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ORIGINAL RESEARCH

Mittag-Leffler conditions and projectively coresolved Gorenstein flat tilting modules

Jingrui Su · Yanjiong Yang

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Abstract We first make an approach to the Mittag-Leffler condition in the theory of relative Ext-orthogonal classes and establish the balance of the relative derived functors of the Hom and tensor product functors with respect to the classes of projectively coresolved Gorenstein flat modules and Gorenstein injective modules. Then, we introduce the concepts of PGF-tilting modules and PGF-weak tilting modules. By means of Mittag-Leffler conditions, we explore the connection between PGF-tilting modules and PGF-weak tilting modules. As an application, we study when Gorenstein tilting modules are Gorenstein weak tilting.

Keywords Mittag-Leffler condition · Balanced pair · PGF-tilting module · PGF-weak tilting module · Gorenstein tilting module

Mathematics Subject Classification 16E30 · 18G10 · 18G25

1 Introduction

To ensure the exactness of the inverse limit of countable inverse system, the Mittag-Leffler condition was introduced by Grothendieck in [16]. In the past few years, Mittag-Leffler conditions on modules have some successful applications in homological algebra. For instance, it plays a crucial role in in solving the Baer splitting problem [1], as well as in proving that all tilting modules are of finite type [4,5]. Emmanouil and Talelli [10,11,15] found some interesting applications of such conditions in studying of some (co) homological invariants of modules (groups). These applications show that Mittag-Leffler conditions are closely related to the vanishing of the functor $\mathrm{Ext}^1_R(-,-)$. Motivated by these results, we study the relation between Mittag-Leffler conditions and relative Ext-functors. This allows us to extend the applications of Mittag-Leffler conditions to the relative tilting theory.

Recently, infinitely generated tilting theory was investigated by many authors in the context of Gorenstein homological algebra. Using the Gorenstein Ext-functor, Yan et al. introduced in [25] the notion of Gorenstein tilting modules over Gorenstein rings, as a generation of infinitely generated tilting modules in [2,8]. Subsequently, Moradifar and Yassemi [21] developed the tools of relative approximation theory, and made a systematic approach to Gorenstein tilting classes and Gorenstein tilting cotorsion pairs over virtually Gorenstein rings. In [19], Mao introduced the definition of Gorenstein weak tilting modules, which is a Gorenstein analogue of weak tilting modules. It is known that all tilting modules are weak tilting modules [27]. This leads us to consider the

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Y. Yang (✉)· J. Su
Department of Mathematics, Taizhou University, Taizhou 225300, People's Republic of China
E-mail: yanjiong1019@126.com

J. Su
E-mail: sjr-sujingrui@163.com

question when Gorenstein tilting modules are Gorenstein weak tilting. However, this question is related to the open problem whether all Gorenstein projective modules are Gorenstein flat. In [23], Šaroch and Šťovíček introduced projectively coresolved Gorenstein flat modules (PGF-modules for short) and showed that PGF-modules are Gorenstein projective and also Gorenstein flat. They suggested that PGF-modules could serve as an alternative of Gorenstein projective modules. Inspired by these, we introduce the notions of projectively coresolved Gorensten flat tilting modules (PGF-tilting modules for short) and projectively coresolved Gorensten flat weak tilting modules (PGF-weak tilting modules for short). We explore the relation between PGF-tilting modules and PGF-weak tilting modules by using Mittag-Leffler conditions. Furthermore, we employ our investigations to discuss when Gorenstein tilting modules are Gorenstein weak tilting modules.

The paper is organized as follows. We first collect some known notions and results in Sect. 2, including relative homological groups, balanced pairs, relative approximations and Gorenstein modules.

In Sect. 3, we establish the connection between the Mittag-Leffler condition and the relative derived functors of the Hom functors and examine the condition when the class of PGF-modules forms the left-hand class of a Hom-balanced pair. Denote by $\mathcal{PGF}$ and $\mathcal{GI}$ the class of projectively coresolved Gorenstein flat modules and Gorenstein injective modules, respectively. We prove that $\mathcal{PGF}$ and $\mathcal{GI}$ form a Hom-balanced pair if and only if $\mathcal{PGF}^{\perp} = {}^{\perp}\mathcal{GI}$ (see Theorem 3.6).

In Sect. 4, We introduce the notions of PGF-tilting modules and PGF-weak tilting modules (see Definitions 4.1 and 4.3). Employing Mittag-Leffler conditions, we give a characterization of PGF-weak n-tilting module in Theorem 4.7. As a further application, we prove that if $\mathfrak{B}(= (\mathcal{PGF}, \mathcal{GI}))$ is a Hom-balanced pair and all PGF-modules are pure projective, then all $\mathfrak{B}$-countable type PGF-tilting modules are PGF-weak tilting (see Theorem 4.9). This implies that if all Gorenstein tilting modules are pure projective, then all Gorenstein n-tilting modules of $\mathfrak{B}$-countable type are Gorenstein weak tilting over n-FC rings (Corollary 4.10).

Throughout this paper, R is an associative ring with an identity. R-Mod (Mod-R) denote the category of left (right) R-modules. The class of projective (injective, flat) modules is denoted by $\mathcal{P}$ ($\mathcal{I}$, $\mathcal{F}$). Let $\mathcal{C}$ be a class of modules. We denote by Add($\mathcal{C}$) the class of all direct summands of arbitrary direct sums of modules from $\mathcal{C}$, and by Prod($\mathcal{C}$) the class of all direct summands of arbitrary direct products of modules from $\mathcal{C}$. For a single module C, Add(C) (Prod(C)) is the class of all direct summands of direct sums (products) of copies of C. For a R-module M, we denote by $\mathrm{Hom}_Z(M, Q/Z)$ its character module.

2 Preliminaries

In this section, we recall some basic notions and facts which will be required in the sequel.

Let $\mathcal{C}$ be a class of R-modules and M a R-module. By [17], a map $f : C \to M$ with $C \in \mathcal{C}$ is a $\mathcal{C}$-precover if for any $C' \in \mathcal{C}$, $\mathrm{Hom}_R(C', f) : \mathrm{Hom}_R(C', C) \to \mathrm{Hom}_R(C', M)$ is surjective. $f : C \to M$ is a special precover if f is surjective and $\mathrm{Ext}^1_R(C, \mathrm{Ker} f) = 0$ for any $C \in \mathcal{C}$. $\mathcal{C}$ is a (special) precovering class if each module admits a (special) $\mathcal{C}$-precover. Dually, we have the notions of a (special) preenvelope and a (special) preenveloping class.

Let $\mathcal{X}$, $\mathcal{Y}$ be classes of left R-modules. Given a left R-module M. Following [12, Definition 8.1.2], a left $\mathcal{X}$-resolution of M is an $\mathrm{Hom}_R(\mathcal{X}, -)$ exact complex

$$\mathbb{X}_M : \cdots \to X_1 \to X_0 \to M \to 0$$

with each $X_i \in \mathcal{X}$. Dually, a right $\mathcal{Y}$-resolution of M is an $\mathrm{Hom}_R(-, \mathcal{Y})$-exact complex

$$\mathbb{Y}^M : 0 \to M \to Y^0 \to Y^1 \to \cdots$$

with each $Y^i \in \mathcal{Y}$. We denote by $\mathcal{X}$-$dim(M)$ the minimum length of left $\mathcal{X}$-resolution of M. By [12, Definition 8.2.13], $\mathrm{Hom}_R(-, -)$ is right balanced on R-Mod$\times R$-Mod by $\mathcal{X} \times \mathcal{Y}$, provided that any $M \in R$-Mod admit a $\mathrm{Hom}_R(-, \mathcal{Y})$ exact left $\mathcal{X}$-resolution and a $\mathrm{Hom}_R(\mathcal{X}, -)$ exact right $\mathcal{Y}$-resolution. In this case, $(\mathcal{X}, \mathcal{Y})$ is said to be a (right) Hom-balanced pair. $(\mathcal{X}, \mathcal{Y})$ is called an admissible Hom-balanced pair (see [21, Section 2]) if $\mathcal{X}$ is precovering and $\mathcal{X}$-precovers are surjective, or $\mathcal{Y}$ is preenveloping and $\mathcal{Y}$-preenvelopes are injective.

Notice that $- \otimes_R -$ is covariant in both variable, we say $- \otimes_R -$ is left balanced on Mod-$R \times R$-Mod by $\mathcal{X}^{op} \times \mathcal{X}$, provided that any $M \in$ Mod-R admits a $- \otimes_R \mathcal{X}$ exact left $\mathcal{X}^{op}$-resolution and any $N \in R$-Mod admits a $\mathcal{X}^{op} \otimes_R -$ exact left $\mathcal{X}$-resolution, where $\mathcal{X}^{op}$ is the class of right R-modules corresponding to the class $\mathcal{X}$.

The following result follows from [6, Proposition 2.2].

Lemma 2.1 *Let $\mathfrak{B} = (\mathcal{X}, \mathcal{Y})$ be a Hom-balanced pair. A complex $\cdots \to C_{n+1} \to C_n \to C_{n-1} \cdots$ is $Hom_R(\mathcal{X}, -)$ exact if and only if it is $Hom_R(-, \mathcal{Y})$ exact.*

The complex in Lemma 2.1 is said to be $\mathfrak{B}$-exact, provided that it is $\mathrm{Hom}_R(\mathcal{X}, -)$ exact and $\mathrm{Hom}_R(-, \mathcal{Y})$ exact. $\mathfrak{B}$-exact complexes must be exact if the Hom-balanced pair $\mathfrak{B}$ is admissible (cf. [6, Proposition 2.2 and Corollary 2.3]). Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a $\mathfrak{B}$-exact sequence, f is said to be $\mathfrak{B}$-monomorphism, g is said to be $\mathfrak{B}$-epimorphism. It is clear that $(\mathcal{P}, \mathcal{I})$ is the standard (admissible) Hom-balanced pair. If $\mathfrak{B} = (\mathcal{P}, \mathcal{I})$, $\mathfrak{B}$-exact sequence is the standard exact sequence.

Let $\mathfrak{B} = (\mathcal{X}, \mathcal{Y})$ be an admissible Hom-balanced pair. By [21], the derived functor $\mathrm{Ext}^n_{\mathfrak{B}}(M, N)$ of $\mathrm{Hom}_R(M, N)$ can be computed using a left $\mathcal{X}$-resolution of M or a right $\mathcal{Y}$-resolution of N. Let $\widehat{\mathbb{X}}_M = \cdots \to X_1 \to X_0 \to 0$ be the deleted complex associated with left $\mathcal{X}$-resolution of M. and $\widehat{\mathbb{Y}}^N = \cdots \to Y^1 \to Y^0 \to 0$ the deleted complex associated with left $\mathcal{Y}$-resolution of N. We set

$$\mathrm{Ext}^n_{\mathfrak{B}}(M, N) = H^n\big(\mathrm{Hom}_R\big(\widehat{\mathbb{X}}_M, N\big)\big) = H^n\big(\mathrm{Hom}_R\big(M, \widehat{\mathbb{Y}}^N\big)\big).$$

Let A be a right R-module and B a left R-module. If $-\otimes_R-$ is left balanced on Mod-$R \times R$-Mod by $\mathcal{X}^{op} \times \mathcal{X}$, Then the derived functor $\mathrm{Tor}^{\mathfrak{B}}_n(A, B)$ of $A \otimes_R B$ can be computed using left $\mathcal{X}^{op}$-resolution of A or left $\mathcal{X}$-resolution of B. We set

$$\mathrm{Tor}^n_{\mathfrak{B}}(A, B) = H_n(\widehat{\mathbb{Y}}_A \otimes B) = H_n(A \otimes \widehat{\mathbb{Y}}_B).$$

where $\widehat{\mathbb{Y}}_A$ is the deleted left $\mathcal{X}^{op}$-resolution and $\widehat{\mathbb{Y}}_B$ is the deleted left $\mathcal{X}$-resolution.

It is obvious that $\mathrm{Ext}^n_{\mathfrak{B}}(M, -) = \mathrm{Ext}^n_{\mathfrak{B}}(-, N) = 0$ for any $M \in \mathcal{X}$ and $N \in \mathcal{Y}$. For a class $\mathcal{C}$ of modules, we write ${}^{\perp_{\mathfrak{B}}}\mathcal{C} = \{M : \mathrm{Ext}^1_{\mathfrak{B}}(M, C) = 0 \text{ for all } C \in \mathcal{C}\}$, ${}^{\perp_{\mathfrak{B}\infty}}\mathcal{C} = \{M : \mathrm{Ext}^i_{\mathfrak{B}}(M, C) = 0 \text{ for all } C \in \mathcal{C} \text{ and all } i \geq 1\}$, $\mathcal{C}^{\perp_{\mathfrak{B}}} = \{M : \mathrm{Ext}^1_{\mathfrak{B}}(C, M) = 0 \text{ for all } C \in \mathcal{C}\}$, $\mathcal{C}^{\perp_{\mathfrak{B}\infty}} = \{M : \mathrm{Ext}^i_{\mathfrak{B}}(C, M) = 0 \text{ for all } C \in \mathcal{C} \text{ and all } i \geq 1\}$, ${}^{\top_{\mathfrak{B}}}\mathcal{C} = \{M : \mathrm{Tor}^{\mathfrak{B}}_1(C, M) = 0 \text{ for all } C \in \mathcal{C}\}$, ${}^{\top_{\mathfrak{B}\infty}}\mathcal{C} = \{M : \mathrm{Tor}^{\mathfrak{B}}_i(C, M) = 0 \text{ for all } C \in \mathcal{C} \text{ and all } i \geq 1\}$.

Definition 2.2 (see [21, Section 4]) Let $\mathfrak{B} = (\mathcal{X}, \mathcal{Y})$ be an admissible Hom-balanced pair.

(1) Let $\mathcal{C}$ be a class of modules and $0 \to N \xrightarrow{f} C \xrightarrow{g} M \to 0$ a $\mathfrak{B}$-exact sequence with $C \in \mathcal{C}$. $g : C \to M$ is called a $\mathfrak{B}$-special $\mathcal{C}$-precover of M if $N \in \mathcal{C}^{\perp_{\mathfrak{B}}}$. $f : N \to C$ is called a $\mathfrak{B}$-special $\mathcal{C}$-preenvelope of N if $M \in {}^{\perp_{\mathfrak{B}}}\mathcal{C}$. $\mathcal{C}$ is called a $\mathfrak{B}$-special $\mathcal{C}$-precovering class if every R-module admits a $\mathfrak{B}$-special $\mathcal{C}$-precover. Dually, $\mathcal{C}$ is called a $\mathfrak{B}$-special $\mathcal{C}$-preenveloping class if every R-module admits a $\mathfrak{B}$-special $\mathcal{C}$-preenvelope.
(2) A class $\mathcal{C}$ of modules is said to be $\mathfrak{B}$-resolving if $\mathcal{X} \subseteq \mathcal{C}$ and for any $\mathfrak{B}$-exact sequence $0 \to A \to B \to C \to 0$ with $C \in \mathcal{C}$, $A \in \mathcal{C}$ if and only if $B \in \mathcal{C}$. $\mathcal{C}$ is said to be $\mathfrak{B}$-coresolving if $\mathcal{Y} \subseteq \mathcal{C}$ and for any $\mathfrak{B}$-exact sequence $0 \to A \to B \to C \to 0$ with $A \in \mathcal{C}$, $B \in \mathcal{C}$ if and only if $C \in \mathcal{C}$.
(3) Let $\mathcal{L}$ and $\mathcal{R}$ are classes of modules. $(\mathcal{L}, \mathcal{R})$ is said to be a $\mathfrak{B}$-cotorsion pair if $\mathcal{L}^{\perp_{\mathfrak{B}}} = \mathcal{R}$ and ${}^{\perp_{\mathfrak{B}}}\mathcal{R} = \mathcal{L}$. $(\mathcal{L}, \mathcal{R})$ is called $\mathfrak{B}$-hereditary if $\mathcal{L}$ is $\mathfrak{B}$-resolving or $\mathcal{R}$ is $\mathfrak{B}$-coresolving. $(\mathcal{L}, \mathcal{R})$ is called $\mathfrak{B}$-complete if $\mathcal{L}$ is $\mathfrak{B}$-special precovering or $\mathcal{R}$ is $\mathfrak{B}$-special preenveloping.

Remark 2.3 Let $(\mathcal{L}, \mathcal{R})$ be a $\mathfrak{B}$-cotorsion pair. From [18], $\mathcal{L}$ is $\mathfrak{B}$-resolving if and only if $\mathcal{R}$ is $\mathfrak{B}$-coresolving, $\mathcal{L}$ is $\mathfrak{B}$-special precovering if and only if $\mathcal{R}$ is $\mathfrak{B}$-special preenveloping. We call the $\mathfrak{B}$-cotorsion pair $(\mathcal{L}, \mathcal{R})$ generated by a set $\mathcal{S}$ if $\mathcal{R} = \mathcal{S}^{\perp_{\mathfrak{B}}}$. It is clear that any $\mathfrak{B}$-cotorsion pair generated by a set is $\mathfrak{B}$-complete by Relative Eklof–Trlifaj Completeness Theorem [21].

Gorenstein modules were introduced by Enochs et al. in [13,14], as generations of projective, flat and injective modules. Recall that a left R-module M is called Gorenstein flat, provided that there is an exact sequence $\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$ of flat modules with $M = \mathrm{Ker}(F^0 \to F^1)$, which is $E \otimes -$ exact for any injective right R-module E. M is called Gorenstein projective, provided that there is an exact sequence $\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$ of projective modules with $M = \mathrm{Ker}(P^0 \to P^1)$, which is $\mathrm{Hom}_R(-, P)$ exact for any projective left R-module P. M is called Gorenstein injective if there is an exact sequence $\cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots$ of injective left R-modules with $M = \mathrm{Ker}(E^0 \to E^1)$, which is $\mathrm{Hom}_R(I, -)$ exact for any injective left R-module I. Projectively coresolved Gorenstein flat modules were introduced by Šaroch and Šťovíček in [23] as follows.

Definition 2.4 Let R be a ring. A left R-module M is called a projectively coresolved Gorenstein flat module, or a PGF-module for short, provided that there is an exact sequence

$$\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

of projective left R-module with $M = \mathrm{Ker}(P^0 \to P^1)$, which is $E \otimes_R -$ exact for any injective right R-module E.

The class of PGF-modules is denoted by $\mathcal{PGF}$. We denote the class of Gorenstein projective, flat and injective modules by $\mathcal{GP}, \mathcal{GF}$ and $\mathcal{GI}$ respectively. It is clear that $\mathcal{PGF} \subseteq \mathcal{GF}$. It was proved in [23] that $\mathcal{PGF} \subseteq \mathcal{GP}$, $(\mathcal{PGF}, \mathcal{PGF}^{\perp})$ is a complete hereditary cotorsion pair and $\mathcal{PGF} \cap \mathcal{PGF}^{\perp} = \mathcal{P}$, $({}^{\perp}\mathcal{GI}, \mathcal{GI})$ is a perfect hereditary cotorsion pair and ${}^{\perp}\mathcal{GI} \cap \mathcal{GI} = \mathcal{I}$, $(\mathcal{GF}, \mathcal{GF}^{\perp})$ is a complete hereditary cotorsion pair.

3 Mittag-Leffler conditions and balanced pairs

In this section, we study Mittag-Leffler conditions in the setting of relative orthogonal classes and discuss when $(\mathcal{PGF}, \mathcal{GI})$ forms a Hom-balanced pair.

Recall that an inverse system of abelian groups $(A_\alpha, u_{\alpha\beta})_{\alpha,\beta\in I}$ with $A = \varprojlim A_\alpha$ is said to satisfy the Mittag-Leffler condition if, for each $\alpha \in I$, there exists an index $\gamma = \gamma(\alpha)$ with $\gamma \geq \alpha$, such that $u_{\alpha\beta}(A_\beta) = u_{\alpha\gamma}(A_\gamma)$ for any $\beta \geq \gamma$. Furthermore, $(A_\alpha, u_{\alpha\beta})_{\alpha,\beta\in I}$ is said to satisfy the strict Mittag-Leffler condition if $u_{\alpha\beta}(A_\beta) = u_\alpha(A)$ for any $\beta \geq \gamma$ where u_α denotes the canonical map $A \to A_\alpha$. For a countable inverse system, the Mittag-Leffler condition equivalent to the strict Mittag-Leffler condition (see [3] for detail).

Let M be a left R-module. Then there is a direct system of finitely presented modules $(F_\alpha, u_{\beta\alpha})_{\alpha,\beta\in I}$ such that $M = \varinjlim F_\alpha$. For a left R-module N, M is strict N-stationary if the inverse system

$$(\mathrm{Hom}_R(F_\alpha, N), \mathrm{Hom}_R(u_{\beta\alpha}, N))_{\alpha,\beta\in I}$$

satisfies the strict Mittag-Leffler condition. Given a class $\mathcal{N}$ of left R-modules, M is called strict $\mathcal{N}$-stationary if for any $N \in \mathcal{N}$, M is strict N-stationary. The following characterization of strict N-stationary module is useful in the sequel. (see [3, Theorem 8.11] or [15, Theorem 1.3] for its proof).

Theorem 3.1 *Let M and N be left R-modules. The following statements are equivalent:*

(1) M is a strict N-stationary module.
(2) For any divisible abelian group D, the natural transformation

$$\Phi : Hom_Z(N, D) \otimes_R M \longrightarrow Hom_Z(Hom_R(M, N), D)$$

defined by $\Phi(f \otimes m) : g \mapsto f(g(m))$, $f \in Hom_Z(N, D), m \in M$ and $g \in Hom_R(M, N)$ is a monomorphism.

Remark 3.2 (1) By Theorem 3.1, it is easy to see that the class of strict N-stationary modules is closed under direct sums and direct summands. The natural transformation Φ in Theorem 3.1 is an isomorphism for any left R-module N, provided that M is finitely presented. Thus, all projective modules and pure projective modules are strict N-stationary for any left R-module N.

(2) Let M be a countably presented module. Then there is a countable direct system $\{F_i\}_{i\in\mathbb{N}}$ of F_i finitely presented modules such that $M = \varinjlim F_i$. By [17, Corollary 2.23], there is a pure exact sequence $0 \to \bigoplus_{i\in\mathbb{N}} F_i \to \bigoplus_{i\in\mathbb{N}} F_i \to M \to 0 \quad (\star)$.

The following proposition is inspired by [4, Theorem 5.1].

Proposition 3.3 *Let $\mathfrak{B} = (\mathcal{X}, \mathcal{Y})$ be an admissible Hom-balanced pair. and M a countably presented left R-module. If $(\star)$ is $\mathfrak{B}$-exact and $\mathrm{Ext}^1_{\mathfrak{B}}(M, P^{(\mathbb{N})}) = 0$ for a left R-module P. Then M is strict P-stationary.*

Proof According to [4, Theorem 5.1], M is strict P-stationary if and only if the pure exact sequence $(\star)$ is $\mathrm{Hom}_R(-, P^{(\mathbb{N})})$ exact. Applying $\mathrm{Hom}_R(-, P^{(\mathbb{N})})$ to the exact sequence $(\star)$, we get that $0 \to \mathrm{Hom}_R(M, P^{(\mathbb{N})}) \to \mathrm{Hom}_R(\bigoplus_{i\in\mathbb{N}} F_i, P^{(\mathbb{N})}) \to \mathrm{Hom}_R(\bigoplus_{i\in\mathbb{N}} F_i, P^{(\mathbb{N})}) \to \mathrm{Ext}^1_{\mathfrak{B}}(M, P^{(\mathbb{N})})$. Since $\mathrm{Ext}^1_{\mathcal{X}}(M, P^{(\mathbb{N})}) = 0$, M is strict P-stationary. □

Let $\mathfrak{B} = (\mathcal{X}, \mathcal{Y})$ be an admissible Hom-balanced pair. We now recall the notion of the $\mathfrak{B}$-filtration in [21]. Let M be a module and $\mathcal{C}$ a class of modules. Let κ be an ordinal. If $\{M_\alpha\}_{\alpha \le \kappa}$ is a family of modules satisfies that $M_0 = 0$, $M_\alpha \subseteq M_{\alpha+1}$ for all $\alpha < \kappa$, and $M_\alpha = \bigcup_{\beta<\alpha} M_\alpha$ for any limit ordinal $\alpha \le \kappa$. Then $\{M_\alpha\}_{\alpha\le\kappa}$ is said to be a continuous chain of modules (see [17, Definition 6.1]). Furthermore, $\{M_\alpha\}_{\alpha\le\kappa}$ is said to be $\mathcal{B}$-proper if for any $\alpha < \kappa$, $0 \longrightarrow M_\alpha \longrightarrow M_{\alpha+1} \longrightarrow M_{\alpha+1}/M_\alpha \longrightarrow 0$ is $\mathfrak{B}$-exact. M is called $\mathfrak{B}$-properly $\mathcal{C}$-filtered, provided that $\{M_\alpha\}_{\alpha\le\kappa}$ is a $\mathcal{B}$-proper continuous chain submodules of M and $M_\kappa = M$, $M_{\alpha+1}/M_\alpha$ isomorphic to some element of $\mathcal{C}$. We call $\{M_\alpha\}_{\alpha\le\kappa}$ a $\mathfrak{B}$-properly $\mathcal{C}$-filtration of M (see [21, Section 3]).

The following result is a relative version of [3, Proposition 8.13].

Proposition 3.4 *Let $\mathfrak{B} = (\mathcal{X}, \mathcal{Y})$ be an admissible Hom-balanced pair and $\mathcal{S}$ a class of left R-module. If both $\mathcal{X}$ and $\mathcal{S}$ are strict $\mathcal{A}$-stationary and $\mathcal{A} \subseteq \mathcal{S}^{\perp_{\mathfrak{B}}}$. Then any module isomorphic to a direct summand of a $\mathfrak{B}$-properly $(\mathcal{S} \cup \mathcal{X})$-filtered module is strict $\mathcal{A}$-stationary.*

Proof Since the class of strict $\mathcal{A}$-stationary modules is closed under direct summands by Remark 3.2, it suffices to prove that $\mathfrak{B}$-properly $(\mathcal{S}\cup\mathcal{X})$-filtered modules are strict $\mathcal{A}$-stationary. Let M be a $\mathfrak{B}$-properly $(\mathcal{S}\cup\mathcal{X})$-filtered left R-module. Then there is an ordinal κ such that $\{M_\alpha\}_{\alpha\le\kappa}$ is a $\mathfrak{B}$-properly $(\mathcal{S}\cup\mathcal{X})$-filtration of M. We will prove that M is strict $\mathcal{A}$-stationary by induction on $\alpha \le \kappa$.

Step 1. $M_0 = 0$, so M_0 is strict $\mathcal{A}$-stationary.

Step 2. Assume that $\alpha < \kappa$ and M_α is strict $\mathcal{A}$-stationary. Consider $\mathfrak{B}$-exact sequence $0 \to M_\alpha \to M_{\alpha+1} \to M_{\alpha+1}/M_\alpha \to 0$. We notice that $\mathcal{A} \subseteq \mathcal{S}^{\perp_{\mathfrak{B}}} = (\mathcal{S}\cup\mathcal{X})^{\perp_{\mathfrak{B}}}$. For any divisible abelian group D and $A \in \mathcal{A}$, we have the following commutative diagram

$$\begin{array}{ccccc}
 & & \mathrm{Hom}_Z(A, D) \otimes_R M_\alpha & \longrightarrow & \mathrm{Hom}_Z(A, D) \otimes_R M_{\alpha+1} \\
 & & \downarrow{\scriptstyle \Phi_\alpha} & & \downarrow{\scriptstyle \Phi_{\alpha+1}} \\
0 & \longrightarrow & \mathrm{Hom}_Z(\mathrm{Hom}_R(M_\alpha, A), D) & \longrightarrow & \mathrm{Hom}_Z(\mathrm{Hom}_R(M_{\alpha+1}, A), D)
\end{array}$$

$$\begin{array}{cccc}
\longrightarrow & \mathrm{Hom}_Z(A, D) \otimes_R M_{\alpha+1}/M_\alpha & \longrightarrow & 0 \\
 & \downarrow{\scriptstyle \Phi} & & \\
\longrightarrow & \mathrm{Hom}_Z(\mathrm{Hom}_R(M_{\alpha+1}/M_\alpha, A), D) & \longrightarrow & 0.
\end{array}$$

Since both Φ_α and Φ are injective, we conclude that $\Phi_{\alpha+1}$ is injective by diagram chasing. So $M_{\alpha+1}$ is strict $\mathcal{A}$-stationary by Theorem 3.1.

Step 3. Assume that α is a limit ordinal and any $M_\beta(\beta < \alpha)$ is strict $\mathcal{A}$-stationary. Since M_α/M_β is a $\mathfrak{B}$-properly $\mathcal{S}\cup\mathcal{X}$-filtered module, $\mathcal{S}\cup\mathcal{X} \subseteq {}^{\perp_{\mathfrak{B}}}\mathcal{A}$ implies that $M_\alpha/M_\beta \in {}^{\perp_{\mathfrak{B}}}\mathcal{A}$ by [21, Relative Ekolf Lemma].

It is obvious that $0 \to M_\beta \to M_\alpha \to M_\alpha/M_\beta \to 0$ is $\mathfrak{B}$-exact for any $\beta < \alpha$. We consider the following commutative diagram

$$\begin{array}{ccccc}
 & & \mathrm{Hom}_Z(A, D) \otimes_R M_\beta & \xrightarrow{\epsilon_\beta} & \mathrm{Hom}_Z(A, D) \otimes_R M_\alpha \\
 & & \downarrow{\scriptstyle \Phi_\beta} & & \downarrow{\scriptstyle \Phi_\alpha} \\
0 & \longrightarrow & \mathrm{Hom}_Z(\mathrm{Hom}_R(M_\beta, A), D) & \longrightarrow & \mathrm{Hom}_Z(\mathrm{Hom}_R(M_\alpha, A), D).
\end{array}$$

Notice that for any β, $M_\beta \to M_\alpha$ is the canonical inclusion. For any $x \in \mathrm{Ker}\Phi_\alpha$, there exists $\beta < \alpha$ and $y \in \mathrm{Hom}_Z(A, D) \otimes_R M_\beta$ such that $\epsilon_\beta(y) = x$. Since Φ_β is injective, $\Phi_\alpha\epsilon_\beta$ is injective. This shows that $y = 0$. So $x = \epsilon_\beta(y) = 0$, Φ_α is injective. By Theorem 3.1, M_α is strict A-stationary.

This completes the proof of that $M(= M_\tau)$ is strict $\mathcal{A}$-stationary. □

Remark 3.5 ${}^{\perp_{\mathfrak{B}}}(\mathcal{S}^{\perp_{\mathfrak{B}}})$ consists of all direct summands of $\mathfrak{B}$-properly $\mathcal{S}\cup\mathcal{X}$-filtered modules. (see Relative Eklof–Trlifaj Completeness Theorem [21]). Proposition 3.4 shows that any module in ${}^{\perp_{\mathfrak{B}}}(\mathcal{S}^{\perp_{\mathfrak{B}}})$ is strict $\mathcal{A}$-stationary.

Now we discuss the condition for $(\mathcal{PGF}, \mathcal{GI})$ to be a Hom-balanced pair. It is clear that $(\mathcal{PGF}, \mathcal{GI})$ is admissible if it is Hom-balanced.

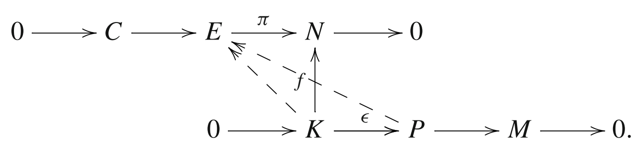
Theorem 3.6 *Let R be a ring. Then the following statements are equivalent.*

(1) $(\mathcal{PGF}, \mathcal{GI})$ is a Hom-balanced pair.
(2) $\mathcal{PGF}^{\perp} = {}^{\perp}\mathcal{GI}$.

Proof (1) $\Rightarrow$ (2) Let $M \in \mathcal{PGF}^{\perp}$. Since $(\mathcal{PGF}, \mathcal{PGF}^{\perp})$ is a complete hereditary cotorsion pair, $\mathcal{PGF}$-precover of M yields a $\mathcal{PGF}$-exact sequence $0 \to K \to G \to M \to 0$ with $K \in \mathcal{PGF}^{\perp}$ and $G \in \mathcal{PGF} \cap \mathcal{PGF}^{\perp} = \mathcal{P}$. Since $(\mathcal{PGF}, \mathcal{GI})$ is a Hom-balanced pair, $0 \to K \to G \to M \to 0$ is $\mathrm{Hom}_R(-, \mathcal{GI})$ exact by Lemma 2.1. So $M \in {}^{\perp}\mathcal{GI}$, $\mathcal{PGF}^{\perp} \subseteq {}^{\perp}\mathcal{GI}$.

Let $N \in {}^{\perp}\mathcal{GI}$. Consider $\mathcal{GI}$-preenvelope of N, there is an exact sequence $0 \to N \to E \to C \to 0$ of modules with $E \in {}^{\perp}\mathcal{GI} \cap \mathcal{GI} = \mathcal{I}$. Since $0 \to N \to E \to C \to 0$ is $\mathrm{Hom}_R(\mathcal{PGF}, -)$ exact, $N \in \mathcal{PGF}^{\perp}$. So $\mathcal{PGF}^{\perp} = {}^{\perp}\mathcal{GI}$.

(2) $\Rightarrow$ (1) Given a left R-module M, $\mathcal{PGF}$-precover of M yields an exact sequence $o \to K \xrightarrow{\epsilon} P \to M \to 0$ with $P \in \mathcal{PGF}$ and $K \in \mathcal{PGF}^{\perp}$. Let $N \in \mathcal{GI}$. There is an exact sequence $0 \to C \to E \xrightarrow{\pi} N \to 0$ with $C \in \mathcal{GI}$ and $E \in \mathcal{I}$. Consider the following diagram



Given any $f : K \to N$, since $\mathcal{PGF}^{\perp} = {}^{\perp}\mathcal{GI}$, there exists a map $g : K \to E$ such that $f = \pi g$. Since E is injective, there is a map $h : P \to E$ such that $g = h\epsilon$. Thus $\pi h : P \to N$ satisfies that $(\pi h)\epsilon = \pi(h\epsilon) = \pi g = f$. This shows that $o \to K \xrightarrow{\epsilon} P \to M \to 0$ is $\mathrm{Hom}_R(-, N)$ exact. Thus any left $\mathcal{PGF}$-resolution of M is $\mathrm{Hom}_R(-, \mathcal{GI})$ exact.

Dually, it is not difficult to verify that any right $\mathcal{GI}$-resolution is $\mathrm{Hom}_R(\mathcal{PGF}, -)$ exact. Therefore $(\mathcal{PGF}, \mathcal{GI})$ is Hom-balanced. □

It is easy to see that the conditions (1) and (2) in 3.6 are equivalent for both left and right R-modules. Recall that a ring is called n-FC [9], provided that it is two-sided coherent with self-FP-injective dimension at most n on both side. We denote by $\mathcal{F}_n$ the class of modules of flat dimension at most n.

Corollary 3.7 *Let R be an n-FC ring. Then*

(1) $(\mathcal{F}_n, \mathcal{GI})$ is a perfect and hereditary cotorsion pair.
(2) $(\mathcal{PGF}, \mathcal{F}_n)$ is a complete and hereditary cotorsion pair.
(3) $(\mathcal{PGF}, \mathcal{GI})$ is an admissible Hom-balanced pair.

Proof (1) By [26, Corollary 3.8].

(2) We see that $(\mathcal{F}_n, \mathcal{GI})$ is a perfect and hereditary cotorsion pair by (1). Let $M \in \mathcal{PGF}^{\perp}$. Special $\mathcal{PGF}$-precover of M yields an exact sequence $0 \to K_1 \to P_0 \to M \to 0$ where $P_0 \in \mathcal{PGF} \cap \mathcal{PGF}^{\perp} = \mathcal{P}$ and $K_1 \in \mathcal{PGF}^{\perp}$. Continuing to consider special $\mathcal{PGF}$-precover of K_1, and so on, we obtain a long exact sequence $0 \to K_n \to P_{n-1} \cdots \to P_1 \to P_0 \to M \to 0$ with each P_i projective and $K_n \in \mathcal{PGF}^{\perp}$. This yields an exact sequence $0 \to M^+ \to P_0^+ \to P_1^+ \to \cdots \to P_{n-1}^+ \to K_n^+ \to 0$. By [26, Corollary 3.8], K_n^+ is Gorenstein injective. Since R is coherent, K_n is Gorenstein flat. This shows that $K_n \in \mathcal{GF} \cap \mathcal{PGF}^{\perp}$. By [23, Theorem 4.11], K_n is flat. So M has flat dimension at most n. Thus $\mathcal{PGF}^{\perp} \subseteq \mathcal{F}_n$. The converse inclusion is clear. Thus $(\mathcal{PGF}^{\perp}, \mathcal{F}_n)$ is a complete hereditary cotorsion pair by [23, Theorem 4.9].

(3) This is a direct consequence of Theorem 3.6 combined with (1) and (2). □

We denote by $\mathcal{PGF}^{op}$ and $\mathcal{GI}^{op}$ the class of projectively coresolved Gorenstein flat right R-modules and Gorenstein injective right R-modules, respectively. From Corollary 3.7, it is easy to see that $(\mathcal{PGF}^{op}, \mathcal{GI}^{op})$ is also an admissible Hom-balanced pair over n-FC rings. Now we discuss the balance of $- \otimes_R -$ relative to the class of PGF-modules.

In the rest of this paper, we set $\mathfrak{B} = (\mathcal{PGF}, \mathcal{GI})$ and assume that $(\mathcal{PGF}, \mathcal{GI})$ and $(\mathcal{PGF}^{op}, \mathcal{GI}^{op})$ are admissible Hom-balanced pairs over the ground rings (for instance, n-FC rings). If there is no danger of confusion, we will also denote $(\mathcal{PGF}^{op}, \mathcal{GI}^{op})$ by $\mathfrak{B}$.

Proposition 3.8 $- \otimes_R -$ *is left balanced on Mod-R $\times$ R-Mod by $\mathcal{PGF}^{op} \times \mathcal{PGF}$.*

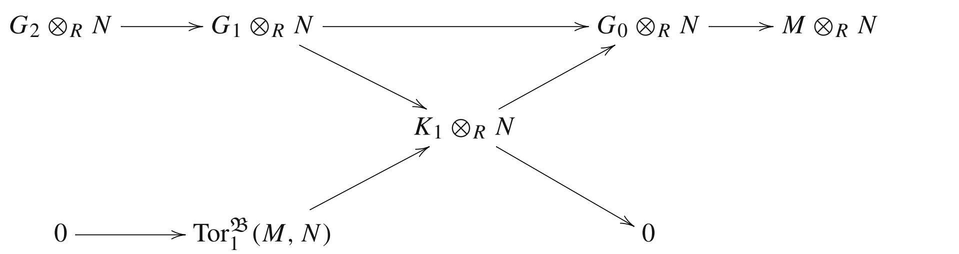
Proof Let M be a right R-module and N a left R-module. Then $(\mathbb{G}_M \otimes_R N)^+ \cong \mathrm{Hom}(\mathbb{G}_M, N^+)$, where $\mathbb{G}_M = \cdots \to G_1 \to G_0 \to M \to 0$ is a left $\mathcal{PGF}^{op}$-resolution. Since N^+ is Gorenstein injective, $\mathrm{Hom}(\mathbb{G}_M, N^+)$ is an exact complex. Thus $\mathbb{G}_M \otimes_R N$ is exact. Similarly, for any left $\mathcal{PGF}$-resolution $\mathbb{G}_N$ of N, $M \otimes_R \mathbb{G}_N$ is exact. □

Remark 3.9 Follows from the proof of 3.8, it is easy to see that $\mathrm{Tor}_n^{\mathfrak{B}}(M, N)^+ \cong \mathrm{Ext}_{\mathfrak{B}}^n(M, N^+) \cong \mathrm{Ext}_{\mathfrak{B}}^n(M^+, N)$ for any $n \geq 1$.

It is clear that $\mathrm{Tor}_n^R(M, N)$ can be computed using projective resolution or flat resolution of M or N. The analogous result for the functor $\mathrm{Tor}_n^{\mathfrak{B}}(M, N)$ as follows.

Proposition 3.10 *Let M be a right R-module and N a left R-module. Then for any left $\mathcal{GF}^{op}$-resolution $\mathbb{G}_M$ of M and left $\mathcal{GF}$-resolution $\mathbb{G}_N$ of N. $\mathrm{Tor}_n^{\mathfrak{B}}(M, N) \cong H_n(\mathbb{G}_M \otimes N) \cong H_n(M \otimes \mathbb{G}_N)$.*

Proof It suffices to prove that $\mathrm{Tor}_n^{\mathfrak{B}}(M, N) \cong H_n(\mathbb{G}_M \otimes N)$. Let $\mathbb{G}_M : \cdots \to G_2 \to G_1 \to G_0 \to M \to 0$ be a left $\mathcal{GF}^{op}$-resolution of M. It is clear that $\mathbb{G}_M$ is $\mathrm{Hom}_R(\mathcal{PGF}, -)$ exact. Take $K_1 = \mathrm{Ker}(G_0 \to M)$, Consider the following commutative diagram.



Given a Gorenstein flat left R-module G, it is clear that G^+ is Gorenstein injective. So $\mathrm{Tor}_i^{\mathfrak{B}}(G, N)^+ \cong \mathrm{Tor}_i^{\mathfrak{B}}(N, G^+) = 0$ by 3.9. Thus $\mathrm{Tor}_i^{\mathfrak{B}}(G, N) = 0$ for any $i \geq 1$. This yields an exact sequence $0 \to \mathrm{Tor}_1^{\mathfrak{B}}(M, N) \to K_1 \otimes_R N \to G_0 \otimes_R N \to M \otimes_R N \to 0$. So $\mathrm{Tor}_1^{\mathfrak{B}}(M, N) \cong \mathrm{Ker}(K_1 \otimes_R N \to G_0 \otimes_R N)$. In view of that $- \otimes_R N$ is right exact, $K_1 \otimes_R N \cong G_1 \otimes_R N / \mathrm{Im}(G_2 \otimes_R N \to G_1 \otimes_R N)$. Thus

$$\begin{aligned}\mathrm{Tor}_1^{\mathfrak{B}}(M, N) &\cong \mathrm{Ker}(K_1 \otimes_R N \to G_0 \otimes_R N)\\ &\cong \mathrm{Ker}\Big(\frac{G_1 \otimes_R N}{\mathrm{Im}(G_2 \otimes_R N \to G_1 \otimes_R N)} \to G_0 \otimes_R N\Big)\\ &\cong \frac{\mathrm{Ker}\big(G_1 \otimes_R N \to G_0 \otimes_R N\big)}{\mathrm{Im}(G_2 \otimes_R N \to G_1 \otimes_R N)} \cong H_1(\mathbb{G}_M \otimes N).\end{aligned}$$

We have that $\mathrm{Tor}_{i+1}^{\mathfrak{B}}(M, N) \cong \mathrm{Tor}_1^{\mathfrak{B}}(K_i, N) = H_1(\mathbb{G}_{K_i} \otimes N) \cong H_{i+1}(\mathbb{G}_M \otimes N)$ where $K_{i+1} = \mathrm{Ker}(G_i \to G_{i-1})$, $i \geq 0$, $G_{-1} = M$. As required. □

4 PGF-tilting modules

In this section, we write $\mathrm{PGext}_R^*(-, -)$ and $\mathrm{PGtor}_*^R(-, -)$, instead of $\mathrm{Ext}_{\mathfrak{B}}^*(-, -)$ and $\mathrm{Tor}_*^{\mathfrak{B}}(-, -)$, respectively.

Definition 4.1 Let R be a ring and n a nonnegative integer. An R-module T is projectively coresolved Gorenstein flat n-tilting, provided that

- (PGT1) $\mathcal{PGF}\text{-dim}(T) \leq n$.
- (PGT2) $\mathrm{PGext}_R^i(T, T^{(\kappa)}) = 0$ for all cardinals κ and all $i \geq 1$.
- (PGT3) For any $X \in \mathcal{PGF}$, there are $r \geq 0$ and a $\mathfrak{B}$-exact sequence $0 \to X \to T_0 \to T_1 \to \cdots \to T_r \to 0$ with each $T_i \in \mathrm{Add(T)}$.

Remark 4.2 (1) For convenience, we always write "PGF-n-tilting", instead of "projectively coresolved Gorenstein flat n-tilting".

(2) Let R be a ring such that $(\mathcal{GP}, \mathcal{GI})$ is an admissible Hom-balanced pair. Recall that an R-module T is Gorenstein tilting [25], provided that

(GT1) $\mathcal{GP}$-dim$(T) \leq n$.
(GT2) $\mathrm{Gext}^i_R(T, T^{(\kappa)}) = 0$ for all cardinals κ and all $i \geq 1$.
(GT3) For any $X \in \mathcal{GP}$, there are $r \geq 0$ and a $\mathfrak{B}$-exact sequence of modules $0 \to X \to T_0 \to T_1 \to \cdots \to T_r \to 0$ with each $T_i \in$ Add(T).

Let R be an n-Gorenstein ring. Then $\mathcal{GP} \subseteq \mathcal{GF}$ by [12, Corollary 10.3.10]. So $\mathcal{GP} \subseteq \mathcal{PGF}$. Clearly $\mathcal{PGF} \subseteq \mathcal{GP}$. This shows that PGF-tilting modules coincide with Gorenstein tilting modules as in [25]. In this case, $(\mathcal{PGF}, \mathcal{GI}) = (\mathcal{GP}, \mathcal{GI})$ is a Hom-balanced pair.

(3) Let T be a PGF-n-tilting module. $T^{\perp_{\mathfrak{B}\infty}}$ is said to be an PGF-n-tilting class. Consider the left $\mathcal{PGF}$-resolution $0 \to G_n \to \cdots \to G_1 \to G_0 \to T \to 0$ of T. Set $S_{i+1} = \mathrm{Ker}(G_i \to G_{i-1})$, $S_0 = G_{-1} = T$, $S_n = G_n$. It is clear that $T^{\perp_{\mathfrak{B}\infty}} = S^{\perp_{\mathfrak{B}}}$, where $S = \bigoplus_{i=0}^{n} S_i$. $({}^{\perp_{\mathfrak{B}}}(T^{\perp_{\mathfrak{B}\infty}}), T^{\perp_{\mathfrak{B}\infty}}) = ({}^{\perp_{\mathfrak{B}}}(S^{\perp_{\mathfrak{B}}}), S^{\perp_{\mathfrak{B}}})$ is said to be a PGF-n-tilting cotorsion pair induced by T. So PGF-n-tilting cotorsion pairs are $\mathfrak{B}$-complete.

We now introduce the definition of PGF-weak tilting modules as follows.

Definition 4.3 Let R be a ring and n a nonnegative integer. A left R-module W is PGF-weak n-tilting, provided that

(PGW1) $\mathcal{PGF}$-dim$(W) \leq n$.
(PGW2) $\mathrm{PGtor}^R_i((W^{(\kappa)})^+, W) = 0$ for all cardinals κ and all $i \geq 1$.
(PGW3) For any $E \in \mathcal{GI}$, there are $r \geq 0$ and a $\mathfrak{B}$-exact sequence $0 \to C_r \to \cdots \to C_1 \to C_0 \to E \to 0$ with each $C_i \in \mathrm{Prod}(W^+)$.

W is called a partial PGF-weak n-tilting module if $(PGW1)$ and $(PGW2)$ hold true. Recall that a left R-module W is called Gorenstein weak n-tilting [19, Definition 3.1], provided that

(GW1) $\mathcal{GF}$-$dim(W) \leq n$;
(GW2) $\mathrm{Gtor}^R_i((W^{(\kappa)})^+, W) = 0$ for all cardinals κ and all $i \geq 1$;
(GW3) For any $E \in \mathcal{GI}$, there are $r \geq 0$ and an exact sequence $0 \to C_r \to \cdots \to C_1 \to C_0 \to E \to 0$ which is $\mathrm{Hom}_R(\mathcal{GP}, -)$ exact, such that $C_i \in \mathrm{Prod}(W^+)$.

The definition of Gorenstein weak tilting modules is based on the assumption that R is an n-Gorenstein ring. In this case, $\mathcal{PGF} = \mathcal{GP}$ and $(\mathcal{PGF}, \mathcal{GI}) = (\mathcal{GP}, \mathcal{GI})$) is a Hom-balanced pair. Thus PGF-weak n-tilting modules are Gorenstein weak n-tilting modules. $\mathrm{Gtor}^R_i((W^{(\kappa)})^+, W)$ can be computed by using left $\mathcal{GP}$-resolution or left $\mathcal{GF}$-resolution of $(W^{(\kappa)})^+$ or W.

The rest of this section is devoted to discuss the relation between PGF-tilting modules and PGF-weak (Gorenstein weak) tilting modules.

Let W be a left R-module, we set $\mathcal{L}_W = {}^{\top_{\mathfrak{B}\infty}}W$ and $\mathcal{R}_W = ({}^{\top_{\mathfrak{B}\infty}}W)^{\perp_{\mathfrak{B}}}$. The following result is akin to [19, Theorem 2.2 (2)].

Lemma 4.4 *Let W be a left R-module. Then $(\mathcal{L}_W, \mathcal{R}_W)$ is a complete and hereditary $\mathfrak{B}$-cotorsion pair.*

Proof It is easily seen that ${}^{\top_{\mathfrak{B}\infty}}W = {}^{\perp_{\mathfrak{B}\infty}}(W^+)$. Consider a right $\mathcal{GI}$-resolution of W^+: $0 \to W^+ \to I^0 \to I^1 \to \cdots$. Put $S = \prod_{i \geq 0} C_i$ where $C_i = \mathrm{Im}(I^{i-1} \to I^i)$, $C_0 = W^+$. Then ${}^{\perp_{\mathfrak{B}\infty}}(W^+) = {}^{\perp_{\mathfrak{B}}}S$. So $(\mathcal{L}_W, \mathcal{R}_W) = ({}^{\perp_{\mathfrak{B}}}S, ({}^{\perp_{\mathfrak{B}}}S)^{\perp_{\mathfrak{B}}})$ is a $\mathfrak{B}$-cotorsion pair. It is clear that the $\mathfrak{B}$-cotorsion pair $(\mathcal{L}_W, \mathcal{R}_W)$ is hereditary. The completeness of $(\mathcal{L}_W, \mathcal{R}_W)$ follows from [7, Theorem 2.4] and [18, Theorem 3.11, Proposition 3.13]. □

Proposition 4.5 *Let W be a PGF-weak n-tilting module. Then $\mathcal{L}_W \cap \mathcal{R}_W = \mathrm{Prod}(W^+)$.*

Proof By $(PGW2)$, $\mathrm{Prod}(W^+) \subseteq \mathcal{L}_W \cap \mathcal{R}_W$. Let $M \in \mathcal{L}_W \cap \mathcal{R}_W$ and $M \to G$ a $\mathcal{GI}$-preenvelope of M. $\mathcal{R}_W$ is $\mathfrak{B}$-coresolving by Lemma 4.4, (PGW3) gives a $\mathfrak{B}$-exact sequence $0 \to K_0 \to C_0 \to G \to 0$ with $K_0 \in \mathcal{R}_W$ and $C_0 \in \mathrm{Prod}(W^+)$. Consider the pullback of $M \to G$ and $C_0 \to G$, we have the following commutative diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_0 & \longrightarrow & D & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_0 & \longrightarrow & C_0 & \longrightarrow & G & \longrightarrow & 0.
\end{array}
$$

It is easy to see that $0 \to K_0 \to D \to M \to 0$ is a $\mathfrak{B}$-exact sequence. Since $\mathrm{PGtor}_1^R(M, K_0) = 0$, $0 \to K_0 \to D \to M \to 0$ splits. M is a direct summand of D, there is a $\mathfrak{B}$-monomorphism $M \to C_0$. Thus, there exists an exact sequence $0 \to M \to P_0 \to C_0 \to 0$ such that $M \to P_0$ is a $\mathrm{Prod}(W^+)$-preenvelope (cf. [22, Corollary 3.5]).

We will show that $0 \to M \to P_0 \to C_0 \to 0$ is a $\mathfrak{B}$-exact sequence. For any $E \in \mathcal{GI}$, we consider the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & C & \xrightarrow{\pi} & E & \longrightarrow & 0 \\
 & & & & \nwarrow & & \uparrow & \nwarrow & \\
 & & 0 & \longrightarrow & & M & \xrightarrow{\epsilon} & P_0 & \longrightarrow C_0 \longrightarrow 0
\end{array}$$

where $0 \to K \to C \to E \to 0$ is an exact sequence with $C \in \mathrm{Prod}(W^+)$ and $K \in \mathcal{R}_W$. Notice that $0 \to K \to C \to E \to 0$ is $\mathrm{Hom}_R(M, -)$ exact. Thus for any $f : M \to E$, there exists $g : M \to C$ such that $f = \pi g$. Since g factors through ϵ, there exists $h : P_0 \to C$ such that $g = h\epsilon$. Put $\varphi = \pi h : P_0 \to E$, we have that $\varphi\epsilon = \pi h\epsilon = \pi g = f$. Thus $0 \to M \to P_0 \to C_0 \to 0$ is $\mathrm{Hom}_R(-, \mathcal{GI})$-exact, it is a $\mathfrak{B}$-exact sequence.

Applying $\mathrm{Hom}_R(-, W^+)$ to the $\mathfrak{B}$-exact sequence $0 \to M \to P_0 \to C_0 \to 0$. We get a long exact sequence $0 \to \mathrm{PGext}_R^1(C_0, W^+) \to \mathrm{PGext}_R^1(P_0, W^+) \to \mathrm{PGext}_R^1(M, W^+) \to \mathrm{PGext}_R^2(C_0, W^+) \to \cdots$. We notice that $\mathrm{PGext}_R^i(P_0, W^+) \cong \mathrm{PGtor}_R^i(P_0, W)^+ = 0$ and $\mathrm{PGext}_R^i(M, W^+) \cong \mathrm{PGtor}_R^i(M, W)^+ = 0$ for any $i \geq 1$, so $\mathrm{PGext}_R^i(C_0, W^+) \cong \mathrm{PGtor}_R^i(C_0, W)^+ = 0$ for any $i \geq 1$. This shows that $C_0 \in \mathcal{L}_W$, $0 \to M \to P_0 \to C_0 \to 0$ splits. Since M is a direct summand of P_0, $M \in \mathrm{Prod}(W^+)$. Thus $\mathcal{L}_W \cap \mathcal{R}_W \subseteq \mathrm{Prod}(W^+)$. □

Lemma 4.6 *Let M and N be left R-modules. If all PGF-modules are strict N-stationary. Then the following assertions are equivalent.*

(1) The map

$$\Phi_M^{(i)} : \mathrm{PGtor}_i^R(Hom_Z(N, D), M) \longrightarrow Hom_Z\big(\mathrm{PGext}_R^n(M, N), D\big)$$

is injective for any divisible abelian group D.

(2) The i-th PGF-syzygy module of M is strict N-stationary.

Proof Let $\cdots \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0$ be a left $\mathcal{PGF}$-resolution of M. Take $K_i = \mathrm{Ker}(G_{i-1} \to G_{i-2})$, $i \geq 1$, $G_{-1} = M$. For any divisible abelian group D, the $\mathcal{B}$-exact sequence $0 \to K_i \to G_{i-1} \to K_{i-1} \to 0$ yields the following commutative diagram with exact rows.

$$\begin{array}{ccccc}
0 & \longrightarrow & \mathrm{PGtor}_1^R(\mathrm{Hom}_Z(N, D), K_{i-1}) & \xrightarrow{\cong} & \mathrm{PGtor}_i^R(\mathrm{Hom}_Z(N, D), M) \\
 & & & & \downarrow{\scriptstyle \Phi_M^i} \\
0 & \longrightarrow & \mathrm{Hom}_Z\big(\mathrm{PGext}_R^1(K_{i-1}, N), D\big) & \xrightarrow{\cong} & \mathrm{Hom}_Z\big(\mathrm{PGext}_R^i(M, N), D\big)
\end{array}$$

$$\begin{array}{ccccc}
\longrightarrow & \mathrm{Hom}_Z(N, D) \otimes_R K_i & \longrightarrow & \mathrm{Hom}_Z(N, D) \otimes_R G_{i-1} \\
 & \downarrow{\scriptstyle \Phi_{K_i}} & & \downarrow{\scriptstyle \Phi_{G_{i-1}}} \\
\longrightarrow & \mathrm{Hom}_Z(\mathrm{Hom}_R(K_i, N), D) & \longrightarrow & \mathrm{Hom}_Z(\mathrm{Hom}_R(G_{i-1}, N), D)
\end{array}$$

We see that G_{i-1} is strict N-stationary, so $\Phi_{G_{i-1}}$ is injective. This shows that Φ_M^i is injective if and only if Φ_{K_i} is injective. The equivalence of (1) and (2) follows from Theorem 3.1. □

Theorem 4.7 *Let T be a PGF-n-tilting module. If all PGF-modules are strict T-stationary. Then the following statements are equivalent.*

(1) T is a PGF-weak n-tilting module.

(2) $\mathcal{L}_T \cap \mathcal{R}_T = \mathrm{Prod}(T^+)$ and every PGF-syzygy of T is strict T-stationary.

Proof (1) $\Rightarrow$ (2) By Proposition 4.5, $\mathcal{L}_T \cap \mathcal{R}_T = \mathrm{Prod}(T^+)$. $(PGW2)$ shows that $\mathrm{PGtor}_i^R(\mathrm{Hom}_Z(T^{(\kappa)}, Q/Z), T) = 0$ for any $i \geq 1$ and any cardinal κ, so for any divisible abelian group D, the following maps

$$\mathrm{PGtor}_i^R(\mathrm{Hom}_Z(T, D), T) \to \mathrm{Hom}_Z\big(\mathrm{PGext}_R^i(T, T), D\big), i \geq 1$$

is injective. Since every PGF-module is strict T-stationary, any PGF-syzygy of T is strict T-stationary by Lemma 4.6.

(2) $\Rightarrow$ (1) It suffices to prove that $(PGW2)$ and $(PGW3)$ are satisfied. We see that every PGF-syzygy of T is strict T-stationary, so these syzygies are strict $T^{(\kappa)}$-stationary for any cardinal κ by [3, Corollary 8.5]. Thus,

$$\mathrm{PGtor}_i^R(\mathrm{Hom}_Z(T^{(\kappa)}, D), T) \to \mathrm{Hom}_Z\big(\mathrm{PGext}_R^i(T, T^{(\kappa)}), D\big)$$

is injective for any divisible abelian group D and any $i \geq 1$. $(PGT2)$ implies that $\mathrm{PGtor}_i^R(\mathrm{Hom}_Z(T^{(\kappa)}, D), T) = 0$ for any $i \geq 1$. T satisfies $(PGW2)$.

Given any $E \in \mathcal{GI}$. Since $\mathcal{GI} \subseteq \mathcal{R}_T$, an iteration of $\mathfrak{B}$-special $\mathcal{L}_T$-precovers (of E etc.) yields a long $\mathfrak{B}$-exact sequence $\cdots \to E_n \to E_{n-1} \cdots \to E_1 \to E_0 \to E \to 0$ with each $E_i \in \mathrm{Prod}(T^+)$. Take $K_i = \mathrm{Ker}(E_{i-1} \to E_{i-2}), i \geq 1, (E_{-1} = E)$, clearly, for any $i \geq 1$, $K_i \in \mathcal{R}_T$, $E_i \to K_i$ is a $\mathfrak{B}$-special $\mathcal{L}_T$-precover of E_i. Since $\mathcal{PGF}\text{-}dim(T) \leq n$, $\mathrm{PGtor}_i^R(K_n, T) \cong \mathrm{PGtor}_{i+1}^R(K_{n-1}, T) \cong \cdots \cong \mathrm{PGtor}_{i+n}^R(E, T) = 0$ for any $i \geq 1$. This shows that $K_n \in \mathcal{L}_T$, hence $K_n \in \mathcal{L}_T \cap \mathcal{R}_T = \mathrm{Prod}(T^+)$. The condition $(PGW3)$ is satisfied. □

Proposition 4.8 *Let T be a PGF-n-tilting module and $T \in \mathcal{PGF}^\perp$. Then the following statements are equivalent.*

(1) T is a partial PGF-weak n-tilting module.
(2) Every PGF-syzygy of T is strict T-stationary.

Proof (2) $\Rightarrow$ (1) Clearly, T satisfies $(PGW1)$.

By our assumption that $(\mathcal{PGF}, \mathcal{GI})$ is an admissible Hom-balanced pair, we get that $\mathcal{PGF}^\perp = {}^\perp\mathcal{GI}$ by Theorem 3.6. Follows from [23, Lemma 5.1], ${}^\perp\mathcal{GI}$ is closed under direct limits, so $\mathcal{PGF}^\perp$ is closed under direct limits. By [24, Theorem 3.5], $(\mathcal{PGF}, \mathcal{PGF}^\perp)$ is a cotorsion pair of countable type and $\mathcal{PGF}^\perp$ is a definable class. Thus, there is a class $\mathcal{S}$ of countably presented PGF-module such that $\mathcal{S}^\perp = \mathcal{PGF}^\perp$. Since $T^{(\kappa)} \in \mathcal{PGF}^\perp$ for any cardinal κ, we conclude that each module in $\mathcal{S}$ is strict T-stationary by Proposition 3.3. So any PGF-module is strict T-stationary by Proposition 3.4 and Remark 3.5. By [3, Corollary 8.5], any PGF-module is strict $T^{(\kappa)}$-stationary.

We see that every PGF-syzygy of T is strict $T^{(\kappa)}$-stationary. It follows from Lemma 4.6 that for any divisible abelian group D,

$$\Phi_T^{(i)} : \mathrm{PGtor}_i^R(\mathrm{Hom}_Z(T^{(\kappa)}, D), T) \longrightarrow \mathrm{Hom}_Z\big(\mathrm{PGext}_R^i(T, T^{(\kappa)}), D\big)$$

is injective for any $i \geq 1$.

Since $\mathrm{PGext}_R^i(T, T^{(\kappa)}) = 0$ for any $i \geq 1$, $\mathrm{PGtor}_i^R(\mathrm{Hom}_Z(T^{(\kappa)}, D), T) = 0$. T satisfies $(PGW2)$. This proves that T is a partial PGF-weak n-tilting module.

(1) $\Rightarrow$ (2) is obvious. □

We denote by $\mathcal{PGF}^{\leq\omega}$ the class of all the modules admitting a left $\mathcal{PGF}$-resolution consisting of countably generated projective modules. Analogous to the notion of countable type of tilting modules, we say a PGF-tilting module T is of G-countable type if there is a set $\mathcal{S} \subseteq \mathcal{PGF}^{\leq\omega}$ such that $T^{\perp_{\mathfrak{B}\infty}} = \mathcal{S}^{\perp_{\mathfrak{B}}}$.

Theorem 4.9 *If all PGF-modules are pure projective. Then every PGF-n-tilting module of $\mathfrak{B}$-countable type is PGF-weak n-tilting.*

Proof Let T be a PGF-n-tilting module of $\mathfrak{B}$-countable type. Put $\mathfrak{C} = (\mathcal{A}, \mathcal{B}) = ({}^{\perp_{\mathfrak{B}}}(T^{\perp_{\mathfrak{B}}}), T^{\perp_{\mathfrak{B}}})$.

There exists $\mathcal{S} \subseteq \mathcal{PGF}^{\leq\omega}$ such that $T^{\perp_{\mathfrak{B}\infty}} = \mathcal{S}^{\perp_{\mathfrak{B}}}$. In view of $T^{(\omega)} \in \mathcal{B}$, we get that $\mathrm{PGext}_R^1(S, T^{(\omega)}) = 0$ for any $S \in \mathcal{S}$. Thus $\mathcal{S}$ is strict T-stationary by Proposition 3.3. By Remark 3.5, $\mathcal{A} = {}^{\perp_{\mathfrak{B}}}(\mathcal{S}^{\perp_{\mathfrak{B}}})$ consists of direct summands of $\mathfrak{B}$-properly $(\mathcal{S} \cup \mathcal{PGF})$-filtered modules, each module in $\mathcal{A}$ is strict T-stationary by Proposition 3.4. So any module in $\mathcal{A}$ is strict $T^{(\kappa)}$-stationary for any cardinal κ.

We notice that every PGF-module is strict $T^{(\kappa)}$-stationary. It is easy to see that any PGF-syzygy of T contained in $\mathcal{A}$, we get that for any divisible abelian group D, there are injective maps

$$\mathrm{PGtor}_i^R(\mathrm{Hom}_Z(T^{(\kappa)}, D), T) \to \mathrm{Hom}_Z\big(\mathrm{PGext}_R^i(T, T^{(\kappa)}), D\big), i \geq 1.$$

So $\mathrm{PGtor}_i^R(\mathrm{Hom}_Z(T^{(\kappa)}, D), T) = 0$ for any $i \geq 1$.

Let $M \in \mathcal{L}_T \cap \mathcal{R}_T$. Clearly, $M \in {}^{\top_{\mathfrak{B}\infty}}T$ implies that $M^+ \in T^{\perp_{\mathfrak{B}\infty}} = \mathcal{B}$. By [20, Theorem 5.2], there is a $\mathfrak{B}$-exact sequence of modules $0 \to K_0 \to T_0 \to M^+ \to 0$ with $T_0 \in \mathrm{Add}(T)$. For any PGF-module G, we obtain the following exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}_R(M^+, G^+) & \longrightarrow & \mathrm{Hom}_R(T_0, G^+) & \longrightarrow & \mathrm{Hom}_R(K_0, G^+) & \longrightarrow & 0 \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\
0 & \longrightarrow & (M^+ \otimes_R G)^+ & \longrightarrow & (T_0 \otimes_R G)^+ & \longrightarrow & (K_0 \otimes_R G)^+ & \longrightarrow & 0 \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\
0 & \longrightarrow & \mathrm{Hom}_R(G, M^{++}) & \longrightarrow & \mathrm{Hom}_R(G, T_0^+) & \longrightarrow & \mathrm{Hom}_R(G, K_0^+) & \longrightarrow & 0.
\end{array}$$

Thus $0 \to M^{++} \to T_0^+ \to K_0^+ \to 0$ is $\mathfrak{B}$-exact. Since all PGF-modules are pure projective, the pure embedding map $M \to M^{++}$ is an $\mathfrak{B}$-monomorphism. This yields a $\mathfrak{B}$-exact sequence $0 \to M \to T_0^+ \to T_0^+/M \to 0$ with $T_0^+ \in \mathrm{Prod}(T^+)$. By [22, Corollary 3.5], there is a $\mathrm{Prod}(T^+)$-preenvelope of M: $f : M \to N$. It is easy to see that f is monomorphic and $0 \to M \xrightarrow{f} N \to N/M \to 0$ is $\mathfrak{B}$-exact. We see that the following exact sequences are isomorphic.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}_R(M, T^+) & \longrightarrow & \mathrm{Hom}_R(N, T^+) & \longrightarrow & \mathrm{Hom}_R(N/M, T^+) & \longrightarrow & 0 \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\
0 & \longrightarrow & (M \otimes_R T)^+ & \longrightarrow & (N \otimes_R T)^+ & \longrightarrow & (N/M \otimes_R T)^+ & \longrightarrow & 0.
\end{array}$$

So $0 \to M \otimes_R T \to N \otimes_R T \to N/M \otimes_R T \to 0$ is exact. Since $N \in \mathrm{Prod}(T^+) \subseteq {}^{\top_{\mathfrak{B}\infty}}T$ and $M \in \mathcal{L}_T = {}^{\top_{\mathfrak{B}\infty}}T$, $N/M \in \mathcal{L}_T$. So $0 \to M \to N \to N/M \to 0$ splits. M is a direct summand of N, $M \in \mathrm{Prod}(T^+)$. This shows that $\mathcal{L}_T \cap \mathcal{R}_T \subseteq \mathrm{Prod}(T^+)$. Since the condition $(PGW2)$ is satisfied, the converse inclusion is clear. The desired result follows from Theorem 4.7. □

Let R be an n-FC ring. By [28, Corollary 3.3], all Gorenstein projective modules are Gorenstein flat. So $\mathcal{GP} = \mathcal{PGF}$. Follows Corollary 3.7, $\mathfrak{B} = (\mathcal{PGF}, \mathcal{GI}) = (\mathcal{GP}, \mathcal{GI})$ is admissible Hom-balanced. In this case, PGF-tilting modules coincide with Gorenstein tilting modules. The following result is an application of Theorem 4.9, which extends [19, Theorem 3.6] to a more general setting.

Corollary 4.10 *Let R be an n-FC ring. If all Gorenstein tilting modules are pure projective. Then every $\mathfrak{B}$-countable type Gorenstein n-tilting module is Gorenstein weak n-tilting.*

Proof Since $\mathcal{PGF} = \mathcal{GP}$ and PGF-weak tilting modules are Gorenstein weak tilting modules, the desired result follows from Theorem 4.9. □

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