Dyn Games Appl (2011) 1: 74–114
DOI 10.1007/s13235-010-0005-0

Some Recent Aspects of Differential Game Theory

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Published online: 5 October 2010

Abstract This survey paper presents some new advances in theoretical aspects of differential game theory. We particular focus on three topics: differential games with state constraints; backward stochastic differential equations approach to stochastic differential games; differential games with incomplete information. We also address some recent development in nonzero-sum differential games (analysis of systems of Hamilton–Jacobi equations by conservation laws methods; differential games with a large number of players, i.e., mean-field games) and long-time average of zero-sum differential games.

Keywords Differential game · Viscosity solution · System of Hamilton–Jacobi equations · Mean-field games · State-constraints · Backward stochastic differential equations · Incomplete information

1 Introduction

This survey paper presents some recent results in differential game theory. In order to keep the presentation at a reasonable size, we have chosen to describe in full details three topics with which we are particularly familiar, and to give a brief summary of some other research directions. Although this choice does not claim to represent all the recent literature on the

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more theoretic aspects of differential game theory, we are pretty much confident that it covers a large part of what has recently been written on the subject. It is clear however that the respective part dedicated to each topic is just proportional to our own interest in it, and not to its importance in the literature.

The three main topics we have chosen to present in detail are:

– Differential games with state constraints,
– Backward stochastic differential equation approach to differential games,
– Differential games with incomplete information.

Before this, we also present more briefly two domains which have been the object of very active research in recent years:

– nonzero-sum differential games,
– long-time average of differential games.

The first section of this survey is dedicated to nonzero-sum differential games. Although zero-sum differential games have attracted a lot of attention in the 80–90's (in particular, thanks to the introduction of viscosity solutions for Hamilton–Jacobi equations), the advances on nonzero-sum differential games have been scarcer, and mainly restricted to linear-quadratic games or stochastic differential games with a nondegenerate diffusion. The main reason for this is that there was very little understanding of the *system* of Hamilton–Jacobi equations naturally attached to these games. In the recent years the analysis of this system has been the object of several papers by Bressan and his co-authors. At the same time, nonzero-sum differential games with a very large number of players have been investigated in the terminology of mean-field games by Lasry and Lions.

In the second section we briefly sum up some advances in the analysis of the large time behavior of zero-sum differential games. Such problems have been the aim of intense research activities in the framework of repeated game theory; it has however only been recently investigated for differential games.

In the third part of this survey (the first one to be the object of a longer development) we investigate the problem of state constraints for differential games, and in particular, for pursuit-evasion games. Even if such class of games has been studied since Isaacs' pioneering work [80], the existence of a value was not known up to recently for these games in a rather general framework. This is mostly due to the lack of regularity of the Hamiltonian and of the value function, which prevents the usual viscosity solution approach to work (Evans and Souganidis [63]): Indeed some controllability conditions on the phase space have to be added in order to prove the existence of the value (Bardi, Koike and Soravia [18]). Following Cardaliaguet, Quincampoix and Saint Pierre [50] and Bettiol, Cardaliaguet and Quincampoix [26] we explain that, even without controllability conditions, the game has a value and that this value can be characterized as the smallest supersolution of some Hamilton–Jacobi equation with discontinuous Hamiltonian.

Next we turn to zero-sum stochastic differential games. Since the pioneering work by Fleming and Souginidis [65] it has been known that such games have a value, at least in a framework of games of the type "nonanticipating strategies against controls". Unfortunately this notion of strategies is not completely satisfactory, since it presupposes that the players have a full knowledge of their opponent's control in all states of the world: It would be more natural to assume that the players use strategies which give an answer to the control effectively played by their opponent. On the other hand it seems also natural to consider nonlinear cost functionals and to allow the controls of the players to depend on events of the past which happened before the beginning of the game. The last two points have been investigated in a series of papers by Buckdahn and Li [35, 36, 39], and an approach more direct than that in [65] has been developed. The first point, together with the two others, will be the object of the fourth part of the survey.

In the last part we study differential games with incomplete information. In such games, one of the parameters of the game is chosen at random according to some probability measure and the result is told to one of the players and not to the other. Then the game is played as usual, players observing each other's control. The main difference with the usual case is that at least one of the players does not know which payoff he is actually optimizing. All the difficulty of this game is to understand what kind of information the informed player has interest in to disclose in order to optimize his payoff, taking thus the risk that his opponent learns his missing information. Such games are the natural extension to differential games of the Aumann–Maschler theory for repeated games [11]. Their analysis has been developed in a series of papers by Cardaliaguet [41, 43–45] and Cardaliaguet and Rainer [51, 52].

Throughout these notes we assume the reader to be familiar with the basic results of differential game theory. Many references can be quoted on this subject: A general introduction for the formal relation between differential games and Hamilton–Jacobi equations (or system) can be found in the monograph Baçar and Olsder [13]. We also refer the reader to the classical monographs by Isaacs [80], Friedman [67] and Krasovskii and Subbotin [83] for early presentations of differential game theory. The recent literature on differential games strongly relies on the notion of viscosity solution: Classical monographs on this subject are Bardi and Capuzzo Dolcetta [17], Barles [19], Fleming and Soner [64], Lions [93] and the survey paper by Crandall, Ishii and Lions [56]. In particular [17] contains a good introduction to the viscosity solution aspects of deterministic zero-sum differential games: the proof of the existence and the characterization of a value for a large class of differential games can be found there. Section 6 is mostly based on the notion of backward stochastic differential equation (BSDE): We refer to El Karoui and Mazliak [60], Ma and Yong [96] and Yong and Zhou [116] for a general presentation. The reader is in particular referred to the work by S. Peng on BSDE methods in stochastic control [101]. Let us finally note that, even if this survey tries to cover a large part of the recent literature on the more theoretical aspects of differential games, we have been obliged to omit some topics: linear-quadratic differential games are not covered by this survey despite their usefulness in applications; however, these games have been already the object of several survey papers. Lack of place also prevented us from describing advances in the domain of Dynkin games.

2 Nonzero-sum Differential Games

In the recent years, the more striking advances in the analysis of nonzero-sum differential games have been directed in two directions: analysis by P.D.E. methods of Nash feedback equilibria for deterministic differential games; differential games with a very large number of small players (mean-field games). These topics appear as the natural extensions of older results: existence of Nash equilibria in memory strategies and of Nash equilibria in feedback strategies for stochastic differential games, which have also been revisited.

2.1 Nash Equilibria in Memory Strategies

Since the work of Kononenko [82] (see also Kleimenov [81], Tolwinski, Haurie and Leitmann [114], Gaitsgory and Nitzan [68], Coulomb and Gaitsgory [55]), it has been known that deterministic nonzero-sum differential games admit Nash equilibrium payoffs in memory strategies: This result is actually the counterpart of the so-called Folk Theorem in repeated game theory [100]. Recall that a memory (or a nonanticipating) strategy for a player is a strategy where this player takes into account the past controls played by the other players. In contrast a feedback strategy is a strategy which only takes into account the present position of the system. Following [82] Nash equilibrium payoffs in memory strategies are characterized as follows: A payoff is a Nash equilibrium payoff if and only if it is reachable (i.e., the players can obtain it by playing some control) and individually rational (the expected payoff for a player lies above its min-max level at any point of the resulting trajectory).

This result has been recently generalized to stochastic differential games by Buckdahn, Cardaliaguet and Rainer [38] (see also Rainer [105]) and to games in which players can play random strategies by Souquière [111].

2.2 Nash Equilibria in Feedback Form

Although the existence and characterization result of Nash equilibrium payoffs in memory strategies is quite general, it has several major drawbacks. Firstly, there are, in general, infinitely many such Nash equilibria, but there exists—at least up to now—no completely satisfactory way to select one. Secondly, such equilibria are usually based on threatening strategies which are often non credible. Thirdly, the corresponding strategies are, in general, not “time-consistent” and in particular cannot be computed by any kind of “backward induction”. For this reason it is desirable to find more robust notions of Nash equilibria. The best concept at hand is the notion of subgame perfect Nash equilibria. Since the works of Case [54] and Friedman [67], it is known that subgame perfect Nash equilibria are (at least heuristically) given by feedback strategies and that their corresponding payoffs should be the solution of a system of Hamilton–Jacobi equations. Up to now these ideas have been successfully applied to linear-quadratic differential games (Case [54], Starr and Ho [113], ...) and to stochastic differential games with non degenerate viscosity term: In the first case, one seeks solutions which are quadratic with respect to the state variable; this leads to the resolution of Riccati equations. In the latter case, the regularizing effect of the non-degenerate diffusion allows us to use fixed point arguments to get either Nash equilibrium payoffs or Nash equilibrium feedbacks. Several approaches have been developed: Borkar and Ghosh [27] consider infinite horizon problems and use the smoothness of the invariant measure associated to the S.D.E; Bensoussan and Frehse [21, 22] and Mannucci [97] build “regular” Nash equilibrium payoffs satisfying a system of Hamilton–Jacobi equations thanks to elliptic or parabolic P.D.E techniques; Nash equilibrium feedbacks can also be built by backward stochastic differential equations methods like in Hamadène, Lepeltier and Peng [75], Hamadène [74], Lepeltier, Wu and Yu [92].

2.3 Ill-posedness of the System of HJ Equations

In a series of articles, Bressan and his co-authors (Bressan and Chen [33, 34], Bressan and Priuli [32], Bressan [30, 31]) have analyzed with the help of P.D.E methods the system of Hamilton–Jacobi equations arising in the construction of feedback Nash equilibria for deterministic nonzero-sum games. In state-space dimension 1 and for the finite horizon problem, this system takes the form

$$\partial \mathbf{V}_i + H_i(x, D\mathbf{V}_1, \dots, D\mathbf{V}_n) = 0 \quad \text{in } \mathbb{R} \times (0, T),\ i = 1, \dots, n,$$

coupled with a terminal condition at time T (here n is the number of players and H_i is the Hamiltonian of player i, $\mathbf{V}_i(t,x)$ is the payoff obtained by player i for the initial condition (t,x)). Setting $p_i = (\mathbf{V}_i)_x$ and deriving the above system with respect to x one obtains the system of conservation laws:

$$\partial_t p_i + \big(H_i(x, p_1, \dots, p_n)\big)_x = 0 \quad \text{in } \mathbb{R} \times (0,T).$$

This system turns out to be, in general, ill-posed. Typically, in the case of two players ($n = 2$), the system is ill-posed if the terminal payoff of the players have an opposite monotonicity. If, on the contrary, these payoffs have the same monotony and are close to some linear payoff (which is a kind of cooperative case), then the above system has a unique solution, and one can build Nash equilibria in feedback form from the solution of the P.D.E [33].

Still in space dimension 1, the case of infinite horizon seems more promising: The system of P.D.E then reduces to an ordinary differential equation. The existence of suitable solutions for this equation then leads to Nash equilibria. Such a construction is carried out in Bressan and Priuli [32], Bressan [30, 31] through several classes of examples and by various methods.

In a similar spirit, the papers Cardaliaguet and Plaskacz [47], Cardaliaguet [42] study a very simple class of nonzero-sum differential games in dimension 1 and with a terminal payoff: In this case it is possible to select a unique Nash equilibrium payoff in feedback form by just imposing that it is Pareto whenever there is a unique Pareto one. However, this equilibrium payoff turns out to be highly unstable with respect to the terminal data. Some other examples of nonlinear-quadratic differential games are also analyzed in Olsder [99] and in Ramasubramanian [106].

2.4 Mean-field Games

Since the system of P.D.Es arising in nonzero-sum differential games is, in general, ill-posed, it is natural to investigate situations where the problem simplifies. It turns out that this is the case for differential games with a very large number of identical players. This problem has been recently developed in a series of papers by Lasry and Lions [87–90, 94] under the terminology of mean-field games (see also Huang, Caines and Malhame [76–79] for a related approach). The main achievement of Lasry and Lions is the identification of the limit when the number of players tends to infinity. The typical resulting model takes the form

$$\begin{cases} \text{(i)} & -\partial_t u - \Delta u + H(x, m, Du) = 0 \quad \text{in } \mathbb{R}^d \times (0,T), \\ \text{(ii)} & \partial_t m - \Delta m - \operatorname{div}\big(D_p H(x, m, Du) m\big) = 0 \quad \text{in } \mathbb{R}^d \times (0,T), \\ \text{(iii)} & m(0) = m_0, \qquad u(x,T) = G\big(x, m(T)\big). \end{cases} \tag{1}$$

In the above system, the first equation has to be understood backward in time while the second one is forward in time. The first equation (a Hamilton–Jacobi one) is associated with an optimal control problem and its solution can be regarded as the value function for a typical small player (in particular the Hamiltonian $H = H(x, m, p)$ is convex with respect to the last variable). As for the second equation, it describes the evolution of the density $m(t)$ of the population.

More precisely, let us first consider the behavior of a typical player. He controls through his control (α_s) the stochastic differential equation

$$dX_t = \alpha_t \, dt + \sqrt{2} B_t$$

(where (B_t) is a standard Brownian motion) and he aims at minimizing the quantity

$$\mathbb{E}\left[\int_0^T \frac{1}{2} L\big(X_s, m(s), \alpha_s\big)\, ds + G\big(X_T, m(T)\big)\right],$$

where L is the Fenchel conjugate of H with respect to the p variable. Note that in this cost the evolving measure $m(s)$ enters as a parameter. The value function of our average player is then given by (1-(i)). His optimal control is—at least heuristically—given in feedback form by $\alpha^*(x,t) = -D_p H(x, m, Du)$. Now, if all agents argue in this way, their repartition will move with a velocity which is due, on the one hand, to the diffusion, and, one the other hand, to the drift term $-D_p H(x, m, Du)$. This leads to the Kolmogorov equation (1-(ii)).

The mean-field game theory developed so far has been focused on two main issues: firstly, investigate equations of the form (1) and give an interpretation (in economics, for instance) of such systems. Secondly, analyze differential games with a finite but large number of players and interpret (1) as their limiting behavior as the number of players goes to infinity.

Up to now the first issue is well understood and well documented. The original works by Lasry and Lions give a certain number of conditions under which (1) has a solution, discuss its uniqueness and its stability. Several papers also study the numerical approximation of this solution: see Achdou and Capuzzo Dolcetta [1], Achdou, Camilli and Capuzzo Dolcetta [2], Gomes, Mohr and Souza [71], Lachapelle, Salomon and Turinici [85]. The mean-field games theory has been used in the analysis of wireless communication systems in Huang, Caines and Malhamé [76], or Yin, Mehta, Meyn and Shanbhag [115]. It seems also particularly adapted to modeling problems in economics: see Guéant [72, 73], Lachapelle [84], Lasry, Lions, Guéant [91], and the references therein.

As for the second part of the program, the limiting behavior of differential games when the number of players tend to infinity has been understood for ergodic differential games [88]. The general case remains mostly open.

3 Long-time Average of Differential Games

Another way to reduce the complexity of differential games is to look at their long-time behavior. Among the numerous applications of this topic let us quote homogenization, singular perturbations and dimension reduction of multiscale systems.

In order to explain the basic ideas, let us consider a two-player stochastic zero-sum differential game with dynamics given by

$$dX_s^{t,\zeta;u,v} = b\big(X_s^{t,\zeta;u,v}, u_s, v_s\big)\, ds + \sigma\big(X^{t,\zeta;u,v}, u_s, v_s\big)\, dB_s, \quad s \in [t, +\infty),$$
$$X_t = \zeta,$$

where B is a d-dimensional standard Brownian motion on a given probability space $(\Omega, \mathcal{F}, P)$, $b : \mathbb{R}^N \times U \times V \to \mathbb{R}^N$ and $\sigma : \mathbb{R}^N \times U \times V \to \mathbb{R}^{N\times d}$, U and V being some metric compact sets. We assume that the first player, playing with u, aims at minimizing a running payoff $\ell : \mathbb{R}^N \times U \times V \to \mathbb{R}$ (while the second players, playing with v, maximizes). Then it is known that, under some Isaacs' assumption, the game has a value $\mathbf{V}_T$ which is the viscosity solution of a second order Hamilton–Jacobi equation of the form

$$\begin{cases} -\partial_t \mathbf{V}_T(t,x) + H\big(x, D\mathbf{V}_T(t,x), D^2\mathbf{V}_T(t,x)\big) = 0 & \text{in } [0,T]\times\mathbb{R}^N, \\ \mathbf{V}_T(T,x) = 0 & \text{in } \mathbb{R}^N. \end{cases}$$

A natural question is the behavior of $\mathbf{V}_T$ as $T \to +\infty$. Actually, since $\mathbf{V}_T$ is typically of linear growth, the natural quantity to consider is the long-time average, i.e., $\lim_{T\to+\infty} \mathbf{V}_T/T$.

Interesting phenomena can be observed under some compactness assumption on the underlying state-space. Let us assume, for instance, that the maps $b(\cdot,u,v)$, $\sigma(\cdot,u,v)$ and $\ell(\cdot,u,v)$ are periodic in all space variables: this actually means that the game takes place in the torus $\mathbb{R}^N/\mathbb{Z}^N$.

In this framework, the long-time average is well understood in two cases: either the diffusion is strongly nondegenerate:

$$\exists \nu > 0, \quad (\sigma\sigma^*)(x,u,v) \geq \nu I_N \quad \forall x,u,v,$$

(where the inequality is understood in the sense of quadratic matrices); or $\sigma \equiv 0$ and $H = H(x,\xi)$ is coercive:

$$\lim_{|\xi|\to+\infty} H(x,\xi) = +\infty \quad \text{uniformly with respect to } x. \tag{2}$$

In both cases the quantity $\mathbf{V}_T(x,0)/T$ uniformly converges to the unique constant $\bar{c}$ for which the problem

$$\bar{c} + H\big(x, D\chi(x), D^2\chi(x)\big) = 0 \quad \text{in } \mathbb{R}^N$$

has a continuous, periodic solution χ. In particular, the limit is independent of the initial condition. Such kind of results has been proved by Lions, Papanicoulaou and Varadhan [95] for first order equations (i.e., deterministic differential games). For second order equations, the result has been obtained by Alvarez and Bardi in [3], where the authors combine fundamental contributions of Evans [61, 62] and of Arisawa and Lions [7] (see also Alvarez and Bardi [4, 5], Bettiol [24], Ghosh and Rao [70]).

For deterministic differential games (i.e., $\sigma \equiv 0$), the coercivity condition (2) is not very natural: Indeed, it means that one of the players is much more powerful than the other one. However, very little is known without such a condition. Existing results rely on a specific structure of the game: see for instance Bardi [16], Cardaliaguet [46]. The difficulty comes from the fact that, in these cases, the limit may depend upon the initial condition (see also Arisawa and Lions [7], Quincampoix and Renault [104] for related issues in a control setting). The existence of a limit for large time differential games is certainly one of the main challenges in differential games theory.

4 Existence of a Value for Zero-sum Differential Games with State Constraints

Differential games with state constraints have been considered since the early theory of differential games: we refer to [23, 28, 66, 69, 80] for the computation of the solution for several examples of pursuit. We present here recent trends for obtaining the existence of a value for a rather general class of differential games with constraints. This question had been unsolved during a rather long period due to problems we discuss now.

The main conceptual difficulty for considering such zero-sum games lies in the fact that players have to achieve their own goal *and to satisfy* the state constraint. Indeed, it is not clear to decide which players has to be penalized if the state constraint is violated. For this reason, we only consider a specific class of *decoupled games* where each player controls independently a part of the dynamics. A second mathematical difficulty comes from the fact that players have to use *admissible controls* i.e., controls ensuring the trajectory to fulfil the state constraint. A byproduct of this problem is the fact that starting from two close initial points it is not obvious to find two close *constrained* trajectories. This also affects the regularity of value functions associated with *admissible controls*: The value functions are, in general, not Lipschitz continuous anymore and, consequently, classical viscosity solutions methods for Hamilton–Jacobi equations may fail.

4.1 Statement of the Problem

We consider a differential game where the first player playing with u, controls a first system

$$\begin{cases} y'(t) = g\big(y(t), u(t)\big), & u(t) \in U, \\ y(t_0) = y_0 \in K_U, \end{cases} \tag{3}$$

while the second player, playing with v, controls a second system

$$\begin{cases} z'(t) = h\big(z(t), v(t)\big), & v(t) \in V, \\ z(t_0) = z_0 \in K_V. \end{cases} \tag{4}$$

For every time t, the first player has to ensure the state constraint $y(t) \in K_U$ while the second player has to respect the state constraint $z(t) \in K_V$ for any $t \in [t_0, T]$. We denote by $x(t) = x[t_0, x_0; u(\cdot), v(\cdot)](t) = (y[t_0, y_0; u(\cdot)](t), z[t_0, z_0; v(\cdot)](t))$ the solution of the systems (3) and (4) associated with an initial data $(t_0, x_0) := (t_0, y_0, z_0)$ and with a couple of controls $(u(\cdot), v(\cdot))$.

In the following lines we summarize all the assumptions concerning with the vector fields of the dynamics:

$$\begin{cases} \text{(i)} & U \text{ and } V \text{ are compact subsets of some finite} \\ & \text{dimensional spaces} \\ \text{(ii)} & f : \mathbb{R}^n \times U \times V \to \mathbb{R}^n \text{ is continuous and} \\ & \text{Lipschitz continuous (with Lipschitz constant } M) \\ & \text{with respect to } x \in \mathbb{R}^n \\ \text{(iii)} & \bigcup_u f(x,u,v) \text{ and } \bigcup_v f(x,u,v) \text{ are convex for any } x \\ \text{(iv)} & K_U = \{y \in \mathbb{R}^l, \phi_U(y) \le 0\} \text{ with } \phi_U \in \mathcal{C}^2(\mathbb{R}^l;\mathbb{R}), \\ & \nabla\phi_U(y) \ne 0 \text{ if } \phi_U(y) = 0 \\ \text{(v)} & K_V = \{z \in \mathbb{R}^m, \phi_V(z) \le 0\} \text{ with } \phi_V \in \mathcal{C}^2(\mathbb{R}^m;\mathbb{R}), \\ & \nabla\phi_V(z) \ne 0 \text{ if } \phi_V(z) = 0 \\ \text{(vi)} & \forall y \in \partial K_U, \exists u \in U \text{ such that } \langle \nabla\phi_U(y), g(y,u)\rangle < 0 \\ \text{(vii)} & \forall z \in \partial K_V, \exists v \in V \text{ such that } \langle \nabla\phi_V(z), h(z,v)\rangle < 0 \end{cases} \tag{5}$$

We need to introduce the notion of admissible controls: $\forall y_0 \in K_U$, $\forall z_0 \in K_V$ and $\forall t_0 \in [0, T]$ we define

$$\mathcal{U}(t_0, y_0) := \big\{u(\cdot) : [t_0, +\infty) \to U \text{ measurable} \mid y[t_0, y_0; u(\cdot)](t) \in K_U \ \forall t \ge t_0\big\}$$
$$\mathcal{V}(t_0, z_0) := \big\{v(\cdot) : [t_0, +\infty) \to V \text{ measurable} \mid z[t_0, z_0; v(\cdot)](t) \in K_V \ \forall t \ge t_0\big\}.$$

Under assumptions (5), the Viability Theorem (see [9, 10]) ensures that for all $x_0 = (y_0, z_0) \in K_U \times K_V$

$$\mathcal{U}(t_0, y_0) \neq \emptyset \quad \text{and} \quad \mathcal{V}(t_0, z_0) \neq \emptyset.$$

Throughout the paper we omit t_0 in the notations $\mathcal{U}(t_0, y_0)$ and $\mathcal{U}(t_0, y_0)$ whenever $t_0 = 0$.

We now describe two quantitative differential games. Let us start with a game with an integral cost:

Bolza Type Differential Game Given a *running cost* $L : [0, T] \times \mathbb{R}^N \times U \times V \to \mathbb{R}$ and a *final cost* $\Psi : \mathbb{R}^N \to \mathbb{R}$, we define the payoff associated to an initial position $(t_0, x_0) = (t_0, y_0, z_0)$ and to a pair of controls $(u, v) \in \mathcal{U}(t_0, y_0) \times \mathcal{V}(t_0, z_0)$ by

$$J\big(t_0, x_0; u(\cdot), v(\cdot)\big) = \int_{t_0}^{T} L\big(t, x(t), u(\cdot), v(\cdot)\big)\, dt + \Psi\big(x(T)\big), \tag{6}$$

where $x(t) = x[t_0, x_0; u(\cdot), v(\cdot)](t) = (y[t_0, y_0; u(\cdot)](t), z[t_0, z_0; v(\cdot)](t))$ denotes the solution of the systems (3) and (4). The first player wants to maximize the functional J, while the second player's goal is to minimize J.

Definition 1 A map $\alpha : \mathcal{V}(t_0, z_0) \to \mathcal{U}(t_0, y_0)$ is a nonanticipating strategy (for the first player and for the point $(t_0, x_0) := (t_0, y_0, z_0) \in \mathbb{R}^+ \times K_U \times K_V$) if, for any $\tau > 0$, for all controls $v_1(\cdot)$ and $v_2(\cdot)$ belonging to $\mathcal{V}(t_0, z_0)$, which coincide a.e. on $[t_0, t_0 + \tau]$, $\alpha(v_1(\cdot))$ and $\alpha(v_2(\cdot))$ coincide almost everywhere on $[t_0, t_0 + \tau]$. Nonanticipating strategies β for the second player are symmetrically defined. For any point $x_0 \in K_U \times K_V$ and $\forall t_0 \in [0, T]$ we denote by $\mathcal{A}(t_0, x_0)$ and by $\mathcal{B}(t_0, x_0)$ the sets of the nonanticipating strategies for the first and the second player respectively.

We are now ready to define the value functions of the game. The lower value $\mathbf{V}^-$ is defined by:

$$\mathbf{V}^-(t_0, x_0) := \inf_{\beta \in \mathcal{B}(t_0, x_0)} \sup_{u(\cdot) \in \mathcal{U}(t_0, y_0)} J\big(t_0, x_0; u(\cdot), \beta\big(u(\cdot)\big)\big), \tag{7}$$

where J is defined by (6). On the other hand we define the upper value function as follows:

$$\mathbf{V}^+(t_0, x_0) := \lim_{\varepsilon \to 0^+} \sup_{\alpha \in \mathcal{A}(t_0, x_0)} \inf_{v(\cdot) \in \mathcal{V}(t_0, z_0)} J_\varepsilon\big(t_0, x_0; \alpha\big(v(\cdot)\big), v(\cdot)\big) \tag{8}$$

with

$$J_\varepsilon\big(t_0, x_0; u(\cdot), v(\cdot)\big) := \int_{t_0}^{T} L\big(t, x(t), u(t), v(t)\big)\, dt + \Psi_\varepsilon\big(x(T)\big),$$

where $x(t) = x[t_0, x_0; u(\cdot), v(\cdot)](t)$ and Ψ_ε is the lower semicontinuous function defined by

$$\Psi_\varepsilon(x) := \inf\big\{\rho \in \mathbb{R} \mid \exists y \in \mathbb{R}^n \text{ with } \big|(y, \rho) - \big(x, \Psi(x)\big)\big| = \varepsilon\big\}.$$

The asymmetry between the definition of the value functions is due to the fact that one assumes that the terminal payoff Ψ is lower semicontinuous. When Ψ is continuous, one can check that $\mathbf{V}^+$ can equivalently be defined in a more natural way as

$$\mathbf{V}^+(t_0, x_0) := \sup_{\alpha \in \mathcal{A}(t_0, x_0)} \inf_{v(\cdot) \in \mathcal{V}(t_0, z_0)} J\big(t_0, x_0; \alpha\big(v(\cdot)\big), v(\cdot)\big).$$

We now describe the second differential game which is a pursuit game with closed target $C \subset K_U \times K_V$.

Pursuit Type Differential Game The hitting time of C for a trajectory $x(\cdot) := (y(\cdot), z(\cdot))$ is:

$$\theta_C\big(x(\cdot)\big) := \inf\big\{t \geq 0 \mid x(t) \in C\big\}.$$

If $x(t) \notin C$ for every $t \geq 0$, then we set $\theta_C(x(\cdot)) := +\infty$. In the pursuit game, the first player wants to maximize θ_C while the second player wants to minimize it. The value functions are defined as follows: The lower optimal hitting-time function is the map $\vartheta_C^- : K_U \times K_V \to \mathbb{R}^+ \cup \{+\infty\}$ defined, for any $x_0 := (y_0, z_0)$, by

$$\vartheta_C^-(x_0) := \inf_{\beta(\cdot)\in\mathcal{B}(x_0)} \sup_{u(\cdot)\in\mathcal{U}(y_0)} \theta_C\big(x\big[x_0, u(\cdot), \beta\big(u(\cdot)\big)\big]\big).$$

The upper optimal hitting-time function is the map $\vartheta_C^+ : K_U \times K_V \to \mathbb{R}^+ \cup \{+\infty\}$ defined, for any $x_0 := (y_0, z_0)$, by

$$\vartheta_C^+(x_0) := \lim_{\varepsilon\to 0^+} \sup_{\alpha(\cdot)\in\mathcal{A}(x_0)} \inf_{v(\cdot)\in\mathcal{V}(z_0)} \theta_{C+\varepsilon B}\big(x\big[x_0, \alpha\big(v(\cdot)\big), v(\cdot)\big]\big).$$

By convention, we set $\vartheta_C^-(x) = \vartheta_C^+(x) = 0$ on C.

Remarks

- Note that here again the definition of the upper and lower value functions are not symmetric: this is related to the fact that the target assumed to be closed, so that the game is intrinsically asymmetric.
- The typical pursuit game is the case when the target coincides with the diagonal: $C = \{(y, z), \mid y = z\}$. We refer the reader to [6, 29] for various types of pursuit games. The formalism of the present survey is adapted from [50].

4.2 Main Result

The main difficulty for the analysis of state-constraint problems lies in the fact that two trajectories of a control system starting from two—close—different initial conditions could be estimated by classical arguments on the continuity of the flow of the differential equation. For constrained systems, it is easy to imagine cases where the *constrained* trajectories starting from two close initial conditions are rather far from each other. So, an important problem in order to get suitable estimates on constrained trajectories, is to obtain a kind of Filippov Theorem with constraints. Namely a result which allows one to approach—in a suitable sense—a given trajectory of the dynamics by a constrained trajectory. Note that similar results exist in the literature. However, we need here to construct a constrained trajectory in a *nonanticipating way* [26] (cf. also [25]), which is not the case in the previous constructions.

Proposition 1 *Assume that conditions* (5) *are satisfied. For any* $R > 0$ *there exist* $C_0 = C_0(R) > 0$ *such that for any initial time* $t_0 \in [0, T]$, *for any* $y_0, y_1 \in K_U$ *with* $|y_0|, |y_1| \leq R$, *there is a nonanticipating strategy* $\sigma : \mathcal{U}(t_0, y_0) \longrightarrow \mathcal{U}(t_0, y_1)$ *with the following property*: *for any* $u_0(\cdot) \in \mathcal{U}(t_0, y_0)$ *and for any* $t \in [t_0, T]$ *we have*

$$\left|y_0(t)-y_1(t)\right|+\int_{t_0}^{t}\left|g\big(y_0(s),u_0(s)\big)-g\big(y_1(s),\sigma\big(u_0(\cdot)\big)(s)\big)\right|ds \le C_0|y_0-y_1|e^{C_0(t-t_0)}, \quad (9)$$

where we have set for simplicity $y_0 = y[t_0, y_0; u_0(\cdot)]$ *and* $y_1 = y[t_0, y_1; \sigma(u_0(\cdot))](t)$.

In particular, if g is affine with respect to the control u, namely

$$g(y,u) = g_1(y)u + g_2(y),$$

where $g_1(y)$ *is an invertible matrix with Lipschitz continuous inverse, then we have*

$$\left|y_0(t)-y_1(t)\right|+\int_{t_0}^{t}\left|u_0(s)-\sigma\big(u_0(\cdot)\big)(s)\right|ds \le C_1|y_0-y_1|e^{C_1(t-t_0)} \quad (10)$$

for some constant $C_1 = C_1(R) > 0$.

A corresponding result for the other player can, of course, be stated and proved similarly. We consider costs satisfying the following conditions:

$$\begin{cases} \text{(i)} & L : [0,T]\times\mathbb{R}^n\times U\times V \longrightarrow \mathbb{R} \text{ is a bounded and Lipschitz} \\ & \text{continuous with respect to all the variables of constant } M; \\ \text{(ii)} & \Psi : \mathbb{R}^n \longrightarrow \mathbb{R} \text{ is bounded and lower semicontinuous.} \end{cases} \quad (11)$$

We also have to assume some structure condition on g, h and L; namely, one of the following conditions has to hold:

Condition 1—$L := L(t,x)$ (L does not depend on u and v)

Condition 2—$L := L(t,x,u,v) = L_0(t,x) + L_1(t,x)u + L_2(t,x)v$,
$g(y,u) = g_1(y)u + g_2(y)$, $h(z,v) = h_1(z)v + h_2(z)$ (12)
where $g_1(y)$ and $h_1(z)$ are invertible bounded matrices
with inverse Lipschitz continuous w.r. to x.

Theorem 1 *Assume that* (5), (11), (12) *hold. Then the Bolza Game has a value, and the associated value function is lower semicontinuous*:

$$\forall (t,x) \in [0,T]\times K_U \times K_V, \quad \mathbf{V}^-(t,x) = \mathbf{V}^+(t,x).$$

The key ingredients for the proof of this Theorem is the reduction of the game to a qualitative differential game and the fact that the Epigraphs of $\mathbf{V}^-$ and $\mathbf{V}^+$ are both equal to a suitable victory domain for this qualitative game. For the principle of the reduction to a qualitative game we refer the reader to [49], for the study of qualitative games we refer to [40]. The detailed proof of Theorem 1 can be found in [26]. We also mention that some kind of constrained differential games with terminal cost can be treated by penalization method; this gives rise to differential games with discontinuous costs (cf. [103, 108]).

Theorem 2 *Assume that conditions* (5) *are fulfilled. Then the pursuit game has a value*:

$$\forall x_0 \in K_U \times K_V, \quad \vartheta_C^-(x_0) = \vartheta_C^+(x_0).$$

4.3 Hamilton–Jacobi–Isaacs Approach

We finally characterize the value function studied above as the unique viscosity solution to a suitable PDE. This method has been introduced by Evans and Souganidis [63] for unconstrained differential games with a continuous value function (cf. also [18] for the extension to some constrained cases). For the sake of shortness we only give results for the Bolza type differential game. Let us first introduce the following Hamilton–Jacobi–Isaacs equation:

$$\begin{cases} -\partial_t W(t,x) + H\big(t,x,\partial_x W(t,x)\big) = 0 & \text{on } (0,T)\times K_U\times K_V, \\ W(T,x) = \Psi(x) & \text{on } K_U\times K_V, \end{cases} \tag{13}$$

where the Hamiltonian function H is given by

$$H(t,x,p) := \max_{v\in V(z)} \min_{u\in U(y)} \big\{-\big\langle f(x,u,v),p\big\rangle - L(t,x,u,v)\big\},$$

and where L is the running cost function. The function H is, in general, discontinuous. Therefore, in order to consider the notion of solution in viscosity sense, we have to use the upper and lower semicontinuous envelopes of H, denoted by H^* and H_* respectively (see for instance [17]):

$$H^*(t,x,p) := \limsup_{(t',x',p')\to(t,x,p)} H(t',x',p'),$$

$$H_*(t,x,p) := \liminf_{(t',x',p')\to(t,x,p)} H(t',x',p').$$

Remark 1 Under assumption (12), the set-valued maps $y \rightsquigarrow U(y)$ and $z \rightsquigarrow V(z)$ are lower semicontinuous and we have

$$H^*(t,x,p) = \max_{v\in V} \min_{u\in U(y)} \big\{-\big\langle f(x,u,v),p\big\rangle - L(t,x,u,v)\big\}$$

and

$$H_*(t,x,p) = \max_{v\in V(z)} \min_{u\in U} \big\{-\big\langle f(x,u,v),p\big\rangle - L(t,x,u,v)\big\}.$$

Definition 2 (Viscosity solution) A viscosity supersolution for the Hamilton–Jacobi–Isaacs equation (13) is a lower semicontinuous function $w:[0,T)\times K_U\times K_V \longrightarrow \mathbb{R}$ with the following property: For any test function $\varphi\in\mathcal{C}^1$ and any $(t_0,x_0)\in[0,T)\times K_U\times K_V$ such that $w-\varphi$ has a local minimum at (t_0,x_0), it holds

$$-\partial_t\varphi(t_0,x_0) + H^*\big(t_0,x_0,\partial_x\varphi(t_0,x_0)\big) \ge 0.$$

An upper semicontinuous function $w:[0,T)\times K_U\times K_V \longrightarrow \mathbb{R}$ is called viscosity subsolution of (13) if, for any test function $\varphi\in\mathcal{C}^1$ and any $(t_0,x_0)\in[0,T)\times K_U\times K_V$ such that $w-\varphi$ achieves a local maximum at (t_0,x_0), we have

$$-\partial_t\varphi(t_0,x_0) + H_*\big(t_0,x_0,\partial_x\varphi(t_0,x_0)\big) \le 0.$$

We say that a continuous function is a viscosity solution of (13) if it is both supersolution and subsolution of (13) at the same time.

Theorem 3 (cf. [26]) *Under assumptions* (5), (11), (12), *the function* $\mathbf{V}^- = \mathbf{V}^+$ *is the smallest lower semicontinuous supersolution to* (13).

It is worth pointing out that we did not prove that the value is a viscosity solution of the Hamilton–Jacobi–Isaacs (HJI) equation. Nevertheless, by proving that upper and lower values are both minimal solutions to the HJI equation, we obtain that they coincide. In general it is not true that the value can be characterized as the unique viscosity solution to the HJI equation (when the Hamiltonian is not convex, uniqueness of discontinuous viscosity solutions does not always hold). However, the characterization can be obtained if the value is regular enough:

Proposition 2 *Under the assumptions* (5), (11), (12), *if the final cost* $\Psi = \Psi(x)$ *is locally Lipschitz continuous*, *then the value function* $\mathbf{V}^- = \mathbf{V}^+$ *is also locally Lipschitz continuous*.

Now recalling that an upper semicontinuous function $w : [0, T) \times K_U \times K_V \longrightarrow \mathbb{R}$ is a viscosity subsolution of (13) if and only if $-w$ is a viscosity supersolution of

$$-\partial_t W(t,x) - H\big(t,x,-\partial_x W(t,x)\big) = 0,$$

we obtain a complete characterization of the value:

Proposition 3 *The Hamilton–Jacobi–Isaacs equation* (13) *admits a unique continuous viscosity solution*, *which is the unique value of the Bolza type game*.

Detailed proofs of results of this section can mainly be found in [26].

5 Stochastic Differential Games. A Backward SDE Approach

In this section, based on the papers by Buckdahn and Li [35–37, 39] the problem of existence of a value for zero-sum two-player stochastic differential games for a nonlinear payoff is revisited. This approach represent an alternative to that in the pioneering paper [65] on two-player zero-sum stochastic differential games stochastic differential games by Fleming and Souganidis. Unlike that approach we allow our control processes for a game over the time interval $[t, T]$ to depend on the past of the driving Brownian motion over the interval $[0, t]$. Once having shown that the upper and lower value functions in our approach are deterministic -which is not evident in our framework- we can prove with the help of Peng's method of backward stochastic differential equations (see [101]) in a rather straight-forward manner, without additional technical notions like r-strategies, and without approximations, first the dynamic programming principle for the upper and the lower value functions, and after, on the basis of the dynamic programming principle, the fact that both functions are the viscosity solutions of their own associated Hamilton–Jacobi–Bellman–Isaacs equation. Both equations coincide under Isaacs' condition and so do the upper and the lower value function, i.e., the game has a value. While the above cited papers consider games of the type "strategy against control", we consider here games of the type "nonanticipating strategy with delay against nonanticipating strategy" by adopting the concept developed in [51, 52].

5.1 Preliminaries and Value Functions

We begin with introducing the framework in which we want to investigate stochastic differential games (in short: SDG). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the classical Wiener space, i.e., given some arbitrarily fixed time horizon $T > 0$, we consider $\Omega = C_0([0, T]; R^d)$ as the space of continuous functions $h : [0, T] \to R^d$ with $h(0) = 0$, endowed with the supremum norm, and we let $\mathbb{P}$ be the Wiener measure on the Borel σ-field $\mathcal{B}(\Omega)$ over Ω, with respect to which the coordinate process $W_t(\omega) = \omega(t)$, $t \in [0, T]$, $\omega \in \Omega$, becomes a d-dimensional Brownian motion. Denoting by $\mathcal{N}_{\mathbb{P}}$ the collection of all $\mathbb{P}$-null sets in Ω, we define $\mathcal{F} = \mathcal{B}(\Omega) \vee \mathcal{N}_{\mathbb{P}}$ as the completion of $\mathcal{B}(\Omega)$ under $\mathbb{P}$. Moreover, we introduce the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ generated by the coordinate process W and completed by all $\mathbb{P}$-null sets: $\mathcal{F}_t = \sigma\{B_s, s \le t\} \vee \mathcal{N}_{\mathbb{P}}$, $t \in [0, T]$.

Let us now introduce the frame for the general two-player zero-sum SDG we want to study. For this, let U and V be two compact metric spaces. We consider as set of admissible controls for the first player the space $\mathcal{U}$ of all $\mathbb{F}$-adapted, U-valued processes, and for the second player the space $\mathcal{V}$ of $\mathbb{F}$-adapted, but now V-valued processes. For any admissible processes $u \in \mathcal{U}$ and $v \in \mathcal{V}$, and for arbitrary initial data $t \in [0, T]$ and $\zeta \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$ we define the controlled state process $X^{t,\zeta;u,v}$ to be the unique solution in $\mathcal{S}^2_{\mathbb{F}}(t, T; \mathbb{R}^d)$ of the following equation:

$$\begin{cases} dX_s^{t,\zeta;u,v} = \sigma\big(X_s^{t,\zeta;u,v}, u_s, v_s\big)\,dW_s + b\big(X_s^{t,\zeta;u,v}, u_s, v_s\big)\,ds, \quad s \in [t, T], \\ X_t^{t,\zeta;u,v} = \zeta. \end{cases} \tag{14}$$

We recall that $\mathcal{S}^2_{\mathbb{F}}(t, T; \mathbb{R}^d)$ denotes the space of $\mathbb{R}^d$-valued, $\mathbb{F}$-adapted continuous processes $(X_s)_{s\in[t,T]}$ such that $\mathbb{E}[\sup_{s\in[t,T]} |X_s|^2] < +\infty$, while $L^2_{\mathbb{F}}(t, T; \mathbb{R}^d)$ is the space of all $\mathbb{R}^d$-valued, $\mathbb{F}$-adapted processes which are square integrable over $[t, T] \times \Omega$. The coefficients $\sigma : \mathbb{R}^d \times U \times V \to \mathbb{R}^{d\times d}$ and $b : \mathbb{R}^d \times U \times V \to \mathbb{R}^d$ are supposed to be continuous in (x, u, v) and Lipschitz in x, uniformly with respect to (u, v). It is well known that under these assumptions SDE (14) possesses a unique solution $X^{t,\zeta;u,v} \in \mathcal{S}^2_{\mathbb{F}}(t, T; \mathbb{R}^d)$, and, for any given $p \ge 1$, there is some constant C_p such that

$$\begin{aligned} &\mathbb{E}\Big[\sup_{s\in[t,T]} \big|X_s^{t,\zeta;u,v} - \zeta\big|^p \mid \mathcal{F}_t\Big] \le C_p\big(1 + |\zeta|^p\big)(T - t)^{p/2}, \\ &\mathbb{E}\Big[\sup_{s\in[t,T]} \big|X_s^{t,\zeta;u,v} - X_s^{t,\zeta';u,v}\big|^p \mid \mathcal{F}_t\Big] \le C_p|\zeta - \zeta'|^p, \end{aligned} \tag{15}$$

for all $t \in [0, T]$, $\zeta, \zeta' \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$, and for all $u \in \mathcal{U}$, $v \in \mathcal{V}$. We now associate with our controlled state process a nonlinear cost functional defined with the help of a backward stochastic differential equation (BSDE). Given a Lipschitz function $\Phi : \mathbb{R}^d \to \mathbb{R}$ describing the terminal cost, and a continuous function $f : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times U \times V \to \mathbb{R}$, Lipschitz in $(x, y, z) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$, uniformly in (u, v), describing the running cost, we consider the BSDE

$$\begin{cases} dY_s^{t,\zeta;u,v} = -f\big(X_s^{t,\zeta;u,v}, Y_s^{t,\zeta;u,v}, Z_s^{t,\zeta;u,v}, u_s, v_s\big)\,ds + Z_s^{t,\zeta;u,v}\,dW_s, \quad s \in [t, T], \\ Y_T^{t,\zeta;u,v} = \Phi\big(X_T^{t,\zeta;u,v}\big); \end{cases} \tag{16}$$

the initial data are here the same as those for the associated SDE (14). By standard estimates combining those for SDE with those for BSDE we get that the existence of some constant

C such that, for all $t \in [0,T]$, $\zeta, \zeta' \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$ and all $u \in \mathcal{U}$, $v \in \mathcal{V}$, $\mathbb{P}$-a.s.,

$$\begin{aligned} &\big|Y_t^{t,\zeta;u,v}\big| \le C\big(1+|\zeta|\big), \\ &\big|Y_t^{t,\zeta;u,v} - Y_t^{t,\zeta';u,v}\big| \le C|\zeta - \zeta'|. \end{aligned} \tag{17}$$

In order to define the game over the time interval $[t,T]$ $(0 \le t \le T)$, we introduce the following subspaces of admissible controls and also the notion of admissible strategies. For this we define first the spaces $U_{t,T} := L^0([t,T]; U)$ and $V_{t,T} := L^0([t,T]; V)$ of all deterministic, measurable functions over $[t,T]$, with values in U and in V, respectively, and we endow them with their Borel-σ-fields, denoted by $\mathcal{B}(U_{t,T})$ and $\mathcal{B}(V_{t,T})$, respectively.

Definition 3 The space $\mathcal{U}_{t,T}$ of controls admissible for the first player for the game over the time interval $[t,T]$ is defined as the space of all controls $u \in \mathcal{U}$ restricted to $[t,T]$: $u|_{[t,T]} := (u_s)_{s\in[t,T]}$. In the same spirit is defined the space of admissible controls for a game on $[t,T]$ for the second player; it is denoted by $\mathcal{V}_{t,T}$.

Definition 4 A nonanticipating strategy with delay (NAD strategy) for player 1 is a measurable mapping $\alpha : \Omega \times [t,T] \times V_{t,T} \longrightarrow U$ satisfying the following properties:

(i) (Progressive measurability) The mapping α is $\mathbb{F}$-progressively measurable, i.e., for all $s \in [t,T]$, the mapping α restricted to $\Omega \times [t,s] \times V_{t,T}$ is $\mathcal{F}_s \otimes \mathcal{B}([t,s]) \otimes \mathcal{B}(V_{t,T}) - \mathcal{B}(U)$-measurable;

(ii) (Strict nonanticipativity) For every $\mathbb{F}$-stopping time $\tau : \Omega \to [t,T]$ it holds for $\mathbb{P}$-almost every $\omega \in \Omega$:

If $v, v' \in V_{t,T}$ coincide a.e. on $[t, \tau(\omega)]$, then also $\alpha(\omega, \cdot, v) = \alpha(\omega, \cdot, v')$ a.e. on $[t, \tau(\omega)]$.

(iii) (Nonanticipativity with delay) There exists an increasing sequence of stopping times $(S_n)_{n\ge0}$ with $t = S_0 \le S_1 \le \cdots \le S_n \le \cdots \le T$ and $\bigcup_{n\ge1}\{S_n = T\} = \Omega$, $\mathbb{P}$-a.s., such that, for any $k \ge 0$ and $\mathbb{P}$-almost every $\omega \in \Omega$, it holds:

If $v, v' \in V_{t,T}$ are such that $v = v'$ a.e. on $[t, S_k(\omega)]$, then $\alpha(\omega, \cdot, v) = \alpha(\omega, \cdot, v')$ on $[t, S_{k+1}(\omega)]$.

The NAD strategies $\beta : \Omega \times [t,T] \times U_{t,T} \longrightarrow V$ for player 2 are defined in a symmetric manner. The space of the NAD strategies for player 1 for games over the time interval $[t,T]$ is denoted by $\mathcal{A}_{t,T}$, that for player 2 by $\mathcal{B}_{t,T}$.

The above definition of NAD strategies is strongly inspired by those given in the pioneering works by Cardaliaguet [41], and by Cardaliaguet and Rainer [51]. The nonanticipativity with delay allows us to prove easily the following statement (for the idea of the proof, see: Lemma 2.2 in [41] or Lemma 1.1 in [51]).

Lemma 1 *For every couple* $(\alpha, \beta) \in \mathcal{A}_{t,T} \times \mathcal{B}_{t,T}$ *there is a unique pair* (u, v) *of control processes in* $\mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$ *such that*

$$\alpha\big(\omega, s, v_{\cdot}(\omega)\big) = u_s(\omega) \quad \textit{and} \quad \beta\big(\omega, s, u_{\cdot}(\omega)\big) = v_s(\omega), \quad ds\mathbb{P}(d\omega)\textit{-a.e. on } [t,T] \times \Omega. \tag{18}$$

Considering now $\zeta = x \in R^d$, the above lemma allows one to define $(X^{t,x;\alpha,\beta}, Y^{t,x;\alpha,\beta}, Z^{t,x;\alpha,\beta})$ by the corresponding triple of processes $(X^{t,x;u,v}, Y^{t,x;u,v}, Z^{t,x;u,v})$ introduced above by (14) and (16), where the couple of control processes (u, v) is associated with $(\alpha, \beta) \in \mathcal{A}_{t,T} \times \mathcal{B}_{t,T}$ by relation (18). In the same spirit we introduce the nonlinear cost functional

$$J(t, x; \alpha, \beta) := Y_t^{t,x;\alpha,\beta} = Y_t^{t,x;u,v},$$

with the help of which we define the lower and the upper value functions of the game over the time interval $[t, T]$:

$$\begin{aligned} W(t,x) &:= \operatorname*{esssup}_{\alpha\in\mathcal{A}_{t,T}} \operatorname*{essinf}_{\beta\in\mathcal{B}_{t,T}} J(t,x;\alpha,\beta), \\ U(t,x) &:= \operatorname*{essinf}_{\beta\in\mathcal{B}_{t,T}} \operatorname*{esssup}_{\alpha\in\mathcal{A}_{t,T}} J(t,x;\alpha,\beta), \quad (t,x)\in[0,T]\times\mathbb{R}^d. \end{aligned} \tag{19}$$

Remark 2 In the above definition of the lower and the upper value functions the essential supremum and the essential infimum over the family of $\mathcal{F}_t$-measurable random variables $J(t, x; \alpha, \beta)$, $(\alpha, \beta) \in \mathcal{A}_{t,T} \times \mathcal{B}_{t,T}$, are understood in the sense of Dunford and Schwartz [59]. Since due to estimate (17) this family is essentially bounded, uniformly with respect to $(\alpha, \beta) \in \mathcal{A}_{t,T} \times \mathcal{B}_{t,T}$, it follows that $W(t, x)$ and $U(t, x)$ are a priori bounded, $\mathcal{F}_t$-measurable random variables. However, the authors of [35] showed that, in fact, W and U are deterministic (see Proposition 3.3 of [35]).

Theorem 4 *For all* $(t, x) \in [0, T] \times \mathbb{R}^d$, *it holds* $W(t, x) = \mathbb{E}[W(t, x)]$, $U(t, x) = \mathbb{E}[U(t, x)]$.

Remark 3 The above statement allows us to identify in all what follows the stochastic fields $W(t, x), U(t, x)$ with their deterministic versions $\mathbb{E}[W(t, x)]$, $\mathbb{E}[U(t, x)]$. Moreover, from estimate (17) we get that the functions $W, U : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ are of linear growth and Lipschitz in x, uniformly with respect to t.

Since the fact that our lower and upper value functions W and U are deterministic, plays a central role in our approach, we will sketch the proof of the theorem. The proof is essentially based on a Girsanov transformation argument. Let $H := \{\int_0^{\cdot} h_s\, ds \mid h \in L^2([0, T]; \mathbb{R}^d)\}$ denote the Cameron–Martin subspace of Ω (Recall that $\Omega = C_0([0, T]; \mathbb{R}^d)$). For $h \in H$ we consider the Girsanov transformation $\tau_h : \Omega \to \Omega$, $\tau_h(\omega) := \omega + h$, $\omega \in \Omega$. Obviously, $\tau_h : \Omega \to \Omega$ is bijective, $\tau_h^{-1} = \tau_{-h}$, and its law is of the form

$$\mathbb{P} \circ [\tau_h]^{-1} = \exp\left\{\int_0^T \dot{h}_s\, dW_s - \frac{1}{2}\int_0^T |\dot{h}_s|^2\, ds\right\} \cdot \mathbb{P}.$$

The following lemma (see, e.g., Lemma 3.4 in [35]) reduces the proof of Theorem 4 to a verification of the invariance of $W(t, x)$ and $U(t, x)$ with respect of the Girsanov transformation introduced above.

Lemma 2 *Let* ζ *be a random variable over the classical Wiener space* $(\Omega, \mathcal{F}, \mathbb{P})$ *such that, for all* $h \in H$: $\zeta \circ \tau_h = \zeta$, $\mathbb{P}$-*a.s. Then there is some constant* $c \in \mathbb{R}$ *such that* $\zeta = c$, $\mathbb{P}$-*a.s.*

Proof of Theorem 4 Let us sketch the proof for $U(t, x)$; that for $W(t, x)$ uses the same arguments. Since $U(t, x)$ is an $\mathcal{F}_t$-measurable random variable over the classical Wiener space $(\Omega, \mathcal{F}, \mathbb{P})$, it suffices due to the above lemma, to show for all $h \in H_t := \{h \in H \mid h(s) = h(t),\ s \in [t, T]\}$ that $U(t, x) \circ \tau_h = U(t, x)$, $\mathbb{P}$-a.s. Let us fix an arbitrary $h \in H_t$. Then, since $\tau_h : \Omega \to \Omega$ is bijective and the law $\mathbb{P} \circ [\tau_h]^{-1}$ is equivalent to $\mathbb{P}$, it can be easily shown that

$$U(t, x) \circ \tau_h = \operatorname*{essinf}_{\beta \in \mathcal{B}_{t,T}} \operatorname*{esssup}_{\alpha \in \mathcal{A}_{t,T}} \big\{ J(t, x; \alpha, \beta) \circ \tau_h \big\}, \quad \mathbb{P}\text{-a.s.}$$

Let us analyze now the expression $J(t, x; \alpha, \beta) \circ \tau_h$. For this we begin with investigating $X^{t,x;\alpha,\beta} \circ \tau_h$. Let $(u, v) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$ be associated with $(\alpha, \beta) \in \mathcal{A}_{t,T} \times \mathcal{B}_{t,T}$ by relation (18). By applying the Girsanov transformation τ_h to SDE (14) (notice that $dW_s(\tau_h) = dW_s$, $s \in [t, T]$, since $h(s) = h(t)$, $s \in [t, T]$) we get the same SDE, but governed by the Girsanov transformed controls $(u \circ \tau_h, v \circ \tau_h)$. Obviously, also $(u \circ \tau_h, v \circ \tau_h)$ belongs to $\mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$, and from the uniqueness of the solution of SDE (14) it follows that $X_s^{t,x;u,v} \circ \tau_h = X_s^{t,x;u(\tau_h),v(\tau_h)}$, $s \in [t, T]$, $\mathbb{P}$-a.s. But,

$$\big(u(\tau_h), v(\tau_h)\big) = (u, v) \circ \tau_h = \big(\alpha(\cdot, \cdot, v), \beta(\cdot, \cdot, u)\big) \circ \tau_h = \big(\alpha\big(\tau_h, \cdot, v(\tau_h)\big), \beta\big(\tau_h, \cdot, u(\tau_h)\big)\big),$$

where with $(\alpha, \beta) \in \mathcal{A}_{t,T} \times \mathcal{B}_{t,T}$ also $(\alpha(\tau_h, \cdot, \cdot), \beta(\tau_h, \cdot, \cdot)) \in \mathcal{A}_{t,T} \times \mathcal{B}_{t,T}$. The proof of the latter fact consists in a direct verification of the definition of the NAD strategies, which uses the observation that, for any $\mathbb{F}$-stopping time $S : \Omega \to [t, T]$, also $S(\tau_h)$ is an $\mathbb{F}$-stopping time. Thus, $\mathbb{P}$-a.s., on $[t, T]$,

$$X^{t,x;\alpha,\beta} \circ \tau_h = X^{t,x;u,v} \circ \tau_h = X^{t,x;u(\tau_h),v(\tau_h)} = X^{t,x;\alpha(\tau_h,\cdot,\cdot),\beta(\tau_h,\cdot,\cdot)}.$$

By using this equality of the processes $X^{t,x;\alpha,\beta} \circ \tau_h$ and $X^{t,x;\alpha(\tau_h,\cdot,\cdot),\beta(\tau_h,\cdot,\cdot)}$ and combining it with the above arguments translated from the SDE framework to the BSDE one, we obtain

$$\big(X^{t,x;\alpha,\beta}, Y^{t,x;\alpha,\beta}, Z^{t,x;\alpha,\beta}\big) \circ \tau_h = \big(X^{t,x;\alpha(\tau_h,\cdot,\cdot),\beta(\tau_h,\cdot,\cdot)}, Y^{t,x;\alpha(\tau_h,\cdot,\cdot),\beta(\tau_h,\cdot,\cdot)}, Z^{t,x;\alpha(\tau_h,\cdot,\cdot),\beta(\tau_h,\cdot,\cdot)}\big).$$

Thus, in particular, with the notation $(\alpha_h, \beta_h) := (\alpha(\tau_h, \cdot, \cdot), \beta(\tau_h, \cdot, \cdot))$ we have

$$J(t, x; \alpha, \beta) \circ \tau_h = Y_t^{t,x;\alpha,\beta} \circ \tau_h = Y_t^{t,x;\alpha_h,\beta_h} = J(t, x; \alpha_h, \beta_h), \quad \mathbb{P}\text{-a.s.}$$

Moreover, since $(\alpha_{-h})_h = \alpha_{-h}(\tau_h, \cdot, \cdot) = \alpha(\tau_h \circ \tau_{-h}, \cdot, \cdot) = \alpha$ and, analogously, $(\beta_{-h})_h = \beta$, for all $\alpha \in \mathcal{A}_{t,T}$, $\beta \in \mathcal{B}_{t,T}$, we see that $\{\alpha_h : \alpha \in \mathcal{A}_{t,T}\} = \mathcal{A}_{t,T}$ and $\{\beta_h : \beta \in \mathcal{B}_{t,T}\} = \mathcal{B}_{t,T}$. Consequently, by summarizing our above arguments we obtain

$$\begin{aligned} U(t, x) \circ \tau_h &= \operatorname*{essinf}_{\beta \in \mathcal{B}_{t,T}} \operatorname*{esssup}_{\alpha \in \mathcal{A}_{t,T}} \big\{ J(t, x; \alpha, \beta) \circ \tau_h \big\} \\ &= \operatorname*{essinf}_{\beta \in \mathcal{B}_{t,T}} \operatorname*{esssup}_{\alpha \in \mathcal{A}_{t,T}} J(t, x; \alpha_h, \beta_h) \\ &= \operatorname*{essinf}_{\beta \in \mathcal{B}_{t,T}} \operatorname*{esssup}_{\alpha \in \mathcal{A}_{t,T}} J(t, x; \alpha, \beta) \\ &= U(t, x), \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

This completes the proof in virtue with what was pointed out already at the beginning of the proof. □

5.2 Dynamic Programming Principle

Once having proved that the lower and the upper value functions W and U are deterministic, we want to show that they satisfy the dynamic programming principle (DPP). Translating Peng's approach for stochastic control problems to the framework of stochastic differential games, Peng's notion of backward stochastic semigroup (see S. Peng [101]) turns out to be also here a powerful and very efficient tool.

For $(t,x)\in[0,T)\times\mathbb{R}^d$ and $(u,v)\in\mathcal{U}_{t,T}\times\mathcal{V}_{t,T}$, we let $X^{t,x;u,v}$ be the unique solution of SDE (14). Given $\delta>0$ with $t+\delta\le T$, and an arbitrary $\eta\in L^2(\Omega,\mathcal{F}_{t+\delta},\mathbb{P})$, we denote by $(Y_s^{\delta,\eta},Z_s^{\delta,\eta})_{s\in[t,t+\delta]}$ the unique solution of the BSDE

$$dY_s^{\delta,\eta}=-f\big(X_s^{t,x;u,v},Y_s^{\delta,\eta},Z_s^{\delta,\eta},u_s,v_s\big)\,ds+Z_s^{\delta,\eta}\,dW_s,\quad s\in[t,t+\delta],\ Y_T^{\delta,\eta}=\eta,$$

and we put

$$G_{s,t+\delta}^{t,x;u,v}[\eta]:=Y_s^{\delta,\eta},\quad s\in[t,t+\delta]. \tag{20}$$

We observe that the such defined backward stochastic semigroup $G_{s,r}^{t,x;u,v}$, $0\le s\le r\le T$, has all the properties of a semigroup. In particular, we have the following lemma which can be derived easily from the uniqueness of the solution of the BSDE which driving coefficient is Lipschitz in (y,z), from BSDE standard estimates and from Peng's comparison theorem for BSDE.

Lemma 3 *For all* $0\le t\le t+\delta\le T$, $x\in\mathbb{R}^d$, *and for all* $(u,v)\in\mathcal{U}_{t,t+\delta}\times\mathcal{V}_{t,t+\delta}$ *it holds*:

(i) (*Semigroup*) $G_{s,r}^{t,x;u,v}\circ G_{r,t+\delta}^{t,x;u,v}(:=G_{s,r}^{t,x;u,v}[G_{r,t+\delta}^{t,x;u,v}(\cdot)])=G_{s,t+\delta}^{t,x;u,v}$, $t\le s\le r\le t+\delta\le T$;

(ii) (*Monotonicity*) $G_{s,t+\delta}^{t,x;u,v}[\eta]\le G_{s,t+\delta}^{t,x;u,v}[\eta']$, $s\in[t,t+\delta]$, $\mathbb{P}$-*a.s.*, *for all* $\eta,\eta'\in L^2(\Omega,\mathcal{F}_{t+\delta},\mathbb{P})$ *with* $\eta\le\eta'$, $\mathbb{P}$-*a.s.*;

(iii) (L^2-*Lipschitzianity*) $\mathbb{E}[\sup_{s\in[t,t+\delta]}|G_{s,t+\delta}^{t,x;u,v}[\eta]-G_{s,t+\delta}^{t,x;u,v}[\eta']|^2\mid\mathcal{F}_t]\le C\mathbb{E}[|\eta-\eta'|^2\mid\mathcal{F}_t]$, $\mathbb{P}$-*a.s.*, *for all* $\eta,\eta'\in L^2(\Omega,\mathcal{F}_{t+\delta},\mathbb{P})$, *where the constant* $C\in\mathbb{R}_+$ *depends only on the Lipschitz constants of* σ, b *and* f *with respect to* x *and* (x,y,z), *respectively.*

For arbitrarily given $(\alpha,\beta)\in\mathcal{A}_{t,t+\delta}\times\mathcal{B}_{t,t+\delta}$ let $(u,v)\in\mathcal{U}_{t,t+\delta}\times\mathcal{V}_{t,t+\delta}$ be associated with by relation (18). Then, in coherence with the definition of $(X^{t,x;\alpha,\beta},Y^{t,x;\alpha,\beta},Z^{t,x;\alpha,\beta})$ we put

$$G_{s,t+\delta}^{t,x;\alpha,\beta}:=G_{s,t+\delta}^{t,x;u,v}.$$

Remark 4 Let us point out two special cases in which the semigroup can be explicitly described:

(1) If the running cost $f(x,y,z,u,v)$ is independent of (y,z), the semigroup has the following form: for $(u,v)\in\mathcal{U}_{t,t+\delta}\times\mathcal{V}_{t,t+\delta}$ and $\eta\in L^2(\Omega,\mathcal{F}_{t+\delta},\mathbb{P})$,

$$G_{s,t+\delta}^{t,x;u,v}[\eta]=\mathbb{E}\bigg[\eta+\int_s^{t+\delta}f\big(X_r^{t,x;u,v},u_r,v_r\big)\,dr\;\Big|\;\mathcal{F}_s\bigg],\quad s\in[t,t+\delta],\ \mathbb{P}\text{-a.s.}$$

(2) For $(u,v)\in\mathcal{U}_{t,T}\times\mathcal{V}_{t,t+\delta}$, let $\eta=Y_{t+\delta}^{t,x;u,v}$. Recalling that $(Y^{t,x;u,v},Z^{t,x;u,v})$ is the unique solution of BSDE (16), we see easily that

$$G_{s,t+\delta}^{t,x;u,v}\big[Y_{t+\delta}^{t,x;u,v}\big]=Y_s^{t,x;u,v},\quad s\in[t,t+\delta],\ \mathbb{P}\text{-a.s.}$$

These both special cases and earlier results on the dynamic programming principle in stochastic control problems but also for stochastic differential games suggest the following:

Theorem 5 *Our upper and our lower value functions* $U(t,x)$ *and* $W(t,x)$ *satisfy the following DPP*: *For all* $0 \le t < t+\delta \le T,\ x \in \mathbb{R}^d$, $\mathbb{P}$-*a.s.*,

$$\begin{aligned} U(t,x) &= \operatorname*{essinf}_{\beta \in \mathcal{B}_{t,t+\delta}} \operatorname*{esssup}_{\alpha \in \mathcal{A}_{t,t+\delta}} G_{t,t+\delta}^{t,x;\alpha,\beta}\big[U\big(t+\delta, X_{t+\delta}^{t,x;\alpha,\beta}\big)\big], \\ W(t,x) &= \operatorname*{esssup}_{\alpha \in \mathcal{A}_{t,t+\delta}} \operatorname*{essinf}_{\beta \in \mathcal{B}_{t,t+\delta}} G_{t,t+\delta}^{t,x;\alpha,\beta}\big[W\big(t+\delta, X_{t+\delta}^{t,x;\alpha,\beta}\big)\big]. \end{aligned} \tag{21}$$

Proof Although being rather straight-forward, the proof turns out to be also a bit technical. In order to give an idea of the techniques, we propose to prove the relation

$$U(t,x) \le U_\delta(t,x) := \operatorname*{essinf}_{\beta \in \mathcal{B}_{t,t+\delta}} \operatorname*{esssup}_{\alpha \in \mathcal{A}_{t,t+\delta}} G_{t,t+\delta}^{t,x;\alpha,\beta}\big[U\big(t+\delta, X_{t+\delta}^{t,x;\alpha,\beta}\big)\big]. \tag{22}$$

For the complete proof, however in a different framework (in that of stochastic differential games of the type "nonanticipating strategies against controls") the reader is referred to [35].

Proof of (22) The initial data $(t,x) \in [0,T) \times \mathbb{R}^d$ as well as $0 < \delta \le T - t$ are fixed during this proof. By introducing the family of $\mathcal{F}_t$-measurable random variables

$$H(\beta_1) := \operatorname*{esssup}_{\alpha_1 \in \mathcal{A}_{t,t+\delta}} G_{t,t+\delta}^{t,x;\alpha_1,\beta_1}\big[U\big(t+\delta, X_{t+\delta}^{t,x;\alpha_1,\beta_1}\big)\big], \quad \beta_1 \in \mathcal{B}_{t,t+\delta},$$

we have, for some sequence $\beta_1^n \in \mathcal{B}_{t,t+\delta}$, $n \ge 1$, that

$$U_\delta(t,x) = \operatorname*{essinf}_{\beta_1 \in \mathcal{B}_{t,t+\delta}} H(\beta_1) = \inf_{n \ge 1} H\big(\beta_1^n\big), \quad \mathbb{P}\text{-a.s.}$$

Given an arbitrarily small $\varepsilon > 0$, it can be easily verified with the help of standard arguments that, with the notation $\Delta_n := \{H(\beta_1^n) \le U_\delta(t,x) + \varepsilon, H(\beta_1^j) > U_\delta(t,x) + \varepsilon, 1 \le j \le n-1\}$, $n \ge 1$, the mapping $\beta_1^\varepsilon(\cdot) := \sum_{n \ge 1} I_{\Delta_n} \beta_1^n(\cdot)$ belongs to $\mathcal{B}_{t,t+\delta}$ and is such that

$$U_\delta(t,x) \ge H\big(\beta_1^\varepsilon\big) - \varepsilon = \operatorname*{esssup}_{\alpha_1 \in \mathcal{A}_{t,t+\delta}} G_{t,t+\delta}^{t,x;\alpha_1,\beta_1^\varepsilon}\big[U\big(t+\delta, X_{t+\delta}^{t,x;\alpha_1,\beta_1^\varepsilon}\big)\big] - \varepsilon, \quad \mathbb{P}\text{-a.s.} \tag{23}$$

Indeed, $\Delta_n \in \mathcal{F}_t$, for all $n \ge 1$, $\sum_{n \ge 1} \Delta_n = \Omega$, $\mathbb{P}$-a.s., and from the uniqueness of the solution of SDE (14) and the definition of the NAD strategies it follows that, for all $\alpha_1 \in \mathcal{A}_{t,t+\delta}$, $X_s^{t,x;\alpha_1,\beta_1^\varepsilon} = \sum_{n \ge 1} I_{\Delta_n} X_s^{t,x;\alpha_1,\beta_1^n}$, $s \in [t, t+\delta]$. This allows to show with the help of the uniqueness of the solution of BSDE (16), that also $G_{t,t+\delta}^{t,x;\alpha_1,\beta_1^\varepsilon}[U(t+\delta, X_{t+\delta}^{t,x;\alpha_1,\beta_1^\varepsilon})] = \sum_{n \ge 1} I_{\Delta_n} G_{t,t+\delta}^{t,x;\alpha_1,\beta_1^n}[U(t+\delta, X_{t+\delta}^{t,x;\alpha_1,\beta_1^n})]$, $\mathbb{P}$-a.s. Consequently, $H(\beta_1^\varepsilon) = \sum_{n \ge 1} I_{\Delta_n} H(\beta_1^n)$, $\mathbb{P}$-a.s., and the wished result follows easily.

We consider now a partition $\{\mathcal{O}_j, j \ge 1\} \subset \mathcal{B}(\mathbb{R}^d)$ of $\mathbb{R}^d$, which elements $\mathcal{O}_j$ are nonempty with diameter less than or equal to ε. Let us fix in every $\mathcal{O}_j$ an element y_j. An argument similar to that developed above shows that, for all $j \ge 1$, there is some $\beta_2^j \in \mathcal{B}_{t+\delta,T}$ such that

$$U(t+\delta, y_j) \geq \operatorname*{esssup}_{\alpha_2 \in \mathcal{A}_{t+\delta,T}} J\big(t+\delta, y_j; \alpha_2, \beta_2^j\big) - \varepsilon, \quad \mathbb{P}\text{-a.s.}, \ j \geq 1.$$

Given an arbitrary $\alpha \in \mathcal{A}_{t,T}$ which we fix, and an arbitrary $v_2 \in V_{t+\delta,T}$, we put $\alpha_1(\cdot,\cdot,v_1) := \alpha(\cdot,\cdot,v_1 \oplus v_2)_{/[t,t+\delta]}$, $v_1 \in V_{t,t+\delta}$. Here the notation $v_1 \oplus v_2$ stands for the process $v \in \mathcal{V}_{t,T}$ which restriction $v_{/[t,t+\delta]}$ to the interval $[t, t+\delta]$ coincides $ds\,d\mathbb{P}$-a.e. with v_1, and which restriction $v_{/(t+\delta,T]}$ to $(t+\delta, T]$ is v_2. Obviously, the such defined mapping α_1 belongs to $\mathcal{A}_{t,t+\delta}$, but without depending on v_2. Thus, there is a unique solution $(u_1^\varepsilon, v_1^\varepsilon) \in \mathcal{U}_{t,t+\delta} \times \mathcal{V}_{t,t+\delta}$ of the relations $\alpha_1(\cdot,\cdot,v_1^\varepsilon) = u_1^\varepsilon$, $\beta_1^\varepsilon(\cdot,\cdot,u_1^\varepsilon) = v_1^\varepsilon$, and we can define the mapping $\alpha_2^\varepsilon \in \mathcal{A}_{t+\delta,T}$ by setting

$$\alpha_2^\varepsilon(\omega,\cdot,v_2) := \alpha\big(\omega,\cdot,v_1^\varepsilon(\omega) \oplus v_2\big)_{/[t+\delta,T]}, \quad v_2 \in V_{t+\delta,T}, \ \omega \in \Omega.$$

Let $\beta_2^\varepsilon(\cdot) := \sum_{j\geq 1} I_{\mathcal{O}_j}(X_{t+\delta}^{t,x;\alpha_1,\beta_1^\varepsilon})\beta_2^j(\cdot)$, and $\beta^\varepsilon(\cdot,\cdot,u) := \beta_1^\varepsilon(\cdot,\cdot,u_1) \oplus \beta_2^\varepsilon(\cdot,\cdot,u_2)$, for $u = u_1 \oplus u_2$, $u_1 \in U_{t,t+\delta}$, $u_2 \in U_{t+\delta,T}$. While it is obvious that $\beta_2^\varepsilon \in \mathcal{B}_{t,t+\delta}$, the proof that $\beta^\varepsilon \in \mathcal{B}_{t,T}$ is a bit more tricky. Let us show the nonanticipativity with delay of β^ε; for its strict nonanticipativity the argument given in Lemma 3.9 of [35] can be adapted to the present framework. Since both β_1^ε and β_2^ε are nonanticipating with delay, there exist two increasing sequences of stopping times $(S_{1,n})_{n\geq 1}$ and $(S_{2,n})_{n\geq 1}$ with $S_{1,0} = t$, $S_{2,0} = t+\delta$, $\bigcup_{n\geq 1}\{S_{1,n} = t+\delta\} = \Omega$ and $\bigcup_{n\geq 1}\{S_{2,n} = T\} = \Omega$, $\mathbb{P}$-a.s., with respect to which the nonanticipativity condition in Definition 4(iii) holds for the intervals $[t, t+\delta]$ and $[t+\delta, T]$, respectively. Then, putting $\tau := \inf\{\ell \geq 1 : S_{1,\ell} = t+\delta\}$, $S_n := S_{1,n} I\{S_{1,n} < t+\delta\} + S_{2,n-\tau} I\{S_{1,n} = t+\delta\}$, $n \geq 0$, is an increasing sequence of stopping times with respect to which β^ε satisfies condition (iii) in Definition 4.

With the strategies introduced above, by using the Lipschitz continuity of $U(t+\delta,\cdot)$ as well as that of $J(t+\delta,\cdot,u,v)$, which is independent of the controls u, v, we obtain, with L denoting the Lipschitz constant,

$$\begin{aligned}
U\big(t+\delta, X_{t+\delta}^{t,x;\alpha_1,\beta_1^\varepsilon}\big) &\geq \sum_{j\geq 1} I_{\mathcal{O}_j}\big(X_{t+\delta}^{t,x;\alpha_1,\beta_1^\varepsilon}\big) U(t+\delta, y_j) - C\varepsilon \\
&\geq \sum_{j\geq 1} I_{\mathcal{O}_j}\big(X_{t+\delta}^{t,x;\alpha_1,\beta_1^\varepsilon}\big) J\big(t+\delta, y_j; \alpha_2^\varepsilon, \beta_2^j\big) - (C+1)\varepsilon \quad \text{(Recall the} \\
&\quad \text{definition of } \beta_2^j,\ j \geq 1) \\
&= \sum_{j\geq 1} I_{\mathcal{O}_j}\big(X_{t+\delta}^{t,x;\alpha_1,\beta_1^\varepsilon}\big) J\big(t+\delta, y_j; \alpha_2^\varepsilon, \beta_2^\varepsilon\big) - (C+1)\varepsilon \\
&\geq J\big(t+\delta, X_{t+\delta}^{t,x;\alpha_1,\beta_1^\varepsilon}; \alpha_2^\varepsilon, \beta_2^\varepsilon\big) - (2C+1)\varepsilon \\
&= Y_{t+\delta}^{t,x;\alpha,\beta^\varepsilon} - (2C+1)\varepsilon, \quad \mathbb{P}\text{-a.s.}
\end{aligned}$$

For the latter relation we have used that, for $(u_2^\varepsilon, v_2^\varepsilon) \in \mathcal{U}_{t+\delta,T} \times \mathcal{V}_{t+\delta,T}$ such that $\alpha_2^\varepsilon(\cdot,\cdot,v_2^\varepsilon) = u_2^\varepsilon$, $\beta_2^\varepsilon(\cdot,\cdot,u_2^\varepsilon) = v_2^\varepsilon$, and for $(u^\varepsilon, v^\varepsilon) := (u_1^\varepsilon, v_1^\varepsilon) \oplus (u_2^\varepsilon, v_2^\varepsilon)$,

$$\begin{aligned}
J\big(t+\delta, X_{t+\delta}^{t,x;\alpha_1,\beta_1^\varepsilon}; \alpha_2^\varepsilon, \beta_2^\varepsilon\big) &= J\big(t+\delta, X_{t+\delta}^{t,x;u_1^\varepsilon,v_1^\varepsilon}; u_2^\varepsilon, v_2^\varepsilon\big) = J\big(t+\delta, X_{t+\delta}^{t,x;u^\varepsilon,v^\varepsilon}; u^\varepsilon, v^\varepsilon\big) \\
&= J\big(t+\delta, X_{t+\delta}^{t,x;\alpha,\beta^\varepsilon}; \alpha, \beta^\varepsilon\big).
\end{aligned}$$

Indeed, $\alpha(\cdot,\cdot,v^\varepsilon) = \alpha_1(\cdot,\cdot,v_1^\varepsilon) \oplus \alpha_2^\varepsilon(\cdot,\cdot,v_2^\varepsilon) = u_1^\varepsilon \oplus u_2^\varepsilon = u^\varepsilon$ and $\beta^\varepsilon(\cdot,\cdot,u^\varepsilon) = \beta_1^\varepsilon(\cdot,\cdot,u_1^\varepsilon) \oplus \beta_2^\varepsilon(\cdot,\cdot,u_2^\varepsilon) = v_1^\varepsilon \oplus v_2^\varepsilon = v^\varepsilon$.

Therefore, from Lemma 3 and Remark 4(2),

$$\begin{aligned} U_\delta(t,x) &\geq G^{t,x;\alpha_1,\beta_1^\varepsilon}_{t,t+\delta}\big[U\big(t+\delta, X^{t,x;\alpha_1,\beta_1^\varepsilon}_{t+\delta}\big)\big] - \varepsilon \quad \text{(Recall (23))} \\ &\geq G^{t,x;\alpha_1,\beta_1^\varepsilon}_{t,t+\delta}\big[Y^{t,x;\alpha,\beta^\varepsilon}_{t+\delta} - (2C+1)\varepsilon\big] - \varepsilon \\ &\geq G^{t,x;\alpha_1,\beta_1^\varepsilon}_{t,t+\delta}\big[Y^{t,x;\alpha,\beta^\varepsilon}_{t+\delta}\big] - \big(C(2C+1)-1\big)\varepsilon \\ &= G^{t,x;\alpha,\beta^\varepsilon}_{t,t+\delta}\big[Y^{t,x;\alpha,\beta^\varepsilon}_{t+\delta}\big] - \big(C(2C+1)-1\big)\varepsilon \\ &= J\big(t,x;\alpha,\beta^\varepsilon\big) - \big(C(2C+1)-1\big)\varepsilon, \quad \mathbb{P}\text{-a.s.,} \end{aligned}$$

i.e.,

$$U_\delta(t,x) \geq J\big(t,x;\alpha,\beta^\varepsilon\big) - \big(C(2C+1)-1\big)\varepsilon, \quad \mathbb{P}\text{-a.s.}$$

Recalling that β^ε has been chosen independently of $\alpha \in \mathcal{A}_{t,T}$, we obtain $U_\delta(t,x) \geq U(t,x) - (C(2C+1)-1)\varepsilon$ by taking first the essential supremum over $\alpha \in \mathcal{A}_{t,T}$. Finally, by letting $\varepsilon \to 0$, we get $U_\delta(t,x) \geq U(t,x)$. Thus, the proof is complete.

□

With the help of the above DPP, but also with standard BSDE estimates it can be shown now that the upper and lower value functions are $1/2$-Hölder continuous in t. More precisely, we have

Proposition 4 *Under our standard assumptions there exists some constant $C \in \mathbb{R}_+$ such that, for all $t, t' \in [0,T]$ and for all $x \in \mathbb{R}^d$,*

$$\big|W(t,x) - W(t',x)\big| + \big|U(t,x) - U(t',x)\big| \leq C\big(1+|x|\big)\sqrt{|t-t'|}.$$

For the proof of this proposition the reader is referred to Theorem 3.10 of [35]. Although in that paper stochastic differential games of the type "strategy against control" are studied, the arguments of the proof can be easily translated to our present framework. The above proposition together with Remark 3 shows that the upper and the lower value functions are continuous and of most linear growth. This continuity property combined with the dynamic programming principle allows to investigate the associated Bellman–Isaacs equations.

5.3 Viscosity Solutions of Bellman–Isaacs' Equations

The goal of this subsection is to associate our lower and upper value functions with the following both Isaacs' equations:

$$\begin{cases} \dfrac{\partial}{\partial t} W(t,x) + H^-\big(x, W, DW, D^2W\big) = 0, & (t,x) \in [0,T) \times \mathbb{R}^n, \\ W(T,x) = \Phi(x), & x \in \mathbb{R}^n, \end{cases} \tag{24}$$

$$\begin{cases} \dfrac{\partial}{\partial t} U(t,x) + H^+\big(x, U, DU, D^2U\big) = 0, & (t,x) \in [0,T) \times \mathbb{R}^n, \\ U(T,x) = \Phi(x), & x \in \mathbb{R}^n, \end{cases} \tag{25}$$

where, for

$$H(x,y,p,A,u,v) := \frac{1}{2}\operatorname{tr}\big(\sigma\sigma^*(x,u,v)A\big) + pb(x,u,v) + f\big(x,y,p\sigma(x,u,v),u,v\big),$$

$(x, y, p, A, u, v) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times S^d \times U \times V$, the Hamiltonians H^+ and H^- are defined as follows:

$$H^-(x, y, p, A) := \sup_{u \in U} \inf_{v \in V} H(x, y, p, A, u, v),$$

$$H^+(x, y, p, A) := \inf_{v \in V} \sup_{u \in U} H(x, y, p, A, u, v).$$

As usual, S^d denotes the set of all symmetric real $d \times d$-matrices.

Under our standard assumptions on the coefficients b, σ, f and Φ, we have the following theorem.

Theorem 6 *The lower value function W is a viscosity solution of* (24), *while the upper value function U is a viscosity solution of* (25). *Moreover, both solutions are unique in the class Θ of continuous functions $V : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ which are such that, for some constant C* (*which, of course, may depend on V*) *the following growth condition is satisfied*:

$$\lim_{|x| \to +\infty} \Big(\sup_{t \in [0,T]} \big|\varphi(t, x)\big| \Big) \exp\big\{-C\big(\ln\big(1 + |x|\big)\big)^2\big\} = 0.$$

For the definition of a viscosity solution, sub- and supersolution we refer the reader to [56]. To the proof that W and U are viscosity solutions, we will come a bit later. This proof will allow to show with which efficiency the BSDE method, introduced by Shige Peng in the framework of stochastic control, works also here in the context of stochastic differential games. Concerning the uniqueness of the viscosity solutions W and V, it is a direct consequence of the comparison principle which we can formulate for (24) and (25). The reader interested in the proof or in more details is referred to [35].

Theorem 7 *Let $V_1 \in \Theta$ be a viscosity subsolution of* (24) (*or of* (25), *respectively*) *and $V_2 \in \Theta$ a viscosity supersolution of* (24) (*or of* (25), *respectively*). *Then $V_1(t, x) \le V_2(t, x)$, $(t, x) \in [0, T] \times \mathbb{R}^d$.*

A comparison theorem using the growth condition, which is contained in the definition of Θ, was established the first time in [20]. This growth condition, which allows the functions in Θ to have a growth more than polynomial but less than exponential, was shown in [20] to be optimal for the heat equation. A direct consequence of the above comparison principle is the existence of a value of the game under

Isaacs' condition: $H^+(x, y, p, A) = H^-(x, y, p, A)$, for all $(x, y, p, A) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times S^d$.

More precisely, we have

Corollary 1 *Under the above Isaac condition the lower and the upper value functions coincide*: $W(t, x) = U(t, x)$, $(t, x) \in [0, T] \times \mathbb{R}^d$.

Indeed, under Isaacs' condition (24) and (25) are the same and, hence, W and U are viscosity solutions of the same equation, both belonging to Θ. Consequently, they must coincide.

In what follows we sketch the proof of the fact that the upper value function U is a viscosity solution of (25). The main idea of the proof consists in translating Shige Peng's BSDE method from the frame of stochastic control to that of stochastic differential games. In [35] this was done for stochastic differential games of the type "strategies against controls", while here games of the type "NAD strategy against NAD strategy" are discussed. This change of the type of the game involves some major modifications in the proof.

Since, due to the definition of the cost functionals $J(t,x;u,v)=Y_t^{t,x;u,v}$ through BSDE (16), the upper value function U coincides at time T with the function Φ, we only have to show that, for any given test function $\varphi\in C^3_{\ell,b}([0,T]\times\mathbb{R}^d)$[1] and any $(t_0,x_0)\in[0,T)\times\mathbb{R}^d$ with

$$U-\varphi\le U(t_0,x_0)-\varphi(t_0,x_0)=0\quad \big(U-\varphi\ge U(t_0,x_0)-\varphi(t_0,x_0)=0,\ \text{respectively}\big)\qquad(26)$$

it holds that

$$\begin{aligned}&\frac{\partial}{\partial t}\varphi(t_0,x_0)+H^+\big(x,\big(\varphi,\nabla\varphi,D^2\varphi\big)(t_0,x_0)\big)\ge 0\\&\left(\frac{\partial}{\partial t}\varphi(t_0,x_0)+H^+\big(x,(\varphi,\nabla\varphi,D^2\varphi)(t_0,x_0)\big)\le 0,\ \text{respectively}\right).\end{aligned}\qquad(27)$$

Indeed, if this is the case, U is a viscosity subsolution (viscosity supersolution, respectively), and if it is both a viscosity sub- as well as a viscosity supersolution, then it is by definition a viscosity solution. In order to verify that we have the both properties in (27), let us arbitrarily fix a test function $\varphi\in C^3_{\ell,b}([0,T]\times\mathbb{R}^d)$ and a point $(t_0,x_0)\in[0,T)\times\mathbb{R}^d$, and let us introduce the function

$$\begin{aligned}F(t,x,y,z,u,v)&=\frac{\partial}{\partial t}\varphi(t,x)+\frac{1}{2}\operatorname{tr}\big(\sigma\sigma^*(x,u,v)D^2\varphi(t,x)\big)+\nabla\varphi(t,x)b(x,u,v)\\&\quad+f\big(x,y+\varphi(t,x),z+\nabla\varphi(t,x)\sigma(x,u,v),u,v\big),\end{aligned}\qquad(28)$$

for all $(t,x,y,z,u,v)\in[t_0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\times U\times V$. We emphasize that verifying (27) is equivalent to showing that under the condition (26) we have

$$\inf_{v\in V}\sup_{u\in U}F(t_0,x_0,0,0,u,v)\ge 0\quad\Big(\inf_{v\in V}\sup_{u\in U}F(t_0,x_0,0,0,u,v)\le 0,\ \text{respectively}\Big).\qquad(29)$$

The relations in (29) will be derived from the DPP (21) with the help of essentially two BSDEs. For any $\delta>0$ with $t_0+\delta\le T$, let us introduce the couple $(Y^{1,\delta,u,v},Z^{1,\delta,u,v})\in\mathcal{S}^2_{\mathbb{F}}(t_0,t_0+\delta;\mathbb{R})\times L^2_{\mathbb{F}}(t_0,t_0+\delta;\mathbb{R}^d)$ as the solution of the BSDE

$$\begin{aligned}&dY_s^{1,\delta,u,v}=-F\big(s,X_s^{t_0,x_0;u,v},Y_s^{1,\delta,u,v},Z_s^{1,\delta,u,v},u_s,v_s\big)\,ds+Z_s^{1,\delta,u,v}\,dW_s,\\&\quad s\in[t_0,t_0+\delta],\\&Y_{t_0+\delta}^{1,\delta,u,v}=0,\end{aligned}\qquad(30)$$

where $X^{t_0,x_0;u,v}$ is the solution of SDE (14), and $(u,v)\in\mathcal{U}_{t_0,t_0+\delta}\times\mathcal{V}_{t_0,t_0+\delta}$. Since the coefficient $F(t,x,y,z,u,v)$ is Lipschitz in (y,z), uniformly with respect to (t,x,u,v), and

[1] $C^3_{\ell,b}([0,T]\times\mathbb{R}^d)$ denotes the space of the three times continuously differentiable functions over $[0,T]\times\mathbb{R}^d$ which derivatives of order 1, 2 and 3 are bounded.

$|F(t,x,0,0,u,v)| \le C(1+|x|^2)$, $(t,x,u,v) \in [0,T] \times \mathbb{R}^d \times U \times V$, for some real constant C, we have the existence and the uniqueness for this BSDE. Moreover, the unique solution $(Y^{1,\delta,u,v}, Z^{1,\delta,u,v})$ of this BSDE satisfies the following relation which can be checked by an easy forward computation (see the proof of Lemma 4.3 in [35]):

Lemma 4 *For all* $\delta > 0$ *and* $s \in [t_0, t_0+\delta]$, *it holds*

$$Y_s^{1,\delta,u,v} = G_{s,t_0+\delta}^{t_0,x_0;u,v}\big[\varphi\big(t_0+\delta, X_{t_0+\delta}^{t_0,x_0;u,v}\big)\big] - \varphi\big(s, X_s^{t_0,x_0;u,v}\big). \tag{31}$$

The above BSDE will allow one to translate the DPP into an inequality involving $Y_t^{1,\delta,u,v}$. However, in order to prove (29), we will have to replace $Y^{1,\delta,u,v}$ by the solution of the following BSDE:

$$\begin{aligned} dY_s^{2,\delta,u,v} &= -F\big(s, x_0, Y_s^{2,\delta,u,v}, Z_s^{2,\delta,u,v}, u_s, v_s\big)\,ds + Z_s^{2,\delta,u,v}\,dW_s, \\ & s \in [t_0, t_0+\delta],\ Y_{t_0+\delta}^{2,\delta,u,v} = 0, \end{aligned} \tag{32}$$

$(u,v) \in \mathcal{U}_{t_0,t_0+\delta} \times \mathcal{V}_{t_0,t_0+\delta}$. The fact that, for $\delta > 0$ small enough, the distance between the process $X_s^{t_0,x_0;u,v}$, $s \in [t_0, t_0+\delta]$, and its initial condition x_0 is of the order $\delta^{1/2}$ (see estimate (15) with $T = t+\delta$ and $\zeta = x$), allows to prove the following by using BSDE standard estimates:

Lemma 5 *There is a constant* $C \in \mathbb{R}_+$ *independent of* $\delta > 0$, *such that for all* $u \in \mathcal{U}_{t_0,t_0+\delta}$ *and* $v \in \mathcal{V}_{t_0,t_0+\delta}$,

$$\big|Y_{t_0}^{1,\delta,u,v} - Y_{t_0}^{2,\delta,u,v}\big| \le C\delta^{3/2}, \quad \mathbb{P}\text{-a.s.} \tag{33}$$

After the above preparation we now can begin with the

Proof that U is a viscosity subsolution of (25) For this we assume that $U - \varphi \le U(t_0,x_0) - \varphi(t_0,x_0) = 0$ and that

$$\inf_{v\in V}\sup_{u\in U} F(t_0, x_0, 0, 0, u, v) < 0. \tag{34}$$

We have to prove that the latter hypothesis leads to a contradiction, i.e., (29) holds true. Under the above hypothesis, thanks to the continuity F, there exists some $\theta > 0$, $R \in (0, T-t_0]$ and $v^* \in V$ such that

$$\sup_{u\in U} F(s, x_0, 0, 0, u, v^*) \le -\theta, \quad \text{for all } s \in [t_0, t_0+R].$$

On the other hand, from $U - \varphi \le U(t_0,x_0) - \varphi(t_0,x_0) = 0$, the DPP (5) for U as well as the Lemmata 3(ii) and 4 it follows that, for all $0 < \delta \le T - t_0$,

$$\begin{aligned} 0 &\le \operatorname*{essinf}_{\beta\in\mathcal{B}_{t_0,t_0+\delta}} \operatorname*{esssup}_{\alpha\in\mathcal{A}_{t_0,t_0+\delta}} \big(G_{t,t+\delta}^{t_0,x_0;\alpha,\beta}\big(\varphi\big(t_0+\delta, X_{t_0+\delta}^{t_0,x_0;\alpha,\beta}\big)\big) - \varphi(t_0,x_0)\big) \\ &= \operatorname*{essinf}_{\beta\in\mathcal{B}_{t_0,t_0+\delta}} \operatorname*{esssup}_{\alpha\in\mathcal{A}_{t_0,t_0+\delta}} Y_{t_0}^{1,\delta;\alpha,\beta}, \quad \mathbb{P}\text{-a.s.}, \end{aligned} \tag{35}$$

where $Y_{t_0}^{1,\delta;\alpha,\beta} := Y_{t_0}^{1,\delta;u,v}$, for $(u,v) \in \mathcal{U}_{t_0,t_0+\delta} \times \mathcal{V}_{t_0,t_0+\delta}$ such that $\alpha(\cdot,\cdot,v) = u$, and $\beta(\cdot,\cdot,u) = v$. Consequently, taking into account that $\beta^*(\omega,s,u) := v^*$, $(\omega,s,u) \in \Omega \times$

$[t_0, t_0+\delta] \times U_{t_0,t_0+\delta}$, defines an element of $\mathcal{B}_{t_0,t_0+\delta}$, (33) yields

$$\begin{aligned}0 &\le \operatorname*{esssup}_{\alpha\in\mathcal{A}_{t_0,t_0+\delta}} Y_{t_0}^{1,\delta;\alpha,\beta^*}\\ &\le \operatorname*{esssup}_{\alpha\in\mathcal{A}_{t_0,t_0+\delta}} Y_{t_0}^{2,\delta;\alpha,\beta^*} + C\delta^{3/2}\\ &= \operatorname*{esssup}_{\alpha\in\mathcal{A}_{t_0,t_0+\delta}} Y_{t_0}^{2,\delta;\alpha(v^*),v^*} + C\delta^{3/2}\\ &\le \operatorname*{esssup}_{u\in\mathcal{U}_{t_0,t_0+\delta}} Y_{t_0}^{2,\delta;u,v^*} + C\delta^{3/2}, \quad \mathbb{P}\text{-a.s.}\end{aligned} \tag{36}$$

Then, by applying a standard argument, we get, for all $\delta > 0$, the existence of a control $u^\delta \in \mathcal{U}_{t_0,t_0+\delta}$ such that

$$\begin{aligned}0 &\le \operatorname*{esssup}_{u\in\mathcal{U}_{t_0,t_0+\delta}} Y_{t_0}^{2,\delta;u,v^*} + C\delta^{3/2}\\ &\le Y_{t_0}^{2,\delta;u^\delta,v^*} + 2C\delta^{3/2}, \quad \mathbb{P}\text{-a.s.}\end{aligned} \tag{37}$$

On the other hand, denoting by L the Lipschitz constant of $F(s, x_0, y, 0, u, v)$ with respect to y, we consider the function $\widetilde{Y}_s^\delta = -\frac{\theta}{L}(1 - e^{-L(t_0+\delta-s)})$, $s \in [t_0, t_0+\delta]$. This function is the unique solution of the (deterministic) backward equation

$$d\widetilde{Y}_s^\delta = -\big(-\theta + L\big|\widetilde{Y}_s^\delta\big|\big)\, ds, \quad s \in [t_0, t_0+\delta],\ \widetilde{Y}_{t_0+\delta}^\delta = 0. \tag{38}$$

Thus, since

$$\begin{aligned}&F(s, x_0, y, 0, u, v^*) \le F(s, x_0, 0, 0, u, v^*) + L|y| \le -\theta + L|y|,\\ &(s, y, u) \in [t_0, t_0+R] \times \mathbb{R} \times U,\end{aligned}$$

the comparison of the backward equations solved by $Y^{2,\delta;u^\delta,v^*}$ and $\widetilde{Y}^\delta$, respectively, yields that

$$Y_{t_0}^{2,\delta;u^\delta,v^*} \le \widetilde{Y}_{t_0}^\delta = -\frac{\theta}{L}\big(1 - e^{-L\delta}\big), \quad \mathbb{P}\text{-a.s., for all } \delta \in (0, R].$$

Combining this latter relation with (37) we obtain

$$-2C\delta^{1/2} \le -\frac{\theta}{L\delta}\big(1 - e^{-L\delta}\big) \longrightarrow -\theta, \quad \text{as } \delta \to 0.$$

However, since the left-hand side converges to 0, when $\delta \to 0$, we have a contradiction which has its origin in hypothesis (34). This proves that U is a viscosity subsolution of (25). □

Let us now tackle the

Proof that U is a viscosity supersolution of (25) Also this proof turns out to be quite different of that in the case of games of the type "strategy against control" studied in [35]. Let us assume that $U-\varphi \ge U(t_0,x_0)-\varphi(t_0,x_0)=0$. Then, in particular, $U\ge \varphi$, and from the DPP (5) for U it follows that, for all $\delta>0$ with $t_0+\delta\le T$,

$$0\ge \operatorname*{essinf}_{\beta\in\mathcal{B}_{t_0,t_0+\delta}} \operatorname*{esssup}_{\alpha\in\mathcal{A}_{t_0,t_0+\delta}} \big(G^{t_0,x_0;\alpha,\beta}_{t_0,t_0+\delta}\big(\varphi\big(t_0+\delta, X^{t_0,x_0;\alpha,\beta}_{t_0+\delta}\big)\big)-\varphi(t_0,x_0)\big),$$

and Lemma 4 yields

$$0\ge \operatorname*{essinf}_{\beta\in\mathcal{B}_{t_0,t_0+\delta}} \operatorname*{esssup}_{\alpha\in\mathcal{A}_{t_0,t_0+\delta}} Y^{1,\delta;\alpha,\beta}_{t_0},\ \mathbb{P}\text{-a.s.}$$

Thus, due to Lemma 5 there is some constant $C>0$ such that, for all $\delta>0$ with $t_0+\delta\le T$,

$$0\ge \operatorname*{essinf}_{\beta\in\mathcal{B}_{t_0,t_0+\delta}} \operatorname*{esssup}_{\alpha\in\mathcal{A}_{t_0,t_0+\delta}} Y^{2,\delta;\alpha,\beta}_{t_0}-C\delta^{3/2},\quad \mathbb{P}\text{-a.s.}$$

Then a standard argument shows that, for all $\delta>0$ with $t_0+\delta\le T$, there is some $\beta^\delta\in\mathcal{B}_{t_0,t_0+\delta}$ such that

$$0\ge \operatorname*{esssup}_{\alpha\in\mathcal{A}_{t_0,t_0+\delta}} Y^{2,\delta;\alpha,\beta^\delta}_{t_0}-2C\delta^{3/2},\quad \mathbb{P}\text{-a.s.} \tag{39}$$

Since $\beta^\delta\in\mathcal{B}_{t_0,t_0+\delta}$, we can find a sequence of stopping times $(S_n)_{n\ge1}$ with $t_0=S_0\le S_1\le\cdots\le S_n\le\cdots\le t_0+\delta$ and $\bigcup_{n\ge1}\{S_n=t_0+\delta\}=\Omega$, $\mathbb{P}$-a.s., such that, for all $n\ge0$, and all $u,u'\in\mathcal{U}_{t_0,t_0+\delta}$ with $u=u'$, $ds\,d\mathbb{P}$-a.e. on the stochastic interval $[[t_0,S_n]]$, it holds $\beta^\delta(u)=\beta^\delta(u')$, $ds\,d\mathbb{P}$-a.e. on $[[t_0,S_{n+1}]]$. Consequently, $v^\delta_s:=\beta^\delta(s,u)$, $s\in[[t_0,S_1]]$, is independent of the special choice of $u\in\mathcal{U}_{t_0,t_0+\delta}$. So we can define u^δ on $[[t_0,S_1]]$ as the process such that $u^\delta_{\cdot\wedge S_1}\in\mathcal{U}_{t_0,t_0+\delta}$ and

$$F\big(s,x_0,0,0,u^\delta_s,v^\delta_s\big)=\sup_{u\in U}F\big(s,x_0,0,0,u,v^\delta_s\big),\quad s\in[[t_0,S_1]].$$

We now set $v^\delta_s:=\beta^\delta(s,u^\delta_{\cdot\wedge S_1})$, $s\in\,]]S_1,S_2]]$, and we extend the control process u^δ to the interval $[[t_0,S_2]]$ by choosing $(u^\delta_{(\cdot\wedge S_2)\vee S_1})\in\mathcal{U}_{t,t+\delta}$ such that

$$F\big(s,x_0,0,0,u^\delta_s,v^\delta_s\big)=\sup_{u\in U}F\big(s,x_0,0,0,u,v^\delta_s\big),\quad s\in\,]]S_1,S_2]].$$

With this choice we have

$$\begin{aligned}F\big(s,x_0,0,0,u^\delta_s,\beta^\delta\big(s,u^\delta_{\cdot\wedge S_2}\big)\big)&=\sup_{u\in U}F\big(s,x_0,0,0,u,v^\delta_s\big)\\&\ge\inf_{v\in V}\sup_{u\in U}F(s,x_0,0,0,u,v),\quad s\in[[t_0,S_2]].\end{aligned}$$

By iterating the above argument and taking into account that $\bigcup_{n\ge1}\{S_n=t+\delta\}=\Omega$, $\mathbb{P}$-a.s., we construct a control $u^\delta\in\mathcal{U}_{t_0,t_0+\delta}$ such that, for all $n\ge1$,

$$\begin{aligned}F\big(s,x_0,0,0,u^\delta_s,\beta^\delta\big(s,u^\delta_{\cdot\wedge S_n}\big)\big)&=\sup_{u\in U}F\big(s,x_0,0,0,u,\beta^\delta\big(s,u^\delta_{\cdot\wedge S_n}\big)\big)\\&\ge\inf_{v\in V}\sup_{u\in U}F(s,x_0,0,0,u,v),\quad s\in[[t_0,S_n]].\end{aligned}$$

Finally, from the nonanticipativity property of the strategy β^δ it follows that

$$F\big(s,x_0,0,0,u^\delta_s,\beta^\delta\big(s,u^\delta\big)\big)\ge\inf_{v\in V}\sup_{u\in U}F(s,x_0,0,0,u,v),\quad s\in[t_0,t_0+\delta]. \tag{40}$$

Since $\alpha^\delta(\omega, s, v) := u_s^\delta(\omega)$, $(\omega, s, v) \in \Omega \times [t_0, t_0 + \delta] \times V_{t_0,t_0+\delta}$ defines an element of $\mathcal{A}_{t_0,t_0+\delta}$, we obtain from (39) that

$$\begin{aligned} 0 &\geq Y_{t_0}^{2,\delta;\alpha^\delta,\beta^\delta} - 2C\delta^{3/2} \\ &= Y_{t_0}^{2,\delta;u^\delta,\beta^\delta(u^\delta)} - 2C\delta^{3/2}, \quad \mathbb{P}\text{-a.s.} \end{aligned} \tag{41}$$

On the other hand, relation (40) allows us to compare the BSDE solved by $Y^{2,\delta;u^\delta,\beta^\delta(u^\delta)}$,

$$dY_s^{2,\delta;u^\delta,\beta^\delta(u^\delta)} = -F\big(s, x_0, Y_s^{2,\delta;u^\delta,\beta^\delta(u^\delta)}, Z_s^{2,\delta;u^\delta,\beta^\delta(u^\delta)}, u_s^\delta, \beta^\delta\big(s, u^\delta\big)\big)\, ds + Z_s^{2,\delta;u^\delta,\beta^\delta(u^\delta)}\, dW_s,$$
$$s \in [t_0, t_0 + \delta],\ Y_{t_0+\delta}^{2,\delta;u^\delta,\beta^\delta(u^\delta)} = 0,$$

with the (deterministic) backward equation

$$dY_s^\delta = -\Big(\inf_{v\in V} \sup_{u\in U} F(s, x_0, 0, 0, u, v) - L\big|Y_s^\delta\big|\Big)\, ds, \quad s \in [t_0, t_0 + \delta],\ Y_{t_0+\delta}^\delta = 0, \tag{42}$$

where L denotes again the Lipschitz constant of $F(s, x_0, y, 0, u, v)$ with respect to y. By the comparison theorem for BSDEs we get that $Y_{t_0}^{2,\delta;u^\delta,\beta^\delta(u^\delta)} \geq Y_{t_0}^\delta$. Thus, from (41) and (42),

$$\delta^{1/2} \geq \frac{1}{\delta} Y_{t_0}^\delta \to \inf_{v\in V} \sup_{u\in U} F(s, x_0, 0, 0, u, v), \quad \text{as } \delta \to 0.$$

Consequently, $\inf_{v\in V} \sup_{u\in U} F(s, x_0, 0, 0, u, v) \leq 0$. Therefore, we have (29), and the proof is complete. □

6 Stochastic Differential Games with Incomplete Information

A differential games with incomplete information is a differential game in which

- at least one of the players has some private knowledge on the structure of the game: For instance, he may know precisely the random payoff or the random initial position of the game, while the other are only aware of the law of the payoff or of the initial position.
- the players observe each other's control perfectly. In this way they can try to guess their missing information by observing the behavior of the other players.

This class of problems is the transposition to differential games for Aumann–Maschler analysis of repeated games with incomplete information [11] (see also Mertens and Zamir [98]). It has been investigated in a series of papers by Cardaliaguet [41, 43, 45] and Cardaliaguet and Rainer [51, 52] for two-player zero-sum differential games. Here we mostly describe the existence and the characterization of the value for a zero-sum two-player differential games with incomplete information on the terminal payoff.

6.1 Description of the Game

Our game involves:

- a terminal time $T > 0$ and an initial time $t_0 \in [0, T]$,

– an initial position $x_0 \in \mathbb{R}^N$ and a stochastic controlled system:

$$dX_s^{t_0,x_0;u,v} = b\big(s, X_s^{t_0,x_0;u,v}, u_s, v_s\big)\,ds + \sigma\big(s, X^{t_0,x_0;u,v}, u_s, v_s\big)\,dB_s, \quad s \in [t,T],$$
$$X_0^{t_0,x_0;u,v} = x_0, \tag{43}$$

where B is a d-dimensional standard Brownian motion on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $b : [0,T] \times \mathbb{R}^N \times U \times V \to \mathbb{R}^N$ and $\sigma : [0,T] \times \mathbb{R}^N \times U \times V \to \mathbb{R}^{N\times d}$, U and V being some metric compact sets,
– a family of types $i \in \{1,\dots,I\}$ for the first player and $j \in \{1,\dots,J\}$ for the second one, and a family of terminal payoffs $g_{ij} : \mathbb{R}^N \to \mathbb{R}$ indexed by the types,
– two probability measures p and q on $\{1,\dots,I\}$ and $\{1,\dots,J\}$ respectively.

The game is played in the following way: At the initial time t_0 the types i and j are chosen randomly accordingly to the probability $p \otimes q$. The type i is announced to the first player, but not to the second one, while the type j is announced to the second player and not to the first one. Then the players control the stochastic differential equation (43) as usual, the first player trying to minimize the terminal payoff $\mathbb{E}[g_{ij}(X_T)]$, while the second one seeking at maximizing it. Note however that the players do not know which payoff there are actually optimizing.

In order to describe rigorously the results, we first state some assumptions and then describe the concept of strategies used along the section. Throughout the section we assume that the functions b and σ are continuous, bounded, and Lipschitz continuous with respect to (t,x), uniformly in $(u,v) \in U \times V$. We also suppose that, for $1 \le i \le I$, $1 \le j \le J$, g_{ij} are Lipschitz continuous and bounded. Finally we assume that Isaacs' condition holds: for all $(t,x) \in [0,T] \times \mathbb{R}^n$, $\xi \in \mathbb{R}^n$, and all $A \in \mathcal{S}_n$ (where $\mathcal{S}_n$ is the set of symmetric $n \times n$ matrices), we have

$$H(t,x,\xi,A) := \inf_u \sup_v \left\{ \langle b(t,x,u,v), \xi \rangle + \frac{1}{2} \operatorname{Tr}\big(A\sigma(t,x,u,v)\sigma^*(t,x,u,v)\big) \right\}$$
$$= \sup_v \inf_u \left\{ \langle b(t,x,u,v), \xi \rangle + \frac{1}{2} \operatorname{Tr}\big(A\sigma(t,x,u,v)\sigma^*(t,x,u,v)\big) \right\}. \tag{44}$$

The description of the notion of strategies takes some time (and some notations). For $R \in \mathbb{N}^*$, we denote by $\Delta(R)$ the set of all $(r_1,\dots,r_R) \in [0,1]^R$ that satisfy $\sum_{n=1}^R r_n = 1$. Elements of $\Delta(R)$ are identified with probability measures over $\{1,\dots,R\}$. We usually use the notation p for elements of $\Delta(I)$ and q for elements of $\Delta(J)$.

For $s \in [t,T]$, we set

$$\mathcal{F}_{t,s} = \sigma\big\{B_r - B_t, r \in [t,s]\big\} \vee \mathcal{N},$$

where $\mathcal{N}$ is the set of all null sets of $\mathbb{P}$. Our definition of admissible control is very close to (but slightly differs from) Definition 3:

Definition 5 An *admissible control* u for player 1 (resp. 2) on $[t,T]$ is a process taking values in U (resp. V), progressively measurable with respect to the filtration $(\mathcal{F}_{t,s}, s \ge t)$.

The set of admissible controls for player 1 (resp. 2) on $[t,T]$ is denoted by $\mathcal{U}(t)$ (resp. $\mathcal{V}(t)$).

Under assumption (H), for all $(t_0,x_0) \in [0,T] \times \mathbb{R}^n$ and $(u,v) \in \mathcal{U}(t) \times \mathcal{V}(t)$, there exists a unique solution to (43) that we denote by $X_\cdot^{t_0,x_0;u,v}$. In this framework of controls we can define the notion of nonanticipating strategies as in Sect. 5:

Definition 6 A *nonanticipating strategy with delay* (NAD strategy) for player 1 is a measurable mapping $\alpha : \Omega \times [t, T] \times \mathcal{V}(t) \longrightarrow U$ satisfying the following properties:

(i) (Progressive measurability) The mapping α is $\mathcal{F}_{t,\cdot}$-progressively measurable, i.e., for all $s \in [t, T]$, the mapping α restricted to $\Omega \times [t, s] \times \mathcal{U}(t)$ is $\mathcal{F}_{t,s} \otimes \mathcal{B}([t, s]) \otimes \mathcal{B}(\mathcal{U}(t)) - \mathcal{B}(U)$-measurable;

(ii) (Strict nonanticipativity) For every $\mathcal{F}_{t,\cdot}$-stopping time $\tau : \Omega \to [t, T]$ it holds for P-almost every $\omega \in \Omega$:

If $v, v' \in \mathcal{U}(t)$ coincide a.e. on $[t, \tau(\omega)]$, then also $\alpha(\omega, \cdot, v) = \alpha(\omega, \cdot, v')$ a.e. on $[t, \tau(\omega)]$.

(iii) (Nonanticipativity with delay) There exists an increasing sequence of stopping times $(S_n)_{n\geq 0}$ with $t = S_0 \leq S_1 \leq \cdots \leq S_n \leq \cdots \leq T$ and $\bigcup_{n\geq 1}\{S_n = T\} = \Omega$, P-a.s., such that, for any $k \geq 0$ and P-almost every $\omega \in \Omega$, it holds:

If $v, v' \in \mathcal{U}(t)$ are such that $v = v'$ a.e. on $[t, S_k(\omega)]$, then $\alpha(\omega, \cdot, v) = \alpha(\omega, \cdot, v')$ on $[t, S_{k+1}(\omega)]$.

The NAD strategies $\beta : \Omega \times [t, T] \times \mathcal{V}(t) \longrightarrow V$ for player 2 are defined in a symmetric manner. The space of the NAD strategies for player 1 for games over the time interval $[t, T]$ is denoted by $\mathcal{A}(t)$, that for player 2 by $\mathcal{B}(t)$.

As in Lemma 1 we have

Lemma 6 *For every couple $(\alpha, \beta) \in \mathcal{A}(t) \times \mathcal{B}(t)$ there is a unique pair (u, v) of control processes in $\mathcal{U}(t) \times \mathcal{V}(t)$ such that*

$$\alpha\big(\omega, s, v_\cdot(\omega)\big) = u_s(\omega) \quad \text{and} \quad \beta\big(\omega, s, u_\cdot(\omega)\big) = v_s(\omega), \quad ds\,P(d\omega)\text{-a.e. on } [t, T] \times \Omega. \tag{45}$$

We denote by $X_\cdot^{t_0,x_0,\alpha,\beta}$ the process $X_\cdot^{t_0,x_0,u,v}$, with (u, v) associated to (α, β) by relation (45).

In the frame of incomplete information it is necessary to introduce random strategies: indeed this is a way for the players to hide their private information.

Definition 7 A *random strategy* $\overline{\alpha}$ for player 1 is given by some $R \in \mathbb{N}^*$ and some R-tuple $\overline{\alpha} = (\alpha^1, \ldots, \alpha^R; r^1, \ldots, r^R)$, with $(\alpha^1, \ldots, \alpha^R) \in (\mathcal{A}(t))^R$, $(r^1, \ldots, r^R) \in \Delta(R)$.

The heuristic interpretation of $\bar{\alpha}$ is that player 1's strategy amounts to choose the pure strategy α^k with probability r^k. We define in a similar way the random strategies for player 2, and denote by $\mathcal{A}_r(t)$ (resp. $\mathcal{B}_r(t)$) the set of all random strategies for player 1 (resp. player 2).

Finally, identifying $\alpha \in \mathcal{A}(t)$ with $(\alpha; 1) \in \mathcal{A}_r(t)$, we can write $\mathcal{A}(t) \subset \mathcal{A}_r(t)$, and the same holds for $\mathcal{B}(t)$ and $\mathcal{B}_r(t)$.

Finally, since the first player knows the index i, a strategy for this player has to depend on i. We denote with a hat the elements of $(\mathcal{A}_r(t))^I$: $\hat{\alpha} = (\overline{\alpha}_1, \ldots, \overline{\alpha}_I)$ which are interpreted as the admissible strategies of the first player. In a symmetric way, $\hat{\beta} = (\overline{\beta}_1, \ldots, \overline{\beta}_J) \in (\mathcal{B}_r(t))^J$ denote the admissible strategies of the second player.

Payoffs and value functions For fixed $(i,j)\in\{1,\dots,I\}\times\{1,\dots,J\}$ and random strategies $(\overline{\alpha},\overline{\beta})\in\mathcal{A}_r(t)\times\mathcal{B}_r(t)$, with $\overline{\alpha}=(\alpha^1,\dots,\alpha^R;r^1,\dots,r^R)$ and $\overline{\beta}=(\beta^1,\dots,\beta^S;s^1,\dots,s^S)$, we set

$$J_{ij}(t,x,\overline{\alpha},\overline{\beta})=\sum_{k=1}^{R}\sum_{l=1}^{S}r^k s^l\mathbb{E}\big[g_{ij}\big(X_T^{t,x,\alpha^k,\beta^l}\big)\big].$$

This is the average of the payoffs with respect to the probability distributions associated to the strategies. Now, given $(p,q)\in\Delta(I)\times\Delta(J)$, with $p=(p_1,\dots,p_I), q=(q_1,\dots q_J)$, and $\hat{\alpha}=(\overline{\alpha}_1,\dots,\overline{\alpha}_I)\in(\mathcal{A}_r(t))^I$ and $\hat{\beta}=(\overline{\beta}_1,\dots,\overline{\beta}_J)\in(\mathcal{B}_r(t))^J$, we use the notation

$$J^{p,q}(t,x,\hat{\alpha},\hat{\beta})=\sum_{i=1}^{I}\sum_{j=1}^{J}p_iq_jJ_{ij}(t,x,\overline{\alpha}_i,\overline{\beta}_j).$$

This is the average of the payoffs with respect to the probability distributions p and q.

We define the value functions for the game by

$$\mathbf{V}^+(t,x,p,q)=\inf_{\hat{\alpha}\in(\mathcal{A}_r(t))^I}\ \sup_{\hat{\beta}\in(\mathcal{B}_r(t))^J}J^{p,q}(t,x,\hat{\alpha},\hat{\beta}),$$

$$\mathbf{V}^-(t,x,p,q)=\sup_{\hat{\beta}\in(\mathcal{B}_r(t))^J}\ \inf_{\hat{\alpha}\in(\mathcal{A}_r(t))^I}J^{p,q}(t,x,\hat{\alpha},\hat{\beta}).$$

Obviously, we have that

$$\mathbf{V}^-(t,x,p,q)\le\mathbf{V}^+(t,x,p,q)$$

and the main part of the work consists in proving the reverse inequality. One can show that $\mathbf{V}^+$ and $\mathbf{V}^-$ are bounded, Lipschitz continuous with respect to x,p,q and Hölder continuous with respect to t. The main structure property of $\mathbf{V}^+$ and $\mathbf{V}^-$ is given by the following:

Lemma 7 *For all $(t,x)\in[0,T]\times\mathbb{R}^n$, the maps $(p,q)\to\mathbf{V}^+(t,x,p,q)$ and $(p,q)\to\mathbf{V}^-(t,x,p,q)$ are convex in p and concave in q.*

Concavity of $\mathbf{V}^+$ with respect to q can easily be understood, because $\mathbf{V}^+$ can be rewritten as

$$\mathbf{V}^+(t,x,p,q)=\inf_{\hat{\alpha}\in(\mathcal{A}_r(t))^I}\sum_{j=1}^{J}q_j\sup_{\overline{\beta}\in\mathcal{B}_r(t)}\sum_{i=1}^{I}p_iJ_{i,j}(t,x,\overline{\alpha}_i,\overline{\beta}).$$

Convexity of $\mathbf{V}^+$ with respect to p is more subtle: It is proved, as in the repeated game theory with incomplete information, by using the so-called splitting method (see Aumann and Maschler [11]).

Because the value functions are convex/concave, it is natural to consider their Fenchel conjugates with respect to p,q. For a general map $w:[0,T]\times\mathbb{R}^N\times\Delta(I)\times\Delta(J)\mapsto\mathbb{R}$, we denote by w^* its convex conjugate with respect to variable p:

$$w^*(t,x,\hat{p},q)=\sup_{p\in\Delta(I)}\hat{p}.p-w(t,x,p,q)\quad\forall(t,x,\hat{p},q)\in[0,T]\times\mathbb{R}^N\times\mathbb{R}^I\times\Delta(J).$$

In particular $\mathbf{V}^{-*}$ and $\mathbf{V}^{+*}$ denote the convex conjugate with respect to the p-variable of the functions $\mathbf{V}^-$ and $\mathbf{V}^+$.

In a symmetric way we denote by $w^\sharp = w^\sharp(t,x,p,\hat{q})$ the concave conjugate with respect to q of w:

$$w^\sharp(t,x,p,\hat{q}) = \inf_{q\in\Delta(J)} \hat{q}.q - w(t,x,p,q) \quad \forall(t,x,p,\hat{q}) \in [0,T]\times\mathbb{R}^N\times\Delta(I)\times\mathbb{R}^J.$$

6.2 Subdynamic Programming Principle and Hamilton–Jacobi Equations

We now describe a striking property of the value functions: the fact that one of their Fenchel conjugates satisfies a dynamic programming principle. Let us point out that this is far from obvious. Indeed, contrary to what usually happens in differential game theory, the players, by observing each other's control, do really learn something on the game along the time. So it cannot be expected that a classical dynamic programming principle holds. However, we have:

Proposition 5 (Subdynamic programming principle for $\mathbf{V}^{-*}$) *For all* $0 \le t_0 \le t_1 \le T$, $x_0 \in \mathbb{R}^n$, $\hat{p} \in \mathbb{R}^I$, $q \in \Delta(J)$, *the following inequality holds*:

$$\mathbf{V}^{-*}(t_0, x_0, \hat{p}, q) \le \inf_{\beta\in\mathcal{B}(t_0)} \sup_{\alpha\in\mathcal{A}(t_0)} \mathbb{E}\big[\mathbf{V}^{-*}\big(t_1, X_{t_1}^{t_0,x_0,\alpha,\beta}, \hat{p}, q\big)\big]. \tag{46}$$

Idea of proof It can first shown that $\mathbf{V}^{-*}$ can be rewritten as:

$$\mathbf{V}^{-*}(t,x,\hat{p},q) = \inf_{\hat{\beta}\in(\mathcal{B}_r(t))^J} \sup_{\alpha\in\mathcal{A}(t)} \max_{i\in\{1,\dots,I\}} \big\{\hat{p}_i - J_i^q(t,x,\alpha,\hat{\beta})\big\}. \tag{47}$$

Now we note that, if the second player plays some strategy independent of j on the time interval $[t_0, t_1]$ in the above expression, then he reveals nothing between t_0 and t_1. So the information structure remains unchanged up to t_1 and, as in the usual dynamic programming principle, the game can be restarted at t_1. Of course, playing in this way on $[t_0,t_1]$ is in general suboptimal for the second player, so that there is just an inequality in (46).

A classical consequence of the subdynamic programming principle for $\mathbf{V}^{-*}$ is that this function is a subsolution of some associated Hamilton–Jacobi equation:

Corollary 2 *For any* $(\hat{p},q) \in \mathbb{R}^I \times \Delta(J)$, $\mathbf{V}^{-*}(\cdot,\cdot,\hat{p},q)$ *is a subsolution in the viscosity sense of*

$$w_t + H^{-*}\big(t,x,Dw,D^2w\big) = 0, \quad (t,x)\in(0,T)\times\mathbb{R}^n,$$

with

$$\begin{aligned} H^{-*}(t,x,p,A) &= -H^-(t,x,-p,-A) \\ &= \inf_{v\in V}\sup_{u\in U}\left\{\big\langle b(t,x,u,v),p\big\rangle + \frac{1}{2}\operatorname{Tr}\big(A\sigma(t,x,u,v)\sigma^*(t,x,u,v)\big)\right\}. \end{aligned} \tag{48}$$

For $\mathbf{V}^+$ we have in a symmetric way:

Proposition 6 (Superdynamic programming principle and HJ equation for $\mathbf{V}^{+\sharp}$) *For all* $0 \le t_0 \le t_1 \le T$, $x_0 \in \mathbb{R}^n$, $p \in \Delta(I)$, $\hat{q} \in \mathbb{R}^J$, *it holds that*

$$\mathbf{V}^{+\sharp}(t_0,x_0,p,\hat{q}) \ge \inf_{\beta\in\mathcal{B}(t_0)} \sup_{\alpha\in\mathcal{A}(t_0)} \mathbb{E}\big[\mathbf{V}^{+\sharp}\big(t_1, X_{t_1}^{t_0,x_0,\alpha,\beta}, p, \hat{q}\big)\big].$$

As a consequence, for any $(p,\hat{q})\in\Delta(I)\times\mathbb{R}^J$, $\mathbf{V}^{+\sharp}(\cdot,\cdot,p,\hat{q})$ *is a supersolution in viscosity sense of*

$$w_t+H^{+*}\big(t,x,Dw,D^2w\big)=0,\quad (t,x)\in(0,T)\times\mathbb{R}^n,$$

where

$$\begin{aligned}H^{+*}(t,x,p,A)&=-H^{+}(t,x,-p,-A)\\&=\sup_{u\in U}\inf_{v\in V}\left\{\langle b(t,x,u,v),p\rangle+\frac{1}{2}\operatorname{Tr}\big(A\sigma(t,x,u,v)\sigma^*(t,x,u,v)\big)\right\}.\end{aligned}\tag{49}$$

6.3 Comparison Principle and Existence of a Value

We now know that $\mathbf{V}^{-*}$ and $\mathbf{V}^{+\sharp}$ satisfy some Hamilton–Jacobi inequalities. Let us point out that these functions cannot be compared at the terminal time T, so one cannot apply the classical comparison principle to deduce some inequality of the form $\mathbf{V}^{-*}\le\mathbf{V}^{+\sharp}$.

In order to show inequality $\mathbf{V}^{-}\ge\mathbf{V}^{+}$, we have to use a comparison principle adapted to the convex/concave structure of the value functions. Let $H:[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathcal{S}_n\times\Delta(I)\times\Delta(J)\to\mathbb{R}$ be continuous and satisfy

$$\begin{aligned}&H(s,y,\xi_2,X_2,p,q)-H(t,x,\xi_1,X_1,p,q)\\&\quad\ge-\omega\big(|\xi_1-\xi_2|+a\big|(t,x)-(s,y)\big|^2+b+\big|(t,x)-(s,y)\big|\big(1+|\xi_1|+|\xi_2|\big)\big),\end{aligned}\tag{50}$$

where ω is continuous and non decreasing with $\omega(0)=0$, for any $a,b\ge0$, $(p,q)\in\Delta(I)\times\Delta(J)$, $s,t\in[0,T]$, $x,y,\xi_1,\xi_2\in\mathbb{R}^n$ and $X_1,X_2\in\mathcal{S}_n$ such that

$$\begin{pmatrix}-X_1&0\\0&X_2\end{pmatrix}\le a\begin{pmatrix}I&-I\\-I&I\end{pmatrix}+bI.$$

Definition 8 We say that a map $w:(0,T)\times\mathbb{R}^n\times\Delta(I)\times\Delta(J)\to\mathbb{R}$ is a supersolution in the dual sense of equation

$$w_t+H\big(t,x,Dw,D^2w,p,q\big)=0\tag{51}$$

if $w=w(t,x,p,q)$ is lower semicontinuous, concave with respect to q and if, for any $\mathcal{C}^2((0,T)\times\mathbb{R}^n)$ function ϕ such that $(t,x)\to w^*(t,x,\hat{p},\bar{q})-\phi(t,x)$ has a maximum at some point $(\bar{t},\bar{x})$ for some $(\hat{p},\bar{q})\in\mathbb{R}^I\times\Delta(J)$ at which $\frac{\partial w^*}{\partial\hat{p}}$ exists, we have

$$\phi_t(\bar{t},\bar{x})-H\big(\bar{t},\bar{x},-D\phi(\bar{t},\bar{x}),-D^2\phi(\bar{t},\bar{x}),\bar{p},\bar{q}\big)\ge0,\quad\text{where }\bar{p}=\frac{\partial w^*}{\partial\hat{p}}(\bar{t},\bar{x},\hat{p},\bar{q}).$$

We say that w is a subsolution of (51) in the dual sense if w is upper semicontinuous, convex with respect to p and if, for any $\mathcal{C}^2((0,T)\times\mathbb{R}^n)$ function ϕ such that $(t,x)\to w^\sharp(t,x,\bar{p},\hat{q})-\phi(t,x)$ has a minimum at some point $(\bar{t},\bar{x})$ for some $(\bar{p},\hat{q})\in\Delta(I)\times\mathbb{R}^J$ at which $\frac{\partial w^\sharp}{\partial\hat{q}}$ exists, we have

$$\phi_t(\bar{t},\bar{x})-H\big(\bar{t},\bar{x},-D\phi(\bar{t},\bar{x}),-D^2\phi(\bar{t},\bar{x}),\bar{p},\bar{q}\big)\le0,\quad\text{where }\bar{q}=\frac{\partial w^\sharp}{\partial\hat{q}}(\bar{t},\bar{x},\bar{p},\hat{q}).$$

A solution of (51) in the dual sense is a map which is sub- and supersolution in the dual sense.

Remark 5 There are several equivalent definitions of the notion of dual solutions (see Cardaliaguet [44]). Here is another very convenient one: a continuous convex/concave map w is a supersolution in the dual sense of (51) if and only if it satisfies, in the viscosity sense,

$$\min\left\{w_t + H(t, x, Dw, D^2w, p, q); \lambda_{\min}\left(\frac{\partial^2 w}{\partial p^2}\right)\right\} \leq 0.$$

It is a dual supersolution of (51) if and only if it satisfies, in the viscosity sense,

$$\max\left\{w_t + H\big(t, x, Dw, D^2w, p, q\big); \lambda_{\max}\left(\frac{\partial^2 w}{\partial q^2}\right)\right\} \geq 0.$$

As usual when dealing with viscosity solutions, the most important result is a comparison principle:

Theorem 8 (Comparison principle) *Let us assume that H satisfies the structure condition* (50). *Let w_1 be a bounded, Hölder continuous subsolution of* (51) *in the dual sense which is uniformly Lipschitz continuous w.r. to q and w_2 be a bounded, Hölder continuous supersolution of* (51) *in the dual sense which is uniformly Lipschitz continuous w.r. to p. Assume that*

$$w_1(T, x, p, q) \leq w_2(T, x, p, q) \quad \forall (x, p, q) \in \mathbb{R}^n \times \Delta(I) \times \Delta(J). \tag{52}$$

Then

$$w_1(t, x, p, q) \leq w_2(t, x, p, q) \quad \forall (t, x, p, q) \in [0, T] \times \mathbb{R}^n \times \Delta(I) \times \Delta(J).$$

Remark 6 The regularity conditions are here for simplicity and can be relaxed.

As a consequence of the comparison principle we have:

Theorem 9 (Existence of a value) *The game has a value*:

$$\mathbf{V}^+(t, x, p, q) = \mathbf{V}^-(t, x, p, q) \quad \forall (t, x, p, q) \in (0, T) \times \mathbb{R}^n \times \Delta(I) \times \Delta(J).$$

Furthermore $\mathbf{V}^+ = \mathbf{V}^-$ is the unique solution in the dual sense of HJ equation (51) *with terminal condition*

$$\mathbf{V}^+(T, x, p, q) = \mathbf{V}^-(T, x, p, q) = \sum_{i=1}^{I} \sum_{j=1}^{J} p_i q_j g_{ij}(x) \quad \forall (x, p, q) \in \mathbb{R}^n \times \Delta(I) \times \Delta(J).$$

Remark 7 The result can be extended to differential games with incomplete information on the running payoff, or on the terminal condition.

The proof of Theorem 9 is a straightforward consequence of Corollary 2, Proposition 6 and Theorem 8.

6.4 An Example of Continuous-time Game Without Dynamics

In order to illustrate the previous results, we now consider a simple class of examples: a two-player zero-sum continuous-time game without dynamics, in which the first player has a private information on the random running payoff. The description of the game involves

(i) an initial time $t_0 \geq 0$ and a terminal time $T > t_0$,
(ii) I integral payoffs (where $I \geq 2$): $\ell_i : [0,T] \times U \times V \to \mathbb{R}$ for $i = 1, \dots, I$ where U and V are compact subsets of some finite dimensional spaces,
(iii) a probability $p = (p_i)_{i=1,\dots,I}$ belonging to the set $\Delta(I)$ of probabilities on $\{1, \dots, I\}$.

As before the game is played in two steps: at time t_0, the index i is chosen at random among $\{1, \dots, I\}$ according to the probability p ; the choice of i is communicated to player 1 only. Then the players choose their respective controls in order, for player 1, to minimize the integral payoff $\int_{t_0}^T \ell_i(s, u(s), v(s))\, ds$, and for player 2 to maximize it. We again assume that both players observe their opponent's control. Note, however, that player 2 does not know which payoff he/she is actually maximizing.

The existence and the characterization of a value for this game is a particular case of the already described results: if Isaacs' condition holds:

$$H(t,p) = \inf_{u \in U} \sup_{v \in V} \sum_{i=1}^{I} p_i \ell_i(t,u,v) = \sup_{v \in V} \inf_{u \in U} \sum_{i=1}^{I} p_i \ell_i(t,u,v) \quad \forall (t,p) \in [0,T] \times \Delta(I), \tag{53}$$

the game has a value $\mathbf{V} = \mathbf{V}(t_0, p)$ given by

$$\begin{aligned} \mathbf{V}(t_0,p) &= \inf_{(\alpha_i) \in (\mathcal{A}_r(t_0))^I} \sup_{\beta \in \mathcal{B}_r(t_0)} \sum_{i=1}^{I} p_i \mathbb{E}_{\alpha_i \beta} \left[\int_{t_0}^T \ell_i\big(s, \alpha_i(s), \beta(s)\big)\, ds \right] \\ &= \sup_{\beta \in \mathcal{B}_r(t_0)} \inf_{(\alpha_i) \in (\mathcal{A}_r(t_0))^I} \sum_{i=1}^{I} p_i \mathbb{E}_{\alpha_i \beta} \left[\int_{t_0}^T \ell_i\big(s, \alpha_i(s), \beta(s)\big)\, ds \right], \end{aligned} \tag{54}$$

for any $(t_0, p) \in [0,T] \times \Delta(I)$. This value can be characterized as the unique dual solution of the HJ equation (51), or, as explained in Remark 5, as the unique solution to

$$\min\left\{ w_t + H(t, Dw, p), \lambda_{\min}\left(\frac{\partial^2 w}{\partial p^2} \right) \right\} = 0.$$

Our first aim is to compute explicitly $\mathbf{V}$ under some conditions.

Example 1 Assume that $I = 2$ and that there exists $h_1, h_2 : [0,T] \to [0,1]$ continuous, $h_1 \leq h_2$, h_1 decreasing and h_2 increasing, such that

$$\operatorname{Vex} H(t,p) = H(t,p) \Leftrightarrow p \in \big[0, h_1(t)\big] \cup \big[h_2(t), 1\big] \tag{55}$$

(where $\operatorname{Vex} H(t,p)$ denote the convex hull of $H(t,\cdot)$) and

$$\frac{\partial^2 H}{\partial p^2}(t,p) > 0 \quad \forall (t,p) \text{ with } p \in \big[0, h_1(t)\big) \cup \big(h_2(t), 1\big]. \tag{56}$$

For instance, if $U=[-1,1]$, $V=[0,2\pi]$ and

$$\ell_1(t,u,v)=u+\alpha(t)\cos(v), \ell_2(t,u,v)=-u+\alpha(t)\sin(v) \quad \forall(u,v)\in U\times V,$$

where the smooth map $\alpha:[0,T]\to\mathbb{R}$ is decreasing and such that $\alpha(t)>2$ for any $t\in[0,T]$, then

$$H(t,p)=-|2p-1|+\alpha(t)\sqrt{p^2+(1-p)^2}$$

satisfies (55) and (56) with $h_1(t)=1/2-1/(2\alpha^2(t)-4)^{\frac{1}{2}}$, $h_2(t)=1/2+1/(2\alpha^2(t)-4)^{\frac{1}{2}}$.

Proposition 7 *Under the assumptions of Example 1*,

$$\mathbf{V}(t,p)=\int_t^T \operatorname{Vex} H(s,p)\,ds \quad \forall(t,p)\in[0,T]\times\Delta(I).$$

Remark 8 The above representation for $\mathbf{V}$ does not hold true in general. For instance let $H(t,p)=\lambda(t)p(1-p)$ where $\lambda:[0,T]\to\mathbb{R}$ is Lipschitz continuous. We set $\Lambda(t)=\int_t^T\lambda(s)\,ds$. If

$$\lambda>0 \quad \text{on } [0,b), \qquad \lambda<0 \quad \text{on } (b,T], \qquad \Lambda(a)=0$$

for some $0<a<b<T$, then it can be easily checked that

$$\mathbf{V}(t,p)=\begin{cases}0 & \text{if } t\in[0,a],\\ \Lambda(t)p(1-p) & \text{if } t\in[b,T].\end{cases}$$

In particular

$$\mathbf{V}(t,p)\neq\int_t^T \operatorname{Vex} H(s,p)\,ds=\Lambda(b)p(1-p) \quad \forall(t,p)\in(a,b)\times(0,1).$$

Proof of Proposition 7 Let $w:[0,T]\times[0,1]\to\mathbb{R}$ be defined by

$$w(t,p)=\int_t^T \operatorname{Vex} H(s,p)\,ds \quad \forall(t,p)\in[0,T]\times\Delta(I).$$

We note that $w(T,p)=0$, $w(t,0)=\mathbf{V}(t,0)$ and $w(t,1)=\mathbf{V}(t,1)$. It can be easily checked that w is a solution of the Hamilton–Jacobi equation

$$\begin{cases}\min\left\{\partial_t w+H(t,p),\ \dfrac{\partial^2 w}{\partial p^2}\right\}=0,\\ w(T,p)=0.\end{cases}$$

Indeed, if $p\in(h_1(t),h_2(t))$, then

$$\frac{\partial^2 w}{\partial p^2}(t,p)=0 \quad \text{and} \quad \partial_t w(t,p)=-H\big(t,h_1(t)\big)\geq -H(t,p).$$

If $p\in(0,h_1(t)]\cup[h_2(t),1)$, then

$$\frac{\partial^2 w}{\partial p^2}(t,p)\geq 0 \quad \text{and} \quad \partial_t w(t,p)=-H(t,p)$$

(where the first equality holds in the viscosity sense since w is convex with respect to p). Therefore $w = \mathbf{V}$. □

In order to describe the strategy of the informed player, we need to provide a further characterization of the value. For this let us introduce some notations: Let $\mathbf{D}(t_0)$ be the set of càdlàg functions (i.e., functions which are right-continuous and have a left limit at each point) from $\mathbb{R} \to \Delta(I)$ which are constant on $(-\infty, t_0)$ and on $[T, +\infty)$, let $t \mapsto \mathbf{p}(t)$ be the coordinate mapping on $\mathbf{D}(t_0)$ and let $\mathcal{G} = (\mathcal{G}_t)$ be the filtration generated by $t \mapsto \mathbf{p}(t)$.

Given $p_0 \in \Delta(I)$, we denote by $\mathbf{M}(t_0, p_0)$ the set of probability measures $\mathbb{P}$ on $\mathbf{D}(t_0)$ such that, under $\mathbb{P}$, $(\mathbf{p}(t), t \in [0, T])$ is a martingale and satisfies

$$\text{for } t < t_0, \quad \mathbf{p}(t) = p_0 \quad \text{and,} \quad \text{for } t \geq T, \quad \mathbf{p}(t) \in \{e_i,\ i = 1, \dots, I\} \quad \mathbb{P}\text{-a.s.}$$

Finally for any measure $\mathbb{P}$ on $\mathbf{D}(t_0)$, we denote by $\mathbb{E}_{\mathbb{P}}[\dots]$ the expectation with respect to $\mathbb{P}$.

The following equality holds:

Theorem 10

$$\mathbf{V}(t_0, p_0) = \min_{\mathbb{P} \in \mathbf{M}(t_0, p_0)} \mathbb{E}_{\mathbb{P}}\left[\int_{t_0}^{T} H\big(s, \mathbf{p}(s)\big)\, ds\right] \quad \forall (t_0, p_0) \in [0, T] \times \Delta(I). \tag{57}$$

Theorem 10 allows to describe the optimal strategy of the informed player: Let $(t_0, p_0) \in [0, T) \times \Delta(I)$ be fixed, $\bar{\mathbb{P}}$ be optimal in the problem (57). Let us set $E_i = \{\mathbf{p}(T) = e_i\}$ and define the probability measure $\bar{\mathbb{P}}_i$ by: $\forall A \in \mathcal{G}$, $\bar{\mathbb{P}}_i(A) := \bar{\mathbb{P}}[A | E_i] = \frac{\bar{\mathbb{P}}(A \cap E_i)}{p_i}$, if $p_i > 0$, and $\bar{\mathbb{P}}_i(A) = P(A)$ for an arbitrary probability measure $P \in \mathbf{M}(t_0, p_0)$ if $p_i = 0$.

We also set

$$\bar{\mathbf{u}}(t) = u^*\big(t, \mathbf{p}(t)\big) \quad \forall t \in \mathbb{R},$$

where $u^* = u^*(t, p)$ is a Borel measurable selection of $\operatorname{argmin}_{u \in U}(\max_{v \in V} \sum_{i=1}^{I} p_i \ell_i(t, u, v))$ and where we denote by $\bar{\mathbf{u}}_i$ the random control $\bar{\mathbf{u}}_i = ((\mathbf{D}(t_0), \mathcal{G}, \bar{\mathbb{P}}_i), \bar{\mathbf{u}}) \in \mathcal{U}_r(t_0)$.

Theorem 11 *The strategy consisting in playing the random control* $(\bar{\mathbf{u}}_i)_{i=1,\dots,I} \in (\mathcal{U}_r(t_0))^I$ *is optimal for* $\mathbf{V}(t_0, p_0)$. *Namely*

$$\mathbf{V}(t_0, p_0) = \sup_{\beta \in \mathcal{B}_r(t_0)} \sum_{i=1}^{I} p_i \mathbb{E}_{\bar{\mathbf{u}}_i}\left[\int_{t_0}^{T} \ell_i\big(s, \bar{\mathbf{u}}_i(s), \beta(\bar{\mathbf{u}}_i)(s)\big)\, ds\right]. \tag{58}$$

Since the computation of the optimal strategy of the informed player can be reduced to the computation of the optimal martingale measure in (57), it would be very interesting to have a full characterization of such measure. Unfortunately, up to now, only some necessary and some sufficient conditions for a measure to be optimal are known. However, in Example 1 it is possible to identify explicitly this optimal martingale measure:

Proposition 8 *Under the assumptions of Example 1, there is a unique optimal martingale measure* $\bar{\mathbb{P}}$. *Under this martingale measure, the process* $\mathbf{p}$ *is purely discontinuous and satisfies*:

$$\mathbf{p}(t^-) = p_0 \quad \forall t \in [t_0, t^*]\bar{\mathbb{P}}\text{-a.s., where } t^* = \inf\big\{t \geq t_0 \mid p_0 \in \big[h_1(t), h_2(t)\big]\big\}$$

and

$$\mathbf{p}(t) \in \{h_1(t), h_2(t)\} \quad \forall t \in [t^*, T)\bar{\mathbb{P}}\text{-}a.s.$$

In particular,

$$\bar{\mathbb{P}}[\mathbf{p}(t) = h_1(t) \mid \mathbf{p}(s) = h_1(s)] = \frac{h_2(t) - h_1(s)}{h_2(t) - h_1(t)} \quad \forall t^* \leq s \leq t < T. \tag{59}$$

6.5 Generalizations and Open Problems

Literature on differential games with information issues (games with incomplete information or games with signals) is rather scarce and the problem is, in general, poorly understood. In most papers the authors construct a strategy for a non-fully informed controller, the other player being seen as a disturbance: see, for instance, the monograph by Baçar and Bernhard [12] and the papers by Baras and Patel [15], Baras and James [14]. In terms of game this means that a kind of worst case design is looked at. In contrast, few works are devoted to the existence of a value for this class of games: Rapaport and Bernhard [107], on the one hand, and Petrosjan [102], on the other hand, analyze this question through some interesting classes of examples. Cardaliaguet and Quincampoix [48] consider a general class of differential games where the players share the same information on the random distribution of the initial position of the system. The associated Hamilton–Jacobi equation turns out to take place in the Wasserstein space of probability measures. Cardaliaguet and Souquière [53] analyze a game where one of the players is blind while the other one has a perfect observation. Here again the value exists and can be characterized as the unique viscosity solution of some HJ equation in the Wasserstein space. However the existence of the value comes from min-max arguments, and not from usual uniqueness result in PDE, because one of the value functions does not seem to satisfy a dynamic programming principle naturally.

The game studied in this chapter is strongly inspired by *repeated games* with lack of information introduced by Aumann and Maschler: see the monographs by Aumann and Maschler [11] and by Sorin [109] for a general presentation. In this framework the *dual approach* was initiated by De Meyer in [57] and later developed by De Meyer and Rosenberg [58] and by Laraki [86].

The first adaptation of the Aumann–Maschler's theory to differential games goes back to Cardaliaguet [41], which deals with deterministic differential games with a terminal payoff, and with games where there is some private information on the initial position of the system. It is generalized to stochastic differential games and to games with running payoffs in Cardaliaguet and Rainer [51]. The infinite horizon problem is considered in As Soulaimani [8]. Examples of such games are analyzed in Cardaliaguet [43], Cardaliaguet and Rainer [52] and Souquière [110], while the construction of ϵ-optimal strategies and approximation of the value function are carried out in Cardaliaguet [45] and Souquière [110].

The construction of the optimal strategy of the informed player in a game without dynamics, as explained in subsection 6.4, is borrowed from Cardaliaguet and Rainer [52]. This result strongly uses the fact that there is no dynamics and the corresponding construction for genuine differential games is unclear. As for the strategy of the informed player, it can be built by using a kind of "approachability technique" combined with a stable bridge approach (Souquière [112]).

Up to now most analysis of differential games with incomplete information has been restricted to zero-sum differential games in which the private information is revealed to the players at the initial time. It would be very interesting to investigate games in which this private information is disclosed all along the game: however this problem is open. Another open question is the generalization of the theory described above to nonzero-sum differential games.

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