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ORIGINAL RESEARCH

A note on Majorana representation of quantum states

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In memory of Professor Kalyanapuram Rangachari Parthasarthy.

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Abstract By the Majorana representation, for any $d > 1$ there is a one-one correspondence between a quantum state of dimension d and $d-1$ qubits represented as $d-1$ points in the Bloch sphere. Using the theory of symmetry class of tensors, we present a simple scheme for constructing $d-1$ points on the Bloch sphere and the corresponding $d-1$ qubits representing a d-dimensional quantum state. Additionally, we demonstrate how the inner product of two d-dimensional quantum states can be expressed as a permanent of a matrix related to their $(d-1)$-qubit state representations. Extension of the result to mixed states is also considered.

Keywords Quantum states · Majorana representation · Principal character · Permanent

AMS Classification 15A90 · 15A60

1 Introduction

Quantum states, represented by unit vectors $\mathbf{a} \in \mathbb{C}^d$, form a fundamental aspect of d-dimensional systems. These vectors are identified up to a phase factor, i.e., $\mathbf{a}$ and $e^{it}\mathbf{a}$ are identified for any $t \in [0, 2\pi)$. In the case of $d = 2$, quantum states are commonly referred to as qubits. For qubits, there exists a one-to-one correspondence between the state $\mathbf{a} = (a_0, a_1)^t$ with $|a_0|^2 + |a_1|^2 = 1$ and a point (c_x, c_y, c_z) on the Bloch sphere:

$$\mathbf{B} = \{(c_x, c_y, c_z) : c_x, c_y, c_z \in \mathbb{R}^3,\ c_x^2 + c_y^2 + c_z^2 = 1\}$$

where $\mathbf{a}\mathbf{a}^*$ corresponds to (c_x, c_y, c_z) by $\frac{1}{2}\begin{pmatrix} 1+c_z & c_x - ic_y \\ c_x + ic_y & 1-c_z \end{pmatrix}$. The correspondence is established using $(c_x, c_y, c_z) = (\Re(\bar{a}_0 a_1), \Im(\bar{a}_0 a_1), |a_0|^2 - |a_1|^2)/2$, ensuring that $c_x^2 + c_y^2 + c_z^2 = 1$.

In [3], Majorana proposed a geometric method to represent a quantum state $\mathbf{a} \in \mathbb{C}^d$ for $d > 1$ using $d-1$ qubits. Consequently, a quantum state in $\mathbb{C}^d$ is associated with $d-1$ points on the Bloch sphere. The Majorana

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representation provides a visual tool to understand the properties and transformations of quantum states. The direct visualization of qubit rotations are useful in the study of different topics of quantum information science such as quantum computation and communication; e.g., see [6] and its references.

In this note, we establish a connection between the Majorana representation and symmetry class of tensors in $\mathbb{V}^{\otimes(d-1)}$ for $\mathbb{V} = \mathbb{C}^2$ associated with the principal character ξ. Using this connection, we provide an easy scheme to determine $v_1, \ldots, v_{d-1} \in \mathbb{C}^2$ associated with a given vector $(a_0, \ldots, a_{d-1}) \in \mathbb{C}^d$. Additionally, we present a simple formula for the inner product of **a** and **b** in $\mathbb{C}^d$ in terms of their $(d-1)$-qubit presentations. Numerical examples are given to illustrate the result. Extension of the result to mixed states is also considered.

2 Results

2.1 Preliminary

Let us present the following standard set up of a symmetry class of tensors in the $(d-1)$-fold tensor space $\mathbb{V}^{\otimes(d-1)}$. In our study we focus on $\mathbb{V} = \mathbb{C}^2$ and the principal character ξ on the symmetric group S_{d-1} of degree $d-1$ such that $\xi(\sigma) = 1$ for all $\sigma \in S_{d-1}$. Define the symmetrizer on the tensor space $\mathbb{V}^{\otimes(d-1)}$ by

$$T(v_1 \otimes \cdots \otimes v_{d-1}) = \frac{1}{(d-1)!} \sum_{\sigma \in S_{d-1}} \xi(\sigma) v_{\sigma^{-1}(v_1)} \otimes \cdots \otimes v_{\sigma^{-1}(d-1)}.$$

Then $\mathbb{V}_\xi^{(d-1)} = T(\mathbb{V}^{\otimes(d-1)})$ is a subspace of $\mathbb{V}^{\otimes(d-1)}$ known as the symmetry class of tensors over $\mathbb{V}$ associated with ξ on S_{d-1}. The elements in $\mathbb{V}_\xi^{(d-1)}$ of the form $T(v_1 \otimes \cdots \otimes v_m)$ are called decomposable tensors and are denoted by $v^\bullet = v_1 \bullet \cdots \bullet v_m$. One may see [2,4] for some general background. In fact, researchers have used decomposable tensors to model boson states; see [1].

Let $\{e_0, e_1\}$ be the standard orthonormal basis of $\mathbb{V} = \mathbb{C}^2$ using the standard inner product $\langle u, v\rangle = u^* v$, where X^* denotes the conjugate transpose of X if X is a complex vector or matrix. Then $\mathbb{V}_\xi^{(d-1)}$ is the subspace of $\mathbb{V}^{\otimes(d-1)}$ spanned by the orthogonal basis

$$\mathcal{S} = \{e_{i_1} \bullet \cdots \bullet e_{i_{d-1}} : 0 \le i_1 \le \cdots \le i_{d-1} \le 1\}$$

using the induced inner product on decomposable tensor $u_1 \bullet \cdots \bullet u_{d-1}$ and $v_1 \bullet \cdots \bullet v_{d-1}$ so that

$$\langle u_1 \bullet \cdots \bullet u_{d-1}, v_1 \bullet \cdots \bullet v_{d-1}\rangle = \frac{1}{(d-1)!}\mathrm{per}(\langle u_i, v_j\rangle),$$

where

$$\mathrm{per}(X) = \sum_{\sigma \in S_k} \prod_{j=1}^{k} x_{j\sigma(j)} \quad \text{for } X = (x_{ij}) \in \mathbb{M}_k$$

is the permanent of $X \in \mathbb{M}_k$; see e.g., [5] for basic properties of the permanent. If $j_1 = \cdots = j_\ell = 0$ and $j_{\ell+1} = \cdots = j_{d-1} = 1$, then

$$\langle e_{j_1} \bullet \cdots \bullet e_{j_{d-1}}, e_{j_1} \bullet \cdots \bullet e_{j_{d-1}}\rangle = \mathrm{per}\left(J_\ell \oplus J_{d-1-\ell}\right)/(d-1)! = \binom{d-1}{\ell}^{-1},$$

where $J_r \in M_r$ has all entries equal to 1. Thus, after normalization $\mathcal{S}$ becomes an orthonormal basis $\{f_0^\bullet, \ldots, f_{d-1}^\bullet\}$. Let

$$C_j = [\underbrace{e_0 \ \cdots \ e_0}_{d-1-j} \ \underbrace{e_1 \ \cdots \ e_1}_{j}] \in \mathbb{M}_{2,d-1}, \qquad j = 0, \ldots, d-1. \tag{1}$$

Suppose $v_1, \ldots, v_{d-1} \in \mathbb{C}^2$. Then the decomposable tensor

$$v^\bullet = v_1 \bullet \cdots \bullet v_{d-1} = a_0 f_0^\bullet + \cdots + a_{d-1} f_{d-1}^\bullet \tag{2}$$

with

$$a_j = \langle f_j^\bullet, v^\bullet \rangle = \frac{1}{(d-1)!}\sqrt{\binom{d-1}{j}}\mathrm{per}(C_j^*[v_1 \cdots v_{d-1}]), \qquad j = 0, \ldots, d-1. \tag{3}$$

Moreover,

(a) $\gamma(v_1 \bullet \cdots \bullet v_{d-1}) = \mu_1 v_1 \bullet \cdots \bullet \mu_{d-1} v_{d-1}$ if $\mu_1, \ldots, \mu_{d-1}, \gamma \in \mathbb{C}$ satisfy $\mu_1 \cdots \mu_{d-1} = \gamma$,
(b) $v_1 \bullet \cdots \bullet v_{d-1} = v_{\sigma(1)} \bullet \cdots \bullet v_{\sigma(d-1)}$ if $\sigma \in S_{d-1}$ is a permutation of $(1, \ldots, d-1)$.

We will also use the following fact about the zeros of a complex polynomial. Let $E_k(\mu_1, \ldots, \mu_{d-1})$ be the kth elementary symmetric function for $\mu_1, \ldots, \mu_{d-1}$, i.e.,

$$E_k(\mu_1, \ldots, \mu_{d-1}) = \sum_{1 \le j_1 < \cdots < j_k \le d-1} \mu_{j_1} \cdots \mu_{j_k}, \qquad k = 1, \ldots, d-1.$$

Let

$$\begin{aligned} g(z) &= c_0 z^{d-1} + c_1 z_{d-2} + \cdots + c_{d-1} = c_0 \left(z^{d-1} + \frac{c_1}{c_0} z_{d-2} + \cdots + \frac{c_{d-1}}{c_0} \right) \\ &= c_0 (z - \mu_1) \cdots (z - \mu_{d-1}), \end{aligned}$$

where $c_0 \neq 0$. Then

$$\frac{c_j}{c_0} = (-1)^j E_j(\mu_1, \ldots, \mu_{d-1}), \qquad j = 1, \ldots, d-1.$$

2.2 Main result and examples

Theorem 2.1 *Let $\mathbb{V} = \mathbb{C}^2$ and $\{f_0^\bullet, \ldots, f_{d-1}^\bullet\}$ be the standard orthonormal basis for $\mathbb{V}_\xi^{(d-1)}$. If $(a_0, \ldots, a_{d-1})^t \in \mathbb{C}^d$ is nonzero and $r \ge 0$ is the smallest integer such that $a_r \neq 0$, then for $\gamma_r = a_r \sqrt{\binom{d-1}{r}}$*

$$\frac{1}{\gamma_r}(a_0 f_0^\bullet + \cdots + a_{d-1} f_{d-1}^\bullet) = v_1 \bullet \cdots \bullet v_{d-1}$$

so that

$$a_0 f_0^\bullet + \cdots + a_{d-1} f_{d-1}^\bullet = v_1 \bullet \cdots \bullet v_{d-2} \bullet (\gamma_r v_{d-1})$$

with $v_1 = \cdots = v_r = (0, 1)^t$, and $v_j = (1, \mu_j)^t$ for $j = r+1, \ldots, d-1$, where $\mu_{r+1}, \ldots, \mu_{d-1}$ are the zeros of the Majorana polynomial

$$g(z) = \sum_{j=0}^{d-1} (-1)^j a_j \sqrt{\binom{d-1}{j}} z^{d-1-j}.$$

If $b_0 f_0^\bullet + \cdots + b_{d-1} f_{d-1}^\bullet = u_1 \bullet \cdots \bullet u_{d-1}$ and $c_0 f_0^\bullet + \cdots + c_{d-1} f_{d-1}^\bullet = w_1 \bullet \cdots \bullet w_{d-1}$, then

$$\sum_{j=0}^{d-1} \bar{b}_j c_j = \langle u_1 \bullet \cdots \bullet u_{d-1}, w_1 \bullet \cdots \bullet w_{d-1} \rangle = \mathrm{per}(\langle u_i, w_j \rangle)/(d-1)!.$$

By Theorem 2.1, every vector $f^\bullet \in \mathbb{V}_\xi^{(d-1)}$ admits a representation of the form $u_1 \bullet \cdots \bullet u_{d-1}$. In particular, if $(a_0, \ldots, a_{d-1})^t \in \mathbb{C}^d$ is a quantum state, a unit vector, then

$$a_0 f_0^\bullet + \cdots + a_{d-1} f_{d-1}^\bullet = \frac{1}{\gamma} \left(\frac{v_1}{\|v_1\|} \bullet \cdots \bullet \frac{v_{d-1}}{\|v_{d-1}\|} \right),$$

where $v_1, \ldots, v_{d-1}$ are defined as in Theorem 2.1 and

$$\gamma = \left\| \left(\frac{v_1}{\|v_1\|} \bullet \cdots \bullet \frac{v_{d-1}}{\|v_{d-1}\|} \right) \right\| = \prod_{j=1}^{d-1} \|v_j\| \left\{ \frac{1}{(d-1)!} \mathrm{per}(\langle v_i, v_j \rangle) \right\}^{1/2}.$$

Proof of Theorem 2.1 If $r = d-1$, then clearly $\frac{1}{\gamma_r}(a_r f_{d-1}^\bullet) = \frac{1}{a_r}(a_r f_{d-1}^\bullet) = e_1 \bullet \cdots \bullet e_1$.

Suppose $r < d-1$. Construct the vectors $v_1, \ldots, v_{d-1}$ as described. We will show that

$$\frac{1}{a_r\sqrt{\binom{d-1}{r}}}(a_0 f_0^\bullet + \cdots a_{d-1} f_{d-1}^\bullet) = v_1 \bullet \cdots \bullet v_{d-1}.$$

Let $\mathbf{1}_k \in \mathbb{C}^k$ has all entries equal to 1, C_j be defined as in (1), and $Q \in \mathbb{M}_{2,d-1}$ have columns $v_1, \ldots, v_{d-1}$. Then

$$C_j^* Q = \begin{pmatrix} 0_{d-1-j,r-1} & \mathbf{1}_{d-1-j}\mathbf{1}_{d-r}^t \\ \mathbf{1}_j\mathbf{1}_{r-1}^t & \mathbf{1}_j(\mu_r, \ldots, \mu_{d-1}) \end{pmatrix}.$$

By a direct computation, say, using the Laplace expansion formula for permanent and induction, we have the following. For $j = 0, \ldots, r-1$, we have $\mathrm{per}(C_j^* Q) = 0$ and hence

$$\langle f_j^\bullet, v_1 \bullet \cdots \bullet v_{d-1}\rangle = \sqrt{\binom{d-1}{j}}\mathrm{per}(C_j^* Q)/(d-1)! = 0.$$

For $j = r, \ldots, d-1$, we have $\mathrm{per}(C_j^* Q) = j!(d-1-j)!E_{j-r}(\mu_{r+1}, \ldots, \mu_{d-1})$, and hence

$$\begin{aligned}\langle f_j^\bullet, v_1 \bullet \cdots \bullet v_{d-1}\rangle &= \sqrt{\binom{d-1}{j}}\mathrm{per}(C_j^* Q)/(d-1)! \\ &= E_{j-r}(\mu_{r+1}, \ldots, \mu_{d-1})/\sqrt{\binom{d-1}{j}}.\end{aligned}$$

Since $\mu_r, \ldots, \mu_{d-1}$ are the zeros of $g(x)$,

$$E_{j-r}(\mu_{r+1}, \ldots, \mu_{d-1})/\sqrt{\binom{d-1}{j}} = \frac{a_j}{a_r\sqrt{\binom{d-1}{r}}} = \frac{a_j}{\gamma_r}.$$

Hence,

$$\frac{1}{\gamma_r}(a_0 f_0^\bullet + \cdots + a_{d-1} f_{d-1}^\bullet) = v_1 \bullet \cdots \bullet v_{d-1}.$$

The last statement is clear. □

The following numerical examples illustrate Theorem 2.1.

Example 2.2 Suppose $d = 5$ and $\mathbf{a} = (a_0, a_1, a_2, a_3, a_4)^t \in \mathbb{C}^5$ be a nonzero vector. Let $r \geq 0$ be the smallest integer such that $a_r \neq 0$. Then for $\gamma_r = a_r\sqrt{\binom{4}{r}}$,

$$\frac{1}{\gamma_r}(a_0 f_0^\bullet + a_1 f_1^\bullet + a_2 f_2^\bullet + a_3 f_3^\bullet + a_4 f_4^\bullet) = v_1 \bullet v_2 \bullet v_3 \bullet v_4,$$

with $v_1 = \cdots = v_r = (0, 1)^t$ and $v_j = (1, \mu_j)^t$ for $j = r+1, \ldots, 4$, where $\mu_{r+1}, \ldots, \mu_4$ are the zeros of the Majorana polynomial

$$g(z) = \sqrt{\binom{4}{0}}a_0 z^4 - \sqrt{\binom{4}{1}}a_2 z^3 + \sqrt{\binom{4}{2}}a_2 z^2 - \sqrt{\binom{4}{3}}a_3 z + \sqrt{\binom{4}{4}}a_4.$$

(i) Let $\mathbf{a} = (1, 3, 13/\sqrt{6}, 6, 4)^t \in \mathbb{C}^5$. Then $g(z) = z^4 - 6z^3 + 13z^2 - 12z + 4 = (z-1)^2(z-2)^2$ so that $\mathbf{a}$ corresponds to $u_1 \bullet u_2 \bullet u_3 \bullet u_4$ with $u_1 = u_2 = (1, 1)^t$ and $u_3 = u_4 = (1, 2)^t$.

(ii) Let $\mathbf{b} = (0, 1/2, \sqrt{6}, 11/2, 6)^t$. Then $g(z) = z^3 - 6z^2 + 11z - 6 = (z-1)(z-2)(z-3)$ so that $\mathbf{b}$ corresponds to $v_1 \bullet \cdots \bullet v_4$ with $v_1 = (0, 1)^t$ and $v_2 = (1, 1)^t$, $v_3 = (1, 2)^t$ and $v_4 = (1, 3)^t$.

(iii) Let $\mathbf{c} = (0, 0, 1/\sqrt{6}, 1, 1)^t$. Then $g(z) = z^2 - 2z + 1 = (z-1)^2$ so that $\mathbf{c}$ corresponds to $w_1 \bullet \cdots \bullet w_4$ with $w_1 = w_2 = (0, 1)^t$, and $w_3 = w_4 = (1, 1)^t$.

We have $\langle \mathbf{a}, \mathbf{b}\rangle = \mathrm{per}([u_1u_2u_3u_4]^*[v_1v_2v_3v_4])/4! = 143/2$,
$\langle \mathbf{a}, \mathbf{c}\rangle = \mathrm{per}([u_1u_2u_3u_4]^*[w_1w_2w_3w_4])/4! = 12 + 1/6$, and
$\langle \mathbf{b}, \mathbf{c}\rangle = \mathrm{per}([v_1v_2v_3v_4]^*[w_1w_2w_3w_4])/4! = 25/2$.

2.3 Mixed states

Recall that a general quantum state is called a mixed state and is represented by a density matrix ρ, which is a positive semi-definite matrix with trace 1. If ρ is rank one, then ρ is pure state. If $\mathbf{a}$ corresponds to $u_1 \bullet \cdots \bullet u_{d-1} \in \mathbb{V}_\xi^{d-1}$, then by (3) the corresponding density matrix $\rho = \mathbf{a}\mathbf{a}^* \in \mathbb{M}_d$ has (r, s) entry equal to

$$a_r \bar{a}_s = \sqrt{\binom{d-1}{r}\binom{d-1}{s}} \frac{\operatorname{per}(C_r^*[u_1 \cdots u_{d-1}])\operatorname{per}([u_1 \cdots u_{d-1}]^* C_s])}{(d-1)!(d-1)!}, \qquad 0 \le r, s \le d-1,$$

where C_j is defined as in (1).

There has been interest in finding the Majorana representation for mixed states; e.g., see [6]. A general mixed state can be written as $\rho = \sum_{j=1}^r p_j \rho_j \in \mathbb{M}_d$, where $(p_1, \ldots, p_r)$ is a probability vector and $\rho_1, \ldots, \rho_r$ are pure states. We can apply Theorem 2.1 to each ρ_j, and express it in terms of $d-1$ qubit states. Then the mixed state ρ can be associated with a collection of r sets of qubit states each has $d-1$ elements and a probability vector $(p_1, \ldots, p_r)$.

Alternatively, by purification one may express ρ as the partial trace of a pure stat $|\psi\rangle\langle\psi|$ with $|\psi\rangle \in \mathbb{C}^{d^2}$, which admits a Majorana representation of $d^2 - 1$ qubits.

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