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ORIGINAL RESEARCH

An inertial parallel iterative method for solving generalized mixed equilibrium problems and common fixed point problem in reflexive Banach spaces

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Abstract By combining the shrinking projection method with the parallel splitting-up technique and the inertial term, we introduce a new inertial parallel iterative method for finding common solutions of a finite system of generalized mixed equilibrium problems and common fixed points of a finite family of Bregman totally quasi-asymptotically nonexpansive mappings. After that, we prove a strong convergence result for the proposed iteration in reflexive Banach spaces. By this theorem, we obtain some convergence results for generalized mixed equilibrium problems in reflexive Banach spaces. In addition, we give a numerical example to illustrate the proposed iterations. The obtained results are improvements and extensions to some known results in this area.

Keywords Bregman totally quasi-asymptotically nonexpansive mapping · Parallel iterative method · Generalized mixed equilibrium problem · Reflexive Banach space

1 Introduction

The equilibrium problem (EP) was introduced by Muu and Oettli [1] in 1992. Later, some sufficient condition for the existence of a the solution for (EP) was studied by Blum and Oettli [2], Noor and Oettli [3]. The equilibrium problem consists of finding $u \in U$ such that

$$f(u, v) \geq 0, \forall v \in U,$$

where U is a nonempty, closed, convex subset of a Banach space W, and $f : U \times U \longrightarrow \mathbb{R}$ is a bifunctional mapping satisfying $f(u, u) = 0$ for all $u \in U$. The set

$$EP(f) = \{u \in U : f(u, v) \geq 0, \forall v \in U\}$$

denotes the set of solutions of (EP). The equilibrium problem had a great influence in the development of some branches of pure and applied sciences. The equilibrium problem theory provides a natural and novel approach for some problems arising in nonlinear analysis, physics and engineering, image reconstruction, economics, finance, game theory and optimization. In 2008, Peng and Yao [4] extended the equilibrium problem (EP) to the generalized mixed equilibrium problem $(GMEP)$. Assume that $\langle u^*, v\rangle$ is the value of the function of u^* at

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$v \in W$, $A : U \longrightarrow \mathbb{R}$ is a real valued function and $B : U \longrightarrow W^*$ is a nonlinear mapping. Then the generalized mixed equilibrium problem $(GMEP)$ is to find $u \in U$ such that

$$f(u, v) + \langle B(u), v - u\rangle + A(v) - A(u) \geq 0, \forall v \in U.$$

The symbol $GMEP(f, A, B) = \{u \in U : f(u, v) + \langle B(u), v - u\rangle + A(v) - A(u) \geq 0, \forall v \in U\}$ denotes the set of solutions of $(GMEP)$. In particular, if $B \equiv 0$, then $(GMEP)$ is reduced into the mixed equilibrium problem (MEP) which is to find $u \in U$ such that

$$f(u, v) + A(v) \geq A(u), \forall v \in U.$$

If $A \equiv 0$, then $(GMEP)$ is reduced into the generalized equilibrium problem (GEP) which is to find $u \in U$ such that

$$f(u, v) + \langle B(u), v - u\rangle \geq 0, \forall v \in U.$$

If $f \equiv 0$, then $(GMEP)$ is reduced into the mixed variational inequality (MVI) of Browder type which is to find $u \in U$ such that

$$\langle B(u), v - u\rangle + A(v) \geq A(u), \forall v \in U.$$

If $B \equiv 0$ and $A \equiv 0$, then $(GMEP)$ is reduced into the equilibrium problem (EP).

In recent times, the authors have studied many iterative methods for solving the equilibrium problem and its generalizations in the setting of Hilbert spaces and Banach spaces. Furthermore, some authors proposed certain iterative methods for finding common solutions of the equilibrium problem or its generalizations and fixed point problem for nonexpansive mappings or generalized nonexpansive mappings in Hilbert spaces and Banach spaces. In 2007, using the hybrid projection method, Tada and Takahashi [5] proposed the following hybrid iterative method for finding common elements of an equilibrium problem and fixed point problem for a nonexpansive mapping S in Hilbert space W.

$$\begin{cases} z_1 \in U \\ u_n \in U \text{ such that } f(u_n, v) + \dfrac{1}{r_n}\langle v - u_n, u_n - z_n\rangle \geq 0, \forall v \in U \\ v_n = b_n z_n + (1 - b_n)Su_n \\ C_n = \{u \in W : \|v_n - u\| \leq \|z_n - u\|\} \\ Q_n = \{u \in W : \langle z_n - u, z_n - z_1\rangle \leq 0\} \\ z_{n+1} = P_{C_n \cap Q_n}(z_1), \forall n \geq 1, \end{cases} \tag{1}$$

where $\{b_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$. In addition, under the suitable conditions, the authors proved that the sequence $\{z_n\}$ strongly converges to $p = P_{F(S)\cap EP(f)}(z_1)$, where $F(S)$ denotes the set of fixed points of the mapping S. In 2016, Alizadeh and Moradlou [6] generalized the main results in [5] by proposing the following hybrid iterative method for solving an equilibrium problem and fixed point problem for a generalized hybrid mapping S in Hilbert space W.

$$\begin{cases} z_1 \in U \\ y_n = a_n z_n + (1 - a_n)Sz_n \\ u_n \in U \text{ such that } f(u_n, v) + \dfrac{1}{r_n}\langle v - u_n, u_n - z_n\rangle \geq 0, \forall v \in U \\ v_n = b_n y_n + (1 - b_n)Su_n \\ C_n = \{u \in U : \|v_n - u\| \leq \|z_n - u\|\} \\ Q_n = \{u \in U : \langle z_n - u, z_n - z_1\rangle \leq 0\} \\ z_{n+1} = P_{C_n \cap Q_n}(z_1), \forall n \geq 1, \end{cases} \tag{2}$$

where $\{a_n\}, \{b_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$. Furthermore, the authors proved that the sequence $\{z_n\}$ strongly converges to $p = P_{F(S)\cap EP(f)}(z_1)$. Note that if $a_n = 1$ for all $n \in \mathbb{N}$, then the iteration (2) becomes the iteration (1).

An interesting work naturally raised is to extend and improve the convergence results of the iterative methods for equilibrium problems and fixed point problems from a Hilbert space to a Banach space. The fact that some characteristic properties and results in Hilbert spaces are not available in more general Banach spaces. To overcome these difficulties, the authors combined the normalized duality mapping, the Lyapunov functional and the generalized projection to construct some iterative methods for equilibrium problems and fixed point problems in smooth Banach spaces [7,8]. In another approach, some authors used the Bregman distance and the Bregman projection in reflexive Banach spaces instead of the norm and the metric projection in Hilbert spaces. By these ways, some authors introduced many iterative methods for finding common elements of the solutions set of the equilibrium problems and the fixed point set of mappings with respect to the Bregman distance in reflexive Banach spaces [9,10] and the references therein.

In 2014, Chang *et al.* [11] introduced the notion of a Bregman totally quasi-asymptotically nonexpansive mapping as a generalization of a Bregman strongly nonexpansive mapping. After that, some convergence results for the equilibrium problems and the fixed point problem for Bregman totally quasi-asymptotically nonexpansive mappings in reflexive Banach spaces were established [12,13].

In 2014, Anh and Chung [14] introduced a parallel splitting-up technique to construct two parallel hybrid methods for finding a common fixed point of a finite family of relatively nonexpansive mappings. By this idea, some authors proposed many parallel iterative methods for finite system of equilibrium problems and a finite family of generalized nonexpansive mappings [15,16]. In 2017, Tuyen [17] proposed some parallel iterative methods for solving a system of generalized mixed equilibrium problems.

$$\begin{cases} z_1 \in W, U_1 = W \\ u_n^{(k)} = \mathrm{Res}_{f_k, A_k, B_k}(z_n) \\ k_n = \mathrm{argmax}\{D_g(u_n^{(k)}, z_n) : k = 1, 2, \ldots, M\}, \overline{u}_n = u_n^{(k_n)} \\ U_{n+1} = \left\{u \in U_n : D_g(u, \overline{u}_n) \leq D_g(u, z_n)\right\} \\ u_{n+1} = P^g_{U_{n+1}}(z_1), \forall n \geq 1. \end{cases} \tag{3}$$

Recently, there were many methods for constructing new iteration processes which generalize some previous ones. In 2008, Mainge [18] proposed the inertial Mann iteration by combining the Mann iteration process and the inertial extrapolation. In 2018, Chidume *et al.* [19] introduced an inertial algorithm for approximating a common fixed point for a countable family of relatively nonexpansive mappings in uniformly convex and uniformly smooth Banach spaces.

Motivated by the mentioned works, we introduce a new inertial parallel iterative method for finding common solutions of a finite system of generalized mixed equilibrium problems and common fixed points of a finite family of Bregman totally quasi-asymptotically nonexpansive mappings. After that, we prove a strong convergence result for the proposed iteration in reflexive Banach spaces. By this theorem, we obtain some convergence results for generalized mixed equilibrium problems in reflexive Banach spaces. In addition, we give a numerical example to illustrate the obtained results.

2 Preliminaries

Assume that W is a real reflexive Banach space, U is a nonempty, closed and convex subset of W, W^* is the dual space of W. Throughout this paper, we suppose that $g : W \longrightarrow (-\infty, +\infty]$ is a proper, lower semi-continuous and convex function. The set $\mathrm{dom} g = \{u \in W : g(u) < +\infty\}$ denotes the domain of g. For any $u \in \mathrm{int}(\mathrm{dom} g)$ and $v \in W$, we denote by $g'(u, v)$ *the right-hand derivative of* g *at* u *in the direction* v, that is

$$g'(u, v) = \lim_{\lambda \downarrow 0} \frac{g(u + \lambda v) - g(u)}{\lambda}. \tag{4}$$

The function g is called *Gâteaux differentiable at* u if the limit (4) exists for any v. In this case, the gradient of g at u is the function $\nabla g(u)$, which is defined by $\langle \nabla g(u), v \rangle = g'(u, v)$ for all $v \in W$. The function g is called *Gâteaux differentiable* on $\mathrm{int}(\mathrm{dom} g)$ if it is Gâteaux differentiable at each $u \in \mathrm{int}(\mathrm{dom} g)$. The function g is called *Fréchet differentiable* at u if the limt (4) is attained uniformly in $\|v\| = 1$. The function g is called *uniformly Fréchet differentiable on a subset* U of W if the limit (4) is attained uniformly for $u \in U$ and $\|v\| = 1$.

Proposition 1 [20, Proposition 1] *Let W be a real reflexive Banach space, and $g : W \longrightarrow (-\infty, +\infty]$ be uniformly Fréchet differentiable and bounded on bounded subsets of W. Then ∇g is uniformly continuous on bounded subsets of W from the strong topology of W to the strong topology of W^*.*

Let $u \in \text{int}(\text{dom}g)$, the subdifferential g at $u \in W$ is defined by

$$\partial g(u) = \{u^* \in W^* : g(u) + \langle u^*, v - u\rangle \le g(v), \forall v \in W\},$$

and the Fenchel conjugate of g is the function $g^* : W^* \longrightarrow (-\infty, +\infty]$ defined by

$$g^*(u^*) = \sup\{\langle u^*, u\rangle - g(u) : u \in W\}, \forall u^* \in W^*.$$

Definition 1 [11, Definition 2.2] Suppose that W is a real reflexive Banach and $g : W \longrightarrow (-\infty, +\infty]$ is a function. Then g is called *Legendre* if the following two conditions are satisfied.

1. $\text{Int}(\text{dom}g) \neq \emptyset$, g is Gâteaux differentiable on $\text{int}(\text{dom}g)$ and $\text{dom}(\nabla g) = \text{int}(\text{dom}g)$.
2. $\text{Int}(\text{dom}g^*) \neq \emptyset$, g^* is Gâteaux differentiable on $\text{int}(\text{dom}g^*)$ and $\text{dom}(\nabla g^*) = \text{int}(\text{dom}g^*)$.

Remark 1 [21] Let W be a real reflexive Banach space and $g : W \longrightarrow (-\infty, +\infty]$ be a Legendre function. Then

1. g is a Legendre function if and only if g^* is a Legendre function.
2. $(\partial f)^{-1} = \partial g^*$.
3. $\nabla g = (\nabla g^*)^{-1}$, $\text{ran}(\nabla g) = \text{dom}(\nabla g^*) = \text{int}(\text{dom}g^*)$ and $\text{ran}(\nabla g^*) = \text{dom}(\nabla g) = \text{int}(\text{dom}g)$, where $\text{ran}(\nabla g)$ denotes the range of ∇g.
4. g and g^* are strictly convex on the interior of their respective domains.

Definition 2 [22, p. 234] Assmue that W is a real reflexive Banach space and $g : W \longrightarrow (-\infty, +\infty]$ is a Gâteaux differentiable function. Then the function $D_g : \text{dom}g \times \text{int}(\text{dom}g) \longrightarrow [0, +\infty)$, defined by $D_g(u, v) = g(u) - g(v) - \langle \nabla g(v), u - v\rangle$ is called the *Bregman distance* with respect to g.

Notice that the Bregman distance is not a distance in the usual sense of the term. In general, $D_g(u, u) = 0$, but $D_g(u, v) = 0$ may not imply $u = v$; D_g is not symmetric and does not satisfy the triangle inequality. By the definition of the Bregman distance, we have $D_g(u, v) + D_g(v, w) - D_g(u, w) = \langle \nabla g(w) - \nabla g(v), u - v\rangle$ for all $u \in \text{dom}g$ and $v, w \in \text{int}(\text{dom}g)$. Note that from [23, p.7], for all $u \in W$, we have

$$D_g\Big(u, \nabla g^*\big(\sum_{n=1}^{m} \lambda_n \nabla g(u_n)\big)\Big) \le \sum_{n=1}^{m} \lambda_n D_g(u, u_n), \tag{5}$$

where $\{u_n\}_{n=1}^m \subset W$ and $\{\lambda_n\}_{n=1}^m \subset [0, 1]$ with $\sum\limits_{n=1}^{m} \lambda_n = 1$.

Definition 3 [24, p. 69] Let W be a real reflexive Banach space, $g : W \longrightarrow (-\infty, +\infty]$ is a convex and Gâteaux differentiable function, and U be a nonempty, closed and convex subset of $\text{int}(\text{dom}g)$. The *Bregman projection* of $u \in \text{int}(\text{dom}g)$ onto U is the unique vector $P_U^g(u) \in U$ such that $D_g\big(P_U^g(u), u\big) = \inf\big\{D_g(v, u) : v \in U\big\}$.

Remark 2 [12, Remark 2.2] Let W be a smooth, strictly convex Banach space and $g(u) = \|u\|^2$ for all $u \in W$. Then $\nabla g(u) = 2Ju$ for all $u \in W$ and J is the normalized duality mapping which is defined by $J(u) = \{u^* \in W^* : \langle u, u^*\rangle = \|u\|^2 = \|u\|^*\}$ for all $u \in W$. Therefore, Bregman distance $D_g(u, v)$ is reduced into $\phi(u, v)$, where $\phi(u, v)$ is a Lyapunov function which is defined by $\phi(u, v) = \|u\|^2 - 2\langle u, Jv\rangle + \|v\|^2$. Thus, the Bregman projection $P_U^g(u)$ is reduced into the generalized projection $\Pi_U(u)$ in smooth Banach which is defined by $\phi\big(\Pi_U(u), u\big) = \min\big\{\phi(v, u) : v \in U\big\}$.

If W is a Hilbert space and $g(u) = \|u\|^2$ for all $u \in W$, then $U(u, v) = \|u - v\|^2$ for all $u, v \in W$, and J is the identity mapping. Therefore, the Bregman projection $P_U^g(u)$ is reduced into the metric projection from W onto U.

Definition 4 [25, p. 1] Let W be a real reflexive Banach space, $g : W \longrightarrow (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. Then

1. g is called *totally convex* at $u \in \text{int}(\text{dom}g)$ if any $t > 0$, we have

$$v_g(u, t) := \inf\big\{D_g(v, u) : v \in \text{dom}g, \|v - u\| = t\big\} > 0.$$

2. g is called *totally convex* if g is totally convex at every point $u \in \text{int}(\text{dom}g)$.

3. g is called *totally convex on bounded subsets of* W if any nonempty bounded subset B of W and $t > 0$, we have $v_g(B, t) := \inf\{v_g(u, t) : u \in B \cap \mathrm{dom}g\} > 0$.

Proposition 2 [24, Lemma 2.1.2] *Let* W *be a real reflexive Banach space,* $g : W \longrightarrow (-\infty, +\infty]$ *be a convex and Gâteaux differentiable function. Then* g *is totally convex on bounded subsets of* W *if and only if any sequence* $\{u_n\} \subset \mathrm{int}(\mathrm{dom}g)$ *and* $\{v_n\} \subset \mathrm{dom}g$ *such that* $\{u_n\}$ *is bounded and* $\lim_{n\to\infty} D_g(v_n, u_n) = 0$*, we have* $\lim_{n\to\infty} \|v_n - u_n\| = 0$.

Proposition 3 [26, Proposition 2.3] *Let* W *be a real Banach space,* $g : W \longrightarrow \mathbb{R}$ *be Legendre such that* ∇g^* *is bounded on bounded subsets of* $\mathrm{int}(\mathrm{dom}g^*)$*,* $u \in W$ *and* $\{u_n\} \subset W$ *satisfying* $\{D_g(u, u_n)\}$ *is bounded. Then the sequence* $\{u_n\}$ *is bounded.*

Proposition 4 [27, Corollary 4.4] *Let* W *be a real reflexive Banach space,* $g : W \longrightarrow (-\infty, +\infty]$ *be Gâteaux differentiable and totally convex on* $\mathrm{int}(\mathrm{dom}g)$*,* U *be a nonempty, closed and convex subset of* $\mathrm{int}(\mathrm{dom}g)$ *and* $u \in \mathrm{int}(\mathrm{dom}g)$*. Then the following statements are equivalent.*

1. $w = P_U^g(u)$.
2. w *is the unique vector such that* $\langle \nabla g(u) - \nabla g(w), w - v\rangle \geq 0$ *for all* $v \in U$.
3. w *is the unique vector such that* $D_g(v, w) + D_g(w, u) \leq D_g(v, u)$ *for all* $v \in U$.

Definition 5 [28] Let W be a Banach space and denote by $S_1 = \{u \in W : \|u\| < 1\}$ and $B_\varepsilon = \{u \in W : \|u\| \leq \varepsilon\}$ for some $\varepsilon > 0$. Then $g : W \longrightarrow \mathbb{R}$ is called *uniformly convex on bounded subsets of* W if $\rho_\varepsilon(t) > 0$ for all $t, \varepsilon > 0$, where the function $\rho_\varepsilon : [0, \infty) \longrightarrow [0, \infty)$ is defined by

$$\rho_\varepsilon(t) = \inf_{u,v\in B_\varepsilon, \|u-v\|=t, \eta\in(0,1)} \frac{\eta g(u) + (1-\eta)g(v) - g(\eta u + (1-\eta)v)}{\eta(1-\eta)}.$$

Note that the notion of an uniformly smooth on bounded subset for a mapping, we can find in [28]. Furthermore if g is uniformly convex, then the function ρ_ε is nondecreasing mapping. In addition, $\rho_\varepsilon(t) = 0$ if and only if $t = 0$ ([28, p. 203]).

Remark 3 [29, p. 6] The function g is totally convex on bounded subsets of W if and only if g is uniformly convex on bounded subsets of W.

Definition 6 [30, Definition 1.3.7] Let W be a Banach space and $g : W \longrightarrow (-\infty, +\infty]$ be a function. Then g is called *strongly coercive* if $\lim_{\|u\|\to+\infty} \dfrac{g(u)}{\|u\|} = +\infty$.

By using [29, Lemma 2.2], we get the following lemma. The proof of this lemma is easy and is omitted.

Lemma 1 *Let* W *be a real reflexive Banach space,* $g : W \longrightarrow \mathbb{R}$ *be a Legendre, strongly coercive function which is uniformly Fréchet differentiable and bounded on bounded subsets of* W*. Then*

$$D_g\Big(u, \nabla g^*\big(\sum_{n=1}^{m} a_n \nabla g(u_n)\big)\Big) \leq \sum_{n=1}^{m} a_n D_g(u, u_n) - a_i a_j \rho_\varepsilon^*\big(\|\nabla g(u_i) - \nabla g(u_j)\|\big),$$

where $i, j \in \{1, 2, \ldots, m\}$*,* $\nabla g(u_n) \in B_\varepsilon^* = \{u \in X^* : \|u\| \leq \varepsilon\}$ *and* $a_n \in [0, 1]$ *such that* $\sum_{n=1}^{m} a_n = 1$*, and the* ρ_ε^* *is defined as in Definition 5.*

Let us denote by $F(S) = \{u \in W : Su = u\}$ the set of fixed points of the mapping $S : W \longrightarrow W$.

Definition 7 Let W be a real reflexive Banach space, $g : W \longrightarrow \mathbb{R}$ be a Gâteaux differentiable function and $S : W \longrightarrow W$ be a mapping. Then

1. ([31], Definition 2) S is called a *Bregman quasi-nonexpansive mapping* if $F(S) \neq \emptyset$ and for all $u \in W$ and $p \in F(S)$, we have $D_g(p, Su) \leq D_g(p, u)$.
2. ([32], Definition 2.10) S is called a *Bregman quasi-asymptotically nonexpansive mapping* if $F(S) \neq \emptyset$ and there exists a real sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that $D_g(p, S^n u) \leq k_n D_g(p, u)$ for all $u \in W$ and $p \in F(S)$.

3. ([11], Definition 2.10) S is called a *Bregman totally quasi-asymptotically nonexpansive mapping* if $F(S) \neq \emptyset$ and there exist nonnegative real sequences $\{\eta_n\}, \{\mu_n\}$ with $\lim\limits_{n\to\infty} \eta_n = \lim\limits_{n\to\infty} \mu_n = 0$ and a strictly increasing continuous function $\xi : [0, \infty) \longrightarrow [0, \infty)$ with $\xi(0) = 0$ such that
$$D_g(u, S^n x) \leq D_g(u, x) + \eta_n \xi(D_g(u, x)) + \mu_n$$
for all $u \in W$ and $p \in F(S)$.
4. ([31], Definition 2) S is called a *Bregman firmly nonexpansive mapping* if for all $u, v \in W$, we have $\langle \nabla g(Su) - \nabla g(Sv), Su - Sv\rangle \leq \langle \nabla g(u) - \nabla g(v), Su - Sv\rangle$.
5. S is called *closed* if any sequence $\{u_n\}$ in W such that $\lim\limits_{n\to\infty} u_n = u \in W$ and $\lim\limits_{n\to\infty} Su_n = v \in W$, we have $Su = v$.
6. ([33], p.3877) S is called *uniformly asymptotically regular* on W if for any bounded subset U of W, we have $\lim\limits_{n\to\infty} \sup\limits_{u\in U} \|S^{n+1}u - S^n u\| = 0$.

Remark 4 1. Every Bregman quasi-asymptotically nonexpansive mapping is a Bregman totally quasi-asymptotically nonexpansive mapping with $\xi(t) = t$ for all $t \geq 0$, $\eta_n = k_n - 1$ with $k_n \geq 1$ satisfying $\lim\limits_{n\to\infty} k_n = 1$, and $\mu_n = 0$, but the converse is not true.
2. Every Bregman firmly nonexpansive mapping is a Bregman quasi-nonexpansive mapping.

Lemma 2 [11, Lemma 2.16] *Suppose that W is a real reflexive Banach space, $g : W \longrightarrow (-\infty, +\infty]$ is a Legendre function which is totally convex on bounded subsets of W, and U is a nonempty, closed and convex subset of* int(dom$g$). *Let $S : U \longrightarrow U$ be a closed and Bregman totally quasi-asymptotically nonexpansive mapping. Then $F(S)$ is a closed and convex subset of U.*

For solving the problem $(GMEP)$, let us assume that f, A, B satisfy the following conditions.

(C1) $f(u, u) = 0$ for all $u \in U$.
(C2) f is monotone, that is, $f(u, v) + f(v, u) \leq 0$ for all $u, v \in U$.
(C3) For all $u, v, w \in U$, we have $\limsup\limits_{t\downarrow 0} f(tw + (1-t)u, v) \leq f(u, v)$.
(C4) For each $u \in U$, $v \longmapsto f(u, v)$ is convex and lower semi-continuous.
(C5) $A : U \longrightarrow \mathbb{R}$ is a lower semi-continuous and convex function.
(C6) $B : U \longrightarrow W^*$ is a continuous monotone mapping.

In order to find the solution of the problem $(GMEP)$, Darvish [9] introduced the notion of mixed resolvent of f. Later, this notion was studied in [17].

Definition 8 [9, Definition 2.4] Let W be a real reflexive Banach space, U be a nonempty, closed and convex subset of W, $g : W \longrightarrow (-\infty, +\infty]$ be a Gâteaux differentiable function. Assume that $f : U \times U \longrightarrow \mathbb{R}$, $A : U \longrightarrow \mathbb{R}$ and $B : U \longrightarrow W^*$ satisfy the conditions (C_1) - (C_6). *The mixed resolvent of f* is the operator $\mathrm{Res}^g_{f,A,B} : W \longrightarrow 2^U$ defined by
$$\mathrm{Res}^g_{f,A,B}(u) = \Big\{w \in U : f(w, v) + A(v) + \langle B(u), v - w\rangle + \langle \nabla g(w) - \nabla g(u), v - w\rangle \geq A(w) \text{ for all } v \in U\Big\}.$$

After that by using a similar idea of [10, Lemma 1], the author of [9] proved that if $g : W \longrightarrow (-\infty, +\infty]$ is strongly coercive and Gâteaux differentiable, then $\mathrm{dom}\big(\mathrm{Res}^g_{f,A,B}\big) = W$. We find that the formula of the function $\mathrm{Res}^g_{f,A,B}$ contains the term $B(u)$ for all $u \in W$. Since $\mathrm{dom}B = U \subset W$, the value $B(u)$ does not exist for all $u \in W \setminus U$. Motivated by this confusion, we revise the formula of the function $\mathrm{Res}^g_{f,A,B}$ by replacing the term $B(u)$, $u \in W$ by $B(w)$, $w \in U$. Then the formula of the mixed resolvent $\mathrm{Res}^g_{f,A,B}$ becomes the following formula.
$$\mathrm{Res}^g_{f,A,B}(u) = \Big\{w \in U : f(w, v) + A(v) + \langle B(w), v - w\rangle + \langle \nabla g(w) - \nabla g(u), v - w\rangle \geq A(w) \text{ for all } v \in U\Big\}. \tag{6}$$

Note that the idea of the formula (6) was pointed out in [12, Lemm 2.5].Next,byusingtheideaof[10, Lemma1], we will prove that $\mathrm{dom}\big(\mathrm{Res}^g_{f,A,B}\big) = W$ under some suitable conditions, where the function $\mathrm{Res}^g_{f,A,B}$ is defined by (6). The proof of following lemma is easy by using [2, Theorem 1 & p.130-131] and is omitted.

Lemma 3 *Let W be a real reflexive Banach space, U be a nonempty, closed and convex subset of W, $g : W \longrightarrow (-\infty, +\infty]$ be a strongly coercive and Gâteaux differentiable function. Assume that $f : U \times U \longrightarrow \mathbb{R}$, $A : U \longrightarrow \mathbb{R}$ and $B : U \longrightarrow W^*$ satisfy the conditions (C_1) - (C_6). Then $\operatorname{dom}\big(\operatorname{Res}^g_{f,A,B}\big) = W$.*

The following lemma presents some properties of the mixed resolvent $\operatorname{Res}^g_{f,A,B}$ which is defined by (6). The proof of this lemma is similar to the proof [9, Lemma 2.8]. Furthermore, these results have been studied in [12, Lemm 2.5].

Lemma 4 [9, Lemma 2.8] *Let W be a real reflexive Banach space, U be a nonempty, closed and convex subset of W, $g : W \longrightarrow (-\infty, +\infty]$ be a Legendre function. Assume that $f : U \times U \longrightarrow \mathbb{R}$, $A : U \longrightarrow \mathbb{R}$ and $B : U \longrightarrow W^*$ satisfy the conditions (C_1) - (C_6). Then*

1. $\operatorname{Res}^g_{f,A,B}$ *is a single-valued.*
2. $\operatorname{Res}^g_{f,A,B}$ *is a Bregman firmly nonexpansive mapping.*
3. $F\big(\operatorname{Res}^g_{f,A,B}\big) = GMEP(f, A, B)$ *with* $F\big(\operatorname{Res}^g_{f,A,B}\big) = \{u \in U : \operatorname{Res}^g_{f,A,B}(u) = u\}$.
4. $GMEP(f, A, B)$ *is a closed and convex subset of W.*
5. *For all $p \in F\big(\operatorname{Res}^g_{f,A,B}\big)$ and $u \in W$, we have*

$$D_g\big(p, \operatorname{Res}^g_{f,A,B}(u)\big) + D_g\big(\operatorname{Res}^g_{f,A,B}(u), u\big) \le D_g(p, u).$$

3 Main results

Let $S_i : W \longrightarrow W$ be Bregman totally quasi-asymptotically nonexpansive mappings with nonnegative real sequences $\{\eta_n^{(i)}\}$ and $\{\mu_n^{(i)}\}$ satisfying $\lim\limits_{n\to\infty} \eta_n^{(i)} = \lim\limits_{n\to\infty} \mu_n^{(i)} = 0$ and strictly increasing continuous functions $\xi^{(i)} : [0, \infty) \longrightarrow [0, \infty)$ with $\xi^{(i)}(0) = 0$ for each $i \in I := \{1, 2, ..., N\}$ with $N \in \mathbb{N}$. Put

$$\eta_n = \max\{\eta_n^{(i)} : i \in I\}, \mu_n = \max\{\mu_n^{(i)} : i \in I\}, \text{and } \xi(t) = \max\{\xi^{(i)}(t) : i \in I\}$$

for all $t \ge 0$. Then $\lim\limits_{n\to\infty} \eta_n = \lim\limits_{n\to\infty} \mu_n = 0$, $\xi(0) = 0$, and we have

$$D_g(p, S_i^n u) \le D_g(p, u) + \eta_n \xi(D_g(p, u)) + \mu_n$$

for all $u \in W$, $p \in \bigcap\limits_{i\in I} F(S_i)$ and for all $i \in I$.

Theorem 5 *Suppose that W is a real reflexive Banach space, and U is a nonempty, closed and convex subset of W. Let $g : W \longrightarrow \mathbb{R}$ be Legendre, strongly coercive on W, and g be bounded, totally convex, uniformly Fréchet differentiable on bounded subsets of W. For each $k \in K := \{1, 2, \ldots, M\}$ with $M \in \mathbb{N}$, $f_k : U \times U \longrightarrow \mathbb{R}$, $A_k : U \longrightarrow \mathbb{R}$ and $B_k : U \longrightarrow W^*$ satisfy the conditions (C_1) - (C_6). For each $i \in I$, $S_i : W \longrightarrow W$ is a closed, uniformly asymptotically regular and Bregman totally quasi-asymptotically nonexpansive mapping with nonnegative real sequences $\{\eta_n^{(i)}\}$ and $\{\mu_n^{(i)}\}$ satisfying $\lim\limits_{n\to\infty} \eta_n^{(i)} = \lim\limits_{n\to\infty} \mu_n^{(i)} = 0$ and strictly increasing continuous function $\xi^{(i)} : [0, \infty) \longrightarrow [0, \infty)$ with $\xi^{(i)}(0) = 0$ such that $\mathcal{F} = \Big(\bigcap\limits_{i\in I} F(S_i)\Big) \bigcap \Big(\bigcap\limits_{k\in K} GMEP(f_k, A_k, B_k)\Big)$ is nonempty and bounded. Let $\{z_n\}$ be a sequence generated by*

$$\begin{cases} z_1, z_2 \in U, U_1 = U_2 = U \\ x_n = z_n + c_n(z_n - z_{n-1}) \text{ for all } n \ge 2 \\ y_n^{(i)} = \nabla g^*\Big(a_n \nabla g(x_n) + (1 - a_n)\nabla g(S_i^n(x_n))\Big) \text{ for all } i \in I \\ i_n = \operatorname{argmax}\{D_g(x_n, y_n^{(i)}) : i \in I\}, \ \overline{y}_n = y_n^{(i_n)} \\ u_n^{(k)} = \operatorname{Res}^g_{f_k, A_k, B_k}(x_n) \text{ for all } k \in K \\ k_n = \operatorname{argmax}\{D_g(x_n, u_n^{(k)}) : k \in K\}, \ \overline{u}_n = u_n^{(k_n)} \\ {}_n^{(j)} = \nabla g^*\Big(b_n \nabla g(\overline{y}_n) + (1 - b_n)\nabla g(S_j^n(\overline{u}_n))\Big) \text{ for all } j \in I \\ j_n = \operatorname{argmax}\{D_g(x_n, v_n^{(j)}) : j \in I\}, \ \overline{v}_n = v_n^{(j_n)} \\ U_{n+1} = \big\{u \in U_n : D_g(u, \overline{v}_n) \le D_g(u, x_n) + \gamma_n\big\} \\ z_{n+1} = P^g_{U_{n+1}}(z_1), \end{cases} \tag{7}$$

where $\gamma_n = \eta_n \sup\{\xi(D_g(u, x_n)) : u \in \mathcal{F}\} + \mu_n$, *and* $\{a_n\}, \{b_n\}, \{c_n\} \subset [0, 1]$ *such that* $\lim_{n\to\infty} a_n = 1$ *and* $\liminf_{n\to\infty} b_n(1 - b_n) > 0$, *and the function* $\mathrm{Res}^g_{f_k, A_k, B_k}$ *is defined as in* (6). *Then the sequence* $\{z_n\}$ *strongly converges to* $p = P^g_{\mathcal{F}}(z_1)$.

Proof The proof of Theorem 5 is divided into following six steps.

Step 1. *We claim that* $P^g_{\mathcal{F}}(z_1)$ *is well-defined.* Indeed, we conclude from Lemma 2 and Lemma 4 that $F(S_i)$ and $GMEP(f_k, A_k, B_k)$ are closed and convex sets for all $i \in I$ and $k \in K$. This proves that

$$\mathcal{F} = \Big(\bigcap_{i\in I} F(S_i)\Big) \bigcap \Big(\bigcap_{k\in K} GMEP(f_k, A_k, B_k)\Big)$$

is a closed and convex subset of U. Since $\mathcal{F}$ is a nonempty set, we find that $\mathcal{F}$ is a nonempty, closed and convex subset of U. This fact ensures that $P^g_{\mathcal{F}}(z_1)$ is well-defined.

Step 2. *We claim that* $P^g_{U_{n+1}}(z_1)$ *is well-defined.* Indeed, we first show that U_n is closed and convex for all $n \geq 2$ by mathematical induction. Obviously, we have $U_2 = U$ is closed and convex. Now, we assume that U_m is closed and convex for some $m \geq 2$. It follows from the definition of U_{m+1}, we get that

$$\begin{aligned} U_{m+1} = \big\{u \in U_m : \langle \nabla g(x_m), u - x_m\rangle - \langle \nabla g(\overline{v}_m), u - \overline{v}_m\rangle \\ \leq g(\overline{v}_m) - g(x_m) + \gamma_m\big\}. \end{aligned} \tag{8}$$

Then by directly checking, we conclude that U_{m+1} is convex. Furthermore, it follows from (8) and the continuity of $\nabla g(.)$ that U_{m+1} is closed. Therefore, U_{m+1} is closed and convex, and hence U_n is closed and convex for all $n \geq 2$. Combining this with $U_1 = U_2$ is closed and convex, we get that U_n is closed and convex for all $n \in \mathbb{N}$.

Next, we claim that $\mathcal{F} \subset U_n$ for all $n \geq 2$ by mathematical induction. Obviously, we obtain $\mathcal{F} \subset U = U_2$. Suppose that $\mathcal{F} \subset U_m$ for some $m \geq 2$. Now, we prove that $\mathcal{F} \subset U_{m+1}$. Assume that $u \in \mathcal{F}$. It follows from $\mathcal{F} \subset U_m$ that $u \in U_m$. By using (5) and the fact that S_{i_m} is a Bregman totally quasi-asymptotically nonexpansive mapping, we get

$$\begin{aligned} D_g(u, \overline{y}_m) &= D_g(u, y_m^{(i_m)}) \\ &= D_g\Big(u, \nabla g^*\big(a_m \nabla g(x_m) + (1 - a_m)\nabla g(S^m_{i_m} x_m)\big)\Big) \\ &\leq a_m D_g(u, x_m) + (1 - a_m) D_g(u, S^m_{i_m} x_m) \\ &\leq a_m D_g(u, x_m) + (1 - a_m)[D_g(u, x_m) + \eta_n \xi(D_g(u, x_m)) + \mu_m] \\ &\leq D_g(u, x_m) + \gamma_m. \end{aligned} \tag{9}$$

From Lemma 4, we find that $\mathrm{Res}^g_{f_{k_m}, A_{k_m}, B_{k_m}}$ is a Bregman firmly nonexpansive mapping and hence it is a Bregman quasi-nonexpansive mapping for each $k_m \in K$. Then, by Remark 4(2), we conclude that $\mathrm{Res}^g_{f_{k_m}, A_{k_m}, B_{k_m}}$ is a Bregman quasi nonexpansive mapping. Therefore, we have

$$D_g(u, \overline{u}_m) = D_g(u, u_m^{(k_m)}) = D_g\big(u, \mathrm{Res}^g_{f_{k_m}, A_{k_m}, B_{k_m}}(x_m)\big) \leq D_g(u, x_m). \tag{10}$$

It follows from the strictly increasing property of ξ and (10) that

$$\xi\big(D_g(u, \overline{u}_m)\big) \leq \xi\big(D_g(u, x_m)\big). \tag{11}$$

By combining (9), (10) and (11), we find that

$$\begin{aligned} D_g(u, \overline{v}_m) &= D_g(u, v_m^{(j_m)}) \\ &= D_g\Big(u, \nabla g^*\big(b_m \nabla g(\overline{y}_m) + (1 - b_m)\nabla g(S^m_{j_m}(\overline{u}_m))\big)\Big) \\ &\leq b_m D_g(u, \overline{y}_m) + (1 - b_m) D_g(u, S^m_{j_m}(\overline{u}_m)) \\ &\leq b_m D_g(u, \overline{y}_m) + (1 - b_m)[D_g(u, \overline{u}_m) + \eta_m \xi(D_g(u, \overline{u}_m)) + \mu_m] \\ &\leq b_m[D_g(u, x_m) + \gamma_m] + (1 - b_m)[D_g(u, x_m) + \eta_m \xi(D_g(u, x_m)) + \mu_m] \\ &\leq D_g(u, x_m) + \gamma_m. \end{aligned} \tag{12}$$

This leads to $u \in U_{m+1}$ and hence $\mathcal{F} \subset U_{m+1}$. This implies that $\mathcal{F} \subset U_n$ for all $n \geq 2$. It follows from $U_1 = U_2$ that $\mathcal{F} \subset U_n$ for all $n \in \mathbb{N}$. Since $\mathcal{F}$ is nonempty, we obtain U_n is nonempty.

By the above, we obtain that U_n is nonempty, closed and convex. Therefore, we conclude that $P^g_{U_{n+1}}(z_1)$ is well-defined.

Step 3. *We claim that* $\{z_n\}$ *is bounded and the limit* $\lim_{n\to\infty} D_g(z_n, z_1)$ *exists.* Indeed, we conclude from $z_n = P^g_{U_n}(z_1)$ and Proposition 4 that

$$D_g(y, z_n) + D_g(z_n, z_1) \leq D_g(y, z_1) \tag{13}$$

for all $y \in U_n$. Let $u \in \mathcal{F}$. It follows from $\mathcal{F} \subset U_n$ that $u \in U_n$. By choosing $y = u$ in (13), we get

$$D_g(u, z_n) + D_g(z_n, z_1) \leq D_g(u, z_1). \tag{14}$$

This proves that $D_g(z_n, z_1) \leq D_g(u, z_1) - D_g(u, z_n) \leq D_g(u, z_1)$. It means that $\{D_g(u_n, u_1)\}$ is bounded. By [34, Lemma 1], we conclude that $\{u_n\}$ is bounded.

Next, from the definition of U_n, we get $z_{n+1} = P^g_{U_{n+1}}(z_1) \in U_{n+1} \subset U_n$. By choosing $y = z_{n+1}$ in (13), we obtain $D_g(z_{n+1}, z_n) + D_g(z_n, z_1) \leq D_g(z_{n+1}, z_1)$. This leads to $D_g(z_n, z_1) \leq D_g(z_{n+1}, z_1) - D_g(z_{n+1}, z_n) \leq D_g(z_{n+1}, z_1)$. Therefore, $\{D_g(z_n, z_1)\}$ is a nondecreasing sequence. By using the boundedness of the sequence $\{D_g(z_n, z_1)\}$, we find that the limit $\lim_{n\to\infty} D_g(z_n, z_1)$ exists.

Step 4. *We claim that* $\lim_{n\to\infty} z_n = p \in U$. Indeed, for all $m > n$, it follows from the definition of z_m that $z_m = P^g_{U_m}(z_1) \in U_m \subset U_n$. Therefore, by taking $y = z_m$ in (13), we obtain $D_g(z_m, z_n) + D_g(z_n, z_1) \leq D_g(z_m, z_1)$. This leads to

$$0 \leq D_g(z_m, z_n) \leq D_g(z_m, z_1) - D_g(z_n, z_1). \tag{15}$$

Taking the limit (15) as $m, n \to \infty$, and using the existence of the limit $\lim_{n\to\infty} D_g(z_n, z_1)$, we have

$$\lim_{m,n\to\infty} D_g(z_m, z_n) = 0. \tag{16}$$

Then, it follows from (16), the boundedness of $\{z_n\}$ and Proposition 2 that

$$\lim_{m,n\to\infty} \|z_m - z_n\| = 0. \tag{17}$$

This proves that $\{z_n\}$ is a Cauchy sequence in U. Since W is a Banach space and U is a closed subset of W, there exists $p \in U$ such that $\lim_{n\to\infty} z_n = p$.

Step 5. *We claim that* $p \in \mathcal{F}$. First, we will prove that $p \in \bigcap_{i\in I} F(S_i)$. Indeed, by choosing $m = n + 1$ in (16) and (17), we get

$$\lim_{n\to\infty} D_g(z_{n+1}, z_n) = \lim_{n\to\infty} \|z_{n+1} - z_n\| = 0. \tag{18}$$

It follows from $z_{n+1} = P^g_{U_{n+1}}(z_1) \in U_{n+1} \subset U_n$ and the definition of U_n that

$$D_g(z_{n+1}, \overline{v}_n) \leq D_g(z_{n+1}, x_n) + \gamma_n. \tag{19}$$

Furthermore, we have $\|x_n - z_n\| = c_n\|z_n - z_{n-1}\|$. By combining this with (18) and the boundedness of $\{c_n\}$, we find that $\lim_{n\to\infty} \|x_n - z_n\| = 0$. Since $\lim_{n\to\infty} z_n = p$, we get that $\lim_{n\to\infty} x_n = p$. Therefore, from (18) and $\lim_{n\to\infty} \|x_n - z_n\| = 0$, we get $\lim_{n\to\infty} \|z_{n+1} - x_n\| = 0$. By using the definition of D_g, we find that

$$\begin{aligned} |D_g(z_{n+1}, x_n)| &= |g(z_{n+1}) - g(x_n) - \langle \nabla g(x_n), z_{n+1} - x_n\rangle| \\ &\leq |g(z_{n+1}) - g(x_n)| + \|z_{n+1} - x_n\|.\|\nabla g(x_n)\|. \end{aligned} \tag{20}$$

Furthermore, by [24, Proposition 1.1.10 & Proposition 1.1.11], we find that ∇g is bounded on bounded subsets of W. Then, by combining this with the boundedness of $\{x_n\}$, $\lim_{n\to\infty} \|z_{n+1} - x_n\| = 0$ and (20), we obtain

$$\lim_{n\to\infty} D_g(z_{n+1}, x_n) = 0. \tag{21}$$

Let $u \in \mathcal{F}$. By using the definition of D_g, we get

$$\begin{aligned}|D_g(u, x_n)| &= |g(u) - g(x_n) - \langle \nabla g(x_n), u - x_n\rangle| \\ &\le |g(u) - g(x_n)| + \|u - x_n\|.\|\nabla g(x_n)\| \\ &\le |g(u)| + |g(x_n)| + (\|u\| + \|x_n\|).\|\nabla g(x_n)\|. \end{aligned} \tag{22}$$

It follows from (22), the boundedness of $\mathcal{F}$ and $\{x_n\}$, the uniform continuity of g and the boundedness on bounded subsets of ∇g that $|D_g(u, x_n)| < \infty$. This proves that the sequence $\{D_g(u, x_n)\}$ is bounded. By using $\lim_{n\to\infty} \eta_n = \lim_{n\to\infty} \mu_n = 0$, we find that

$$\lim_{n\to\infty} \gamma_n = \lim_{n\to\infty} \Big(\eta_n \sup\big\{\xi\big(D_g(u, x_n)\big) : u \in \mathcal{F}\big\} + \mu_n\Big) = 0.$$

By combining (19), (21) and $\lim_{n\to\infty} \gamma_n = 0$, we find that $\lim_{n\to\infty} D_g(z_{n+1}, \overline{v}_n) = 0$. Furthermore, by (12) and the boundedness of $\{D_g(u, x_n)\}$, we obtain that $\{D_g(u, v_n^{(j)})\}$ is bounded. Now, by [28, Proposition 3.6.4], we find that g^* is bounded on bounded subsets of W^*. This implies that ∇g^* is bounded on bounded subsets of W^*. By combining this with the boundedness of $\{D_g(u, v_n^{(j)})\}$ and using Proposition 3, we find that $\{v_n^{(j)}\}$ is bounded. This implies that $\{\overline{v}_n\}$ is bounded. By combining this with $\lim_{n\to\infty} D_g(z_{n+1}, \overline{v}_n) = 0$ and using Proposition 2, we find that

$$\lim_{n\to\infty} \|z_{n+1} - \overline{v}_n\| = 0. \tag{23}$$

By combining (23) and $\lim_{n\to\infty} \|z_{n+1} - x_n\| = 0$, we get $\lim_{n\to\infty} \|x_n - \overline{v}_n\| = 0$. Then, by using the same proof as in that of (21), we find that $\lim_{n\to\infty} D_g(x_n, \overline{v}_n) = 0$. By the definition $\overline{v}_n$, we get that

$$\lim_{n\to\infty} D_g(x_n, v_n^{(j)}) = 0. \tag{24}$$

Next, by using the same proofs as in that of (23), we find that

$$\lim_{n\to\infty} \|x_n - v_n^{(j)}\| = 0. \tag{25}$$

Since g is uniformly continuous and ∇g is uniformly continuous on bounded sets, from (25), we find that

$$\lim_{n\to\infty} \|g(x_n) - g(v_n^{(j)})\| = \lim_{n\to\infty} \|\nabla g(x_n) - \nabla g(v_n^{(j)})\| = 0. \tag{26}$$

Next, by using the same proofs as in that of (9) and (10), we conclude that

$$D_g(u, \overline{y}_n) \le D_g(u, x_n) + \gamma_n \tag{27}$$

and

$$D_g(u, \overline{u}_n) \le D_g(u, x_n). \tag{28}$$

Then, from (27), (28) and using the boundedness of $\{D_g(u, x_n)\}$, $\{\gamma_n\}$, we find that $\{D_g(u, \overline{y}_n)\}$ and $\{D_g(u, \overline{u}_n)\}$ are bounded. Note that ∇g^* is bounded on bounded subsets of W^*. By combining this with the boundedness of $\{D_g(u, \overline{y}_n)\}$, $\{D_g(u, \overline{u}_n)\}$ and using Proposition 3, we find that $\{\overline{y}_n\}$ and $\overline{u}_n\}$ are bounded. Furthermore, for each $j \in I$, we have

$$D_g(u, S_j^n(\overline{u}_n)) \le D_g(u, \overline{u}_n) + \eta_n \xi\big(D_g(u, \overline{u}_n)\big) + \mu_n. \tag{29}$$

By (29) and the boundedness of $\{D_g(u, \overline{u}_n)\}$, we obtain that $\{D_g(u, S_j^n(\overline{u}_n))\}$ is bounded. By Proposition 3, we find that $\{S_j^n(\overline{u}_n)\}$ is bounded.

Since $\{\overline{y}_n\}$ and $\{S_j^n(\overline{u}_n)\}$ are bounded and ∇g is bounded on bounded subsets, we find that $\{\nabla g(\overline{y}_n)\}$ and $\{\nabla g(S_j^n(\overline{u}_n))\}$ are bounded. Put

$$\varepsilon = \max\{\sup_{n\in\mathbb{N}} \|\nabla g(\overline{y}_n)\|, \sup_{n\in\mathbb{N}} \|\nabla g(S_j^n(\overline{u}_n))\|\}.$$

This leads to $\nabla g(\overline{y}_n), \nabla g(S_j^n(\overline{u}_n)) \in B_\varepsilon^*$. Therefore, by using Lemma 1, we find that

$$D_g(u, v_n^{(j)})$$

$$= D_g\Big(u, \nabla g^*\big(b_n \nabla g(\overline{y}_n) + (1-b_n)\nabla g(S_j^n(\overline{u}_n))\big)\Big)$$
$$\leq b_n D_g(u, \overline{y}_n) + (1-b_n) D_g(u, S_j^n(\overline{u}_n)) - b_n(1-b_n)\rho_\varepsilon^*\big(\|\nabla g(\overline{y}_n) - \nabla g(S_j^n(\overline{u}_n))\|\big)$$
$$\leq b_n D_g(u, \overline{y}_n) + (1-b_n)[D_g(u, \overline{u}_n) + \eta_n \xi(D_g(u, \overline{u}_n)) + \mu_n]$$
$$-b_n(1-b_n)\rho_\varepsilon^*\big(\|\nabla g(\overline{y}_n) - \nabla g(S_j^n(\overline{u}_n))\|\big). \quad (30)$$

It follows from (28), (30) and the strictly increasing property of ξ, we find that

$$D_g(u, v_n^{(j)}) \leq b_n D_g(u, \overline{y}_n) + (1-b_n)[D_g(u, x_n) + \eta_n \xi(D_g(u, x_n)) + \mu_n]$$
$$-b_n(1-b_n)\rho_\varepsilon^*\big(\|\nabla g(\overline{y}_n) - \nabla g(S_j^n(\overline{u}_n))\|\big). \quad (31)$$

By combining (27) and (31), we find that

$$D_g(u, v_n^{(j)}) \leq b_n[D_g(u, x_n) + \gamma_n] + (1-b_n)[D_g(u, x_n) + \eta_n \xi(D_g(u, x_n)) + \mu_n]$$
$$-b_n(1-b_n)\rho_\varepsilon^*\big(\|\nabla g(\overline{y}_n) - \nabla g(S_j^n(\overline{u}_n))\|\big)$$
$$\leq D_g(u, x_n) + \gamma_n - b_n(1-b_n)\rho_\varepsilon^*\big(\|\nabla g(\overline{y}_n) - \nabla g(S_j^n(\overline{u}_n))\|\big).$$

This leads to

$$b_n(1-b_n)\rho_\varepsilon^*\big(\|\nabla g(\overline{y}_n) - \nabla g(S_j^n(\overline{u}_n))\|\big) \leq D_g(u, x_n) - D_g(u, v_n^{(j)}) + \gamma_n. \quad (32)$$

Moreover, by using the property of the function D_g, we obtain

$$|D_g(u, x_n) - D_g(u, v_n^{(j)})|$$
$$= |-D_g(x_n, v_n^{(j)}) + \langle \nabla g(v_n^{(j)}) - \nabla g(x_n), u - x_n\rangle|$$
$$\leq |D_g(x_n, v_n^{(j)})| + \|\nabla g(v_n^{(j)}) - \nabla g(x_n)\|.\|u - x_n\|. \quad (33)$$

It follows from (24), (26) and (33) that $\lim_{n\to\infty} |D_g(u, x_n) - D_g(u, v_n^{(j)})| = 0$. By combining this with (32), $\lim_{n\to\infty} \gamma_n = 0$ and $\liminf_{n\to\infty} b_n(1-b_n) > 0$, we conclude that

$$\lim_{n\to\infty} \rho_\varepsilon^*\big(\|\nabla g(\overline{y}_n) - \nabla g(S_j^n(\overline{u}_n))\|\big) = 0. \quad (34)$$

Suppose that $\lim_{n\to\infty} \|\nabla g(\overline{y}_n) - \nabla g(S_j^n(\overline{u}_n))\| > 0$. Then, there exist $r > 0$ and a subsequence $\{k(n)\}$ of n such that $\|\nabla g(\overline{y}_{k(n)}) - \nabla g(S_j^{k(n)}(\overline{u}_{k(n)}))\| \geq r$. It follows from the nondecreasing property of ρ_ε^* that

$$\rho_\varepsilon^*(\|\nabla g(\overline{y}_{k(n)}) - \nabla g(S_j^{k(n)}(\overline{u}_{k(n)}))\|) \geq \rho_\varepsilon^*(r) \quad (35)$$

for all $n \in \mathbb{N}$. By taking the limit as $n \to \infty$ in (35) and using (34), we obtain that $0 \geq \rho_\varepsilon^*(r)$. This contradicts the fact that $\rho_\varepsilon^*(r) > 0$. Therefore,

$$\lim_{n\to\infty} \|\nabla g(\overline{y}_n) - \nabla g(S_j^n(\overline{u}_n))\| = 0. \quad (36)$$

Note that ∇g^* is uniformly continuous on bounded subsets. By combining this with $\nabla g = (\nabla g^*)^{-1}$ and (36), we conclude that

$$\lim_{n\to\infty} \|\overline{y}_n - S_j^n(\overline{u}_n)\| = 0. \quad (37)$$

Now, by the definition of $\overline{y}_n$ and $\nabla g = (\nabla g^*)^{-1}$, we get

$$\nabla g(\overline{y}_n) = \nabla g\Big(\nabla g^*\big(a_n \nabla g(x_n) + (1-a_n)\nabla g(S_i^n(x_n))\big)\Big) = a_n \nabla g(x_n) + (1-a_n)\nabla g(S_{i_n}^n(x_n)).$$

This leads to

$$\|\nabla g(\overline{y}_n) - \nabla g(x_n)\| = (1-a_n)\|\nabla g(S_{i_n}^n(x_n)) - \nabla g(x_n)\|. \quad (38)$$

It follows from (37), (38), $\lim_{n\to\infty} a_n = 1$ and the boundedness of $\{x_n\}$ that

$$\lim_{n\to\infty} \|\nabla g(\overline{y}_n) - \nabla g(x_n)| = 0. \quad (39)$$

Then, by using the uniform continuous on bounded subsets of ∇g^* , $\nabla g = (\nabla g^*)^{-1}$ and (39), we obtain

$$\lim_{n\to\infty} \|\overline{y}_n - x_n| = 0. \tag{40}$$

It follows from (37) and (40) that

$$\lim_{n\to\infty} \|S_j^n(\overline{u}_n) - x_n\| = 0. \tag{41}$$

By combining (41) and $\lim_{n\to\infty} x_n = p$, we have $\lim_{n\to\infty} S_j^n(\overline{u}_n) = p$. Moreover, we have

$$\|S_j^{n+1}(\overline{u}_n) - p\| \le \|S_j^{n+1}(\overline{u}_n) - S_j^n(\overline{u}_n)\| + \|S_j^n(\overline{u}_n) - p\|. \tag{42}$$

Then, we conclude from (42), $\lim_{n\to\infty} S_j^n(\overline{u}_n) = p$ and the asymptotically regular property of S_j that $\lim_{n\to\infty} S_j^{n+1}(\overline{u}_n) = p$ for all $j \in I$. This proves that $\lim_{n\to\infty} S(S_j^n(\overline{u}_n)) = p$. Since S_j is closed, we find that $S_j(p) = p$ for all $j \in I$ and $p \in \bigcap_{i\in I} F(S_i)$.

Next, we prove that $p \in \bigcap_{k\in K} GMEP(f_k, A_k, B_k)$. Indeed, for each $k \in K$, we have $u_n^{(k)} = \mathrm{Res}^g_{f_k, A_k, B_k}(\overline{y}_n)$. It follows from (6) that

$$f_k(u_n^{(k)}, v) + A_k(v) + \langle B_k(u_n^{(k)}), v - u_n^{(k)}\rangle + \langle \nabla g(u_n^{(k)}) - \nabla g(\overline{y}_n), v - u_n^{(k)}\rangle \ge A_k(u_n^{(k)}) \text{ for all } v \in U.$$

By using the condition (C_2), we get

$$\begin{aligned} &\langle B_k(u_n^{(k)}), v - u_n^{(k)}\rangle + \langle \nabla g(u_n^{(k)}) - \nabla g(\overline{y}_n), v - u_n^{(k)}\rangle + A_k(v) - A_k(u_n^{(k)}) \\ &\ge -f_k(u_n^{(k)}, v) \ge f_k(v, u_n^{(k)}). \end{aligned} \tag{43}$$

Since g is uniformly continuous and ∇g is uniformly continuous on bounded sets, by (41), we obtain

$$\lim_{n\to\infty} \|g(x_n) - g(S_j^n(\overline{u}_n))\| = \lim_{n\to\infty} \|\nabla g(x_n) - \nabla g(S_j^n(\overline{u}_n))\| = 0. \tag{44}$$

Let $u \in \mathcal{F}$. By using the property of the Bregman distance, we have

$$\begin{aligned} &|D_g(u, x_n) - D_g(u, S_j^n(\overline{u}_n))| \\ &= | - D_g(x_n, S_j^n(\overline{u}_n)) + \langle \nabla g(S_j^n(\overline{u}_n)) - \nabla g(x_n), u - x_n\rangle| \\ &\le |D_g(x_n, S_j^n(\overline{u}_n))| + \|\nabla g(S_j^n(\overline{u}_n)) - \nabla g(x_n)\|.\|u - x_n\| \\ &\le |g(x_n) - g(S_j^n(\overline{u}_n))| + \|\nabla g(S_j^n(\overline{u}_n))\|.\|x_n - S_j^n(\overline{u}_n)\| \\ &\quad + \|\nabla g(S_j^n(\overline{u}_n)) - \nabla g(x_n)\|.\|u - x_n\|. \end{aligned} \tag{45}$$

It follows from (41), (44) and (45) that

$$\lim_{n\to\infty} |D_g(u, x_n) - D_g(u, S_j^n(\overline{u}_n))| = 0. \tag{46}$$

Furthermore, by $\overline{u}_n = \mathrm{Res}^g_{f_{k_n}, A_{k_n}, B_{k_n}}(x_n)$, Lemma 4 and (29), we find that

$$\begin{aligned} D_g(x_n, \overline{u}_n) &\le D_g(u, x_n) - D_g(u, \overline{u}_n) \\ &\le D_g(u, x_n) - D_g(u, S_j^n(\overline{u}_n)) + \eta_n \xi\big(D_g(u, \overline{u}_n)\big) + \mu_n. \end{aligned} \tag{47}$$

Then, by (28) and the strictly increasing property of ξ, the inequality (47) becomes

$$\begin{aligned} D_g(x_n, \overline{u}_n) &\le D_g(u, x_n) - D_g(u, S_j^n(\overline{u}_n)) + \eta_n \xi\big(D_g(u, x_n)\big) + \mu_n \\ &\le D_g(u, x_n) - D_g(u, S_j^n(\overline{u}_n)) + \gamma_n. \end{aligned} \tag{48}$$

It follows from (46), (48) and $\lim_{n\to\infty} \gamma_n = 0$ that $\lim_{n\to\infty} D_g(x_n, \overline{u}_n) = 0$. By the definition $\overline{u}_n$, we get that

$$\lim_{n\to\infty} D_g(x_n, u_n^{(k)}) = 0. \tag{49}$$

Then, by using the same proof as in that of (23), we find that $\lim_{n\to\infty} \|x_n - u_n^{(k)}\| = 0$. By combining this with $\lim_{n\to\infty} x_n = p$, we get that

$$\lim_{n\to\infty} u_n^{(k)} = p. \tag{50}$$

Moreover, it follows from (40) and $\lim_{n\to\infty} \|x_n - u_n^{(k)}\| = 0$ that $\lim_{n\to\infty} \|u_n^{(k)} - \overline{y}_n\| = 0$. Since ∇g is uniformly continuous on bounded subsets, we obtain $\lim_{n\to\infty} \|\nabla g(u_n^{(k)}) - \nabla g(\overline{y}_n)\| = 0$. This implies that

$$\lim_{n\to\infty} |\langle \nabla g(u_n^{(k)}) - \nabla g(\overline{y}_n), v - u_n^{(k)}\rangle| = 0. \tag{51}$$

Since A_k is lower semi-continuous and (50), we find that

$$\liminf_{n\to\infty} A_k(u_n^{(k)}) \geq A_k(p). \tag{52}$$

By the condition (C_4), we get that f_k is lower semi-continuous in the second variable for each $k \in K$. It follows from (50) that

$$\liminf_{n\to\infty} f_k(v, u_n^{(k)}) \geq f_k(v, p). \tag{53}$$

We also have

$$\begin{aligned}
&|\langle B_k(u_n^{(k)}), v - u_n^{(k)}\rangle - \langle B_k(p), v - p\rangle| \\
&= |\langle B_k(u_n^{(k)}) - B_k(p), v\rangle - \langle B_k(u_n^{(k)}), u_n^{(k)}\rangle + \langle B_k(p), p\rangle| \\
&\leq |\langle B_k(u_n^{(k)}) - B_k(p), v\rangle| + |\langle B_k(u_n^{(k)}), u_n^{(k)} - p\rangle| + |\langle B_k(u_n^{(k)}) - B_k(p), p\rangle| \\
&\leq |\langle B_k(u_n^{(k)}) - B_k(p), v\rangle| + \|B_k(u_n^{(k)})\|.\|u_n^{(k)} - p\| + |\langle B_k(u_n^{(k)}) - B_k(p), p\rangle|.
\end{aligned} \tag{54}$$

It follows from (50), (54), the continuity of B_k and $B_k(u_n^{(k)}) \in W^*$ that

$$\lim_{n\to\infty} \langle B_k(u_n^{(k)}), v - u_n^{(k)}\rangle = \langle B_k(p), v - p\rangle. \tag{55}$$

Then, by (43), (51), (52), (53) and (55), we find that

$$\langle B_k(p), v - p\rangle + A_k(v) - A_k(p) \geq f_k(v, p) \tag{56}$$

for all $v \in U$. For all $t \in (0, 1]$, put $v_t = tv + (1-t)p$. Due to $y, p \in U$ and U is convex, we have $v_t \in U$. Then, by replacing y by v_t in (56), we conclude that

$$f_k(v_t, p) + \langle B_k(p), p - v_t\rangle + A_k(p) - A_k(v_t) \leq 0. \tag{57}$$

By using the condition (C_1), the convexity in the second variable of f_k and the convexity of A_k and (57), we conclude that

$$\begin{aligned}
0 = f_k(v_t, v_t) &= f_k(v_t, v_t) + \langle B_k(p), v_t - v_t\rangle + A_k(v_t) - A_k(v_t) \\
&\leq t f_k(v_t, y) + (1-t) f_k(v_t, p) + t\langle B_k(p), y - v_t\rangle + (1-t)\langle B_k(p), p - v_t\rangle \\
&\qquad + t A_k(y) + (1-t) A_k(p) - A_k(v_t) \\
&= t\big[f_k(v_t, v) + \langle B_k(p), v - v_t\rangle + A_k(v) - A_k(v_t)\big] \\
&\qquad + (1-t)\big[f_k(v_t, p) + \langle B_k(p), p - v_t\rangle + A_k(p) - A_k(v_t)\big] \\
&\leq t\big[f_k(v_t, y) + \langle B_k(p), v - v_t\rangle + A_k(v) - A_k(v_t)\big].
\end{aligned} \tag{58}$$

It follows from (58) and $t > 0$ that

$$f_k(v_t, v) + \langle B_k(p), v - v_t\rangle + A_k(v) - A_k(v_t) \geq 0. \tag{59}$$

Therefore, by the condition (C_3), we have

$$\limsup_{t\downarrow 0} f_k(v_t, v) = \limsup_{t\downarrow 0} f_k(tv + (1-t)p, v) \leq f_k(p, v). \tag{60}$$

Since A_k is lower semi-continuous, we get that $-A_k$ is upper semi-continuous. From $\lim_{t\to 0} v_t = \lim_{t\to 0}(tv + (1-t)p) = p$, we find that

$$\limsup_{t\to 0}[-A_k(v_t)] \leq -A_k(p). \tag{61}$$

By (59), (60), (61) and $\lim_{t\to 0} v_t = p$, we find that

$$f_k(p, v) + \langle B_k(p), v - p\rangle + A_k(v) - A_k(p) \geq 0.$$

This implies that $p \in \bigcap_{k\in K} GMEP(f_k, A_k, B_k)$. By the above, we conclude that

$$p \in \mathcal{F} = \Big(\bigcap_{i\in I} F(S_i)\Big) \bigcap \Big(\bigcap_{k\in K} GMEP(f_k, A_k, B_k)\Big).$$

Step 6. *We claim that* $p = P^g_{\mathcal{F}}(z_1)$. Indeed, we put $z = P^g_{\mathcal{F}}(z_1)$. We will show that $z = p$. From $z_n = P^g_{U_n}(z_1)$ and Definition 3, we find that

$$D_g(z_n, z_1) \leq D_g(v, z_1) \tag{62}$$

for all $v \in U_n$. It follows from $z = P^g_{\mathcal{F}}(z_1) \in \mathcal{F}$ and $\mathcal{F} \subset U_n$ that $z \in U_n$. Therefore, by taking $v = z$ in (62), we obtain

$$D_g(z_n, z_1) \leq D_g(z, z_1). \tag{63}$$

Furthermore, we have

$$\begin{aligned} |D_g(z_n, z_1) - D_g(p, z_1)| &= |g(z_n) - g(p) + \langle \nabla g(z_1), p - z_n\rangle| \\ &\leq |g(z_n) - g(p)| + \|\nabla g(z_1)\|.\|p - z_n\|. \end{aligned} \tag{64}$$

Taking the limit as $n \longrightarrow \infty$ in (64) and using $\lim_{n\to\infty} u_n = p$, the uniform continuity of g and the boundedness on bounded subsets of ∇g that $\lim_{n\to\infty} D_g(z_n, z_1) = D_g(p, z_1)$. Then, it follows from (63) that $D_g(p, z_1) \leq D_g(z, z_1)$. By the definition of z and $p \in \mathcal{F}$, we find that $p = z = P^g_{\mathcal{F}}(z_1)$. □

In Theorem 5, by choosing $a_n = 1$, we obtain the following result. Note that iteration (65) is an improvements to iteration (1) which was presented in [5].

Corollary 1 *Suppose that W is a real reflexive Banach space, and U is a nonempty, closed and convex subset of W. Let $g : W \longrightarrow \mathbb{R}$ be Legendre, strongly coercive on W, and g be bounded, totally convex, uniformly Fréchet differentiable on bounded subsets of W. For each $k \in K := \{1, 2, \ldots, M\}$ with $M \in \mathbb{N}$, $f_k : U \times U \longrightarrow \mathbb{R}$, $A_k : U \longrightarrow \mathbb{R}$ and $B_k : U \longrightarrow W^*$ satisfy the conditions (C_1) - (C_6). For each $i \in I$, $S_i : W \longrightarrow W$ is a closed, uniformly asymptotically regular and Bregman totally quasi-asymptotically nonexpansive mapping with non-negative real sequences $\{\eta^{(i)}_n\}$ and $\{\mu^{(i)}_n\}$ satisfying $\lim_{n\to\infty} \eta^{(i)}_n = \lim_{n\to\infty} \mu^{(i)}_n = 0$ and strictly increasing continuous function $\xi^{(i)} : [0, \infty) \longrightarrow [0, \infty)$ with $\xi^{(i)}(0) = 0$ such that $\mathcal{F} = \Big(\bigcap_{i\in I} F(S_i)\Big) \bigcap \Big(\bigcap_{k\in K} GMEP(f_k, A_k, B_k)\Big)$ is nonempty and bounded. Let $\{z_n\}$ be a sequence generated by*

$$\begin{cases} z_1, z_2 \in U, U_1 = U_2 = U \\ x_n = z_n + c_n(z_n - z_{n-1}) \text{ for all } n \geq 2 \\ u^{(k)}_n = \mathrm{Res}^g_{f_k, A_k, B_k}(x_n) \text{ for all } k \in K \\ k_n = \operatorname{argmax}\{D_g(x_n, u^{(k)}_n) : k \in K\},\ \overline{u}_n = u^{(k_n)}_n \\ v^{(j)}_n = \nabla g^*\Big(b_n \nabla g(x_n) + (1 - b_n)\nabla g(S^n_j(\overline{u}_n))\Big) \text{ for all } j \in I \\ j_n = \operatorname{argmax}\{D_g(x_n, v^{(j)}_n) : j \in I\},\ \overline{v}_n = v^{(j_n)}_n \\ U_{n+1} = \big\{u \in U_n : D_g(u, \overline{v}_n) \leq D_g(u, x_n) + \gamma_n\big\} \\ z_{n+1} = P^g_{U_{n+1}}(z_1), \end{cases} \tag{65}$$

where $\gamma_n = \eta_n \sup\big\{\xi\big(D_g(u, x_n)\big) : u \in \mathcal{F}\big\} + \mu_n$, and $\{b_n\}, \{c_n\} \subset [0, 1]$ such that $\liminf_{n\to\infty} b_n(1 - b_n) > 0$, and the function $\mathrm{Res}^g_{f_k, A_k, B_k}$ is defined as in (6). Then the sequence $\{z_n\}$ strongly converges to $p = P^g_{\mathcal{F}}(z_1)$.

Remark 5 In Theorem 5 and Corollary 1, by choosing $S_i = S$ for all $i \in I$, $f_k = f$, $A_k = A$ and $B_k = B$ for all $k \in K$, we get two convergence results for a generalized mixed equilibrium problem and a Bregman totally quasi-asymptotically nonexpansive mapping in reflexive Banach spaces.

In Theorem 5, when S_i is an identity mapping for all $i \in I$, we obtain the following corollary. Note that the iteration process (66) is an improvement of the the iteration process (3) in the sense of adding the inertial extrapolation. Therefore, Corollary 2 is a generalization of the main result in [17].

Corollary 2 *Suppose that W is a real reflexive Banach space, and U is a nonempty, closed and convex subset of W. Let $g : W \longrightarrow \mathbb{R}$ be Legendre, strongly coercive on W, and g be bounded, totally convex, uniformly Fréchet differentiable on bounded subsets of W. For each $k \in K := \{1, 2, \ldots, M\}$ with $M \in \mathbb{N}$, $f_k : U \times U \longrightarrow \mathbb{R}$, $A_k : U \longrightarrow \mathbb{R}$ and $B_k : U \longrightarrow W^*$ satisfy the conditions (C_1) - (C_6) such that $\mathcal{F}_1 = \bigcap_{k\in K} GMEP(f_k, A_k, B_k)$ is nonempty and bounded. Let $\{z_n\}$ be a sequence generated by*

$$\begin{cases} z_1, z_2 \in U, U_1 = U_2 = U \\ x_n = z_n + c_n(z_n - z_{n-1}) \text{ for all } n \geq 2 \\ u_n^{(k)} = \mathrm{Res}^g_{f_k, A_k, B_k}(x_n) \\ k_n = \operatorname{argmax}\{D_g(u_n^{(k)}, x_n) : k \in K\}, \overline{u}_n = u_n^{(k_n)} \\ v_n = \nabla g^*\Big(b_n \nabla g(x_n) + (1 - b_n)\nabla g(\overline{u}_n)\Big) \\ U_{n+1} = \big\{u \in U_n : D_g(u, v_n) \leq D_g(u, x_n) + \gamma_n\big\} \\ z_{n+1} = P^g_{U_{n+1}}(z_1), \end{cases} \tag{66}$$

where $\gamma_n = \eta_n \sup\big\{\xi\big(D_g(u, x_n)\big) : u \in \mathcal{F}\big\} + \mu_n$, and $\{b_n\}, \{c_n\} \subset [0, 1]$ such that $\liminf_{n\to\infty} b_n(1 - b_n) > 0$, and the function $\mathrm{Res}^g_{f_k, A_k, B_k}$ is defined as in (6). Then the sequence $\{z_n\}$ strongly converges to $p = P^g_{\mathcal{F}_1}(z_1)$.

Remark 6
1. Note that every Bregman quasi-asymptotically nonexpansive mapping is a Bregman totally quasi-asymptotically nonexpansive mapping with $\xi(t) = t$ for all $t \geq 0$, $\eta_n = k_n - 1$ with $k_n \geq 1$ satisfying $\lim_{n\to\infty} k_n = 1$, and $\mu_n = 0$ for all $n \in \mathbb{N}$. Therefore, the conclusions of Theorem 5, Corollary 1 and Corollary 2 hold when S_i is a Bregman quasi-asymptotically nonexpansive mapping for all $i \in I$ and $\gamma_n = (k_n - 1)\sup\big\{D_g(u, x_n) : u \in \mathcal{F}\big\}$ for all $n \in \mathbb{N}$.
2. The conclusions of Theorem 5, Corollary 1 and Corollary 2 are satisfied when $(GMEP)$ is replaced by (MEP), (GEP), (MVI) and (EP).

Finally, we give a numerical example to illustrate for the convergence of iteration (66) and iteration (3).

Example 1 Let $W = \mathbb{R}$, $U = [0, 1]$, $g(u) = u^4$, $S_i(u) = \dfrac{u}{2^i}$ for all $u \in W$ and $i = 1, 2$. Let $B_k(u) = ku$, $A_k(u) = ku^2$ and $f_k(u, v) = k(-2u^2 + uv + v^2)$ for all $u, v \in U$ and $k = 1, 2$. Then

1. By directly calculating, we have $\nabla g(u) = 4u^3$ for all $u \in W$, $g^*(w) = 3\sqrt[3]{\left(\frac{w}{4}\right)^4}$ and $\nabla g^*(w) = \sqrt[3]{\frac{w}{4}}$ for all $w \in W$.
2. For all $u, v \in W$, we have $D_g(u, v) = g(u) - g(v) - \langle \nabla g(v), u - v\rangle = u^4 + 3v^4 - 4uv^3$.
3. For each $i = 1, 2$, we obtain $F(S_i) = \{0\}$. Therefore, for all $p \in F(S_i)$ and $u \in U$, we find that $D_g(p, S_i^n u) = 3(S_i^n u)^4 = 3\left(\frac{u}{2^{ni}}\right)^4 \leq 3(u)^4 = D_g(0, u) = D_g(p, u)$. This proves that S_i is a Bregman totally quasi-asymptotically nonexpansive mapping with $\eta_n^{(i)} = \mu_n^{(i)} = 0$ for all $n \in \mathbb{N}$.
4. By directly checking, we find that f_k, A_k, B_k satisfies the conditions (C_1) - (C_6).
5. Now, we will find the formula of $\mathrm{Res}^g_{f_k, A_k, B_k}$ as in (6). Indeed, $w = \mathrm{Res}^g_{f_k, A_k, B_k}(u)$ for all $u \in W$ if and only if

$$f_k(w, v) + A_k(v) + \langle B_k(w), v - w\rangle + \langle \nabla g(w) - \nabla g(u), v - w\rangle \geq A_k(w) \tag{67}$$

for all $v \in U$. By substituting f_k, A_k, B_k into (67) and by directly calculating, we find that $kv^2 + (kw + 2w^3 - 2u^3)v + 2u^3 w - 2w^4 - 2kw^2 \geq 0$. Put

$$h(v) = kv^2 + (kw + 2w^3 - 2u^3)v + 2u^3 w - 2w^4 - 2kw^2.$$

Then, we have $\Delta = (3kw + 2w^3 - 2u^3)^2$. We consider the following two cases.
Case 1. $\Delta > 0$. Then the quadratic equation $h(v) = 0$ have two solutions as follows.

$$v_1 = w \quad \text{and} \quad v_2 = \frac{2u^3 - 2w^3 - 2kw}{k}.$$

In oder to $h(v) \geq 0$ for all $v \in \Omega$, we have the following cases.
Case 1.1. $v_1 = 1$ and $v_2 > v_1$. Then $w = v_1 = 1$, and $v_2 = \dfrac{2u^3 - 2k - 2}{k} > 1$ and hence $u > \sqrt[3]{\dfrac{3k+2}{2}}$.
Case 1.2. $v_1 = 0$ and $v_2 < v_1$. Then $w = v_1 = 0$, and $v_2 = \dfrac{2u^3}{k} < 0$ and hence $u < 0$.
Case 2. $\Delta \leq 0$. Then $3kw + 2w^3 = 2u^3$ and $h(v) \geq 0$ for all $v \in U$. Note that $3kw + 2w^3 = 2u^3$ if and only if $w = \dfrac{\left(\sqrt[3]{\sqrt{4u^6 + 2k^3} + 2u^3}\right)^2 - \sqrt[3]{2}k}{\sqrt[3]{4}\sqrt[3]{\sqrt{4u^6 + 2k^3} + 2u^3}}$. Since $w \in U$, we have $0 \leq 3kw + 2w^3 = 2u^3 \leq 3k + 2$ and hence $0 \leq u \leq \sqrt[3]{\dfrac{3k+2}{2}}$. Therefore,

$$\mathrm{Res}^g_{f_k, A_k, B_k}(u) = w = \begin{cases} 0 & \text{if } u < 0 \\ \dfrac{\left(\sqrt[3]{\sqrt{4u^6 + 2k^3} + 2u^3}\right)^2 - \sqrt[3]{2}k}{\sqrt[3]{4}\sqrt[3]{\sqrt{4u^6 + 2k^3} + 2u^3}} & \text{if } 0 \leq u \leq \sqrt[3]{\dfrac{3k+2}{2}} \\ 1 & \text{if } u > \sqrt[3]{\dfrac{3k+2}{2}}. \end{cases}$$

By the above, all assumptions in Theorem 5 are satisfied with the given functions f_k, A_k, B_k, S_i. Therefore, by Theorem 5, the sequence $\{z_n\}$ which is defined by (7) converges to $0 \in \left(\bigcap_{i=1}^{2} F(S_i)\right) \bigcap \left(\bigcap_{k=1}^{2} GMEP(f_k, A_k, B_k)\right)$.

Next, we compare the rate of convergence of the iteration process (3) and the iteration process (66) to 0 which is a solution of a finite system of $(GMEP)$. Numerical results of the mentioned iteration processes with the initial point $z_1 = 1, z_2 = 0.8, b_n = \dfrac{1}{10000n}$ and the different choices of c_n are presented in the following table.

Table 1 shows that for given mappings, the iteration process (66) has a better convergence rate and requires a small number of iterations than the iteration process (3).

4 Conclusions

In this paper, a new inertial parallel iterative method was proposed for finding common solutions of a finite system of generalized mixed equilibrium problems and common fixed points of a finite family of Bregman totally quasi-asymptotically nonexpansive mappings. A strong convergence result for the proposed iteration in reflexive Banach spaces was established and proved. From this theorem, some convergence results for generalized mixed equilibrium problems in reflexive Banach spaces were given. In addition, a numerical example was provided to demonstrate the proposed iterations. On comparing our results with the main result of [5,6,17], we find that

1. Theorem 5 is a generalization of [6, Theorem 3.1] from an equilibrium problem and a generalized hybrid mapping in Hilbert spaces to a finite system of generalized mixed equilibrium problems and a finite family of Bregman totally quasi-asymptotically nonexpansive mappings in reflexive Banach spaces.
2. Corollary 1 is an improvement of [5, Theorem 3.1] from an equilibrium problem and a nonexpansive mapping in Hilbert spaces to a generalized mixed equilibrium problem and a Bregman totally quasi-asymptotically nonexpansive mapping in reflexive Banach spaces.
3. Corollary 2 is an extension of [17, Theorem 3.6] in the sense of adding the inertial extrapolation to the iteration process (3). Furthermore, Example 1 was given to prove the efficiency of iteration (66) which has a better convergence rate and requires a small number of iterations than the iteration process (3) which was presented in [17].

Table 1 Number of iterations of the processes (3) and (66)

n	Iteration (3)	Iteration (66)		
		$c_n = \frac{1}{n}$	$c_n = \frac{1}{2}$	$c_n = \frac{9n+2}{10n+2}$
1	1.000000	1.000000	1.000000	1.000000
2	0.767589	0.800000	0.800000	0.800000
3	0.579151	0.525025	0.525025	0.463658
4	0.434901	0.325035	0.290662	0.119140
5	0.326252	0.206283	0.130114	0.119140
⋮	⋮	⋮	⋮	⋮
21	0.003270	0.001056	0.000008	0.000001
22	0.002452	0.000778	0.000004	0.
23	0.001839	0.000574	0.000001	0.
24	0.001379	0.000424	0.	0.
⋮	⋮	⋮	⋮	⋮
43	0.000005	0.000001	0.	0.
44	0.000004	0.000001	0.	0.
45	0.000003	0.	0.	0.
⋮	⋮	⋮	⋮	⋮
49	0.000001	0.	0.	0.
50	0.	0.	0.	0.

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