

Cooperative Advertising in a Dynamic Retail Market Oligopoly

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Abstract Cooperative advertising is a key incentive offered by a manufacturer to influence retailers' promotional decisions. We study cooperative advertising in a dynamic retail oligopoly where a manufacturer sells his product through N competing retailers. We model the problem as a Stackelberg differential game in which the manufacturer announces his shares of advertising costs of the N retailers or his subsidy rates, and the retailers in response play a Nash differential game in choosing their optimal advertising efforts over time. We obtain the feedback equilibrium solution consisting of the optimal advertising policies of the retailers and manufacturer's subsidy rates. We identify key drivers that influence the optimal subsidy rates and, in particular, obtain the conditions under which the manufacturer will not support the retailers. For the special case of two retailers we obtain insights on some key supply chain issues. First, we analyze its impact on profits of channel members and the extent to which it can coordinate the channel. Second, we investigate the case of an anti-discrimination act which restricts the manufacturer to offer equal subsidy rates to the two retailers.

Keywords Cooperative advertising · Nash differential game · Stackelberg differential game · Sales-advertising dynamics · Sethi model · Feedback Stackelberg equilibrium · Retail level competition · Channel coordination · Robinson–Patman act

1 Introduction

Cooperative advertising is a common means by which a manufacturer incentivizes its retailers to advertise its product to increase its sales. In a typical arrangement, the manufacturer

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contributes a percentage of a retailer's advertising expenditures to promote the product. We consider a marketing channel involving a manufacturer and N retailers. We model the problem of the channel as a Stackelberg differential game in which the manufacturer acts as the leader by announcing its subsidy rate, i.e. its share of retailer's advertising cost to each of the N retailers who act as followers and play a Nash differential game to obtain their optimal advertising efforts in response to the support offered by the manufacturer.

Cooperative advertising is a fast increasing activity in retailing amounting to billions of dollars a year. Nagler [21] found that the total expenditure on cooperative advertising in 2000 was estimated at \$15 billion, compared with \$900 million in 1970, and according to some recent estimates, it was higher than \$25 billion in 2007. Cooperative advertising can be a significant part of the manufacturer's expense according to [7], and as many as 25–40 % of local advertisements and promotions are cooperatively funded. In addition, Dutta et al. [9] report that the subsidy rates differ from industry to industry: it is 88.38 % for consumer convenience products, 69.85 % for other consumer products, and 69.29 % for industrial products.

Many researchers in the past have used static models to study cooperative advertising. Berger [3] modeled cooperative advertising in the form of a wholesale price discount offered by the manufacturer to its retailer as an advertising allowance. He concluded that both the manufacturer and the retailer can do better with cooperative advertising. Dant and Berger [7] extended the Berger's model to incorporate demand uncertainty and considered a scenario when the manufacturer and its retailer have a different opinion on anticipated sales. Kali [19] examined cooperative advertising subsidy with a threshold minimum advertised price by the retailer, and found that the channel can be coordinated in this case. Huang et al. [15] allowed for advertising by the manufacturer in addition to cooperative advertising, and justified their static model by making a case for short-term effects of promotion.

Jørgensen et al. [16] formulated a dynamic model with cooperative advertising, as a Stackelberg differential game between a manufacturer and a retailer with the manufacturer as the leader. They considered short-term as well as long-term forms of advertising efforts made by the retailer as well as the manufacturer. They showed that the manufacturer's support of both types of retailer advertising benefits both channel members more than support of just one type, which in turn is more beneficial than no support at all. Jørgensen et al. [17] modified the above model by introducing decreasing marginal returns to goodwill and studied two scenarios: a Nash game without advertising support and a Stackelberg game with support from the manufacturer as the leader. Jørgensen et al. [18] explored the possibility of advertising cooperation even when the retailer's promotional efforts may erode the brand image. Karray and Zaccour [20] extended the above model to consider both the manufacturer's national advertising and the retailer's local promotional effort. All of these papers use the Nerlove and Arrow [23] model, in which the rate of change of goodwill w.r.t. time increases linearly in advertising and decreases linearly in goodwill, and there is no interacting term between sales and advertising effort in the dynamics of sales.

He et al. [12] solved a manufacturer–retailer Stackelberg game with cooperative advertising using the stochastic sales-advertising model proposed by Sethi (see [26]), in which the effectiveness of advertising in increasing sales decreases as sales increase. The Sethi model was validated empirically by [5] and [22]. Despite the presence of the interactive term involving sales and advertising, [12] were able to obtain a feedback Stackelberg solution for the retailer's optimal advertising effort and the manufacturer's subsidy rate, and provided a condition for positive subsidy by the manufacturer. This paper extends the work of [12] to allow for retail level competition and provides useful managerial insights on the impact of this competition on the manufacturer's decision. It contributes to the cooperative advertising literature in the following ways.

Table 1 A comparison of various research papers based on issues addressed

Issues addressed	Papers										
	[3]	[7]	[19]	[15]	[16]	[17]	[18]	[20]	[12]	[13]	This paper
Dynamic model					✓	✓	✓	✓	✓	✓	✓
Advertising interacting with sales									✓		✓
Stochastic demand	✓			✓					✓		
Feedback strategy					✓	✓	✓	✓	✓	✓	✓
Retail level competition			✓							✓	✓
Channel performance & coordination	✓	✓	✓	✓	✓				✓		✓
Anti-discrimination legislation											✓

Firstly, most of the cooperative advertising literature uses a one manufacturer, one retailer setting, with the exception of [13, 14]. He et al. [13] study a duopoly retail market with two symmetric retailers and [14] consider the oligopoly case and retailer asymmetry. Their formulations are based on the Lanchester-type extensions of the Sethi model (see [27] and [24]), in which the two competitors split the total market. Moreover, in [13, 14], the manufacturer sells through only one retailer and the remaining retailer(s) act as outside competition. We use Erickson’s [10] oligopolistic extension of the Sethi model, in which the manufacturer sells through N independent and competing retailers and the competitors could increase their shares of a given total potential market at the same time. In [13, 14], only one retailer is supported by the manufacturer at any time. In our model, however, more than one retailer can be supported simultaneously. We formulate the model as a Stackelberg differential game between the manufacturer as the leader and the retailers as the followers. Furthermore, the retailers competing for market share play a Nash game between themselves. While our model is considerably more complicated, we are still able to obtain, like in [12], a feedback equilibrium solution, sometimes explicitly and sometimes by numerical means. We also explore the threshold conditions under which no retailer will be supported by the manufacturer and a non-cooperative solution is optimal. We consider the case when only one retailer, say retailer 1, sells from this manufacturer and the other $N - 1$ retailers compete with retailer 1 and the manufacturer for the market share. In this case we investigate the impact of retail level competition on the manufacturer’s tendency to support retailer 1.

Secondly, for a case of retailer duopoly ($N = 2$), we investigate in detail the issue of supply chain coordination with cooperative advertising. We study the extent to which the supply chain can be coordinated with cooperative advertising, and its effect on the profits of all the parties in the supply chain. We find that cooperative advertising can coordinate the channel, but interestingly, if the difference between the subsidy rates to the two retailers is large enough, the channel can perform worse than without cooperative advertising. Finally, for the case of two retailers, we contribute by investigating the case of anti-discrimination legislation (such as the Robinson–Patman Act against price discrimination), under which the manufacturer is required to offer equal subsidy rates to the two retailers. We obtain feedback Stackelberg equilibrium in this case and compare the optimal common subsidy rate with the two optimal subsidy rates without such an act. We also investigate the impact of such acts on the profits of manufacturer, the retailers and the overall supply chain, and subsequently, its role in coordinating the supply chain. Table 1 summarizes a comparison of several research papers in the cooperative advertising area, including ours, and positions our paper on the basis of various issues that are self-explanatory.

The rest of the paper is organized as follows. We describe our model in Sect. 2 and present preliminary results in Sect. 3. We obtain explicit analytical results for a special case of identical retailers in Sect. 4. We perform numerical analysis for the general case in Sect. 5. In the next two sections we discuss supply chain issues like channel performance and coordination, and impact of an anti-discrimination legislation. Due to the analytical complexity of the general model, it is very difficult to analyze these issues with N retailers. We therefore present these segments for retailer duopoly. In Sect. 6, for a retailer duopoly, we discuss the issue of channel coordination brought about by cooperative advertising and analyze its effect on the manufacturer's and retailers' profits. In Sect. 7, for a retailer duopoly, we present an extension in which the manufacturer is required to offer equal subsidy rates, if any, to both retailers. We conclude the paper in Sect. 8. Proofs of some of the results are relegated to Appendices A, B and C.

2 The Model

We consider a dynamic market channel where a manufacturer sells its product through N competing retailers, labeled $1, 2, \dots, N$. The manufacturer may choose to subsidize the advertising expenditures of the retailers. The subsidy, expressed as a fraction of a retailer's total advertising expenditure, is referred to as the manufacturer's subsidy rate for that retailer. We use the following notation in the paper:

t	Time $t \in [0, \infty)$;
i	Indicates retailer i , $i = 1, 2, \dots, N$, when used as a subscript;
$x_i(t) \in [0, 1]$	Retailer i 's proportional market share;
$\bar{x}(t)$	$= \sum_{j=1}^N x_j(t)$, Combined market share of N retailers;
$X(t)$	$\equiv (x_1(t), x_2(t), \dots, x_N(t))$, Market share vector of N retailers at time t ;
$u_i(t)$	Retailer i 's advertising effort rate at time t ;
$\theta_i(t) \geq 0$	Manufacturer's subsidy rate for retailer i at time t ;
$\Theta(X(t))$	$\equiv (\theta_1(X(t)), \dots, \theta_N(X(t)))$, Subsidy rate vector in feedback form at time t ;
$\rho_i > 0$	Advertising effectiveness parameter of retailer i ;
$\delta_i \geq 0$	Market share decay parameter of retailer i ;
$r > 0$	Discount rate of the manufacturer and the retailers;
$m_i \geq 0$	Gross margin of retailer i ;
$M_i \geq 0$	Gross margin of the manufacturer from retailer i ;
V_i, V_m	Value functions of retailer i and of the manufacturer, respectively;
V	Value function of the integrated channel.

Also, $V_{ix_j} = \partial V_i / \partial x_j$, $V_{mx_i} = \partial V_m / \partial x_i$, and $V_{x_i} = \partial V / \partial x_i$, for $i = 1, 2, \dots, N$, $j = 1, 2, \dots, N$. Furthermore, for simplicity, $X(t)$ and $\Theta(X(t))$ are also written as X , and $\Theta(X)$, respectively.

The state of the system is represented by the market share vector (x_1, x_2, \dots, x_N) , so that the state at time t is $(x_1(t), x_2(t), \dots, x_N(t))$, which is also written as $X(t)$, or simply X . The sequence of events is as follows. First, the manufacturer announces the subsidy rate $\theta_i(t)$ for retailer i , $i = 1, 2, \dots, N$, $t \geq 0$. In response, the retailers choose their respective advertising efforts $u_i(t)$, $i = 1, 2, \dots, N$, in order to compete for market share. This situation is modeled as a Stackelberg game between the manufacturer as the leader and the retailers as followers and a Nash differential game between the retailers; see Fig. 1. The solution concept we employ is that of a feedback Stackelberg equilibrium. The cost of advertising is assumed to be quadratic in the advertising effort, signifying a marginal diminishing effect of advertising.

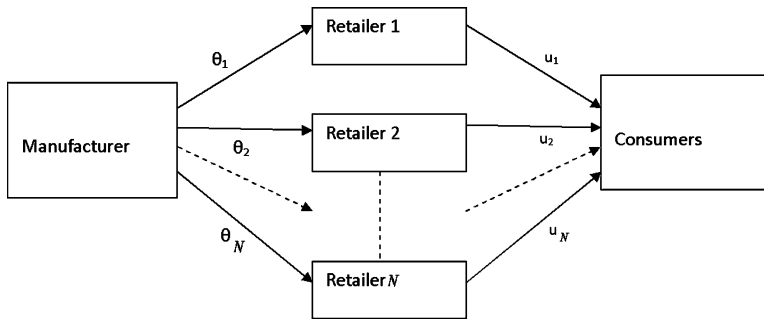


Fig. 1 Sequence of events

Given the subsidy rates θ_i , the retailer i 's advertising expenditure is $(1 - \theta_i)u_i^2$. The total advertising expenditure for the manufacturer is $\sum_{i=1}^N \theta_i u_i^2$. The quadratic cost function is common in the literature (see, e.g., [4, 8, 10, 11, 16, 24] and [12]).

To model the effect of advertising on sales over time, we use an oligopolistic extension of the Sethi model (see [26]), proposed by [10]. This extension is different from the duopolistic extensions of the Sethi model studied by [24, 25, 27] and [13], where the competitors split a given total market. Here, a gain in the market share of one retailer comes from an equal loss of the market share of the other. In contrast, the Erickson extension permits even a simultaneous increase of the retailers' shares of a given total market potential. Moreover, [10] used his extension to study the competition between Anheuser-Busch, SABMiller, and Molson Coors in the beer industry.

We use the Erickson extension as our market share dynamics:

$$\dot{x}_i(t) = \rho_i u_i \sqrt{1 - \bar{x}(t)} - \delta_i x_i(t), \quad x_i(0) = x_i \in [0, 1], \tag{1}$$

where, for $i = 1, 2, \dots, N$, $x_i(t)$ is the fraction of the total market captured by retailer i at time t , $u_i(t)$ is retailer i 's advertising effort at time t , ρ_i is the effectiveness of retailer i 's advertising effort, and δ_i is the rate at which market share is lost by retailer i due to factors such as competition, obsolescence, customers switching to other substitutable products, forgetting, etc.

Because the total market share captured by the manufacturer is $x_1(t) + x_2(t) + \dots + x_N(t) = \bar{x}(t)$ at any time t , the advertising effort of a retailer acts upon the *square-root* of the uncaptured market potential, i.e., $(1 - \bar{x}(t))$. This is a key distinguishing feature of the models which are extensions of the Sethi model, from the classical Vidale–Wolfe model (see [28]) where the advertising effort acts simply upon the uncaptured market potential. Some justification of the square-root feature and its empirical validation can be found in [5, 22, 23, 27], and [10, 11]. Furthermore, the advertising effort u_i is subject to marginally diminishing returns modeled by having its cost as u_i^2 , $i = 1, 2, \dots, N$. Finally, the subsidy rates do not affect the market share dynamics, as they simply reflect the internal cost sharing arrangements between the manufacturer and the retailers.

Since we are interested in obtaining a feedback Stackelberg solution, the manufacturer announces his subsidy rate policy $\Theta(X)$ as functions of the market share vector X . This means that the subsidy rates at time $t \geq 0$ are $\theta_i(x_1(t), x_2(t), \dots, x_N(t))$, respectively, for $i = 1, 2, \dots, N$. The retailers then choose their optimal advertising efforts by solving their respective optimization problems in order to maximize the present value of their respective

profit streams over the infinite horizon. Thus, retailer i 's optimal control problem is

$$V_i(X) = \max_{u_i(t) \geq 0, t \geq 0} \int_0^\infty e^{-rt} (m_i x_i(t) - (1 - \theta_i(X)) u_i^2(t)) dt, \quad i = 1, 2, \dots, N, \quad (2)$$

subject to (1), where we stress that $x_i, i = 1, 2, \dots, N$, are initial conditions, which can be any given values satisfying $x_i \geq 0, \forall i = 1, 2, \dots, N$, and $\sum_{i=1}^N x_i \leq 1$. Since retailer i 's problem is an infinite horizon optimal control problem, we can define $V_i(x_1, x_2, \dots, x_N) \equiv V_i(X)$ as his so-called value function. In other words, $V_i(x_1, x_2, \dots, x_N)$ also denotes the optimal value of the objective function of retailer i at a time $t \geq 0$, so long as $x_i(t) = x_i, i = 1, 2, \dots, N$, at that time. It should also be mentioned that the problem (1)–(2) is a Nash differential game, whose solution will give retailer i 's feedback advertising effort, expressed with a slight abuse of notation as $u_i(x_1, x_2, \dots, x_N | \theta_1(x_1, x_2, \dots, x_N), \dots, \theta_N(x_1, x_2, \dots, x_N)) \equiv u_i(X | \Theta)$, respectively, for $i = 1, 2, \dots, N$.

The manufacturer anticipates the retailers' optimal responses and incorporates these into his optimal control problem, which is also a stationary infinite horizon problem. Thus, the manufacturer's problem is given by

$$V_m(X) = \max_{\substack{0 \leq \theta_i(t) \leq 1, i=1, 2, \dots, N \\ t \geq 0}} \int_0^\infty e^{-rt} \sum_{j=1}^N [M_j x_j(t) - \theta_j(t) [u_j(X(t) | \theta_1(t), \dots, \theta_N(t))]^2] dt, \quad (3)$$

subject to, for $i = 1, 2, \dots, N$,

$$\dot{x}_i(t) = \rho_i u_i(X(t) | \theta_1(t), \dots, \theta_N(t)) \sqrt{1 - \bar{x}} - \delta_i x_i(t), x_i(0) = x_i \in [0, 1]. \quad (4)$$

Here, with an abuse of notation, $\theta_i(t), i = 1, 2, \dots, N$, denote the subsidy rates at time $t \geq 0$ to be obtained. Solution of the control problem (3)–(4) yields the optimal subsidy rates policy in feedback form expressed as $\theta_i^*(x_1, x_2, \dots, x_N) \equiv \theta_i^*(X), i = 1, 2, \dots, N$. Furthermore, we can express retailer i 's feedback advertising policy, with an abuse of notation, as $u_i^*(X) \equiv u_i^*(X | \theta_1^*(X), \dots, \theta_N^*(X)) \equiv u_i^*(X | \Theta^*(X)), i = 1, 2, \dots, N$.

The policies $\theta_i^*(X)$ and $u_i^*(X), i = 1, 2, \dots, N$, constitute a feedback Stackelberg equilibrium of the problem (1)–(4), which is time consistent, as opposed to an open-loop Stackelberg equilibrium, which, in general, is not. Substituting these policies into the state equations in (1) yields the market share process $X^*(t) \equiv (x_1^*(t), x_2^*(t), \dots, x_N^*(t)), t \geq 0$, and the respective decisions, with notational abuses, as $\theta_i^*(t) = \theta_i^*(X^*(t))$ and $u_i^*(t) = u_i^*(X^*(t)), t \geq 0, i = 1, 2, \dots, N$.

3 Preliminary Results

We first solve the problem of retailer i to find the optimal advertising policy $u_i^*(X | \Theta(X))$, given the subsidy policies $\theta_j(X), j = 1, 2, \dots, N$, announced by the manufacturer. The Hamilton–Jacobi–Bellman (HJB) equations for the value functions $V_i(X), i = 1, 2, \dots, N$, are

$$r V_i(X) = \max_{u_i \geq 0} \left[m_i x_i - (1 - \theta_i(X)) u_i^2 + \sum_{j=1}^N V_{ix_j} (\rho_j u_j \sqrt{1 - \bar{x}} - \delta x_j) \right], \quad (5)$$

where V_{ix_j} can be interpreted as the marginal increase in the total discounted profit of retailer $i, i = 1, 2, \dots, N$, with respect to an increase in the market share of retailer $j, j = 1, 2, \dots, N$.

Remark 1 Before proceeding further, we note that while we restrict $\theta_i(X), i = 1, 2, \dots, N$, to be nonnegative, it should be obvious that $0 \leq \theta_i(X) < 1, \forall i = 1, 2, \dots, N$. This is because, were the optimal $\theta_j(X) \geq 1$, for any $j = 1, 2, \dots, N$, retailer j would set $u_j(X)$ to be infinitely large, resulting in the value function of the manufacturer to be $-\infty$. This would mean that the manufacturer, who is the leader, would have even less profit than he would by setting $\theta_j(X) = 0$. This leads to a contradiction, and proves that $\theta_i(X) < 1, \forall i = 1, 2, \dots, N$. Thus, in what follows, any positive $\theta_i(X)$ will be an interior solution satisfying $\theta_i(X) < 1, i = 1, 2, \dots, N$.

We can now prove the following result characterizing the optimal advertising policy given the subsidy rates of the manufacturer.

Proposition 1 *For a given subsidy rate policy $\theta_i(X), i = 1, 2, \dots, N$, the optimal feedback advertising decision of retailer i is*

$$u_i^* = u_i^*(X | \Theta) = \frac{V_{ix_i} \rho_i \sqrt{1 - \bar{x}}}{2(1 - \theta_i(X))}, \quad i = 1, 2, \dots, N, \tag{6}$$

and the value function $V_i(X)$ satisfies

$$rV_i(X) = m_i x_i - \sum_{j=1}^N \delta_j x_j V_{ix_j} + \frac{V_{ix_i}^2 (-1 + \bar{x}) \rho_i^2}{4(-1 + \theta_i(X))} + \sum_{j \neq i} \left[\frac{V_{jx_j} V_{ix_j} (-1 + \bar{x}) \rho_j^2}{2(-1 + \theta_j(X))} \right]. \tag{7}$$

Proof Since the cost of advertising effort u_i is u_i^2 , it is clear that there is no sense in having a negative $u_i, i = 1, 2, \dots, N$. Thus, we can use the first-order conditions w.r.t. $u_i, i = 1, 2, \dots, N$, in (5) to obtain (6), and use (6) in (5) to obtain (7). We can also see that $V_i(X)$ given by (5) is concave in u_i , thus satisfying the second-order condition w.r.t. u_i .

We see that the advertising effort by retailer i is proportional to the marginal benefit of his own market share, $i = 1, 2, \dots, N$. Moreover, the higher is the uncaptured market $(1 - \bar{x})$, the greater is the advertising effort by retailer i .

After each retailer decides on an optimal response to the manufacturer’s announced subsidy rates policy, the manufacturer solves his problem, taking into account the retailers’ choices, and decides the optimal subsidy rates. The HJB equation for the manufacturer’s value function $V_m(X)$ is

$$rV_m(X) = \max_{\theta_i \geq 0, i=1,2,\dots,N} \sum_{i=1}^N [M_i x_i - \theta_i u_i^{*2} + V_{mx_i} (\rho_i u_i^* \sqrt{1 - \bar{x}} - \delta_i x_i)].$$

Using (6), we can rewrite the above HJB equation as

$$rV_m(X) = \max_{\theta_i \geq 0, i=1,2,\dots,N} \sum_{i=1}^N \left[M_i x_i - \delta_i x_i V_{mx_i} + \frac{V_{ix_i} (1 - \bar{x}) (2V_{mx_i} (1 - \theta_i) - V_{ix_i} \theta_i) \rho_i^2}{4(-1 + \theta_i)^2} \right]. \tag{8}$$

We can now state the following result characterizing the manufacturer’s optimal subsidy rates policy. □

Proposition 2 *The manufacturer’s optimal subsidy rates are*

$$\theta_i^*(X) = \max\{\hat{\theta}_i(X), 0\}, \quad i = 1, 2, \dots, N, \tag{9}$$

where

$$\hat{\theta}_i(X) = \frac{2V_{mx_i} - V_{ix_i}}{2V_{mx_i} + V_{ix_i}}, \quad i = 1, 2, \dots, N, \tag{10}$$

and the manufacturer’s value function $V_m(X)$ satisfies

$$rV_m(X) = \sum_{j=1}^N \left[M_j x_j - \delta_j x_j V_{mx_j} + \frac{V_{jx_j}(1 - \bar{x})(2V_{mx_j}(1 - \theta_j^*(X)) - V_{jx_j}\theta_j^*(X))\rho_j^2}{4(-1 + \theta_j^*(X))^2} \right]. \tag{11}$$

Proof The first-order conditions w.r.t. $\theta_i, i = 1, 2, \dots, N$, in (8) give a unique solution, i.e., $\hat{\theta}_i$ as shown in (10). We then, in view of Remark 1, characterize the optimal subsidy rate policy as in (9). Using (9) in (8) gives (11). To verify the second-order conditions w.r.t. $\theta_i, i = 1, 2, \dots, N$, we first find the Hessian matrix, i.e., $\frac{\partial^2 V_m(X)}{\partial \theta_i \partial \theta_j}, i = 1, 2, \dots, N, j = 1, 2, \dots, N$, from (8). We see that $\frac{\partial^2 V_m(X)}{\partial \theta_i \partial \theta_j} = 0$ whenever $i \neq j$, and $\frac{\partial^2 V_m(X)}{\partial \theta_i^2} < 0$ when we use $\theta_i = \hat{\theta}_i, i = 1, 2, \dots, N$. We thus see that this Hessian is negative definite when $\theta_i = \hat{\theta}_i, i = 1, 2, \dots, N$. Thus, the optimal subsidy rates policy given by (9)–(10) is a maximum.

A number of important insights follow from our analysis thus far. Equation (10) says that the optimal subsidy rate offered by the manufacturer to a retailer increases as the manufacturer’s marginal profit, with respect to the market share of that retailer increases. Thus, the manufacturer provides more support to the retailer who offers a higher marginal profit from his market share to the manufacturer. The increased subsidy from the manufacturer would also increase the retailer’s advertising effort. On the other hand, as a retailer’s own marginal profit from his own market share increases, then the subsidy rate offered by the manufacturer to that retailer decreases. The intuition behind this result is that the manufacturer would lower his subsidy to the retailer from the knowledge that the retailer has his own incentive to increase his very profitable market share anyway by advertising at a higher rate. Thus, we can expect that the advertising effort by the retailer will increase with the marginal profit of the retailer as well as that of the manufacturer from that retailer. This is confirmed by the following relation, which is derived from (6) and (10):

$$u_i^*(X) = \frac{\rho_i(2V_{mx_i} + V_{ix_i})\sqrt{1 - \bar{x}}}{4}.$$

In the dynamic programming equations (7) and (11), we see that their right-hand sides are linear in $x_i, i = 1, 2, \dots, N$, except for $\theta_i^*(X), i = 1, 2, \dots, N$, to be determined. Taking a cue from [23], we shall look for linear value functions. That is, we use the forms

$$V_i(X) = \alpha_i + \beta_i x_i + \sum_{j \neq i} \gamma_{ij} x_j, \quad i = 1, 2, \dots, N, \quad j = 1, 2, \dots, N, \quad j \neq i, \tag{12}$$

$$V_m(X) = \alpha + \sum_{j=1}^N B_j x_j, \tag{13}$$

and then try to solve for the coefficients $\alpha_i, \beta_i, \gamma_{ij}, \alpha$ and $B_i, i = 1, 2, \dots, N, j = 1, 2, \dots, N, j \neq i$. With these, we see that

$$\beta_i = V_{ix_i}, \gamma_{ij} = V_{ix_j}, \quad B_i = V_{mx_i}, \quad i = 1, 2, \dots, N, \quad j = 1, 2, \dots, N, \quad j \neq i, \tag{14}$$

and therefore $\hat{\theta}_i(X)$ and $\theta_i^*(X), i = 1, 2, \dots, N$, given in (9) and (10) would be constants. Thus, from here on, we shall simply denote them as $\hat{\theta}_i$ and θ_i^* , respectively, $i = 1, 2, \dots, N$. We substitute (12)–(13) into (7) and (11), set the coefficients of $x_i, i = 1, 2, \dots, N$, and the constant terms equal to zero in the resulting equations, and obtain the following system of equations to be solved for the coefficients in (12)–(13): For $i = 1, 2, \dots, N, j = 1, 2, \dots, N, j \neq i$,

$$4r\alpha_i = -\frac{\beta_i^2 \rho_i^2}{(-1 + \theta_i^*)} + \sum_{k \neq i} \frac{2\beta_k \gamma_{ik} \rho_k^2}{(-1 + \theta_k^*)}, \tag{15}$$

$$4r\beta_i = 4m_i - 4\beta_i \delta_i + \sum_{k \neq i} \frac{2\beta_k \gamma_{ik} \rho_k^2}{(-1 + \theta_k^*)} + \frac{\beta_i^2 \rho_i^2}{(-1 + \theta_i^*)}, \tag{16}$$

$$4r\gamma_{ij} = -4\gamma_{ij} \delta_j + \sum_{k \neq i} \frac{2\beta_k \gamma_{ik} \rho_k^2}{(-1 + \theta_k^*)} + \frac{\beta_i^2 \rho_i^2}{(-1 + \theta_i^*)}, \tag{17}$$

$$r\alpha = -\sum_{k=1}^N \frac{\beta_k (2B_k (-1 + \theta_k^*(X)) + \beta_k \theta_k^*(X)) \rho_k^2}{4(-1 + \theta_k^*)^2}, \tag{18}$$

$$rB_i = M_i - B_i \delta_i + \sum_{k=1}^N \frac{\beta_k (2B_k (-1 + \theta_k^*(X)) + \beta_k \theta_k^*(X)) \rho_k^2}{4(-1 + \theta_k^*)^2}, \tag{19}$$

$$\theta_i^* = \max \left\{ \frac{2B_i - \beta_i}{2B_i + \beta_i}, 0 \right\}. \tag{20}$$

Our analysis also reveals a condition under which the manufacturer will support his retailers. To explore this, we let

$$P_i = 2V_{mx_i} - V_{ix_i} = 2B_i - \beta_i, \quad i = 1, 2, \dots, N. \tag{21}$$

Then, the subsidy rate for retailer i (given by (9) and (10)) depends on the sign of $P_i, i = 1, 2, \dots, N$. Thus, when $P_i > 0$, the manufacturer supports retailer i , otherwise he does not.

When $P_i \leq 0, \forall i = 1, 2, \dots, N$, no retailer receives any advertising support from the manufacturer. In this case, $\theta_i^* = 0, \forall i = 1, 2, \dots, N$, and Eqs. (15)–(17) can be solved independently of (18)–(19). Using $\theta_i^* = 0, \forall i = 1, 2, \dots, N$, in (15)–(19), we can solve for the coefficients $\alpha_i, \beta_i, \gamma_{ij}, \alpha$, and $B_i, i = 1, 2, \dots, N, j = 1, 2, \dots, N, j \neq i$. We can then write the conditions for a non-cooperative solution, i.e., $P_i \leq 0, \forall i = 1, 2, \dots, N$, in terms of the parameters $m_i, M_i, \rho_i, \delta_i, i = 1, 2, \dots, N$. It is also clear that in this case, our model reduces to the oligopoly model of advertising competition studied by [10].

The explicit solution of the system of equations (15)–(20) is difficult to obtain in general, and therefore, the condition $P_i \leq 0, i = 1, 2, \dots, N$, for non-cooperation cannot be given explicitly.

In the special case of *identical* retailers, defined when $m_i = m$, $\rho_i = \rho$, and $\delta_i = \delta, \forall i = 1, 2, \dots, N$, some explicit results can be obtained. In addition to this, when $M_i = M, \forall i = 1, 2, \dots, N$, which we refer to as the case of *symmetric* retailers, we can obtain additional explicit results. Nevertheless, even in the general case, it is easy to solve the system numerically. \square

4 Identical Retailers

Let $m_i = m, \rho_i = \rho$, and $\delta_i = \delta, \forall i = 1, 2, \dots, N$. Recall that $P_i \leq 0, \forall i = 1, 2, \dots, N$, ensures that no retailer will be supported by the manufacturer. In order to obtain the required condition for no cooperation at all, we set $\theta_i^* = 0, \forall i = 1, 2, \dots, N$, in the system of equations (15)–(19), and then explicitly solve for all of the coefficients. Then, using (21) we obtain, for $i = 1, 2, \dots, N$,

$$\begin{aligned}
 P_i = & \frac{M_i}{(r + \delta)} \left[\frac{m(N - 1)\rho^2 + \sqrt{4(r + \delta)^4 + 4mN(r + \delta)^2\rho^2 + m^2(N - 1)^2\rho^4}}{(r + \delta)^2 + mN\rho^2} \right] \\
 & - \frac{M_{-i}}{(r + \delta)} \left[\frac{2(r + \delta)^2 + m(N + 1)\rho^2 - \sqrt{4(r + \delta)^4 + 4mN(r + \delta)^2\rho^2 + m^2(N - 1)^2\rho^4}}{(r + \delta)^2 + mN\rho^2} \right] \\
 & - \frac{m(N - 1)}{(r + \delta)(2N - 1)} + \frac{2(r + \delta)^2 - \sqrt{4(r + \delta)^4 + 4mN(r + \delta)^2\rho^2 + m^2(N - 1)^2\rho^4}}{(r + \delta)(2N - 1)\rho^2},
 \end{aligned} \tag{22}$$

where, $M_{-i} = \sum_{j \neq i} M_j$. The calculation of (22) is explained in Appendix A.

Proposition 3 *When $P_i \leq 0, \forall i = 1, 2, \dots, N$, where $P_i, i = 1, 2, \dots, N$, is given by (22), we have a non-cooperative equilibrium in which it is optimal for the manufacturer to not support any retailer. Furthermore, if $P_k > 0$, for some $k \in 1, 2, \dots, N$ and $P_j \leq 0, \forall j = 1, 2, \dots, N, j \neq k$, then only retailer k is supported and all others are not.*

One can observe that $P_i, i = 1, 2, \dots, N$, are linear in M_1, M_2, \dots, M_N . In P_i , the coefficient of M_i is positive and that of $M_j, j \neq i$, is negative. Thus, P_i increases as the margin of the manufacturer from retailer i increases, and it decreases as that from any other retailer increases. When $P_k \leq 0, \forall k = 1, 2, \dots, N$, as retailer i offers a higher margin to the manufacturer, he gets closer to the point of getting advertising support. Moreover, this increase in margin from retailer i further hampers the case of retailer $j, j \neq i$, in getting support from the manufacturer. Indeed, from (22), it can be seen that

$$P_i - P_j = \frac{2(M_i - M_j)}{(r + \delta)}, \quad i = 1, 2, \dots, N, \quad j = 1, 2, \dots, N, \quad i \neq j, \tag{23}$$

which means that $M_i > M_j$ implies $P_i > P_j$. Thus, in the case when no retailer is supported, i.e., when $P_k \leq 0, \forall k = 1, 2, \dots, N$, as long as we have $M_i > M_j, j \neq i$, we will never see a scenario when a change in the parameters ($M_k, k = 1, 2, \dots, N, m, \rho, \delta, r$) will induce the manufacturer to start supporting retailer j and not retailer i . In other words, if we assume $M_1 > M_2 > \dots > M_N$, retailer 1 will be the first to start receiving a positive subsidy rate (if at all any retailer does get supported), whenever changes in the parameters take place, as P_1 would change sign from negative to positive before P_2, P_3, \dots, P_N .

The expression for $P_i, i = 1, 2, \dots, N$, in (22) is still quite complicated. It simplifies a great deal, however, if we assume that the decay coefficient and the discount rate are very small, i.e., $r + \delta \approx 0$. Under this condition, after taking appropriate limits, we get

$$\text{sgn}(P_i) = \text{sgn} \left[M_i - \frac{M_{-i}}{(N - 1)} - \frac{mN}{(2N - 1)} \right], \tag{24}$$

where $\text{sgn}(y)$ represents the sign of y .

Our analysis of the identical retailers case with small r and small δ reveals some important insights. First of all, we can interpret the term $M_{-i}/(N - 1)$ as the average margin of the manufacturer from all the retailers except retailer i , where $M_{-i} = \sum_{j \neq i} M_j$. We then see from (24) that if $M_i < M_{-i}/(N - 1)$, then retailer i does not get any advertising support. Thus, we can say that retailer i will not be supported if it offers to the manufacturer a margin which is less than the average margin from all the other retailers. Retailer i will be supported only when the his margin to the manufacturer is at least $mN/(2N - 1)$ more than the average margin the other retailers offer to the manufacturer.

This last observation clearly and explicitly brings out the effect of retail level competition on the manufacturer’s subsidy rate policy, when compared to the results obtained in [12] which dealt with a one-manufacturer-one-retailer channel. For small r and δ , they concluded that if the manufacturer’s margin from the retailer is greater than the retailer’s margin, then the retailer will be supported. In our case, the manufacturer will support retailer i only when $M_i - \frac{M_{-i}}{(N-1)}$ exceeds $\frac{(N)}{(2N-1)}$ times retailers’ margin. To make the comparison even clearer, if we assume that only one retailer, say retailer 1, sells the manufacturer’s product and all the other $N - 1$ retailers buy from another manufacturer and are simply outside competitors to retailer 1 and the manufacturer, then $M_2 = M_3 = \dots = M_N = 0$, hence $M_{-1} = 0$, and manufacturer supports retailer 1 when $M_1 > mN/(2N - 1)$. Thus, the threshold condition for support for $N = 2$ is $M_1 > 2m/3$, for $N = 3$ is $M_1 > 3m/5$, and so on. If the number of competing retailers is very large, i.e., $N \rightarrow \infty$, then the manufacturer supports his retailer when his margin is at least half of the retailer’s margin. If we assume $N = 1$, then the threshold condition is $M_1 > m$, which is consistent with the result found by [12].

An interesting and related question is, how will the manufacturer’s subsidy rate decision change as the number of retailers selling his product changes? In other words, would the manufacturer tend to support more when he has more retailers, or less? While it is analytically difficult to answer this in the general case, the following segment does it for a case of symmetric retailers.

4.1 N Symmetric Retailers ($M_1 = M_2 = \dots = M_N = M$)

In this case, (22) $\forall i = 1, 2, \dots, N$ reduces to

$$P_i = \frac{M}{(r + \delta)} \times \left[\frac{N\sqrt{4(r + \delta)^4 + 4mN(r + \delta)^2\rho^2 + m^2(N - 1)^2\rho^4} - (N - 1)(Nm\rho^2 + 2(r + \delta)^2)}{(r + \delta)^2 + mN\rho^2} \right] - \frac{m(N - 1)}{(r + \delta)(2N - 1)} + \frac{2(r + \delta)^2 - \sqrt{4(r + \delta)^4 + 4mN(r + \delta)^2\rho^2 + m^2(N - 1)^2\rho^4}}{(r + \delta)(2N - 1)\rho^2}. \tag{25}$$

After a simple algebraic analysis, it can be shown that P_i in (25) decreases as N increases for $N \geq 2$. This leads to the following result.

Proposition 4 *When all the retailers are symmetric in nature, the manufacturer's support will be equal for all the retailers and his tendency to support his retailers decreases as the number of retailers increases.*

A possible explanation of this result is that as N increases, the competition among retailers for the market share increases and the retailers themselves have an incentive to advertise on their own accord to increase their respective sales. Then, the manufacturer need not provide as much advertising support. Furthermore, when $(r + \delta) \approx 0$, then $P_i < 0, \forall i, i = 1, 2, \dots, N$; and no retailer gets any support, which can also be verified by (24).

In the special case of two symmetric retailers, i.e., $M_1 = M_2$, along with $m_1 = m_2 = m$, $\rho_1 = \rho_2 = \rho$, and $\delta_1 = \delta_2 = \delta$, we can solve the system of equations (15)–(20) completely and explicitly. Moreover, it can be shown that this solution is unique. We can now state the following.

Proposition 5 *In the case of two symmetric retailers, the solution to the Stackelberg–Nash cooperative advertising game, obtained by solving the system of equations (15)–(20), is unique.*

See Appendix B for proof.

In the general case, however, the system of equations (15)–(20) is too difficult to solve explicitly and a proof of uniqueness cannot be established. Even when we have N symmetric retailers, proving uniqueness is extremely difficult. However, while performing the numerical analysis for a general case, always a unique solution was found.

5 Non-identical Retailers: Numerical Analysis

The system of equations (15)–(20) constitutes a total of $(2N + 1) + N(N + 1)$ nonlinear equations that can be solved numerically to get insights on the optimal advertising and subsidy rates policy. We carried out the numerical analysis for the case of two as well as three retailers, but for clarity we present the analysis for two retailers in this section, as the insights that follow are similar for $N = 2$ and $N = 3$. We perform numerical analysis to study the dependence of the manufacturer's subsidy rates on the manufacturer's margins (M_1, M_2) from retailers 1 and 2, respectively, the retailers' margins (m_1, m_2), the advertising effectiveness coefficients (ρ_1, ρ_2), and the decay coefficients (δ_1, δ_2). In the results presented, the base case is $M_1 = M_2 = 1, m_1 = m_2 = 0.25, \rho_1 = \rho_2 = 1, \delta_1 = \delta_2 = 0.2$, and $r = 0.05$. We then vary each parameter one by one to study how it affects θ_1^* and θ_2^* .

The numerical analysis involves only solving a system of equations, which is fairly straightforward to carry out. In all instances, we find a unique solution to the system. For the case of two symmetric retailers, we prove the uniqueness in Appendix B. We describe the results as follows.

- (a) The effect of the manufacturer's margin (Fig. 2): If the manufacturer's margin from retailer 1 increases, the manufacturer starts offering a higher subsidy rate to retailer 1 and reduces the subsidy rate to retailer 2. Thus, the manufacturer rewards retailer 1 for providing him a higher margin in two ways.

Fig. 2 Subsidy rates vs. M_1

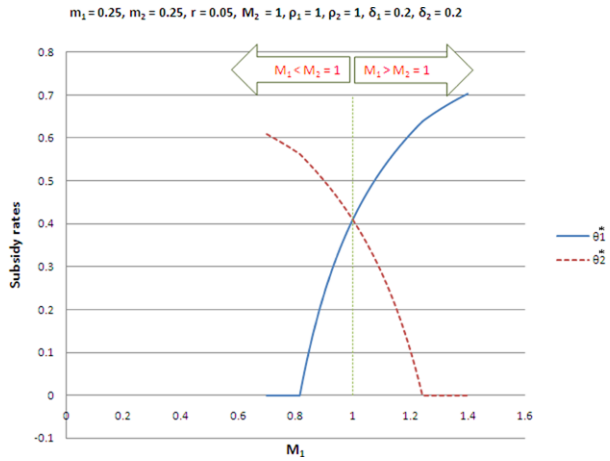
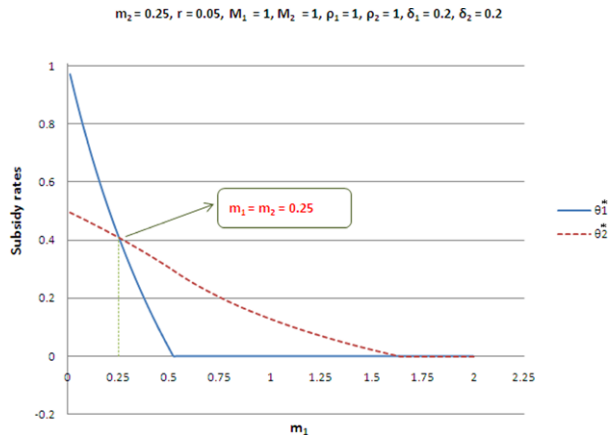


Fig. 3 Subsidy rates vs. m_1



- (b) The effect of a retailer’s margin (Fig. 3): As the margin of retailer 1 increases, both θ_1^* and θ_2^* decrease. The decrease in θ_2^* is smaller compared to that in θ_1^* . Since the manufacturer’s margins are kept constant, we can say that if the margin of retailer 1 increases relative to the margin he offers to the manufacturer, then the manufacturer starts reducing his subsidy to both retailers, and more drastically with retailer 1 than with retailer 2. Thus, if all other parameters remain the same, the retailer with the higher margin gets a lower subsidy rate.
- (c) The effect of the advertising effectiveness parameter (Fig. 4): As the advertising effectiveness of retailer 1 increases, the subsidy rates for both retailers decrease. The rate of decrease is higher for retailer 2 than for retailer 1. All other parameters being the same, the retailer with the more effective advertising gets a higher subsidy rate.
- (d) The effect of the decay coefficient (Fig. 5): As δ_1 increases, the subsidy rates for both retailers increase. However, θ_2^* is more sensitive to increase in δ_1 than θ_1^* .

Fig. 4 Subsidy rates vs. ρ_1

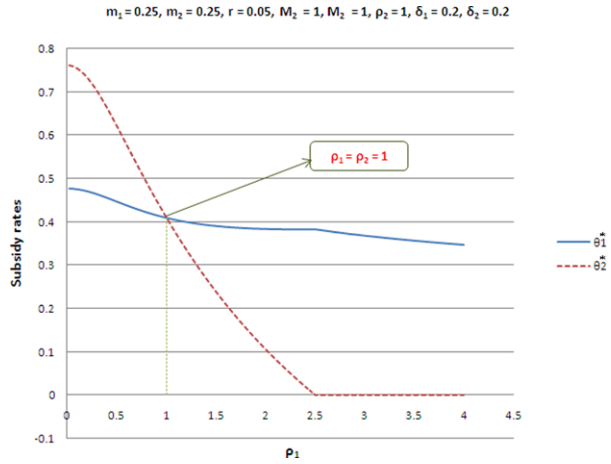
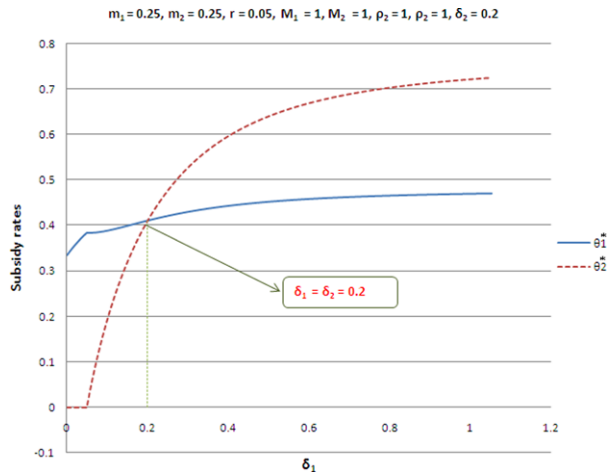


Fig. 5 Subsidy rates vs. δ_1



6 Cooperative Advertising and Channel Coordination: Two Retailers

In this section, for a retailer duopoly, we analyze the role of cooperative advertising as a tool to better coordinate the channel and improve the overall channel profit. We evaluate and compare the channel profit in three cases. First, we consider an integrated channel where the advertising decisions are based on the maximization of total profit for the manufacturer and the retailers together. Second, we consider a decentralized channel with the optimal subsidy rates. Third, we consider a decentralized channel without cooperative advertising.

In the first case, given the retailers’ and manufacturer’s margins (m_1, m_2, M_1, M_2), the optimization problem for the integrated channel can be written as follows:

$$V(x_1, x_2) = \max_{u_1(t) \geq 0, u_2(t) \geq 0, t \geq 0} \int_0^\infty e^{-rt} ((M_1 + m_1)x_1(t) + (M_2 + m_2)x_2(t) - u_1^2(t) - u_2^2(t)) dt \tag{26}$$

subject to

$$\dot{x}_i(t) = \frac{dx_i(t)}{dt} = \rho_i u_i(t) \sqrt{1 - x_1(t) - x_2(t)} - \delta_i x_i(t), \quad x_i(0) = x_i \in [0, 1], \quad i = 1, 2. \tag{27}$$

The HJB equation for the value function V is

$$rV(x_1, x_2) = \max_{u_1 \geq 0, u_2 \geq 0} [(M_1 + m_1)x_1 + (M_2 + m_2)x_2 - u_1^2 - u_2^2 + V_{x_1}\dot{x}_1 + V_{x_2}\dot{x}_2], \tag{28}$$

where \dot{x}_1 and \dot{x}_2 are given by (27). Using (27) in the HJB equation (28) and applying the first-order conditions for maximization w.r.t. u_1 and u_2 give the following result.

Proposition 6 *For the integrated channel, the optimal feedback advertising policies are*

$$u_1^* = \frac{1}{2} \rho_1 V_{x_1} \sqrt{1 - x_1 - x_2}, \quad u_2^* = \frac{1}{2} \rho_2 V_{x_2} \sqrt{1 - x_1 - x_2}, \tag{29}$$

and the integrated channel’s value function satisfies

$$4rV(x_1, x_2) = 4(M_1 + m_1 - V_{x_1}\delta_1)x_1 + 4(M_2 + m_2 - V_{x_2}\delta_2)x_2 + (1 - x_1 - x_2)(V_{x_1}^2\rho_1^2 + V_{x_2}^2\rho_2^2). \tag{30}$$

Once again, we conjecture a linear value function of the form $V(x_1, x_2) = \alpha^I + \beta_1^I x_1 + \beta_2^I x_2$, where $\alpha^I, \beta_1^I = V_{x_1}$ and $\beta_2^I = V_{x_2}$ are constants, and solve the following system of equations:

$$4r\alpha^I = \beta_1^{I2}\rho_1^2 + \beta_2^{I2}\rho_2^2, \tag{31}$$

$$4r\beta_i^I = 4(m_i + M_i) - 4\beta_i^I\delta_i - \beta_i^{I2}\rho_i^2 - \beta_{3-i}^{I2}\rho_{3-i}^2, \quad i = 1, 2. \tag{32}$$

The set of equations (31)–(32) is obtained by comparing the coefficients of x_1 and x_2 , and the constant terms in (30) with $\beta_1^I (= V_{x_1})$, $\beta_2^I (= V_{x_2})$, and α^I , respectively.

In the second case, we have a decentralized channel with cooperative advertising, for which we define the channel value function as $V^c(x_1, x_2) = V_m^c(x_1, x_2) + V_r^c(x_1, x_2)$, where V_m^c is the manufacturer’s value function (given by (13)) and V_r^c is the total value function of both retailers (obtained by (12) for $i = 1, 2$).

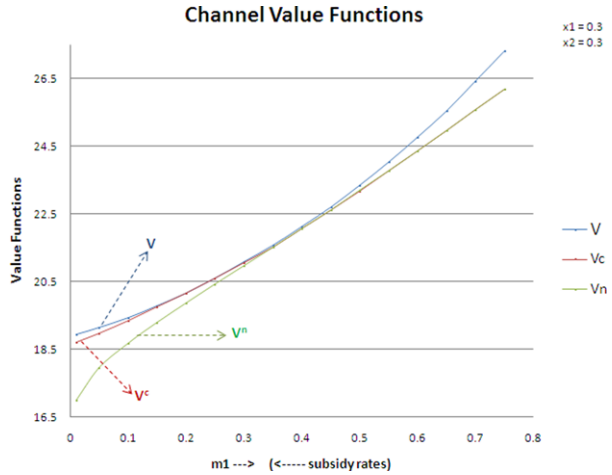
In the third case, namely, a decentralized channel with no cooperation, the channel value function is defined as $V^n(x_1, x_2) = V_m^n(x_1, x_2) + V_r^n(x_1, x_2)$, where V_m^n and V_r^n are the manufacturer’s value function and the sum of the two retailers’ value functions in the non-cooperative setting, respectively. These are computed by simply setting $\theta_1^* = \theta_2^* = 0$ in (15)–(19) and then using (12)–(13).

Before we proceed further, let us observe that the manufacturer is the leader and he obtains his optimal subsidy rates by maximizing his objective function. Therefore, it should be obvious that

$$V_m^c(x_1, x_2) \geq V_m^n(x_1, x_2). \tag{33}$$

Thus, it remains to study the effect of cooperative advertising on the retailers’ profits and the total channel profit. First, we examine this in the simple case of symmetric retailers. We now present the following result.

Fig. 6 Channel value functions in integrated, cooperative, and non-cooperative cases



Proposition 7 *In the case of symmetric retailers, the value functions $V_r^c(x_1, x_2)$ and $V_r^n(x_1, x_2)$ depend only on the sum $(x_1 + x_2)$, and can thus be expressed as $V_r^c(x_1 + x_2)$ and $V_r^n(x_1 + x_2)$, with a slight abuse of notation. Furthermore, we have*

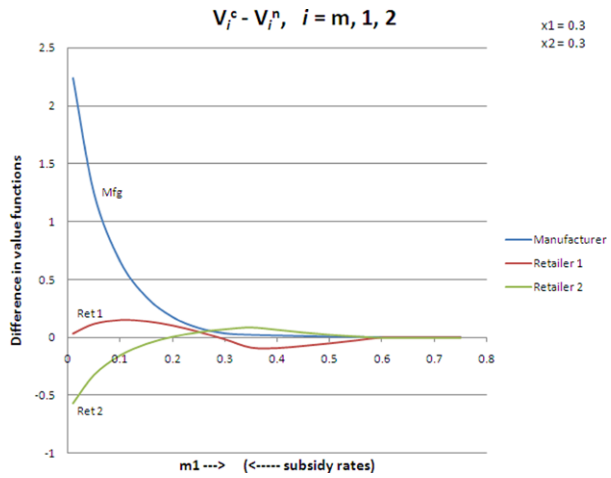
$$V_r^c(x_1 + x_2) \geq V_r^n(x_1 + x_2). \tag{34}$$

The proof of Proposition 7 is provided in Appendix C. It is clear from (33) and (34) that in the symmetric case, cooperative advertising can partially coordinate the channel, and that the manufacturer as well as the retailers are better off with cooperative advertising than without it.

We now return to the general case where explicit analytical relationships between various value functions are difficult to establish. We, therefore, resort to numerical analysis, and report our findings based on the results obtained. We compare V , V^c , and V^n with varying values of the optimal subsidy rates. Since the value functions depend on the state (x_1, x_2) , the analysis was carried out for different values of (x_1, x_2) .

We would like to study V , V^c , and V^n with respect to the changes in the optimal subsidy rates brought about by changes in the model parameters; for this we consider varying retailer 1’s margin. As m_1 increases, we know from Fig. 3 that the subsidy rates for both retailers decrease. As a result, we can compare the various value functions as m_1 increases or, roughly speaking, as subsidy rates decrease. Figure 6 depicts the values of V , V^c , and V^n for $x_1 = x_2 = 0.3$. The data range for the calculations shown are the same as those used for the results shown in Fig. 3. Thus, for any point in Fig. 6, the values of the optimal subsidy rates are the same as the corresponding values in Fig. 3. Recall that as m_1 increases, the overall cooperation by the manufacturer decreases. It is found that under all instances, V is greater than V^c as well as V^n . This is understandable as we expect the channel value function in the integrated case to be higher than in the decentralized case, with or without cooperative advertising. In the scenario when both retailers get advertising support, we find that $V^c > V^n$, indicating that the channel attains partial coordination. Moreover, it is found that the difference $V - V^c$ is minimum at the point when both retailers receive equal positive subsidy rates, and it increases as the difference between the two subsidy rates increases. These results indicate that the level of coordination achieved is maximum when both retailers receive equal positive subsidy rates.

Fig. 7 Difference between the value functions in the cooperative case and the non-cooperative case for each channel member



An interesting, perhaps even counter-intuitive observation is that in the case when it is optimal for the manufacturer to support only one retailer, we find that the overall value function of the channel in the non-cooperative scenario is slightly higher than in the cooperative case. Thus, from the channel’s perspective, it is better in this case not to support any retailer than to support only one of the two.

This analysis was carried out for changes in parameters m_1 , ρ_1 and δ_1 with the corresponding changes in the optimal subsidy rates as shown in Figs. 3, 4 and 5, respectively, as well as for different values of (x_1, x_2) . However, we find that the nature of changes in V , V^c , and V^n with respect to varying optimal subsidy rates and the overall insights do not change. Figure 7 shows the difference in the value functions between the cooperative and non-cooperative settings for the manufacturer and the two retailers, respectively. As expected, the manufacturer always benefits from cooperative advertising. The retailers, however, do not always seem to benefit from cooperative advertising. Furthermore, the manufacturer’s benefit from cooperative advertising increases as his subsidy increases. For the two retailers, it is found that as the difference between the two subsidy rates increases, the retailer getting the lower subsidy rate has a lower value function than in the non-cooperative setting. Moreover, as this difference in the subsidy rates increases further, even the combined value function of the retailers is lower than in the non-cooperative setting. Figure 7 also shows that the region in which both retailers benefit from cooperative advertising is a small “window” around the point where the retailers get the same positive subsidy rate from the manufacturer. In other words, for both retailers to benefit from cooperative advertising, their subsidy rates should not differ significantly. Thus, the retailer getting a significantly lower subsidy rate might not be happy with such a cooperative advertising program.

These observations raise an interesting issue relating to the differential treatment of the retailers by the manufacturer, i.e., when only one retailer is favored, then in some situations the channel makes less profit with cooperative advertising than without it. Indeed, this might be viewed as an additional argument for the non-discriminating practice legislated under such acts as the Robinson–Patman Act of 1936 in the context of price discrimination, for reasons to enhance competition. In view of these results, we study next the case of non-discrimination in the context of cooperative advertising.

7 Equal Subsidy Rate for All Retailers: Two Retailers

We consider the case when the manufacturer is restricted to offer the same subsidy rate to its retailers. As in the previous section, we restrict our analysis to a retailer duopoly due to the analytical complexity of a general N retailer model. This case is motivated by possible legal issues that may arise when the manufacturer discriminates between the two retailers in terms of subsidy rates. Note that discrimination in terms of price, promotions, discounts, etc. is prohibited by the Robinson–Patman Act of 1936. Specifically, the Act proscribes discrimination in price between two or more competing buyers in the sale of commodities of like grade and quality. This and other anti-discrimination acts, such as the Sherman Antitrust Act of 1890, the Clayton Act of 1914, and the Celler–Kefauver Act of 1950, prevent discriminatory policies which might lead to reduced competition and create monopolies in the market.

In our model, different optimal subsidy rates for the two retailers arise from factors such as difference in the manufacturer’s margins relative to the retailers’ margins, which affect the manufacturer’s profit. However, if the manufacturer is not allowed to offer different subsidy rates to the retailers, we need to reformulate our problem so that the manufacturer’s optimization problem has only one subsidy rate decision. In this case, let $V_m^{RP}(x_1, x_2)$, $V_1^{RP}(x_1, x_2)$, $V_2^{RP}(x_1, x_2)$ and V^{RP} denote the value functions of the manufacturer, retailer 1, retailer 2, and the total channel, respectively, with the superscript RP standing for Robinson and Patman. These value functions solve the control problems defined by (1)–(4) with $\theta_1 = \theta_2 = \theta$. Once again, we expect these value functions to be linear in the market share vector, and express them as in (12)–(13), except that the coefficients will now be denoted as $\tilde{\alpha}$, $\tilde{\alpha}_1$, $\tilde{\alpha}_2$, $\tilde{\beta}_1$, $\tilde{\beta}_2$, $\tilde{\gamma}_1$, $\tilde{\gamma}_2$, \tilde{B}_1 , and \tilde{B}_2 . These coefficients will satisfy the system of equations obtained by setting $\theta_1^* = \theta_2^* = \theta^*$ in (15)–(19). Thus, we have the following equation system: For $i = 1, 2$,

$$4r\tilde{\alpha}_i = -\frac{\tilde{\beta}_i^2 \rho_i^2}{(-1 + \theta^*)} + \frac{2\tilde{\beta}_{3-i}\tilde{\gamma}_i \rho_{3-i}^2}{(-1 + \theta^*)}, \tag{35}$$

$$4r\tilde{\beta}_i = 4m_i - 4\tilde{\beta}_i \delta_i + \frac{2\tilde{\beta}_{3-i}\tilde{\gamma}_i \rho_{3-i}^2}{(-1 + \theta^*)} + \frac{\tilde{\beta}_i^2 \rho_i^2}{(-1 + \theta^*)}, \tag{36}$$

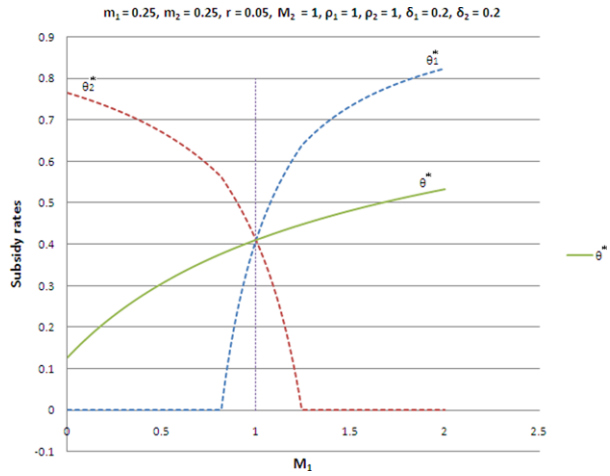
$$4r\tilde{\gamma}_i = -4\tilde{\gamma}_i \delta_{3-i} + \frac{2\tilde{\beta}_{3-i}\tilde{\gamma}_i \rho_{3-i}^2}{(-1 + \theta^*)} + \frac{\tilde{\beta}_i^2 \rho_i^2}{(-1 + \theta^*)}, \tag{37}$$

$$r\tilde{\alpha} = -\frac{\tilde{\beta}_i^2 \rho_i^2 \theta^*}{4(-1 + \theta^*)^2} + \frac{\tilde{\beta}_i \tilde{B}_i \rho_i^2}{2(-1 + \theta^*)} + \frac{\tilde{\beta}_{3-i}^2 \rho_{3-i}^2 \theta^*}{2(-1 + \theta^*)^2} + \frac{\tilde{\beta}_{3-i} \tilde{B}_{3-i} \rho_{3-i}^2}{2(-1 + \theta^*)}, \tag{38}$$

$$r\tilde{B}_i = M_i - \tilde{B}_i \delta_i + \frac{\tilde{\beta}_i^2 \rho_i^2 \theta^*}{4(-1 + \theta^*)^2} + \frac{\tilde{\beta}_i \tilde{B}_i \rho_i^2}{2(-1 + \theta^*)} + \frac{\tilde{\beta}_{3-i}^2 \rho_{3-i}^2 \theta^*}{4(-1 + \theta^*)^2} + \frac{\tilde{\beta}_{3-i} \tilde{B}_{3-i} \rho_{3-i}^2}{2(-1 + \theta^*)}, \tag{39}$$

$$\theta^* = \max \left\{ \begin{array}{l} \tilde{\beta}_1(2\tilde{B}_1 - \tilde{\beta}_1)\rho_1^2 + \tilde{\beta}_2(2\tilde{B}_2 - \tilde{\beta}_2)\rho_2^2 \\ \tilde{\beta}_1(2\tilde{B}_1 + \tilde{\beta}_1)\rho_1^2 + \tilde{\beta}_2(2\tilde{B}_2 + \tilde{\beta}_2)\rho_2^2 \end{array} \right\}, \tag{40}$$

Fig. 8 Subsidy rate vs. M_1



The common threshold condition for no cooperation to be optimal is that

$$P = \tilde{\beta}_1(2\tilde{B}_1 - \tilde{\beta}_1)\rho_1^2 + \tilde{\beta}_2(2\tilde{B}_2 - \tilde{\beta}_2)\rho_2^2 \leq 0. \tag{41}$$

In the case of identical retailers, i.e., $m_1 = m_2 = m$, $\rho_1 = \rho_2 = \rho$, and $\delta_1 = \delta_2 = \delta$, we can solve Eqs. (35)–(39) explicitly when $\theta^* = 0$. The condition for no support by the manufacturer, in this scenario, reduces to

$$\begin{aligned} &9(r + \delta)^2\rho^2(M_1 + M_2) \\ &+ (2(r + \delta)^2 - m\rho^2 - \sqrt{4(r + \delta)^4 + 8m(r + \delta)^2\rho^2 + m^2\rho^4}) \\ &\times \frac{((r + \delta)^2 + m\rho^2 + \sqrt{4(r + \delta)^4 + 8m(r + \delta)^2\rho^2 + m^2\rho^4})}{3(r + \delta)} \leq 0. \end{aligned} \tag{42}$$

In the general case, we performed numerical analysis to see the behavior of θ^* with respect to different model parameters. Figures 8, 9, 10 and 11 show the dependence of θ^* on M_1 , m_1 , ρ_1 and δ_1 , respectively, and compare the θ^* with the optimal subsidy rates for two retailers without any legislation, i.e., θ_1^* and θ_2^* . We find that as M_1 increases, the manufacturer has higher incentive to support retailer 1 and thus θ^* increases, but with a decreasing rate. Rate of increase of θ^* is lower than the rate of increase of θ_1^* , which could be attributed to the fact that the subsidy rates have to be equal for both the retailers even though retailer 2 does not offer any incentive for higher support. The overall impact of parameters m_1 , ρ_1 and δ_1 on manufacturer’s support is similar to the general model, i.e., the subsidy rate θ^* decreases with m_1 , decreases with ρ_1 , and increases with δ_1 . Moreover, as is the case in the unrestricted model, the rate of change of θ^* (in absolute sense) decreases as the value of these parameters increases. It is noticeable though that θ^* lies between θ_1^* and θ_2^* , and the retailer who is less profitable for the manufacturer and would have received a lower subsidy rate in the unrestricted model, benefits from this legislation.

Next, we study the impact of anti-discriminatory act on the profits of all the parties in the supply chain and on the total channel profit. We compare the value functions in three cases: a channel without any cooperative advertising, a channel with no anti-discriminatory act and optimal subsidy rates, and a channel with an anti-discriminatory act and optimal

Fig. 9 Subsidy rate vs. m_1

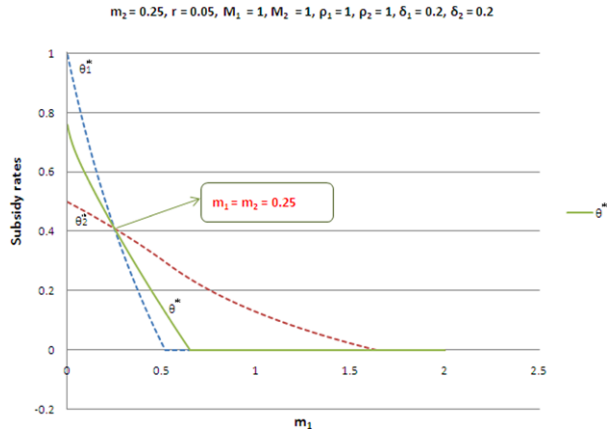


Fig. 10 Subsidy rate vs. ρ_1

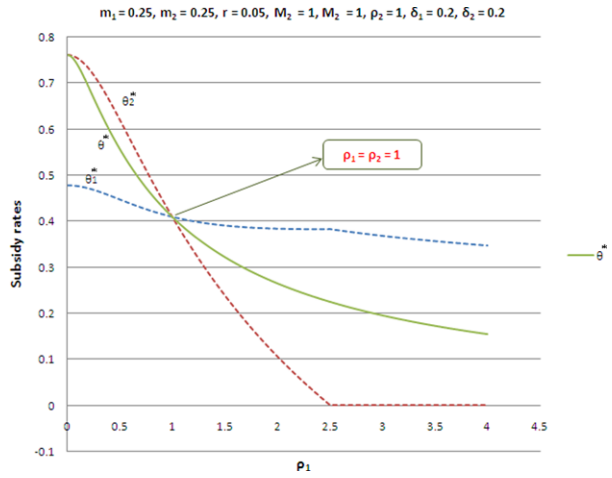


Fig. 11 Subsidy rate vs. δ_1

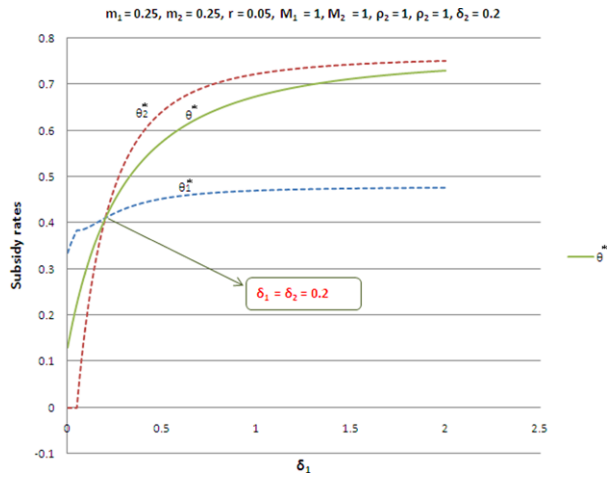


Fig. 12 Value function with a non-discriminatory act minus value function without a non-discriminatory act

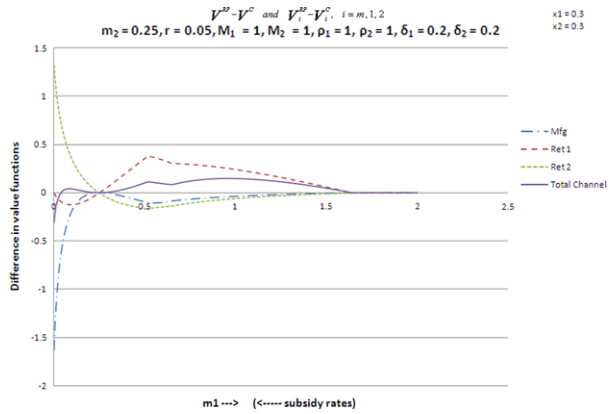
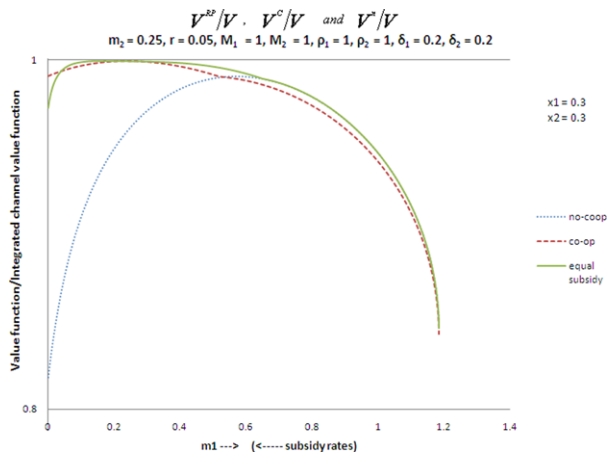


Fig. 13 Value functions in different cases divided by integrated channel value function



common subsidy rate for the two retailers. Figure 12 shows the difference between the value functions in the cases of with and without the non-discriminatory legislation, with changes in subsidy rates brought about by the changes in m_1 . As anticipated, the manufacturer earns less profit because of the added constraint to his optimization problem. The manufacturer’s loss is higher when m_1 is low, i.e., when subsidy rates θ^* , θ_1^* and θ_2^* are high. We find that the retailer who would have obtained a higher subsidy rate in the absence of such legislation also earns less profit, whereas a less efficient retailer who would have earned a lower subsidy rate in an unconstrained problem benefits from this act. When m_1 is very low, the gain of the less efficient retailer is not able to offset the losses of the other two parties, and the supply chain in the whole loses. However, for a larger set of values of m_1 , the inefficient retailer’s gain is more than the losses of the other two parties and the total channel profit is higher. This result indicates that better supply chain coordination can be obtained with the anti-discriminatory act in effect. Figure 13 compares the total profit of an integrated channel (V) with the total channel profit in three cases: no advertising cooperation (V^n), cooperation with no legislation (V^c), and cooperation with equal subsidy rates (V^{RP}). Here again, we see that with the non-discriminatory act, we are able to achieve a higher level of channel coordination in most cases, except when m_1 is too low.

8 Concluding Remarks

We consider a cooperative advertising model with a manufacturer supplying to N retailers, and formulate it as a Stackelberg differential game. We obtain the Stackelberg feedback equilibrium and derive the conditions under which there will or will not be any cooperative advertising. We also provide the sensitiveness of the optimal subsidy rates with respect to the various problem parameters. We study the effect of retail level competition on the manufacturer's subsidy rate decisions. In the case of N identical retailers, when only one retailer sells the manufacturer's product and the remaining $(N - 1)$ retailers are outside competitors, we show that the manufacturer's threshold to start supporting his retailers eases as the number of competing retailers increases. In other words, when the number of competing retailers is higher, the manufacturer supports his retailer under a larger set of conditions. Furthermore, when all of the retailers sell the manufacturer's product, we show that the manufacturer's tendency to provide support to each retailer decreases as the number of retailers increases.

In the case of retail duopoly, we examine some important supply chain issues. We study the issue of channel coordination with cooperative advertising and find that partial coordination can be achieved when both retailers are supported. When only one of the retailers is supported, there are cases when the manufacturer's gain from cooperative advertising does not offset the loss incurred by the retailers. This leads us to also examine the model when the manufacturer is required to offer the same subsidy rates to both retailers, in the spirit of non-discriminating legislations such as the Robinson–Patman Act of 1936. We find that the optimal common subsidy rate lies between the two optimal subsidy rates that would prevail in the absence of any such legislation. We find that the legislation benefits the less efficient retailer, and takes away some profits from the manufacturer and the other retailer. We also see evidence of a higher channel profit and better supply chain coordination with the legislation.

There are a number of ways in which our research can be extended. First, we have assumed margins for the retailers and the manufacturer as exogenous variables. It is possible to extend our model to include wholesale and retail pricing decisions as in [12] and [6]. Second, our problem is modeled as a Stackelberg game with the manufacturer as the leader and the retailers as followers. One may consider modeling the problem in a mixed leadership setting involving several decision variables (see, e.g., [1, 2]) in which the manufacturer is the leader for some decisions and the retailers take the lead in others.

Appendix A: Derivation of P_i , $i = 1, 2, \dots, N$, for N Identical Retailers

For N identical retailers, with $\theta_i^* = 0, \forall i = 1, 2, \dots, N$, by subtracting (17) from (16), we get $\gamma_{ij} = \beta_i - \frac{m}{(r+\delta)}, i = 1, 2, \dots, N, j = i = 1, 2, \dots, N, j \neq i$. Since the retailers are identical, it is not surprising that γ_{ij} is same for all j 's, where $i = 1, 2, \dots, N, j = i = 1, 2, \dots, N, j \neq i$. Using this relation in (16), we can rewrite (16) as

$$4(r + \delta)\beta_i = 4m - \rho^2\beta_i^2 - 2\rho^2 \sum_{k \neq i} \beta_k \left[\beta_i - \frac{m}{(r + \delta)} \right], \quad i = 1, 2, \dots, N.$$

From the above equation we can say that $\beta_1 = \beta_2 = \dots = \beta_N = \beta$, which is intuitive as the retailers are identical. With this we get a quadratic equation in β which gives two roots, one negative and one positive. Clearly it makes sense to have the value function of retailer i , i.e.,

$V_i(X)$, increase with his own market share x_i . Therefore, we expect $\beta_i = \partial V_i(X)/\partial x_i > 0$. Thus, we ignore the negative root and get for $i = 1, 2, \dots, N$,

$$\beta_i = \beta = \frac{(n - 1)m\rho^2 - 2(r + \delta)^2 + \sqrt{4(r + \delta)^4 + 4nm\rho^2(r + \delta)^2 + (n - 1)^4m^2\rho^4}}{(2n - 1)\rho^2(r + \delta)^2}. \tag{43}$$

Using (43) in (19) we solve for $B_i, i = 1, 2, \dots, N$, and then using (21) we get (22).

Appendix B: Uniqueness of Optimal Solution in the Case of Two Symmetric Retailers

The uniqueness of an optimal solution to the problem defined by (6), (7), (9)–(11) is guaranteed by a unique solution of the system of equations (15)–(20). It appears to be difficult to prove the uniqueness in the general case. However, in the special case of symmetric retailers ($M_1 = M_2 = M, m_1 = m_2 = m, \delta_1 = \delta_2 = \delta$ and $\rho_1 = \rho_2$), we can establish the result as follows.

We first look at the signs of α_i, β_i and $\gamma_i, i = 1, 2$. It is expected that $\beta_i > 0$. Now consider γ_i , which can be expressed in terms of β_1 and β_2 by using Eq. (17):

$$\gamma_i = \frac{\beta_i^2(-1 + \theta_{3-i}^*)\rho_i^2}{2(-1 + \theta_i^*)((r + \delta_{3-i})(-1 + \theta_{3-i}^*) - \beta_{3-i}\rho_{3-i}^2)}, \quad i = 1, 2.$$

Since $\beta_i > 0$ and $\theta_i^* < 1, i = 1, 2$, we have $\gamma_i = V_{ix_{3-i}} < 0$, as intuition would suggest on account of the competition between the retailers. We can also use (16) and (17) to write

$$\gamma_i = \frac{-m_i + \beta_i(r + \delta_i)}{(r + \delta_{3-i})}. \tag{44}$$

Since $\gamma_i < 0$, we must have

$$\beta_i < \frac{m_i}{(r + \delta_i)}. \tag{45}$$

Now consider α_i , which is retailer i 's value function when the initial market is zero for both retailers. We now show that this value is positive. By adding (15) and (17), we can conclude that $\alpha_i = -\gamma_i(r + \delta_{3-i})/r$, which is positive since $\gamma_i < 0$. Thus,

$$\alpha_i = \frac{m_i - \beta_i(r + \delta_i)}{r} > 0. \tag{46}$$

Moreover, using (44) in Eq. (15), we can write α_i in terms of β_1 and β_2 , and then rewrite (46) as

$$\alpha_i = \frac{1}{4r} \left[\frac{2\beta_{3-i}(m_i - \beta_i(r + \delta_i))\rho_{3-i}^2}{(r + \delta_{3-i})(-1 + \theta_{3-i})} - \frac{\beta_i^2\rho_i^2}{(-1 + \theta_i)} \right] > 0, \quad i = 1, 2. \tag{47}$$

Clearly, in the symmetric case, we will have $\alpha_1 = \alpha_2 = \alpha, \beta_1 = \beta_2 = \beta, \gamma_1 = \gamma_2 = \gamma, B_1 = B_2 = B$, and hence $\theta_1^* = \theta_2^* = \theta^*$. We can thus rewrite (47) as

$$\frac{\beta(2m - 3\beta(r + \delta))\rho^2}{2r(r + \delta)(-1 + \theta)} > 0.$$

This, along with $\beta > 0$ and $\theta < 1$, gives us

$$\beta > \frac{2m}{3(r + \delta)}. \tag{48}$$

From (45) and (48), we have

$$\frac{2m}{3(r + \delta)} < \beta < \frac{m}{(r + \delta)}. \tag{49}$$

To prove a unique solution to Eqs. (15)–(20), we reduce them into one equation of a single variable β , and then aim for the unique solution of β . We will separately consider the cases of a cooperative equilibrium where $\theta^* > 0$ and a non-cooperative equilibrium where $\theta^* = 0$.

Case I: Cooperative equilibrium ($\theta^ > 0$)* Since $\beta_1 = \beta_2 = \beta$ and $B_1 = B_2 = B$ in the symmetric case, (20) reduces to $\theta^* = (2B - \beta)/(2B + \beta)$. Using this, (44), and (46), we can reduce Eqs. (15)–(20) to two equations in variables β and B , i.e.,

$$4r(r + \delta)\beta = 4(r + \delta)(m - \beta\delta) + (\beta + 2B)(2m - 3\beta(r + \delta))\rho^2, \tag{50}$$

$$4rB = 4M - 4B\delta - (\beta + 2B)^2\rho^2. \tag{51}$$

Using (50), (51) and (14) in (21) and setting $P_1 = P_2 = P$ on account of the case being symmetric, we obtain the participation threshold function

$$P = 2B - \beta = \frac{8(r + \delta)(-m + 2M + (\beta - 2B)\delta) - (\beta + 2B)(2m - (\beta - 4B)(r + \delta))\rho^2}{8r(r + \delta)}. \tag{52}$$

Using (50), we can write B in terms of β as follows:

$$B = \frac{-2m(4(r + \delta) + \beta\rho^2) + \beta(r + \delta)(8(r + \delta) + 3\beta\rho^2)}{2(2m - 3\beta(r + \delta))\rho^2}. \tag{53}$$

Now using (52) and (53), we can rewrite P in terms of β only, and then write the condition of the cooperative equilibrium as

$$P = \frac{-4m((r + \delta)^2 + \beta\rho^2) + 2\beta(r + \delta)(4(r + \delta) + 3\beta\rho^2)}{(2m - 3\beta(r + \delta))\rho^2} > 0. \tag{54}$$

Next, we find the values of β for which the inequality in (54) holds. In order to see how P varies with β , we first find the roots of the equation $P = 0$. The numerator is quadratic in β with the roots denoted as η_1 and η_2 :

$$\eta_1 = \frac{-2(r + \delta)^2 + m\rho^2 + \sqrt{4(r + \delta)^4 + 8m(r + \delta)^2\rho^2 + m^2\rho^4}}{3(r + \delta)\rho^2},$$

$$\eta_2 = \frac{-2(r + \delta)^2 + m\rho^2 - \sqrt{4(r + \delta)^4 + 8m(r + \delta)^2\rho^2 + m^2\rho^4}}{3(r + \delta)\rho^2}.$$

Clearly, $\eta_1 > 0$ and $\eta_2 < 0$. Also, the denominator $(2m - 3\beta(r + \delta))\rho^2$ of (54) changes sign at $\beta = 2m/3(r + \delta)$; the denominator is strictly positive when $\beta < 2m/3(r + \delta)$ and strictly negative when $\beta > 2m/3(r + \delta)$.

We will now compare the value of η_1 with $m/(r + \delta)$ and $2m/3(r + \delta)$. We can see that the difference

$$\eta_1 - \frac{2m}{3(r + \delta)} = \frac{-2(r + \delta)^2 - m\rho^2 + \sqrt{4(r + \delta)^4 + 8m(r + \delta)^2\rho^2 + m^2\rho^4}}{3(r + \delta)\rho^2} > 0,$$

and thus $\eta_1 > 2m/3(r + \delta)$. Furthermore,

$$\eta_1 - \frac{m}{(r + \delta)} = \frac{-2(r + \delta)^2 - 2m\rho^2 + \sqrt{4(r + \delta)^4 + 8m(r + \delta)^2\rho^2 + m^2\rho^4}}{3(r + \delta)\rho^2} < 0,$$

and thus we have

$$\frac{2m}{3(r + \delta)} < \eta_1 < \frac{m}{(r + \delta)}. \tag{55}$$

We can then conclude from (54) that $P > 0$ is satisfied when

$$\beta \in (-\infty, -\eta_2) \quad \text{or} \quad \beta \in \left(\frac{2m}{3(r + \delta)}, \eta_1 \right). \tag{56}$$

Therefore, the conditions (49), (55) and (56) along with the fact that $\beta > 0$ give us the desirable range of the solution for β , i.e.,

$$\beta \in \left(\frac{2m}{3(r + \delta)}, \eta_1 \right). \tag{57}$$

Now using (53) in (51), we can write a single equation in β . After some steps of algebra, for the symmetric retailer case with positive cooperation, this single equation for β can be written as

$$F(\beta) = \frac{8\beta(r + \delta)^3(m - \beta(r + \delta)) - (2m - 3\beta(r + \delta))^2(2M + \beta(r + \delta))\rho^2}{(2m - 3\beta(r + \delta))^2\rho^2} = 0.$$

Thus, a unique cooperative solution in the symmetric retailer case is guaranteed when exactly one root of the equation $F(\beta) = 0$ lies in the range given by (57). The numerator of the above expression, denoted as $N(\beta)$, is cubic in β . Thus, we can rewrite the equation for β as

$$N(\beta) = a\beta^3 + b\beta^2 + c\beta + d = 0, \tag{58}$$

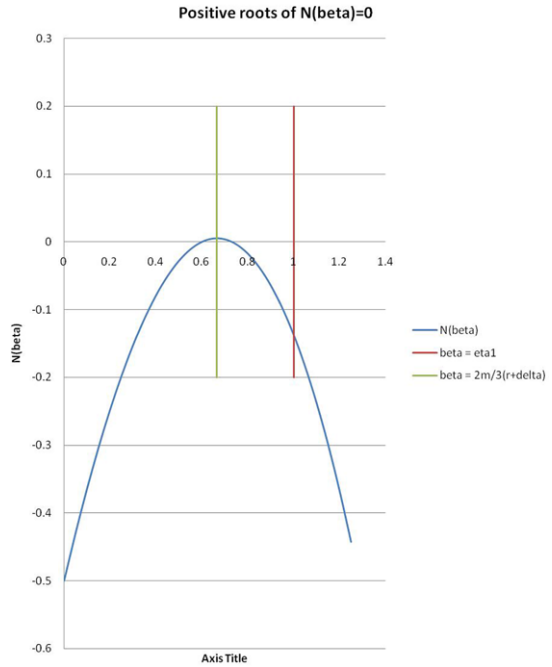
where

$$a = -9(r + \delta)^3\rho^2, \quad b = -8(r + \delta)^4 + 6(2m - 3M)(r + \delta)^2\rho^2, \\ c = 4m(r + \delta)(2(r + \delta)^2 - (m - 6M)\rho^2) \quad \text{and} \quad d = -8m^2M\rho^2.$$

Since the denominator of $F(\beta)$ is positive for all values of β except $2m/3(r + \delta)$, the sign of $F(\beta)$ is the same as that of $N(\beta)$. In what follows, we perform a simple sign analysis of $N(\beta)$ to draw inference about the roots of (58). After a few steps of algebra with the help of Mathematica, the following observations can be made:

$$N(\beta) \rightarrow \infty \quad \text{as} \quad \beta \rightarrow -\infty; \\ N(\beta) = -8m^2M\rho^2 < 0 \quad \text{when} \quad \beta = 0;$$

Fig. 14 Subsidy rates vs. M_1



$$N(\beta) = \frac{16}{9}m^2(r + \delta)^2 > 0 \quad \text{when } \beta = \frac{2m}{3(r + \delta)};$$

$$N(\beta) = -m^2(m + 2M)\rho^2 < 0 \quad \text{when } \beta = \frac{m}{(r + \delta)};$$

$$N(\beta) \rightarrow -\infty \quad \text{as } \beta \rightarrow \infty.$$

These observations make it clear that the equation $N(\beta) = 0$ has three real roots in the following intervals:

$$(-\infty, 0), \quad \left(0, \frac{2m}{3(r + \delta)}\right) \quad \text{and} \quad \left(\frac{2m}{3(r + \delta)}, \frac{m}{(r + \delta)}\right).$$

Moreover, from (55) and (57), we see that there should be exactly one root in the desired interval $(\frac{2m}{3(r+\delta)}, \eta_1)$ for there to be cooperation in the equilibrium solution. In fact, the location of the third root in the interval $(\frac{2m}{3(r+\delta)}, \frac{m}{(r+\delta)})$ w.r.t. η_1 determines whether we will have a cooperative or non-cooperative equilibrium. Figure 14 shows the curve $N(\beta)$ when $\beta > 0$. This curve gives us an idea of when exactly one of the two positive roots of the equation $N(\beta) = 0$ would be in the interval $(\frac{2m}{3(r+\delta)}, \eta_1)$. Note that one root of this equation is negative and is not shown in the figure. It can be easily seen that to attain exactly one root in the interval $(\frac{2m}{3(r+\delta)}, \eta_1)$ and thereby to have a cooperative equilibrium, we must have

$$F(\beta)|_{\beta=\eta_1} < 0, \tag{59}$$

which, when using $\beta = \eta_1$ in $F(\beta)$, gives us

$$\frac{1}{9} \left(\frac{2m^2 \rho^2}{(r + \delta)^2} + \frac{(r + \delta)^2 - 18M\rho^2 - \sqrt{4(r + \delta)^4 + 8m(r + \delta)^2 \rho^2 + m^2 \rho^4}}{\rho^2} + m \left(7 + \frac{2\sqrt{4(r + \delta)^4 + 8m(r + \delta)^2 \rho^2 + m^2 \rho^4}}{(r + \delta)^2} \right) \right) < 0. \tag{60}$$

After a few steps of algebra, one can see that the condition (60) has just the opposite sign to the one that ensures a non-cooperative solution in the case of symmetric retailers, which can be obtained by simply using $M_1 = M_2 = M$ in Proposition 3. In other words, when the parameters m, M, r, δ , and ρ are such that the inequality (60) is not satisfied, then the third root of the equation $N(\beta) = 0$ will be greater than or equal to η_1 , and the optimal solution will be a non-cooperative one. Thus, a unique cooperative equilibrium is guaranteed when (60) is satisfied.

Case II: Non-cooperative Equilibrium We now consider the non-cooperative equilibrium ($\theta^* = 0$) in the symmetric retailer case. As illustrated in Appendix A, the system of equations (15)–(20) can be solved explicitly in the non-cooperative case to get a unique positive solution of β_1 and β_2 , given by (43). Since the symmetric retailer case is a further simplification of the case of identical retailers, i.e., with $M_1 = M_2 = M$, the solution of β is unique and is given by (43). Note that this value of β equals η_1 .

Appendix C: Proof of Proposition 7

As defined in Sect. 7, V_r^c is the combined value function of the two retailers in the cooperative scenario and V_r^n is the same in the non-cooperative scenario. We can write γ_i and α_i in terms of β_i from (44) and (46), respectively. Recall that in the case of symmetric retailers, $\alpha_1 = \alpha_2 = \alpha$, $\beta_1 = \beta_2 = \beta$, and $\gamma_1 = \gamma_2 = \gamma$. Furthermore, when there is no cooperation, $\beta = \eta$, and its value is given by (43). Using $\beta = \eta$, (44), and (46), we can find V_r^n by adding computing (12) for $i = 1, 2$, and adding the two. After a few steps of algebra, we get

$$V_r^n = \frac{2(x_1 + x_2)(m(r + 2\delta) - 2\delta(r + \delta)\eta_1) + (-1 + x_1 + x_2)\eta_1(2m - 3(r + \delta)\eta_1)\rho^2}{2r(r + \delta)}. \tag{61}$$

In the case of symmetric retailers with cooperation, using (44) and (46) from Appendix B, and using the fact that $\alpha_1 = \alpha_2 = \alpha$, $\beta_1 = \beta_2 = \beta$ and $\gamma_1 = \gamma_2 = \gamma$, we get

$$V_r^c = \frac{4(x_1 + x_2)(m(r + 2\delta) - 2\beta\delta(r + \delta)) - (\beta + 2B)(1 - x_1 - x_2)(2m - 3\beta(r + \delta))\rho^2}{4r(r + \delta)}. \tag{62}$$

Clearly, V_r^n and V_r^c depend only on the sum $(x_1 + x_2)$, and this proves the first statement of Proposition 7. Next, we define $\Delta V_r = V_r^c - V_r^n$, which can be computed as follows:

$$\Delta V_r = \frac{-8x\delta(r + \delta)(\beta - \eta_1) + (-1 + x)(2m(\beta + 2B - 2\eta_1) - 3(r + \delta)(\beta^2 + 2B\beta - 2\eta_1^2))\rho^2}{4r(r + \delta)},$$

where $x = x_1 + x_2$. ΔV_r is linear in x , and we will write it as $\Delta V_r(x)$. We can see that $\Delta V_r(1) = 2\delta(\eta_1 - \beta)/r > 0$. This is because we know from (57) that to sustain a cooperative equilibrium, the parameters (m, M, r, δ, ρ) should be such that $\beta < \eta_1$.

Now consider

$$\Delta V_r(0) = \frac{(-2m(\beta + 2B - 2\eta_1) + 3(r + \delta)(\beta^2 + 2B\beta_2 - 2\eta_1^2))\rho^2}{4r(r + \delta)}.$$

Substituting the value of B in terms of β (from (53)) in the above expression, we can write

$$\Delta V_r(0) = \frac{4(r + \delta)(m - \beta(r + \delta)) + \eta(2m - 3(r + \delta)\eta)}{2r(r + \delta)}. \quad (63)$$

It is clear by (63) that a decrease in the value of β (caused by changes in parameters) also decreases the value of $\Delta V_r(0)$. We know that for a cooperative equilibrium, $\beta < \eta_1$, and so a lower bound for $\Delta V_r(0)$ can be obtained by using $\beta = \eta_1$ in (63). This lower bound is

$$\frac{4m(r + \delta) + 2m\eta\rho^2 - (r + \delta)\eta(4(r + \delta) + 3\eta\rho^2)}{2r(r + \delta)}.$$

By using the value of η_1 from Appendix B, and after a few steps of algebra, we can see that the above expression reduces to zero. Therefore, $\Delta V_r(0) > 0$.

Because $\Delta V_r(x)$ is linear in x , $\Delta V_r(0) > 0$, and $\Delta V_r(1) > 0$, we can say that $\Delta V_r(x) > 0$, $\forall x \in [0, 1]$. Thus, $V_r^c(x) > V_r^n(x)$, $\forall x \in [0, 1]$. The equality holds when $\beta = \eta_1$, i.e., when non-cooperation is optimal for the manufacturer.

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