$p_2 = \frac{1}{2}\left(p_1 - \frac{s}{2}x_1 + \frac{1}{2} + \lambda_2\right) \tag{21}$$
$$\frac{s}{2}p_2 + (1 - \frac{s}{2})\lambda_2 - (p_2 - \lambda_2)\frac{\partial p_1}{\partial x_1} = \dot{\lambda}_2 - \rho\lambda_2 \tag{22}$$

Here, (19) and (21) imply that equilibrium prices are of the form:

$$p_1 = \frac{s}{6}x_1 + \frac{1}{2} + \frac{\lambda_2 - 2\lambda_1}{3}$$
$$p_2 = -\frac{s}{6}x_1 + \frac{1}{2} + \frac{2\lambda_2 - \lambda_1}{3}$$

Hence $\frac{\partial p_1}{\partial x_1} = -\frac{\partial p_2}{\partial x_1} = \frac{s}{6}$. Substituting into (20) and (22) we obtain

$$\frac{s}{3}p_2 + \left(1 - \frac{s}{3}\right)\lambda_2 = \dot{\lambda}_2 - \rho\lambda_2$$
$$-\frac{s}{3}p_1 + \left(1 - \frac{s}{3}\right)\lambda_1 = \dot{\lambda}_1 - \rho\lambda_1$$

We solve using method of undetermined coefficients. Assume $\lambda_i = a_i x_1 + b_i$ for $i = 1, 2$. Substitution into the last equation yields

$$-\frac{s}{3}\left(\frac{s}{6}x_1 + \frac{1}{2} + \frac{1}{3}(a_2x_1 + b_2 - 2a_1x_1 - 2b_1)\right) + \left(1 - \frac{s}{3}\right)(a_1x_1 + b_1)$$
$$= a_1\dot{x}_1 - \rho(a_1x_1 + b_1)$$
$$= a_1\left(-x_1\left(1 - \frac{s}{2}\right) + \frac{1}{2} - p_1 + p_2\right) - \rho a_1x_1 - \rho b_1$$
$$= a_1\left(-x_1\left(1 - \frac{s}{2}\right) + \frac{1}{2} - \frac{1s}{3}x_1 + \frac{\lambda_1 + \lambda_2}{3}\right) - \rho a_1x_1 - \rho b_1$$
$$= a_1\left(-x_1\left(1 - \frac{s}{2} + \frac{s}{3}\right) + \frac{1}{2} + \frac{a_1x_1 + b_1 + a_2x_1 + b_2}{3}\right) - \rho a_1x_1 - \rho b_1$$

This results in the following two equations:

$$-\frac{s^2}{18} - \frac{s}{9}(a_2 - 2a_1) + \left(1 - \frac{s}{3}\right) a_1 = -\left(1 - \frac{s}{6}\right) a_1 + \frac{1}{3}(a_1 + a_2)a_1 - \rho a_1 \quad (23)$$

$$-\frac{s}{6} - \frac{s}{9}(b_2 - 2b_1) + b_1\left(1 - \frac{s}{3}\right) = \frac{1}{3}(b_1 + b_2)a_1 + \frac{a_1}{2} - \rho b_1 \quad (24)$$

In a similar fashion, we obtain two additional equations:

$$-\frac{s^2}{18} - \frac{s}{9}(a_1 - 2a_2) + \left(1 - \frac{s}{3}\right) a_2 = -\left(1 - \frac{s}{6}\right) a_2 + \frac{1}{3}(a_1 + a_2)a_2 - \rho a_2 \quad (25)$$

$$\frac{s}{6} + \frac{s}{9}(2b_2 - b_1) + b_2\left(1 - \frac{s}{3}\right) = \frac{1}{3}(b_1 + b_2)a_2 + \frac{a_2}{2} - \rho b_2 \quad (26)$$

The symmetry of Eqs. (23) and (25) imply $a_1 = a_2 = a$ which is a solution to the quadratic equation:

$$g(a) = 2a^2 - 3\left(2 + \rho - \frac{7}{18}s\right) a + \frac{s^2}{6} = 0$$

By adding and substracting (24) and (26) we obtain

$$b_1 + b_2 = \frac{a}{1 + \rho - \frac{2}{3}\left(a + \frac{s}{3}\right)} b_1 - b_2 = \frac{s}{3(1 + \rho)}$$

It follows that

$$b_1 = \frac{1}{2}\left[\frac{s}{3(1 + \rho)} + \frac{a}{1 + \rho - \frac{2}{3}\left(a + \frac{s}{3}\right)}\right]$$

$$b_2 = \frac{1}{2}\left[-\frac{s}{3(1 + \rho)} + \frac{a}{1 + \rho - \frac{2}{3}\left(a + \frac{s}{3}\right)}\right]$$

The equilibrium pricing strategies can be written as:

$$p_1 = \frac{x_1}{3}\left(\frac{s}{2} - a\right) + \frac{1}{2} - \frac{1}{2}\left(\frac{s}{3(1 + \rho)} + \frac{a}{1 + \rho - \frac{2}{3}\left(a + \frac{s}{3}\right)}\right)$$

$$p_2 = -\frac{x_1}{3}\left(\frac{s}{2} - a\right) + \frac{1}{2} - \frac{1}{2}\left(\frac{s}{3(1 + \rho)} - \frac{a}{1 + \rho - \frac{2}{3}\left(a + \frac{s}{3}\right)}\right),$$

which concludes the proof.

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