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Oligopoly Pricing and Advertising in Isoelastic Adoption Models

Kurt Helmes · **Rainer Schlosser**

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Abstract This paper deals with deterministic dynamic pricing and advertising differential games which are stylized models of special durable-good oligopoly markets. We analyze infinite horizon models with constant price and advertising elasticities of demand in the cases of symmetric and asymmetric firms. In particular, we consider general saturation/adoption effects. These effects are modeled as transformations of the sum of the cumulative sales of all competing firms. We specify a necessary and sufficient condition such that a unique Markovian Nash equilibrium for such games exist. For two classes of models we derive solution formulas of the optimal policies and of the value functions, and we show how to compute the evolution of the cumulative sales of each firm. The analysis of these games reveals that the existence of the Nash equilibrium relies on the possibility to separate a component, which is specific for each firm, from a [market] component, which is the same for all firms. The common factor is a function of the decreasing untapped market size. The individual factor of each firm reflects its individual market power and has an impact on equilibrium prices; each such coefficient depends on the price elasticities, unit costs, arrival rates, and discount factors of all competing companies. Formulas for these coefficients reveal how equilibrium prices depend on the number of competing firms, and how the entry or exit of a firm affects the price structure of the oligopoly.

Keywords Dynamic pricing and advertising · Infinite horizon · Oligopoly competition · Constant demand elasticities · Deterministic differential games

1 Introduction

In survey articles, Mahajan et al. [15], [16], [17], and Peres et al. [20] have emphasized the importance of normative results for product growth models of durable-goods and nondurable ones in oligopolistic markets. Since the ’70s a number of models of advertising competition

K. Helmes (✉) · R. Schlosser
Institute of Operations Research, Humboldt University of Berlin, Spandauer Str. 1, 10178 Berlin , Germany
e-mail: helmes@wiwi.hu-berlin.de

have been proposed, and optimal advertising policies have been derived, see, for example, the series of papers by G. Erickson spanning almost thirty years, e. g. [4], [5] and references therein, the many papers by G. Fruchter, see [6] and [7] to cite but a few, and the various contributions by S. Sethi and co-authors, e. g. [21], [23] and [13]. The recent review article by Huang et al. [11], and the books by Dockner et al. [2] and by Jørgensen and Zaccour [12] provide rich bibliographies and excellent accounts of many of the accomplishments in this research area. The papers by Teng and Thompson [24] and by Dockner and Feichtinger [1] are classical papers which analyze models of dynamic advertising combined with dynamic pricing in a competitive environment. The papers [13], [23] and [4] are recent articles where optimal marketing-mix strategies in dynamic competitive markets are analyzed; for further references see the bibliographies of the articles and books referred to above. The article [13] by Krishnamoorthy et al. is most important for the present manuscript.

In [13], the authors analyze a deterministic dynamic duopoly game. They study a cumulative sales model with particular dynamics and objective function. The authors show how the competing firms should dynamically adjust their advertising spending and how they should [dynamically] set their prices. The specific model is an extension of the monopoly problem analyzed in [23]. The infinite horizon duopoly game takes into account the evolution of cumulative sales of two brands/firms of a product category. As far as the demand function is concerned, they analyze the case, when the demand depends linearly on p, and the case with constant price elasticity. In the case with constant price elasticity, the dynamics of each brand of their model is specified by, $i = 1, 2$,

$$\dot{x}_i(t) = u_i w_i(t) p_i(t)^{-\varepsilon_i} \sqrt{N - x_1(t) - x_2(t)},$$

where $x_i(t)$ denotes the cumulative sales of firm/brand i, N is the category potential, $w_i(t)$ denotes the advertising effort of firm i, u_i the effectiveness of its advertising activity, and $p_i(t)$ the price it sets for one unit of its product. The parameter ε_i, $i = 1, 2$, is the price elasticity of demand of product i. The factor $\sqrt{N - x_1(t) - x_2(t)}$ describes a particular friction (inertia) of the market.

Each firm chooses its advertising and price to maximize its discounted infinite horizon profit, where the profit rate is $e^{-r_i t}((p_i(t) - c_i)\dot{x}_i(t) - k_i w_i(t)^2)$, and r_i is the discount rate of firm i, c_i is the marginal cost of production of firm i, $k_i w_i(t)^2$ is the cost of firm's i advertising, and k_i is a (positive) factor of proportionality.

If marginal costs are positive, Krishnamoorthy et al. determine the feedback Nash equilibrium strategies of both players and they derive analytical expressions of their optimal advertising and pricing policies. This particular duopoly game suggests several interesting research questions. For example, even in the case of a duopoly it is not clear whether or not a Nash equilibrium exists if the marginal cost of one of the players is zero. Thus, the question of existence and uniqueness of an equilibrium in the case of many firms, where some firms have marginal cost zero, is a natural one. Furthermore, for asymmetric market situations it is not at all clear how the entry or exit of a firm to the market will affect the price equilibrium. Specifically, the impact of the number of competing firms on equilibrium prices, should such prices exist, is most important for practical applications. Moreover, even for the duopoly problem considered by Krishnamoorthy et al. it is not obvious how the optimal market share of each firm develops over time. These questions and related ones have motivated our research and will be answered in Sects. 3 and 4.

In this paper, we extend the model for the particular duopoly game to two [more general] classes of differential games with (i) any finite number of heterogeneous firms and (ii) more general dynamics; further details will be given below. The basic setting of both classes of extended dynamic games is closely related to the monopoly model proposed in [23], which was generalized in [10]. In the case of an oligopoly, it is assumed that each firm is selling its brand of a [category] durable-good and is facing a [constant] brand-specific elasticity of demand. The rate of sales of each firm is postulated to be multiplicative in its price, in [its] advertising effort—a power expression—and involves two additional terms: a [constant] firm-specific arrival intensity factor and a factor which depends on the total sales of all brands. The latter, which is the same for all firms, reflects the way the companies compete in the market. A general system function makes it possible to capture varied adoption and saturation effects and allows the modeling of different [direct] network externalities.

The models of both classes are primarily motivated by marketing applications, for example, the evolution of the market share of the premium car segment of each of the three main competitors in Germany. However, the models are also related to dynamic oligopoly games studied in the context of extracting and pricing natural resources, e. g. petroleum, copper, etc., see [8], [14] and references therein. The brands/firms within a product category should be equated with different crude-oil producing countries/regions, or types of petroleum characterized by their specifications, e. g. West Texas Intermediate and Brent Blend (low sulfur crudes), or Oman Crude (high sulfur content); the size of the untapped market of a category of a specific good corresponds to the known reserves of the commodity.

Both classes of oligopoly models to be analyzed in this paper are closely related to the monopoly model analyzed in Helmes et al. [10], see also [9]. The oligopoly models differ from the monopoly model as far as the following aspects are concerned. In the competitive case, we only consider infinite horizon problems. Moreover, we have to restrict ourselves to time-independent arrival intensities in order to be able to prove the existence of a unique Nash equilibrium. On the other hand and in contrast to [10], each firm is assigned a [constant] *nonnegative* unit cost.

It turns out that the number of firms with zero unit costs and the characteristics of these firms determine whether or not a Markovian Nash equilibrium of the differential game exists. The equilibrium result is a corollary to a tailor-made existence theorem of solutions (in the positive orthant of $\mathbb{R}^n$) of a special nonlinear system of n equations. We prove the existence and uniqueness of a solution of a particular system of equations determined by the differential game assuming that a fundamental condition holds true. Inspired by the expression "tragedy of the commons," we call this necessary and sufficient condition "the condition of the commons." The term refers to the fact that not "too many" firms should have access to a free nonrenewable resource. The condition is satisfied in any monopoly market; in an oligopoly market the condition is satisfied should all companies have positive marginal costs, or if a fair number of price *insensitive* customers are attracted by the products of firms with zero unit costs. In the special case of homogeneous firms with zero marginal costs the condition is equivalent to a bound on the number of competing companies. This bound depends on the price elasticity of demand, and on a ratio associated with advertising costs and advertising effectiveness, see Sect. 2.2 for details.

We shall derive explicit solutions for both classes of differential games referred to above. Models of Class I (zero unit-cost models) are characterized by *general* adoption/friction functions—cf. [10] for the case of a monopolist—but a *special* market structure. The market environment is supposed to be such that all firms face an identical price elasticity of demand and have zero variable unit costs. Typical applications of such situations include, for example, selling digital goods, end-of-year sales of retail fashion goods, or the situation of car-dealers at the end of a model year. More generally, situations when n competing firms are selling a fixed number of similar assets, and costs are sunk, can be cast as a Class I model. The sales of digital goods nicely fit the model assumptions, since a typical customer does not need more than one copy of a movie, of a song or a piece of software. Thus, like with durable-goods, there is typically no repurchasing and the market potential depletes over time.

The second class of models (Class II) extends the particular problem analyzed by Krishnamoorthy et al. and includes their problem as a special case. We consider the situation of any finite number of competing firms. We allow for (fixed) unit costs and allow that all other firm-specific characteristics, e.g., price elasticities, financing rates, arrival rates, etc., differ as well. In contrast to Case I models friction/system functions are restricted to special saturation functions of power type. This class of system functions includes the square root function which is traditionally considered in the literature, cf. [4], [5] and [13]. An important special case which we can also handle is a linear system function. It characterizes a pure pricing model with a linear friction term. A pure pricing model assumes customers to arrive due to intrinsic motives. The general advertising-and-pricing model postulates the necessity of an extrinsic stimulus which might be costly. Paying for commercials to inform and attract customers, and to boost arrival rates this way, is the prototypical example of (explicit) advertising expenses. Paying higher rent to set up shop at a prime location is an example of (implicit) advertising costs.

In addition to the analytical results to be derived, the dependence of profits, of prices, etc., on arrival rates, price elasticities, marginal costs, and the number of competing firms will also be illustrated by looking at numerical examples. These (simple) numerical studies complement the general theorems which comprise explicit formulas of value functions and of optimal marketing-mix policies, as well as sensitivity results.

Besides the marketing application, e.g., in the car industry mentioned above, there are numerous other [similar] applications which arise in different industries, and for which the models of either Class I or Class II are applicable. For example, selling life insurance, homeowners insurance or work-disability insurance to specific cohorts of customers is just another such application. The evolution of sales of cigarette brands is a classical example of an oligopoly market where the sales dynamics are well described by models of Class II, and for which the results of our analysis, see Sect. 4, nicely fit observations: prices [without taxes] are fairly stable over the years, but advertising is dynamic. The [light] beer market is a more recent example of that kind.

A collection of formulas and abbreviations to which we shall regularly refer to is given in an Appendix. Technical proofs, especially the lengthy proof of Lemma 1, and some additional tables related to the numerical study described in Sect. 3 are all relegated to the Appendix. A Table of variables and parameters which are pertinent to the model description, see Sect. 2, can be found at the beginning of the Appendix.

2 The Deterministic Oligopoly Model

In this section, we precisely describe the adoption models with (constant) isoelastic demand functions which will be analyzed in the sequel. Let $p_i > 0$ denote a price to be set by firm i, $i = 1, 2, \ldots, n$, and $w_i \geqslant 0$ the advertising effort (per unit of time) by a firm. Let $x_i(t)$ be the (accumulated) sales of company i by time t, and let $x(t) := \sum_{j=1}^{n} x_j(t)$. Thus, $x(t)$ represents the number of all customer who have adopted a brand of a product category by time t. Let $N \in \mathbb{R}^+$ be the number of potential customers in the market, and let $y(t) = N - x(t)$; $y(t)$ is the number[1] of all customers who have not yet adopted the product at time t. The

[1] We assume the number of customers is large enough so that it is a valid approximation to treat N, x and y as continuous variables.

value $y = N$ indicates that no unit of any brand has yet been sold; if $y(0) = N$, then $x(0) = 0$. Throughout, we assume the rate of sales λ_i of each firm i is of the form, $0 < y$, $\psi : (0, N) \to \mathbb{R}^+$,

$$\lambda_i(p_i, w_i, y) := u_i w_i^\delta p_i^{-\varepsilon_i} \psi(y), \qquad \delta \geqslant 0,\ \varepsilon_i > 1,\ u_i > 0, \tag{1}$$

and λ_i equals zero if $y = 0$. The arrival intensity vector $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ has positive components, while the vector $\boldsymbol{\varepsilon}$ of price elasticities of the different firms has components ε_i which are bigger than 1. The nonnegative advertising elasticity $\delta \geqslant 0$ is assumed to be the same for all firms. The property that all arrival rates u_i are positive is a fundamental assumption of the market models to be considered. The assumption implies that each firm has a loyal group of customers, and no matter how large a company's product price will be, there will always be some buyers. Moreover, each firm has its individual discount parameter $r_i > 0$ and unit cost $c_i \geqslant 0$.

The nonnegative real-valued function ψ captures adoption and saturation effects. Typical examples of ψ are power functions y^b, b positive or negative, the Bass function $\psi(y) = \Omega y + \Gamma y(1 - y)$, $\Omega, \Gamma \geqslant 0$, and variants thereof. For Class I models, we allow for general functions ψ; they only have to satisfy a minor technical condition, s. Lemma 2. Advertising cost functions are assumed to be of the form $k_i w_i^a$, where $k_i > 0$, and a is a fixed (common) parameter larger than δ. We prefer the parametrization (δ, a), $0 \leqslant \delta < a$, over a 1-dimensional parametrization given by δ/a. This way, the different interpretations of the control value w, i.e., w represents the control effort like in [13], or w represents the amount spent on advertising (per unit of time) as in [19], can be dealt with in a unified way. The special case $\delta = 1$ and $a = 2$ is treated in [13]. As far as the mathematical formulas are concerned, s. below, only the ratio δ/a matters. Observe, should a be less than or equal to δ, then the cost of advertising spending w_i^a would grow at the same rate or more slowly (in w_i) than the factor w_i^δ of λ_i, the i-th rate of sale, and revenue would tend to infinity. The firm-specific proportionality parameters k_i can be interpreted as effectiveness factors of individual advertising campaigns, or as tax multipliers (surcharges or discounts).

We postulate that each firm decides on its price and advertising rate by exploiting the common knowledge $\mathbf{x}(t) := (x_1(t), \ldots, x_n(t))$ and the values of all parameters of the model. At each time point t, $0 \leqslant t < \infty$, each firm i, $i = 1, 2, \ldots, n$, chooses a positive price $p_i(t)$ and a non-negative advertising rate $w_i(t)$. The choice of control values is further restricted as described below. For each pair $\big(p_i(t), w_i(t)\big)$, the state of the game $\mathbf{x}(t)$ evolves according to a system of differential equations of the form,

$$\dot{x}_i(t) = \lambda_i\big(p_i(t), w_i(t), y(t)\big), \quad x_i(0) \text{ given;} \tag{2}$$

for abbreviation, we will sometimes denote the right hand side of (2) by $\lambda_i(t)$.

The objective of each firm is to maximize its discounted profit J_i which depends on its choice of $\big(p_i(t), w_i(t)\big)$ and on the evolution of the whole market (which depends on the activities of all players):

$$J_i := \int_0^\infty e^{-r_i t}\Big(\big(p_i(t) - c_i\big)\lambda_i(t) - k_i w_i(t)^a\Big)dt. \tag{3}$$

To be able to identify optimal policies of each firm, we shall restrict the maximization of (3) to the class of *admissible* policies. From now on, control policies $\big(p_i(t), w_i(t)\big)$, $0 \leqslant t < \infty$, $i = 1, 2, \ldots, n$, will be called *admissible* iff there exists vector-valued functions $\Phi_i = \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^2$, which are a Markovian Nash equilibrium for the dynamic game (2) and (3); see [2], p. 86, for all details and specificities of the definition. Furthermore, we assume that all integrals which involve the feedback policies are well defined and finite. To simplify notation, we shall write $p_i(t, \mathbf{x})$ and $w_i(t, \mathbf{x})$ for the coordinate functions of any Markovian Nash equilibrium $(\Phi_1, \dots, \Phi_n)$. In [2], Section 4.2, a set of sufficient conditions for an equilibrium to exist is described. These conditions, see [2], Theorem 4.1, will be verified for the problems under consideration.

For any possible state $\mathbf{x} \in \mathbb{R}^n$, $0 \leqslant x_j$ and $x := \sum_{j=1}^n x_j \leqslant N$, let $W_i(\mathbf{x})$ denote the largest discounted profit of player i when the infinite horizon game starts at $\mathbf{x}$. To identify a solution of the differential game, see [2] pp. 92, we are looking for equilibrium strategies $\left(p_j^*(\mathbf{x}), w_j^*(\mathbf{x})\right)_{1\leqslant j\leqslant n}$, and bounded functions $W_i(\mathbf{x})$ such that the system of partial differential equations, $x < N$, $i = 1, 2, \dots, n$,

$$r_i W_i(\mathbf{x}) = \sup_{p_i>0, w_i\geqslant 0} \Bigg\{ \lambda_i \Bigg(p_i, w_i, N - \sum_{j=1}^n x_j \Bigg) \Bigg(p_i - c_i + \frac{\partial W_i}{\partial x_i}(\mathbf{x}) \Bigg) - k_i w_i^a$$
$$+ \sum_{j\neq i} \lambda_j \Bigg(p_j^*(\mathbf{x}), w_j^*(\mathbf{x}), N - \sum_{\ell=1}^n x_\ell \Bigg) \frac{\partial W_i}{\partial x_j}(\mathbf{x}) \Bigg\}, \tag{4}$$

together with the boundary conditions $W_i(\mathbf{x}) = 0$, if $x = N$, has a solution. The structure of the rates λ_i, and the identity $y = N - \sum_{j=1}^n x_j$, $0 \leqslant y \leqslant N$, suggests that solutions $W_i(\mathbf{x})$ are of a special form: subject to given functions $(p_j^*(y), w_j^*(y))$, $j = 1, 2, \dots, n$, $V_i'(y) := \frac{dV_i}{dy}(y)$,

$$W_i(\mathbf{x}) = V_i(y), \tag{5}$$

where the functions $V_i(y)$ satisfy the system of ordinary differential equations, $y > 0$,

$$r_i V_i(y) = \sup_{p_i>0, w_i\geqslant 0} \Bigg\{ \lambda_i(p_i, w_i, y)\big(p_i - c_i - V_i'(y)\big) - k_i w_i^a$$
$$- \sum_{j\neq i} \lambda_j \big(p_j^*(y), w_j^*(y), y\big) V_i'(y) \Bigg\}, \tag{6}$$

together with the boundary conditions $V_i(0) = 0$. Note, if (5) holds, then $\frac{\partial W_i}{\partial x_j}(\mathbf{x}) = -V_i'(y)$ for all i and j. In the sequel, we shall verify that under appropriate conditions, the system (6) has a unique solution $(V_i(y))_{1\leqslant i\leqslant n}$, and so does (4).

2.1 Optimality Conditions

The Bellman equations (4) yield optimality conditions which the equilibrium (feedback) pricing and advertising decisions $p_i^*(y)$ and $w_i^*(y)$ of each firm i have to satisfy. Taking partial derivatives with respect to p_i and w_i of the function showing up on the i-th right hand side of (6), simple algebra yields a formula of $p_i^*(y)$ in terms of ε_i, c_i, and $V_i'(y)$, see (23); the formula of w_i^* in feedback form is given by (24). The formula of w_i^* involves ψ, p_i^* and the parameters ε_i, a, δ, k_i, and u_i. We shall see in Subsection 2.2, cf. formulas (9), (10) and (14), that V_i' is positive for all $i = 1, \dots, n$. From now on, we use the abbreviating notation $\lambda_i^*(y) := \lambda_i\left(p_i^*(y), w_i^*(y), y\right)$, $0 < y \leqslant N$, where $p_i^*(y)$ and $w_i^*(y)$ are the equilibrium pricing and advertising policies. The optimality conditions imply that a dynamic Dorfman-Steiner identity holds for each firm. The classical Dorfman-Steiner theorem provides the theoretical underpinning of the empirical fact (see Schmalensee [22]) that in different markets (monopoly, oligopoly) advertising strategies of firms are very often based on a constant percent of sales rule; for more details and additional references see Dockner and Feichtinger [1].

Proposition 1 *For the oligopoly problem described in Sect. 2.1, a Dorfman-Steiner identity holds for each firm,* $i = 1, 2, \ldots, n$*:*

$$\frac{w_i^*(y)^a}{p_i^*(y)\lambda_i^*(y)} \equiv \frac{\delta}{k_i a \varepsilon_i}, \qquad y \in (0, N], \tag{7}$$

i. e. optimal advertising expenditure and revenue are pointwise proportional.

Proof The identities (7) are an immediate implication of (23) and (24). □

Since the Dorfman-Steiner identity holds for every $y \in (0, N)$, we can evaluate (7) along the optimal trajectory $y(s)$, $0 \leqslant s < \infty$. This way, for any $t \geqslant 0$, we easily prove identities which connect the accumulated revenue (from t onwards) $\bar{U}_i(t)$,

$$\bar{U}_i(t) := \int_t^\infty e^{-r_i(s-t)} p_i^*(y(s))\lambda_i^*(y(s))ds,$$

with the production/purchasing cost $\bar{C}_i(t)$ and the accumulated advertising expenditure $\bar{W}_i(t)$, see Proposition 2; the last two quantities are defined by

$$\bar{W}_i(t) := k_i \int_t^\infty e^{-r_i(s-t)} w_i^*(y(s))^a ds, \quad \bar{C}_i(t) := c_i \int_t^\infty e^{-r_i(s-t)} \lambda_i^*(y(s))ds,$$

and $\bar{V}_i(t) = \bar{U}_i(t) - \bar{W}_i(t) - \bar{C}_i(t)$ holds true. For the two classes of models described in the Introduction, we shall derive explicit expressions for $\bar{V}_i(t)$ and $\bar{C}_i(t)$, see Sects. 3 and 4. Employing the following proposition, we shall then obtain explicit expressions for the important characteristics $\bar{U}_i(t)$ and $\bar{W}_i(t)$.

Proposition 2 *Let* $y(s)$ *denote the optimal path of (category wide) unsold items,* $0 \leqslant s < \infty$. *For any* i, $i = 1, 2, \ldots, n$, *and* $t \geqslant 0$,

(i) $\bar{W}_i(t)/\bar{U}_i(t) = \delta/(a \cdot \varepsilon_i)$,
(ii) $\bar{W}_i(t) = \frac{\delta}{a\varepsilon_i - \delta}\Big(\bar{V}_i(t) + \bar{C}_i(t)\Big)$, *and* $\bar{U}_i(t) = \frac{a\varepsilon_i}{a\varepsilon_i - \delta}\Big(\bar{V}_i(t) + \bar{C}_i(t)\Big)$.

In the sequel, in order to shorten some expressions we shall use the following abbreviations throughout the paper. For each firm i, $i = 1, 2, \ldots, n$, let

$$\gamma_i := \frac{a\varepsilon_i - \delta}{a - \delta} = \frac{\varepsilon_i - \delta/a}{1 - \delta/a},$$

see also (26), and let $\eta_i = \eta_i(k_i, a, \delta, \varepsilon_i, u_i)$ be the constants defined by (27). Except for sensitivity studies, the explicit formula of η_i will not be important. However, it is useful to remember that η_i is increasing in the intensity parameter u_i and is decreasing in k_i.

We shall call the parameter γ_i the leveraged (due to advertising) price elasticity of demand of firm i. If the price is decreased by 1%, then the demand will increase by $\sim \gamma_i$%. Note, γ_i is larger than ε_i! Thus, for each firm i, the parameter γ_i quantifies the benefit of [informative] advertising.

2.2 Fundamental Results

Using the expressions for the equilibrium strategies $w_i^*(y)$ and $p_i^*(y)$ in terms of derivatives of the value functions $V_i(y)$, see (23) and (24), the Bellman equations (4) turn into a system of 1st order nonlinear differential equations. This system, together with the [natural] boundary conditions $V_i(0) = 0$ for each i, determines the value functions V_i. The system can be written as, $0 < y \leqslant N$, $i = 1, 2, \ldots, n$, $V_i' = V_i'(y)$, $\psi = \psi(y)$, etc.,

$$r_i V_i = \left(\eta_i \left(c_i + V_i'\right)^{1-\gamma_i} - \sum_{j \neq i} (\gamma_j - 1) \eta_j \left(c_j + V_j'\right)^{-\gamma_j} \cdot V_i' \right) \psi^{\frac{a}{a-\delta}}. \tag{8}$$

To solve (8), we look for solutions V_i which are given as the product of a firm specific factor α_i and a function $\beta(y)$ which is common to all firms:

$$V_i(y) := \alpha_i \beta(y). \tag{9}$$

The constants α_i are assumed to be positive numbers, and β is a positive increasing differentiable function of the variable y, the untapped market size. The common factor $\beta(y)$ reflects the value of a market of size y. The number α_i quantifies the market power of each firm i.

The separable "Ansatz" implies that the constants α_i and the function β have to satisfy the system of equations (28). To identify both factors, we will first prove two lemmas. The first one, Lemma 1, characterizes the solution of a special nonlinear system of algebraic equations, see (10) below. The solution values are the numbers α_i. Lemma 2 characterizes the function $\beta(y)$. The proofs of both results will be given in the Appendix.

Lemma 1 *(i) For positive variables α_i, let $z_i := (\gamma_i - 1)\eta_i(c_i + \alpha_i)^{-\gamma_i}$, $i = 1, 2, \ldots, n$, and $Z := \sum_{i=1}^n z_i$. The system of equations in the unknowns α_i, $i = 1, 2, \ldots, n$,*

$$r_i = \eta_i (c_i + \alpha_i)^{-\gamma_i} \frac{c_i + \alpha_i}{\alpha_i} - \sum_{j \neq i} z_j, \tag{10}$$

which is equivalent to $\left(\frac{1}{\gamma_i - 1} \cdot \frac{c_i + \alpha_i}{\alpha_i} + 1\right) z_i - r_i = Z$, has a unique positive solution vector $\boldsymbol{\alpha}^* = (\alpha_i^*)_{1 \leqslant i \leqslant n}$ if and only if the "condition of the commons",

$$1 > \sum_{i=1:c_i=0}^{n} \frac{\gamma_i - 1}{\gamma_i}, \tag{11}$$

holds true.

(ii) Let $(n + 1)$ companies compete against each other, cf. the model description at the beginning of this section. Let the condition of the commons be satisfied for the enlarged system of Eq. (10), i.e., we consider a system like (10) with $n + 1$ equations and an additional variable. Let $\boldsymbol{\alpha}^*(n)$, $\boldsymbol{\alpha}^*(n + 1)$ resp., denote the unique positive solution vector of the system with n equations, $n + 1$ equations resp. Then, for $i = 1, \ldots, n$,

$$\alpha_i^*(n) \leqslant \alpha_i^*(n + 1). \tag{12}$$

If the condition of the commons is violated for the oligopoly market with n firms, then no equilibrium exists for the market with $(n + 1)$ firms, no matter what the characteristics of the entering firm might be.

(iii) Let $c_i = 0$, $i = 1, 2, \ldots, n$, and assume $n - 1 < \sum_{j=1}^n \gamma_j^{-1}$. Then, there are explicit expressions of the unique solution values α_i^* of (10), see (41).

In the general case, $c_i \geqslant 0$, $i = 1, 2, \ldots, n$, the unique positive solution vector $\boldsymbol{\alpha}$ of (10) can be computed as described in Steps 1–3, see (13). To this end, define n real-valued functions f_i on the positive line, $i = 1, 2, \ldots, n$, $f_i(\xi) := \left(\frac{c_i}{\xi+\gamma_i}\right)\eta_i(c_i+\xi)^{-\gamma_i} - r_i, \xi > 0$. Moreover, without loss of generality, we select the function f_1 for our analysis, and use α_1 as the pivoting variable. For any positive number ξ and $j = 1, 2, \ldots, n$, let $\alpha_j > 0$ denote the unique positive solution value of the equation $f_j(\alpha_j) = f_1(\xi)$, see Appendix; we call the function $\hat{\alpha_j}(\xi) := \alpha_j$ the j-th reaction function. Then,

Step 1 Determine the reaction functions $\hat{\alpha_j}(\xi)$ for $j = 1, \ldots, n, \xi > 0$.

Step 2 Solve the 1-dimensional equation (in the positive unknown α_1):

$$f_1(\alpha_1) = \sum_{j=1}^{n}(\gamma_j - 1)\eta_j(c_j + \hat{\alpha}_j(\alpha_1))^{-\gamma_j}. \tag{13}$$

Let $\alpha_1^* > 0$ denote the unique solution of (13).
Step 3 For $j = 2, \ldots, n$, compute $\alpha_j^* := \hat{\alpha}_j(\alpha_1^*)$. The vector $\boldsymbol{\alpha}^* := (\alpha_1^*, \alpha_2^*, \ldots, \alpha_n^*)$ is the unique positive solution of (10).

An explanation of the construction and the details of the proof are given in the Appendix.

To get a first understanding of the importance of Lemma 1, and to have an economic interpretation of the quantities z_i, choose $\delta = 0$, $a = 1$, and let all c_i be positive. Jumping ahead, cf. Theorem 4 in Sect. 4, we define $p_i := \frac{\varepsilon_i}{\varepsilon_i - 1}(c_i + \alpha_i)$. Then, see formulas (23) and (24), $z_i = u_i p_i^{-\varepsilon_i}$, and z_i is the demand/output rate of firm i should it set its price as defined above. Later on, see Sect. 4, we shall elaborate on this observation.

The next lemma is about the market size value β, cf. (9). The lemma characterizes β as the solution of a particular Bernoulli differential equation. It is a slight extension of Lemma 3.1 in [10].

Lemma 2 *Let $\psi(y)^{1/(\varepsilon-1)}$ be a nonnegative integrable function on $[0, N]$. The solution of the Bernoulli differential equation, $0 < y < N$,*

$$\beta'(y) = \beta(y)^{\frac{1}{\gamma-1}}\psi(y)^{\frac{1}{\varepsilon-1}} \tag{14}$$

and $\beta(0) = 0$, is given by $\beta(y) = B(y)^{(\gamma-1)/\gamma}$, where

$$B(y) := \frac{\gamma}{\gamma - 1}\int_0^y \psi(s)^{\frac{1}{\varepsilon-1}} ds. \tag{15}$$

If $\psi(y)$ is positive on $(0, N)$ then $\beta(y)$ and $B(y)$ are strictly increasing functions of y. Furthermore, if $\psi'(y)\psi(y)^{-\frac{\varepsilon}{\varepsilon-1}}B(y) \leqslant 1 - \delta/a$, then $\beta(y)$ is concave. If $\psi(y) = 1$, i.e. in the case of no demand learning effects/externalities, this condition is always satisfied and $\beta(y) = y^{(\gamma-1)/\gamma}$.

Proof See Appendix; expressions (29) are equivalent formulations of (14). □

In Sects. 3 and 4, we will be using Lemma 1 and Lemma 2 to solve special cases of the oligopoly problem described in this section. In contrast to dynamic advertising games analyzed by Prasad and Sethi [21] and Erickson [4,5], where value functions of all players are linear in the state variable(s), the classes of dynamic games under consideration lead to nonlinear value functions, see formula (9).

3 A Special Market Structure but a General Adoption Function

In this section, we shall consider models with general adoption functions ψ, where $\psi^{1/(\varepsilon-1)}$ is positive and integrable on $(0, N)$, but we assume that (i) all unit costs are zero, (ii) the price elasticities of all firms are identical, i. e. $\varepsilon_i \equiv \varepsilon > 1$, and (iii) condition (11) is satisfied. Dynamic games with evolution Eq. (2), which satisfy these properties, will be called Case I models.

For Case I models, the system (28) separates into an algebraic system of equations and a differential equation; exploit (41), (30) and the Bernoulli differential Equation (29). If all unit costs are zero and $\varepsilon_i \equiv \varepsilon$, the solution of the algebraic system (10) is given by

$$\alpha_i^* = \left(\frac{1}{\eta_i \gamma}\left(r_i + \frac{\sum_{j=1}^n r_j}{\gamma/(\gamma-1)-n}\right)\right)^{\frac{-1}{\gamma}}. \tag{16}$$

Obviously, the values α_i^* are independent of ψ. In the symmetric case, i. e. all firms have the same characteristics, $u_i \equiv u$, $k_i \equiv k$, and $r_i \equiv r$, $i = 1, 2, \ldots, n$, all α_i^* are equal to the value α_n^{sym}, where

$$\alpha_n^{sym} := \left(\frac{\eta}{r}(\gamma - n(\gamma-1))\right)^{\frac{1}{\gamma}}.$$

The value α_n^{sym} is positive iff $n < 1 + 1/(\gamma-1)$, cf. (11). This inequality imposes an upper bound on the number of firms such that an equilibrium point exists. Expressed differently, if n is given, then the inequality $\gamma < n/(n-1)$ imposes an upper bound on the elasticity ε in order that an equilibrium exists, viz. $\varepsilon < 1 + (1-\delta/a)/(n-1)$. Hence, if the unit cost of (homogeneous) firms is zero, a Markovian Nash equilibrium of the game with n firms exists if and only if consumers are not "too" price sensitive. In the case of a monopoly, i. e. $n = 1$, the condition $n < 1 + 1/(\gamma-1)$ is always satisfied.

Furthermore, for Case I models there is an explicit expression of each value function $V_i(y)$ as a product of the solution $\beta(y)$ of the differential Eq. (29), and α_i^*. The two factors are defined by (14) and (16). Using the optimality conditions (23) and (24), one obtains a formula of the optimal category rate of sales $\lambda^*(y)$. Thus, we are able to compute and characterize the evolution of the untapped market $y(t)$. Expressions of the individual rates and accumulated sales of each firm are implied by these formulas. The proof of the next result is given in the Appendix, see also Lemma 2.

Theorem 1 *For Case I models, the equilibrium rate of sales $\lambda^*(y)$ in feedback form is*

$$\lambda^*(y) = B(y)\psi(y)^{\frac{-1}{\varepsilon-1}} Z.$$

The equilibrium y-trajectory satisfies the equation

$$B(y(t)) = B(N)e^{\frac{-\gamma}{\gamma-1}Zt} \iff y(t) = B^{-1}\left(B(N)e^{\frac{-\gamma}{\gamma-1}Zt}\right), \tag{17}$$

where B^{-1} denotes the inverse function of B. For each firm i, $i = 1, 2, \ldots, n$, its equilibrium rate of sales equals

$$\lambda_i^*(y(t)) = z_i \frac{\beta(y(t))}{\beta'(y(t))}. \tag{18}$$

The accumulated sales (up to time t) of each firm are given by, $x_i(0) = 0$,

$$x_i(t) = \frac{z_i}{Z}(N - y(t)).$$

Expressions (15) and (16) together with Theorem 1, yield solution formulas of the value functions and the equilibrium strategies in feedback form, as well as in open-loop form, i. e. as functions of time, for each firm. The next theorem is a collection of such formulas. The straightforward proofs of the various formulas are based on the optimality conditions (23) and (24), and the characterization of each value function V_i as the product of the number α_i^* and the function β.

Theorem 2 *For Case I models, let $y(t)$ denote the optimal y-path given by (17). The value function (in feedback form) of each firm i, $i = 1, 2, \ldots, n$, is given by*

$$V_i(y) = a_i^* \beta(y), \qquad y \in [0, N];$$

in the time-domain, it is described by

$$\bar{V}_i(t) := V_i(y(t)) = \alpha_i^* \beta(N) e^{-Zt}.$$

The optimal prices are

$$p_i^*(y) = \frac{\varepsilon}{\varepsilon - 1} \alpha_i^* \beta(y)^{\frac{-1}{\gamma-1}} \psi(y)^{\frac{1}{\varepsilon-1}}, \qquad y \in (0, N];$$

$$\bar{p}_i(t) := p_i^*(y(t)) = \frac{\varepsilon}{\varepsilon - 1} \alpha_i^* \beta(N)^{\frac{-1}{\gamma-1}} e^{\frac{1}{\gamma-1} Zt} \psi(y(t))^{\frac{1}{\varepsilon-1}}.$$

The optimal advertising rates are (see (25) for the definition of θ_i)

$$w_i^*(y)^a = \theta_i^a \alpha_i^{*\,1-\gamma} \beta(y), \quad y \in (0, N);$$

$$\bar{w}_i(t)^a := w_i^*(y(t))^a = \theta_i^a \alpha_i^{*\,1-\gamma} \beta(N) e^{-Zt}.$$

The optimal rates of sales are

$$\bar{\lambda}_i(t) := \lambda_i^*(y(t)) = z_i \beta(N)^{\frac{\gamma}{\gamma-1}} e^{-\frac{\gamma}{\gamma-1} Zt} \psi(y(t))^{\frac{-1}{\varepsilon-1}}.$$

For Case I models, see Theorem 2, a firm's value function $\bar{V}_i(t)$ and its optimal advertising rates $\bar{w}_i(t)$ are exponentially decreasing functions of t. The evolution of the optimal price paths $\bar{p}_i(t)$ is determined by a product of three terms: the first factor is a company's market power α_i^*; the second factor is an exponentially increasing function of time, and the third factor is a power expression of the adoption function evaluated along the optimal path $y(t)$. Optimal advertising paths only depend on ψ via the potential B and the initial value $y(0)$. Optimal price paths explicitly depend on ψ and the optimal path $y(t)$. Thus, for Case I models dynamic prices are the major driving forces of the oligopolistic market. For such models, since $\bar{C}_i(t) = 0$, Proposition 2, when combined with the explicit formulas of the value functions $\bar{V}_i(t)$, yields explicit formulas of the evolution of each firm's specific revenue trajectory $\bar{U}_i(t)$ and expenditure function $\bar{W}_i(t)$. Like in a monopoly market, in an oligopoly market, depending on the structure of ψ, optimal pricing strategies of the companies can be skimming policies, market penetration policies, or a combination of both principles, i.e., penetration pricing followed by a long period of declining prices. Like in Helmes et al., detailed [numerical] analyses are possible for different system functions ψ. In particular, studies of the classes of power functions, Bass-functionals, von Bertalanffy dynamics, and NUI-models, cf. [3], provide insight into the many different ways that sales of firms can evolve in an oligopoly, and these studies show the interplay of firms' pricing and advertising policies.

Next, we shall study how the market power and other important characteristics of firms depend on changes in a firm's parameter values r_i (its discount value, which reflects its

Table 1 Case I models: sensitivity of market power α_i^* and other system components as functions of model parameters

	Z	η_i	z_i/Z	α_i^*	w_i^*	λ^*, λ_i^*	
							y
	z_i	θ_i		p_i^*		x_i	
r_i	+	0	+	−	+	+	−
u_i	0	+	0	+	?	0	0
k_i	0	−	0	−	?	0	0
$r_j,\ j \neq i$	+	0	−	−	+	+	−

financing cost), u_i (the arrival rate), and k_i (the proportionality factor of advertising expenses). Moreover, we analyze how changes in a competitor's parameter values influence the output characteristics of a rival firm. The general results, see Table 1, follow from sensitivity results for z_i and Z to changes in the parameter values. *For Case I models,* the values z_i and Z are independent of η_i, see (27), (37) and formula (16). Hence, for Case I models, z_i and Z do not depend on u_i and k_i but only depend on r_i. However, the market power values α_i, and thus the value functions V_i depend on u_i.

The starting point of sensitivity studies is Theorem 2. Table 1, s. below, is a summary of our calculations. In Table 1, entries "+", "−" and "0" indicate that the quantity of a column is monotone increasing (+), is monotone decreasing (−) or is independent of the parameter of a particular row; a question mark "?" indicates that no general statement is possible. For example, if the rate r_i increases (all else equal), i.e., the financing cost of company i goes up, then its market power α_i^* will decrease. In such a situation, the firm's optimal marketing strategy is to lower prices $p_i^*(y)$ but to increase advertising spending $w_i^*(y)$. This way, the company accelerates the growth of its market share.

Higher arrival rates u_i guarantee larger market power α_i^*; higher arrival rates also suggest higher prices and increased advertising spending. Since the values z_i are independent of η_i, optimal (feedback) rates of sales λ^* and λ_i^* are not affected by changes of u_i or k_i.

The ambiguous results (s. both questions marks in the w_i^*- column) are due to the fact that the first two factors of the formula of w_i^*, cf. Theorem 2, are reacting in opposite directions to changes of u_i, and k_i, respectively. For instance, if u_i is increasing, then the first factor θ_i^a is increasing, while the second factor $\alpha_i^{*1-\gamma_i}$, since $\gamma_i > 1$, will be decreasing. Hence, there is no definite result for all parameter constellations.

The following example illustrates Theorem 1 and Theorem 2. We choose the Mansfield functional $\psi(y) = y/N(1 - y/N)$, $N = 100$, as a particular adoption model, cf. [18]. This ψ function captures situations when consumers are either not very well informed about a new product or are somewhat reluctant to buy the product right from the start. This adoption function is a special Bass model with innovation coefficient 0 and imitation coefficient 1. Sales increase due to word-of-mouth. In the example, we assume that firms only differ by their arrival rates, $u_1 = 20$, $u_2 = 30$, and $u_3 = 40$. Note, condition (11) is satisfied and, recall (16), α_i^* is not affected by ψ.

Example 3.1 Let $\psi(y) = y/N(1 - y/N)$, $N = 100$, $n = 3$, $\varepsilon = 1.2$, $\delta = 1$, $a = 2$, $c = 0$, $k = 1$, $r = 0.1$ and $\mathbf{u} = (20, 30, 40)$.

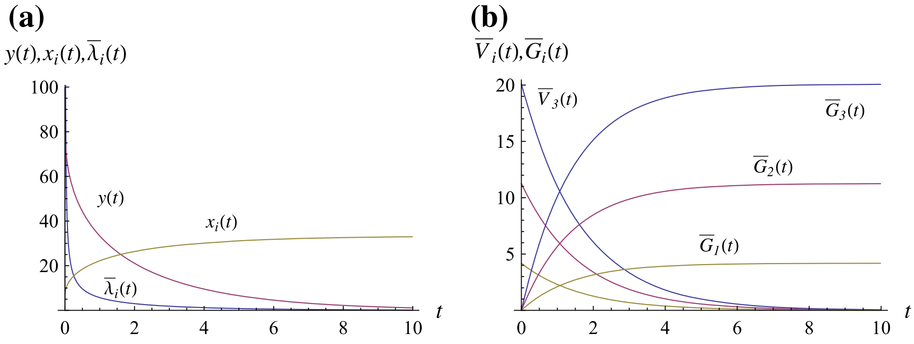

Fig. 1 Sales, market fractions (*left window* (**a**)) and accumulated profits (*right window* (**b**))

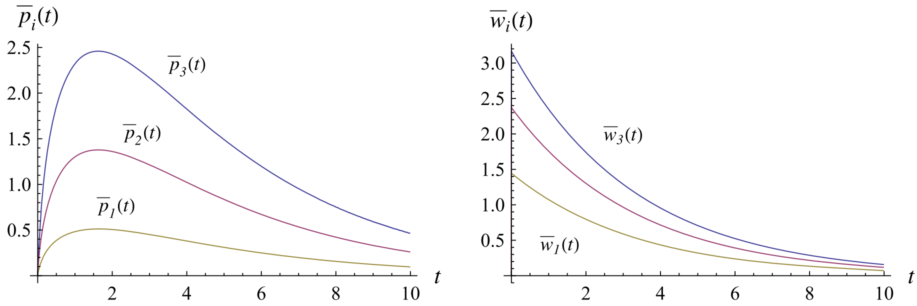

Fig. 2 Optimal price paths (*left window*) and advertising rates (*right window*)

Figures 1 and 2 show the evolution of various characteristics of the dynamic game of a Case I model, namely the evolution of sales, of market shares and accumulated profits; Fig. 1 also shows optimal price paths. If the financing costs of all firms are identical, i.e., $r_i = r$, Fig. 1a illustrates the remarkable fact that the optimal rates of sales of all companies are the same, cf. formula (39); observe, the number of yet uncommitted customers decreases exponentially at rate Z. The individual gains $\bar{G}_i(t) := \bar{V}_i(0) - e^{-rt}\bar{V}_i(t)$, together with $\bar{V}_i(t)$, are shown in Fig. 1b. The optimal price paths and optimal advertising rates are shown in Fig. 2.

All graphs of Figs. 1 and 2 clearly show the impact the arrival rate u_i has on profit and the pricing options of a firm. Since in many applications the arrival rate is related to the location of a business, the graphs illustrate the mantra in marketing: location, location, and location! A higher arrival rate of a firm implies higher prices, higher revenues, and increased market power, but also intensified advertising compared to the advertising levels of other rival firms.

Example 3.1 also illustrates the influence of the adoption function ψ. The optimal price paths of all firms are closely related to the properties of ψ, see Fig. 2a. In case of the special function $\psi(y) = y/N(1 - y/N)$ a market penetration pricing strategy is advised to be used. Such a strategy jump-starts sales and boosts the "word-of-mouth" momentum. Furthermore, if the market is saturated and the adoption effect is small, then optimal prices should go down. In the case of monopoly markets, additional examples and further management recommendations are discussed in [10].

4 Heterogeneous Unit Costs and Price Elasticities, but a Special Class of Adoption Functions

In this section, we consider a second class of n-player differential games. This class is characterized by the property that the ψ-function belongs to a special class of power functions, $\psi(y) = y^{(a-\delta)/a}$, $0 \leqslant \delta < a$; it captures a (new) product adoption with a special saturation effect. The intensity functions λ_i are again given by (1), and each firm i, $i = 1, 2, \ldots, n$, is characterized by individual parameters $c_i \geqslant 0$, $\varepsilon_i > 1$ and positive values r_i, u_i, and k_i. If condition (11) holds, we call this class of differential games Case II models. Recall, if all c_i, $i = 1, 2, \ldots, n$, are positive, then the condition of the commons (11) is always satisfied no matter how many companies are competing for customers.

The special parameter choice $n = 2$, $a = 2$ and $\delta = 1$ specifies the duopoly model analyzed in [13]. Case II models also include the pure pricing model with a linear adoption function $\psi(y) = y$ as a very special case. The pure pricing model can be parameterized choosing $\delta = 0$ and any positive value a. This choice of parameter values implies $w^* = 0$ for all firms i, $i = 1, 2, \ldots, n$.

For Case II models, to find solutions for (8) we try, see [13] and also [10], the linear "Ansatz", $i = 1, 2, \ldots, n$,

$$V_i(y) = \alpha_i y. \tag{19}$$

Since ψ is a very special power function, the coupled system of ODEs (8) simplifies and reduces to the identities

$$r_i \alpha_i y = \eta_i (c_i + \alpha_i)^{-(\gamma_i - 1)} y - \sum_{j \neq i} \underbrace{(\gamma_j - 1)\eta_j (c_j + \alpha_j)^{-\gamma_j}}_{z_j} \alpha_i y. \tag{20}$$

Since y is a common factor of all three terms of (20), Lemma 1 can be applied. It guarantees a unique positive solution of (20). In the special case of n symmetric firms, the algebraic system (20) collapses to one equation in one unknown,

$$\frac{c + \alpha_n^{sym}}{\alpha_n^{sym}} - \frac{r}{\eta}(c + \alpha_n^{sym})^{\gamma} = (n-1)(\gamma - 1). \tag{21}$$

In the case of a monopoly, i. e. $n = 1$, equation (21) becomes

$$\alpha^{mon} r/\eta = (c + \alpha^{mon})^{-(\gamma - 1)}.$$

If $c = 0$, we obtain the formula $\alpha^{mon} = (\eta/r)^{1/\gamma}$, and we have an explicit expression of the value function

$$V^{mon}(y) = (\eta/r)^{1/\gamma} y,$$

see [23] and [10]. In the general case with heterogeneous firms, we are able to numerically compute the solution values α_i of (20), cf. Steps 1–3 in Section 2.2. Moreover, the very special dependence of the value functions $V_i(y)$ on y, see (19), when combined with the formulas of the optimal feedback controls p_i^* and w_i^*, see (23) and (24), makes it possible to compute the optimal rates of sales $\lambda_i^*(y)$ and the accumulated rate $\lambda^*(y)$. Hence, the evolution of the category sales can be easily computed, see the following theorem and its proof in the Appendix.

Theorem 3 *For Case II models, we have, $i = 1, 2, \ldots, n$, $0 \leqslant t < \infty$,*

$$\lambda^*(y) = Zy \text{ and } y(t) = Ne^{-Zt};$$
$$\text{moreover, } \lambda_i^*(y) = z_i y \text{ and } x_i(t) = Nz_i/Z(1 - e^{-Zt}).$$

For Case II models, the rates of sales $\lambda_i^*(y)$ and $\lambda^*(y)$ are linear functions of the untapped market share y. Using the feedback formulas of the optimal controls and Proposition 2, we obtain the evolution of all other quantities of interest of such models.

Theorem 4 *For Case II models, we have, $i = 1, 2, \ldots, n$, $0 \leqslant t < \infty$,*

$$\begin{array}{ll} value functions: & V_i(y) = \alpha_i^* y, \text{ and } \bar{V}_i(t) = \alpha_i^* N e^{-Zt}; \\ optimal prices: & p_i^*(y) \equiv \bar{p}_i(t) \equiv \frac{\varepsilon_i}{\varepsilon_i - 1}(c_i + \alpha_i^*); \\ opt.\ advertising\ rates: & w_i^*(y)^a = \theta_i^a (c_i + \alpha_i^*)^{1-\gamma} y, \text{ and} \\ & \bar{w}_i^a(t) = \theta_i^a (c_i + \alpha_i^*)^{1-\gamma_i} N e^{-Zt}; \\ opt.\ production\ costs: & \bar{C}_i(t) = c_i \frac{N z_i}{Z + r_i} e^{-Zt}; \\ opt.\ revenue: & \bar{U}_i(t) = \frac{a\varepsilon_i}{a\varepsilon_i - \delta} N e^{-Zt} \left(\alpha_i^* + \frac{c_i z_i}{Z + r_i} \right); \\ opt.\ expenditures: & \bar{W}_i(t) = \frac{\delta}{a\varepsilon_i} \bar{U}_i(t). \end{array}$$

Proof All formulas follow from Theorem 1 and Proposition 2 once the necessary optimality conditions (23) and (24) are combined with (19). □

When combined with Lemma 1, the previous two theorems show that in an asymmetric oligopoly market with (special) externalities the *structure* of optimal prices and the value function of each agent are the same as in the case of a duopoly, cf. [13]. If the condition of the commons hold, then optimal prices are constants and the value functions are linear in the cumulated sales of all firms. A crucial difference between the two market environments - the case of a duopoly and a general oligopoly - is the surcharge that the competing firms can impose on top of the basic mark-up $c_i \varepsilon_1/(\varepsilon_i - 1)$. More firms imply lower surcharges, i.e., lower equilibrium prices. On the other hand, should companies exit the market, then an existing equilibrium will prevail, but prices will go up! These most relevant facts all follow from Lemma 1 (ii). Furthermore, taking the quotient of cumulative sales of two rivals i and j shows that in equilibrium this ratio is independent of time, and equals the ratio of the corresponding growth coefficients z_i/z_j.

For a market of homogeneous firms, there are simplifications and refinements of all the results; in particular, there are refined results for the problems of entering firms and exiting firms.

Proposition 3 *For symmetric Case II models, the following properties hold:*

(i) If $u_i(n) = u(n) = u$, $i = 1, \ldots, n$ (more firms attract more customers), then $\eta(n) = \eta$, α_n^{sym} is decreasing and z_n^{sym} is increasing in n. Moreover, $Z(n) = n z_n^{sym}$ increases super-linearly in the number of competing firms.

(ii) If $u(n) = u/n$, $u > 0$ fixed (the customer base is equally shared by all firms), then the product $\eta(n) n^{a/(a-\delta)}$ is independent of n, and α_n^{sym} is decreasing in n.

(iii) If $u(n) = u/n$, then the quantity $Z(n)$ is given by the formula,

$$Z(n) = n(\gamma - 1)\eta(n) \left(c + \alpha_n^{sym} \right)^{-\gamma}. \tag{22}$$

If $\delta = 0$, then $Z(n)$ is increasing in n. If $\delta > 0$, then $Z(n)$ will be decreasing for large values of n.

Proof (i) Note, the left hand side of (21) is a decreasing function in the variable α_n^{sym}. Hence, if the right hand side increases, i.e., more firms are competing against each other, then the solution of Eq. (21) moves to the left, and the first claim follows. If α_n^{sym} is decreasing in n, it follows by definition, see (20), that z_n^{sym} is increasing.

(ii) This time, η depends on n and, by definition, see (27),decreases in n. Taking this property into account, we can repeat the arguments of the proof of (i). Since $z_n^{sym} := (\gamma - 1)\eta(n)(c + \alpha_n^{sym})^{-\gamma}$, formula (22) immediately follows.

(iii) Using (22) and the relationship that $\eta(n)n^{a/(a-\delta)}$ is constant, we deduce the properties of $Z(n)$. □

For the case of symmetric firms, the results (ii) and (iii) of Proposition 3 are most important. They show, for instance, that the optimal equilibrium price and the value of each firm decrease with the number of competing firms should increased competition erodes the customer base of each firm, i.e., $u(n) = u/n$.

The following two examples illustrate these theoretical results and properties which are typical for Case II models. Example 4.1 highlights how the number of competing firms n affects optimal prices, profits, etc., in such a case, see, in particular, the second coloumn and the last one of Table 2. Table 3 illustrates part (iii) of Proposition 3, in particular, the non trivial impact advertising has on market share if the number of competitors is increasing. The saturation function used in both examples has been considered by Sethi et al. [23] in the case of a monopoly, but only for the special case that the unit cost c is zero, see also Helmes et al. [10]. As nicely explained by Sethi et al. [23], this special ψ function is a reasonable approximation of a Bass functional. The second example, cf. Example 4.2, illustrates the case of asymmetric oligopolies.

Example 4.1 Let $\psi(y) = \sqrt{y}$ and $\delta = 1$, $a = 2$. We study how several companies rival for $N = 100$ "batches" of customers. Tables 2 and 3, see below, illustrate—for Case II models—how variations of the parameters affect optimal prices, profits, etc., of each firm. We choose the following symmetric situation as a benchmark for our analysis: $\varepsilon = 1.8$, $c = 10$, $k = 1$, $r = 0.1$ and $u(n) = 30/n$, n the number of competing firms. Note, since the customer base is fixed the arrival rate of customers of each firm decreases with the number of firms competing in the market.

The values of a, δ, and ε imply the leveraged price elasticity of demand γ to be 2.6. The choice $u(n) = 30/n$ assumes the fixed arrival intensity of shoppers, 30, to be equally split among all firms. Hence, if a price p and an advertising rate w are chosen, then the initial rate of sales facing a monopolist equals $30p^{-2.5}w\sqrt{100}$. Should, instead, 10 brands compete in the market, then this rate drops to a meagre $3p^{-2.5}w\sqrt{100}$ for each brand. Example 4.1 is representative of market situations where an increasing number of competing firms does not stimulate the shopping behavior of customers; gas stations are a well-known example.

Before analyzing how the equilibrium varies with the number of brands, it is instructive to exploit Theorem 4 and formula (21) in the case of a monopoly. If $n = 1$ and $\psi(y) = \sqrt{y}$, the monopoly price will be $\frac{\varepsilon}{\varepsilon-1}c$ *plus* the additional mark-up $\frac{\varepsilon}{\varepsilon-1}\alpha^{mon}$, where α^{mon} satisfies the equation $\alpha^{mon} = 10\eta(10 + \alpha^{mon})^{-1.6}$, and $\eta := (25/3)^2 \cdot 2.25^{-1.6} = 18.97$, cf. (27). Thus, $\alpha^{mon} = 3.0956$, $\varepsilon c/(\varepsilon - 1) = 22.5$ and the monopoly price p^{mon} equals 29.47. If the number of firms n is increasing, then the solution value α_n^{sym} – as a function of n – will converge to 0, cf. Proposition 3, and the optimal price approaches $\varepsilon c/(\varepsilon - 1) = 22.5$.

Table 2 illustrates the dependence of profits $V_n^{sym} = \alpha_n^{sym}N$, revenues $\bar{U}_n^{sym}$, production costs $\bar{C}_n^{sym}$, advertising spending $\bar{W}_n^{sym}$, and market prices $\bar{p}_n^{sym}$ (last column of Table 2) on n. The numbers show that these quantities might drop substantially if more firms enter the market and the (fixed) number of consumers is spread equally among all firms.

Table 3 illustrates how the speed of sales, determined by $Z(n)$, see Theorem 3, depends on the number of competing firms and on the parameter δ, $\delta \in \{0, 0.2, 1\}$. The case $\delta = 0$ corresponds to the pure pricing model. It follows from Proposition 3 (iii) that in the case of

Table 2 Dependence of various characteristics of homogeneous firms on the number of competing firms n (symmetric case, $u(n) = 30/n$, Example 4.1)

n	V_n^{sym}	$n \cdot V_n^{sym}$	% Sales	$\bar{U}_n^{sym}$	$\bar{C}_n^{sym}$	$\bar{W}_n^{sym}$	$\bar{p}_n^{sym}$
1	309.56	309.56	100.00	808.59	274.42	224.61	29.47
2	90.07	180.14	50.00	286.39	116.77	79.55	24.53
3	42.98	128.94	33.30	145.15	61.85	40.32	23.47
4	25.24	100.96	25.00	87.42	37.90	24.28	23.07
5	16.62	83.12	20.00	58.32	25.50	16.20	22.87
10	4.43	44.32	10.00	15.84	7.01	4.40	22.60
20	1.15	22.96	5.00	4.12	1.83	1.15	22.53
100	0.05	4.73	1.00	0.17	0.08	0.05	22.50

Table 3 Symmetric Case II model, $u(n) = 30/n$: speed of sales $Z(n)$ and total profits as functions of n; Example 4.1

n	$\delta = 0$		$\delta = 0.2$		$\delta = 1$	
	nV_n^{sym}	$Z(n)$	nV_n^{sym}	$Z(n)$	nV_n^{sym}	$Z(n)$
1	845.64	0.03665	641.81	0.03475	309.56	0.03782
2	809.52	0.05991	586.55	0.05049	180.14	0.03047
3	772.73	0.07312	550.45	0.05707	128.94	0.02278
4	748.38	0.08112	527.72	0.06013	100.96	0.01787
5	731.96	0.08638	512.10	0.06166	83.12	0.01461
6	720.34	0.09007	500.58	0.06244	70.69	0.01233
7	711.73	0.09279	491.61	0.06282	61.52	0.01065
8	705.12	0.09488	484.37	0.06296	54.47	0.00937
9	699.90	0.09653	478.33	0.06297	48.87	0.00835
10	695.67	0.09786	473.19	0.06289	44.32	0.00754
20	676.16	0.10404	443.96	0.06096	22.96	0.00380
100	660.11	0.10916	391.37	0.05275	4.73	0.00076

a pure pricing model, $Z(n)$ is monotone increasing in the number of competing (symmetric) firms. If $\delta > 0$, $Z(n)$ is either a unimodal function (increasing, then decreasing) or a monotone decreasing function. Formula (22) reveals that, if $\delta > 0$, the postulated asymptotic behavior of $Z(n)$ follows from the fact that α_n^{sym} decreases to zero, should n converge to infinity.

The total producers surplus nV_n^{sym} decreases in n, and the rate of decrease is significantly larger, if δ is big. Thus, if advertising is possible, then competition is more intense compared to situations without advertising.

Next, we will study the case of heterogeneous firms.

Example 4.2 Let $\psi(y) = \sqrt{y}$, $\delta = 1$, $a = 2$, $N = 100$, and $n = 3$. If not chosen otherwise, the parameters $\varepsilon = 1.8$, $c = 10$, $k = 1$, $r = 0.1$, and $u = 10$ are the ones of the reference model, see. below.

Table 4 The impact of the brand image: different price elasticities ε_i; Example 4.2

i	ε_i	V_i	% Sales	$\bar{U}_i$	$\bar{C}_i$	$\bar{W}_i$	$\bar{p}_i$
1	1.7	79.46	48.24	244.93	93.43	72.04	26.22
2	1.8	42.55	31.65	143.80	61.30	39.94	23.46
3	1.9	23.03	20.11	84.10	38.94	22.13	21.60

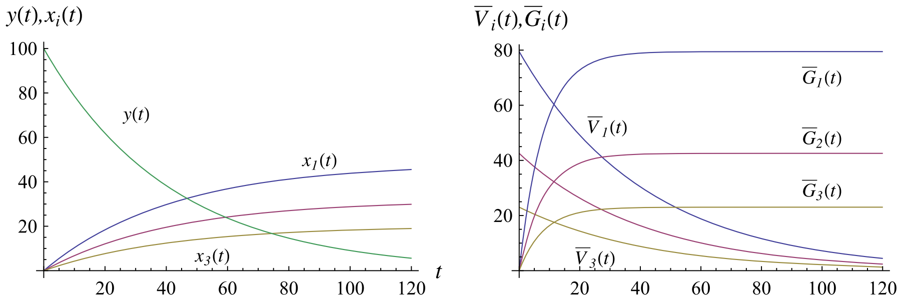

Fig. 3 Evolution of untapped market size y and individual market shares x_i (*left window*); the *right window* shows the evolution of profits $\bar{V}_i(t)$ und $\bar{G}_i(t) = \bar{V}_i(0) - e^{-rt}\bar{V}_i(t)$; Example 4.2 (different price elasticities)

Tables 4, see. below, and Tables 6, 7, 8 and 9, see Appendix, illustrate different scenarios of a 3-firm competition characterized by the saturation effect $\sqrt{y}$. Each table illustrates the dependence of quantities like profit V_i, revenue U_i, etc., on variations of just one of the five characteristics u_i, ε_i, c_i, r_i, and k_i. In all scenarios, Firm 2 represents the reference model, while Firm 1 always enjoys a competitive advantage over the other two firms. Firm 3 is always the one with a handicap.

For example, see Table 4, the price elasticity of Firm 1, $\varepsilon_1 = 1.7$, is smaller than $\varepsilon_2 = 1.8$ and $\varepsilon_3 = 1.9$. The smaller elasticity value 1.7, compared to 1.8 and 1.9, could be due to many reasons, e.g. a stellar image of the brand, good quality reputation, a product with special and attractive features, etc. The arrival intensities at all three locations are the same, $u_1 = u_2 = u_3 = 10$. If not chosen differently, we use the parameters specified above, see Example 4.2; $n = 3$ will be norm.

In the case of Example 4.2 with different elasticities, the firm with the smallest elasticity value will experience the highest profit, see Fig. 3, and will charge the highest price, cf. Fig. 4a. However, Firm 1 will spend more on advertising than its competitors do; it is trying to attract many shoppers to be turned into profitable buyers. Firm 3, at the other end of the spectrum, will set a low price, and its profit, 23.03, is less than a third of the profit of the top brand.

The study of different elasticities describes a typical oligopoly market consisting of a high quality firm, an average quality firm and a discounter. We observe three different (optimal) price levels and matching decreasing advertising expenditures. As expected, the high quality (or very reputable) firm experiences the largest profit as a result of the lowest price elasticity. However, other numerical examples show that the market shares of firms also critically depend on the (relative) magnitude of the production costs. For small values of c, the firm facing the smallest price elasticity will still set the highest price, but Firm 3, the one with the "handicap", might be gaining the biggest share of the market. Such parameter settings and solutions correspond to oligopoly markets where the top brand only sells a small number of

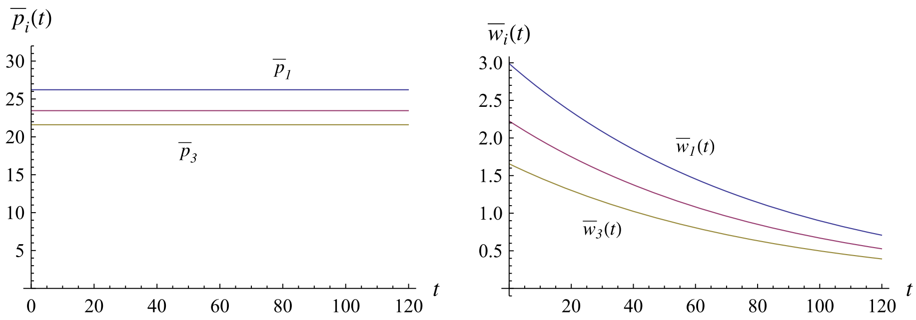

Fig. 4 Optimal prices (*left*) und advertising rates (*right*) over time; Example 4.2 (different elasticities)

high quality products, whereas a "discounter," following a low price strategy, captures most of the market. The business results of the "in between" firm, Firm 2, are usually—as to be expected—somewhere in between the two extremes.

The impact of variations of arrival rates u_i, of unit costs c_i, of discount rates r_i, and variations of advertising efficiency coefficients k_i are summarized in the Appendix, see Tables 6–9.

Remark 1 Case II models arise in many different applied contexts. Due to the non trivial relationship between market power values α_i and market shares z_i/Z, different phenomena can be observed, see. above. Many such phenomena, for instance, long terms market shares of duopolies/triopolies, can be explained by properly chosen parameter settings. Numerical studies reveal the nontrivial interplay of such (parameter) asymmetries. Such studies can be used to calibrate model parameters when analyzing specific market situations.

5 Conclusions

In this paper, we have developed and analyzed dynamic pricing and advertising oligopoly models for two specific market situations. These models allow us to study the competition for sales of (category) brands by any finite number of firms. The two classes of models generalize and complement the duopoly game analyzed by Krishnamoorthy et al. The formulas of the equilibrium (feedback) pricing and advertising strategies which we have derived offer *quantitative* insights into the dynamic of the competition for sales in particular oligopoly markets. Specifically, these insights comprise a full understanding of the impact on equilibrium prices of firms entering (the market) or leaving the market. From the point of view of an entering firm, this understanding puts it in the position to evaluate its chances in noncollusive competitive environments. From the point of view of a monopolist or a collusive oligopoly, the firm(s) can calculate and set a model-based limit price to prevent a firm from entering the market. The open-loop versions of our solution formulas make it possible to simply evaluate different scenarios, and predict the evolution of market share, revenue, cost, etc., over time of each company.

The main theoretical result of the paper is the existence of a unique feedback Nash equilibrium of oligopolies with special structure. The existence of such an equilibrium primarily depends on the number of firms, the price elasticity of demand and whether or not the individual marginal costs of competing firms are zero. We give a necessary and sufficient condition, the 'condition of the commons', for the existence of a unique equilibrium.

Furthermore, we have shown that the value function of each firm depends on a common market factor and a firm-specific coefficient. The market power coefficient reflects the (nontrivial) interplay of the various characteristic parameters of all firms, i.e., brand image, financing, and production costs, as well as technology and location factors. These coefficients can be used to identify and to evaluate competitive strengths and weaknesses of firms for specific applications.

Moreover, sensitivity results were derived. In particular, we have shown how the market power equilibrium is affected by the exit of firms and the entry of new firms. The results explain that the entry of a competitor—in symmetric, as well as asymmetric market situations—leads to lower equilibrium prices and a loss of market power of each firm.

Our analysis also highlights the interplay of dynamic pricing and advertising. Like in the case of a monopoly, in Case I and Case II oligopoly markets price adjustments are synchronized with advertising adjustments, and the benefit of advertisement is quantified by the leveraged price elasticity.

Appendix

see Appendix Tables 5, 6, 7, 8 and 9

Table 5 List of variables and parameters

T	Time horizon	λ_i	Rate of sales
N	Amount to sell/total market size	u_i	Arrival intensity/effectiveness
n	Number of firms ($i = 1, \ldots, n$)	$\psi(y)$	System function; adoption effect
V_i, W_i	Value functions	ε_i	Price elasticity
x_i	Amount sold by firm i	a	Advertising cost exponent
y	Items left to sell ($y = N - \sum x_i$)	δ	Advertising elasticity
r_i	Discount rate	k_i	Advertising cost parameter
p_i	Price asked	γ_i	Leveraged price elasticity
w_i	Advertising rate	α_i	Market power coefficient
c_i	Unit cost	$\beta(y)$	Inventory/market effect

Table 6 The importance of location: different arrival rates u_i; Example 4.2

i	u_i	V_i	% Sales	$\bar{U}_i$	$\bar{C}_i$	$\bar{W}_i$	$\bar{p}_i$
1	12	61.52	45.23	202.69	84.86	56.30	23.88
2	10	42.87	32.89	144.81	61.71	40.22	23.46
3	8	27.49	21.88	94.91	41.05	26.36	23.12

Table 7 Technology: different production costs c_i; Example 4.2

i	c_i	V_i	% Sales	$\bar{U}_i$	$\bar{C}_i$	$\bar{W}_i$	$\bar{p}_i$
1	8	59.91	50.01	194.04	80.23	53.90	19.35
2	10	42.19	30.34	142.66	60.84	39.63	23.45
3	12	31.58	19.65	109.21	47.29	30.34	27.71

Table 8 Financing costs: different discount rates r_i; Example 4.2

i	r_i	V_i	% Sales	$\bar{U}_i$	$\bar{C}_i$	$\bar{W}_i$	$\bar{p}_i$
1	0.08	51.29	32.71	171.28	72.41	47.58	23.65
2	0.10	42.99	33.39	145.20	61.87	40.33	23.47
3	0.12	37.00	33.90	126.01	54.01	35.00	23.33

Table 9 Advertising costs: different parameters k_i; Example 4.2

i	k_i	V_i	% Sales	$\bar{U}_i$	$\bar{C}_i$	$\bar{W}_i$	$\bar{p}_i$
1	0.8	53.41	39.71	177.84	75.03	49.40	23.70
2	1.0	42.80	32.62	144.59	61.62	40.16	23.46
3	1.2	35.70	27.67	121.82	52.27	33.84	23.30

Collection of Formulas

Optimality conditions:

$$p_i^*(y) = \frac{\varepsilon_i}{\varepsilon_i - 1}(c_i + V_i'(y)) \iff p_i^*(y) - c_i - V_i'(y) = \frac{1}{\varepsilon_i} p_i^*(y), \tag{23}$$

$$w_i^*(y) = \left(\frac{\delta}{\varepsilon_i}\frac{u_i}{k_i a}\psi(y) p_i^{-\varepsilon_i+1}\right)^{\frac{1}{a-\delta}} = \theta_i \psi(y)^{\frac{1}{a-\delta}} (c_i + V_i'(y))^{\frac{1-\varepsilon_i}{a-\delta}}, \tag{24}$$

where

$$\theta_i := \left(\delta u_i/(k_i \varepsilon_i a) \cdot \left(\varepsilon_i/(\varepsilon_i - 1)\right)^{-\varepsilon_i+1}\right)^{1/(a-\delta)}. \tag{25}$$

Auxilliary parameters:

$$\gamma_i := \frac{a\varepsilon_i - \delta}{a - \delta} \iff 1 - \gamma_i = \frac{-a\varepsilon_i + a}{a - \delta} \iff \frac{1-\gamma_i}{\gamma_i} = \frac{-a\varepsilon_i + a}{a\varepsilon_i - \delta}, \tag{26}$$

$$\eta_i := k_i \frac{a-\delta}{\delta} \left(\frac{\delta}{\varepsilon_i}\frac{u_i}{k_i a}\right)^{\frac{a}{a-\delta}} \left(\frac{\varepsilon_i}{\varepsilon_i - 1}\right)^{1-\gamma_i}. \tag{27}$$

Characterization of value functions by ODEs: $\beta = \beta(y)$, $\psi = \psi(y)$, etc.

$$r_i \alpha_i \beta = \left(\eta_i \left(c_i + \alpha_i \beta'\right)^{-\gamma_i+1} - \sum_{j \neq i} (\gamma_j - 1)\eta_j \left(c_j + \alpha_j \beta'\right)^{-\gamma_j} \cdot \alpha_i \beta'\right) \psi^{\frac{a}{a-\delta}}. \tag{28}$$

Bernoulli differential equation for $\beta(y)$ (market effect):

$$\beta(y)\beta'(y)^{\gamma-1} = \psi(y)^{\frac{a}{a-\delta}} \iff \beta(y)^{-1}\beta'(y)^{1-\gamma}\psi(y)^{\frac{a}{a-\delta}} \tag{29}$$

$$\iff \frac{\beta(y)}{\beta'(y)} = \beta(y)^{\frac{a\varepsilon-\delta}{a\varepsilon-a}} \psi^{\frac{-1}{\varepsilon-1}} = \psi^{\frac{a}{a-\delta}} \beta'(y)^{-\gamma}.$$

Case I. (Special form of (28)): $z_j = (\gamma - 1)\eta_j \alpha_j^{-\gamma}$,

$$r_i \alpha_i = \left(\eta_i \alpha_i^{1-\gamma} - \alpha_i \sum_{j \neq i} z_j\right) \underbrace{\beta'^{1-\gamma}/\beta\psi^{a/(a-\delta)}}_{1}. \tag{30}$$

Proofs

Proof of Lemma 1 We shall subdivide the proof into four main parts, (i) reducing the system of equations to the analysis of a particular nonlinear equation, (ii) existence of a solution of this particular equation, (iii) uniqueness, and (iv) dependence on n. Each part will be subdivided into several steps. In Part 1, we show how the analysis of the system of equations can be reduced to the analysis of a particular nonlinear equation in one unknown. The final part, Part 5, exhibits explicit solution formulas of the components α_i, if all cost parameters are zero. □

Part 1: Reducing the system to one equation

If there is a positive vector $\boldsymbol{\alpha} = (\alpha_i)_{1\leqslant i\leqslant n}$ such that

$$r_i = \eta_i(c_i+\alpha_i)^{-\gamma_i}(c_i/\alpha_i+1) - \sum_{j\neq i}\underbrace{(\gamma_j-1)\eta_j(c_j+\alpha_j)^{-\gamma_j}}_{z_j}, \tag{31}$$

then simple algebra shows that $\boldsymbol{\alpha}$ satisfies the system of equations,

$$r_i = \left(\frac{c_i}{\alpha_i}+\gamma_i\right)\eta_i(c_i+\alpha_i)^{-\gamma_i} - Z, \qquad i=1,2,\ldots,n, \tag{32}$$

where $Z := \sum_j z_j$, and z_i is defined in Lemma 1, see also (31). We define n real-valued functions f_i on the positive real line, $\xi > 0$, $i = 1, 2, \ldots, n$,

$$f_i(\xi) := \left(\frac{c_i}{\xi}+\gamma_i\right)\eta_i(c_i+\xi)^{-\gamma_i} - r_i. \tag{33}$$

If $\boldsymbol{\alpha}$ satisfies (32), then $f_1(\alpha_1) = f_2(\alpha_2) = \ldots = f_n(\alpha_n) = Z$. Observe, each function f_i is strictly monotone decreasing, $f_i(0+) = +\infty$, and $\lim_{\xi\to\infty} f_i(\xi) = -r_i$. We denote the unique root of each f_i by $\alpha_i^{(0)}$ and we consider the intervals $(0, \alpha_i^{(0)}]$, $i = 1, 2, \ldots, n$; when f_i is restricted to this interval, the range of f_i equals $[0, \infty)$. From now on - without loss of generality - we choose the function f_1 to work with. By definition, for any $\kappa \geqslant 0$, there is a unique ξ in $(0, \alpha_1^{(0)}]$ such that $\kappa = f_1(\xi)$, and, for $i = 2, \ldots, n$, there are unique positive numbers $f_i^{-1}(f_1(\xi)) =: \chi_i(\xi)$ in $(0, \alpha_i^{(0)}]$ such that

$$\kappa = f_1(\xi) = f_i(\chi_i(\xi)).$$

By construction, $\chi_i(\xi)$ are monotone increasing functions on $(0, \alpha_i^{(0)}]$. We are looking for positive vectors $\boldsymbol{\alpha} = (\alpha_i)_{1\leqslant i\leqslant n}$ which solve system (32). If $\boldsymbol{\alpha}$ is positive, then $\mathbf{z} = (z_j)_{1\leqslant j\leqslant n}$ is a positive vector and $Z = \sum_j(\gamma_j-1)\eta_j(c_j+\alpha_j)^{-\gamma_j}$ is a positive number; moreover, the values $f_i(\alpha_i)$ are positive as well. Since $f_i(\xi)$ is nonpositive whenever $\xi \geqslant \alpha_i^{(0)}$, any component α_i of a positive solution vector $\boldsymbol{\alpha}$ needs to be less than $\alpha_i^{(0)}$ Table 5.

To find the first component α_1 of a solution $\boldsymbol{\alpha}$, we define a particular nonnegative monotone decreasing function G on $(0, \alpha_1^{(0)}]$, namely

$$G(\xi) := \sum_j(\gamma_j-1)\eta_j(c_j+\chi_j(\xi))^{-\gamma_j}. \tag{34}$$

If α_1 is the first component of a solution of (32), then the equation $G(\alpha_1) = f_1(\alpha_1)$ has to be satisfied. We shall verify that the equation $G(\xi) = f_1(\xi)$ has at least one solution $\xi > 0$. In Part 3, see below, we will show that the equation has exactly one solution. If α_1 is known, then all other components of a positive solution vector $\boldsymbol{\alpha}$ are given by $\alpha_j = \chi_j(\alpha_1)$, $2 \leqslant j \leqslant n$.

Part 2: Existence (a necessary and sufficient condition)

To prove existence of a positive value α_1, we shall employ the Intermediate Value Theorem. To be specific, we verify that (a),

$$\lim_{\xi\to 0} f_1(\xi) > \lim_{\xi\to 0} G(\xi),$$

i. e. the graph of f_1 lies above the graph of G in a neighbourhood of zero, and (b), G dominates f_1 for large values ξ. The last statement is obvious, since

$$f_1(\alpha_1^{(0)}) = 0 < G(\alpha_1^{(0)}). \tag{35}$$

To see that f_1 dominates G in a neighbourhood of zero we observe that $\lim_{\xi\to 0}\chi_j(\xi) = 0$, $j = 1, 2, \ldots, n$. This last property follows from the fact that $\lim_{\xi\to 0} f_j^{-1}(f_1(\xi)) = 0$. The behavior of G in a neighborhood of zero depends on how many cost coefficients c_j are positive. If all c_j are positive, then $G(0+)$ is finite, while $f_1(0+) = \infty$. If there is at least one parameter c_j which is equal to zero, then $G(0+) = \infty$. To see that G stays below f_1, we analyze the quotient of both functions G and f_1. Applying l'Hopital's rule, we obtain

$$\frac{G'(\xi)}{f_1'(\xi)} = \frac{\sum_j (\gamma_j - 1)\eta_j(-\gamma_j)\big(c_j + f_j^{-1}(f_1(\xi))\big)^{-\gamma_j-1}\frac{f_1'(\xi)}{f_j'\left(f_j^{-1}(f_1(\xi))\right)}}{f_1'(\xi)}.$$

To streamline the expression on the right hand side of the last equation, we introduce the abbreviation $\xi_j := \chi_j(\xi) = f_j^{-1}(f_1(\xi))$. Taking the derivatives of all functions f_j, simple algebra yields

$$\begin{aligned}\frac{G'(\xi)}{f_1'(\xi)} &= \sum_j \frac{(\gamma_j - 1)\eta_j\gamma_j\big(c_j + \xi_j)\big)^{-\gamma_j-1}}{-f_j'(\xi_j)}\\ &= \sum_j \frac{(\gamma_j - 1)\eta_j\gamma_j\big(c_j + \xi_j)\big)^{-\gamma_j-1}}{\frac{c_j}{\xi_j^2}\eta_j(c_j+\xi_j)^{-\gamma_j} + \left(\frac{c_j}{\xi_j} + \gamma_j\right)\eta_j\gamma_j(c_j+\xi_j)^{-\gamma_j-1}}\\ &= \sum_j \frac{(\gamma_j - 1)\gamma_j}{(c_j/\xi_j)^2 + (c_j/\xi_j)(\gamma_j+1) + \gamma_j^2}.\end{aligned} \tag{36}$$

If ξ converges to zero, we obtain

$$\frac{G(0+)}{f_1(0+)} = \lim_{\xi\to 0}\frac{G'(\xi)}{f_1'(\xi)} = \sum_{\substack{j=1:\\ c_j=0}}^{n} \frac{\gamma_j - 1}{\gamma_j} = \begin{cases}\sum_j (1 - 1/\gamma_j)) \ , & \text{all } c_j = 0\\ 0 \ , & \text{all } c_j > 0\end{cases}. \tag{37}$$

Let $J_0 := \{j | c_j = 0, j = 1, ..., n\}$ and $H := \sum_{j\in J_0}(1 - 1/\gamma_j)$. Next, we consider the following two cases: $H \geqslant 1$ and $H < 1$. In the first case, the fact that the sum is greater or equal to one, together with (37), implies, $\xi > 0$,

$$\frac{G'(\xi)}{f_1'(\xi)} = \sum_{j=1}^{n}\frac{\gamma_j - 1}{\gamma_j} \geqslant \sum_{j\in J_0}\frac{\gamma_j - 1}{\gamma_j} = H \geqslant 1.$$

Thus, the (negative) slope of G is steeper than the (negative) slope of f_1. Since G dominates f_1 in a neighborhood of $\alpha_1^{(0)}$, and $G'(\xi) \leqslant f_1'(\xi) < 0$, the difference between G and f_1

will always be positive on $(0, \alpha_1^{(0)}]$. Hence, if $H \geqslant 1$, there will never be a solution of (31). On the other hand, (37) implies that the condition of the commons, $H < 1$, is necessary and sufficient for a solution of $G(\xi) = f_1(\xi)$ to exist. Observe that the strict inequality and (37) imply G to be dominated by f_1 in a neighborhood of zero. Together with (35), at least one positive solution α_1 exists. Recall, if all unit cost coefficients c_j are positive, then the condition of the commons is always satisfied.

Part 3: Uniqueness

Assume the condition of the commons holds true. Part 2 implies that there is at least one solution of the equation

$$G(\alpha_1) = f_1(\alpha_1). \tag{38}$$

Since the graph of f_1 starts above the graph of G and ends below it, the number of intersections of the two functions is odd. Should there be more than one solution of (38), then the functions G and f_1 intersect at least 3 times and, by the Mean Value Theorem, the functions have to have the same slope at two different locations. To see that this last statement contradicts (36), we distinguish between the following two possibilities, (i) all c_j are zero, and (ii) there is at least one positive c_j. If all c_j are zero, then $G'(\xi)/f_1'(\xi) = \sum_{j=1}^{n}(1 - 1/\gamma_j)$, and the slopes of f and G, at any ξ, are never the same if a solution of (38) exists.

Should there be at least one positive cost coefficient, say c_l, then $G'(\xi)/f_1'(\xi)$ is strictly monotone increasing, and there can be only one location where the slopes are the same. To see that the ratio G'/f_1' is a strictly monotone increasing function, observe that each denominator of the terms of the right hand side of (36) is decreasing in the variable $\xi_j = \chi_j(\xi_1)$, and the term involving the variable ξ_l is strictly decreasing in ξ_l. Moreover, $\chi_j(\xi_1)$ is monotone increasing in ξ_1. Hence, the ratio $G'(\xi_1)/f_1'(\xi_1)$ is a strictly increasing function too. Thus, the uniqueness of a positive solution α_1 of (38), as well as the uniqueness of a positive solution vector $\boldsymbol{\alpha}$ of (31) follows.

Part 4: Dependence on the number of firms

If there are $n + 1$ equations of the form (10), we shall add one additional function f_{n+1} to the family of functions $f_1, ..., f_n$, cf. (33). Notice that in Part I of the existence proof, we are free to choose any element of $f_1, ..., f_{n+1}$ to be used for the construction of the solution $\boldsymbol{\alpha}^*(n + 1)$; w.l.o.g. we will choose f_1 for both systems of equations. Observe, the function G, see (34), is isotone in the number of equations, i.e., G based on $n + 1$ terms dominates the sum with only n terms. Thus, $\alpha_1(n + 1) < \alpha_1(n)$. Since the functions χ_j, $j = 2, ..., n$, are strictly monotone increasing, the claim follows.

Part 5: Explicit Solution Formulas (all unit costs $\boldsymbol{c_j}$ are zero)

To conclude, we display the explicit solution of equation (32) if all $c_j = 0$. If all cost coefficients are zero, (32) simplifies, and we get $Z = f_i(\alpha_i) = \gamma_i \eta_i \alpha_i^{-\gamma_i} - r_i = \frac{\gamma_i}{\gamma_i - 1} z_i - r_i$. This system of equations is equivalent to the system

$$z_i = \frac{\gamma_i - 1}{\gamma_i}(Z + r_i), \quad i = 1, 2, \ldots, n. \tag{39}$$

Taking the sum of all z_i, we obtain

$$Z = \sum_i \frac{\gamma_i - 1}{\gamma_i} r_i \Bigg/ \left(1 - \sum_i \frac{\gamma_i - 1}{\gamma_i}\right) > 0. \tag{40}$$

Since all $c_i = 0$, using the definition of α_i in terms of z_i, see above, and using (39) and (40), we can express the unique solution value α_i in terms of the parameter u_i, k_i, and all r_j, γ_j, $j = 1, 2, \ldots, n$:

$$\alpha_i = \left(\frac{z_i}{\eta_i(\gamma_i - 1)}\right)^{-1/\gamma_i} = \left(\frac{Z + r_i}{\eta_i \gamma_i}\right)^{-1/\gamma_i}. \tag{41}$$

□

Proof of Lemma 2 Elementary calculations show that $\beta(y) = B(y)^{1-1/\gamma}$ is a solution of the Bernoulli Eq. (14), see Lemma 2. The facts that β and B are increasing functions are an immediate consequence of formula (13). It remains to show that $\beta(y)$ is concave on $[0, N]$, if the condition $\psi'(y)\psi(y)^{\frac{-\varepsilon}{\varepsilon-1}} B(y) < \frac{1-\delta}{a}$ holds true. Since $\beta'(y) = B(y)^{-\frac{1}{\gamma}} \psi(y)^{\frac{1}{\varepsilon-1}} \psi'(y)$, the second derivative of β is given by $\beta''(y) = \frac{-1}{\gamma-1} B(y)^{\frac{-1-\gamma}{\gamma}} \psi(y)^{\frac{2}{\varepsilon-1}} + B(y)^{\frac{-1}{\gamma}} \frac{1}{\varepsilon-1} \psi(y)^{\frac{2-\varepsilon}{\varepsilon-1}} \psi'(y)$, and the assertion follows. □

Proof of Theorem 1 By definition, see Sect. 2.1, $\dot{y} = -\lambda$, where $\lambda = \sum_{i=1}^n \lambda_i$. Using the optimality conditions (23) and (24), we obtain

$$\begin{aligned}
\lambda_i^*(y) &= u_i w_i^{*\delta}(y) p_i^{*-\varepsilon}(y)\psi(y) \\
&= u_i \left(\frac{\varepsilon}{\varepsilon - 1}\right)^{\frac{-\varepsilon a+\delta}{a-\delta}} \left(\frac{\delta}{\varepsilon} \frac{u_i}{a}\right)^{\frac{\delta}{a-\delta}} \psi(y)^{\frac{a}{a-\delta}} (c_i + \partial_y V_i(y))^{\frac{-\varepsilon_i a+\delta}{a-\delta}} \\
&= \frac{a\varepsilon_i - a}{a - \delta} \eta_i \psi(y)^{\frac{a}{a-\delta}} (c_i + \alpha_i \beta'(y))^{-\gamma} \\
&\overset{Case\ I}{=} (\gamma - 1)\eta_i \psi(y)^{\frac{a}{a-\delta}} \alpha_i^{-\gamma} \beta'(y)^{-\gamma} \\
&= z_i \frac{\beta(y)}{\beta'(y)} = z_i B(y)\psi(y)^{\frac{-1}{\varepsilon-1}}.
\end{aligned}$$

Taking the sum of all $\lambda_i^*(y)$, we obtain

$$\lambda(y) = Z \frac{\beta(y)}{\beta'(y)}. \tag{42}$$

Since the Bernoulli differential equation (14) can be equivalently written as

$$\frac{\beta(y)}{\beta'(y)} = B(y)\psi(y)^{\frac{-1}{\varepsilon-1}},$$

elementary transformations yield the formula of λ. To see (17), evaluate (42) along an optimal trajectory $y(t)$. Multiplying (42) by $\beta'(y(t))$ yields the differential equation

$$\dot{\overbrace{\beta(y(t))}} = \beta'(y(t))\dot{y}(t) = -Z\beta(y(t)).$$

Since $y(0) = N$, we obtain the formula $\beta(y(t)) = \beta(N)e^{-Zt}$. Since B is strictly increasing, (17) follows. Since $-\frac{\dot{y}}{Z} = \frac{\beta(y)}{\beta'(y)}$, cf. (42), integrating the individual rates $\lambda_i^*(y(t))$ yields the accumulated sales of each company

$$x_i(t) = \int_0^t \lambda_i^*(y(s))ds = z_i \int_0^t \frac{\beta(y(s))}{\beta'(y(s))}\, ds = -\frac{z_i}{Z} \int_0^t \dot{y}(s)ds = \frac{z_i}{Z}\,(N - y(t)).$$

□

Proof of Theorem 3 It follows from the proof of Theorem 1 that, $1 \leqslant i \leqslant n$,

$$\lambda_i^*(y) = \frac{a\varepsilon_i - a}{a-\delta}\eta_i \psi(y)^{\frac{a}{a-\delta}}(c_i + \alpha_i \beta'(y))^{\frac{-\varepsilon_i a+\delta}{a-\delta}} \overset{Case\ II}{=} z_i y.$$

Taking the sum of the individual rates of sales implies

$$\lambda^*(y(t)) = y(t)Z = -\dot{y}(t), \quad \text{where } y(0) = N.$$

The solution of this elementary differential equation is $y(t) = e^{-Zt}N$.

When integrating the individual rates $\lambda_i^*(y)$ we get

$$x_i(t) = \int_0^t \lambda_i^*(y(s))ds = z_i \int_0^t y(s)\, ds = z_i \int_0^t e^{-Zs} N\, ds = Nz_i/Z(1 - e^{-Zt}).$$

□

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