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ORIGINAL RESEARCH

Sufficient conditions for component factors in a graph

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Abstract Let G be a graph and $\mathcal{H}$ be a set of connected graphs. A spanning subgraph H of G is called an $\mathcal{H}$–factor if each component of H is isomorphic to a member of $\mathcal{H}$. In this paper, we first present a lower bound on the size (resp. the spectral radius) of G to guarantee that G has a $\{P_2, C_n : n \geq 3\}$–factor (or a perfect k–matching for even k) and construct extremal graphs to show all this bounds are best possible. We then provide a lower bound on the signless laplacian spectral radius of G to ensure that G has a $\{K_{1,j} : 1 \leq j \leq k\}$–factor, where $k \geq 2$ is an integer. Moreover, we also provide some Laplacian eigenvalue (resp. toughness) conditions for the existence of $\{P_2, C_n : n \geq 3\}$–factor, $P_{\geq 3}$–factor and $\{K_{1,j} : 1 \leq j \leq k\}$–factor in G, respectively. Some of our results extend or improve the related existing results.

Keywords Component factor · (Signless Laplacian) Spectral radius · Laplacian eigenvalue · Toughness · Perfect k–matching

1 Introduction

All graphs considered in this paper are undirected, connected and simple. Let $G = (V(G), E(G))$ be a graph of order n and size m. For a subset $S \subseteq V(G)$, we use $G[S]$ and $G - S$ to denote the subgraphs of G induced by S and $V(G)\backslash S$, respectively. We denote by $G_1 \cup G_2$ the disjoint union of G_1 and G_2. The join $G_1 \vee G_2$ is the graph obtained from $G_1 \cup G_2$ by adding all possible edges between $V(G_1)$ and $V(G_2)$. As usual, the star and the cycle of order n are denoted by $K_{1,n-1}$ and C_n, respectively. Let G_1 and G_2 be two vertex–disjoint graphs. Other undefined notations can be found in [4].

The Laplacian and the signless Laplacian matrices of a graph G are defined as $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$, where $A(G)$ and $D(G)$ are the adjacency matrix and the diagonal degree matrix of G,

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respectively. The largest eigenvalues of $A(G)$ and $Q(G)$ are called the spectral radius and the signless Laplacian spectral radius of G, denoted by $\rho(G)$ and $q(G)$, respectively. Let $\mu_i(G)$ (or μ_i for short) be the i-th smallest Laplacian eigenvalues of G. In particular, the second smallest Laplacian eigenvalue $\mu_2(G)$ is also known as the algebraic connectivity of G [8].

For a set of connected graphs $\mathcal{H}$, a spanning subgraph H of G is called an $\mathcal{H}$–factor if each component of H is isomorphic to a member of $\mathcal{H}$. In particular, an $\mathcal{H}$–factor is a $\mathcal{P}_{\geq \ell}$–factor if $\mathcal{H} = \{P_\ell, P_{\ell+1}, \ldots\}$, where P_ℓ is a path with ℓ vertices. Up to now, there have been lots of research work to seek the conditions for the existence of $\mathcal{H}$–factor in a graph, such as $\{P_2, C_n : n \geq 3\}$–factor [9,17], $\{K_{1,j} : 1 \leq j \leq k\}$–factor [1], $P_{\geq 3}$–factor [13], etc. We won't list them all here, but we'll focus primarily on those related to the spectral conditions for the existence of $\mathcal{H}$–factor in a graph. Zhang [20] characterize the extremal graphs with maximum spectral radius among all connected graphs of given order with prescribed minimum degree and without a $\mathcal{P}_{\geq 2}$–factor or a $\mathcal{P}_{\geq 3}$–factor, which generalizes the result in Li and Miao [14]. In [16], Miao and Li give some sufficient conditions (size, the spectral radius, or the distance spectral radius) to ensure that a graph contains a $\{K_{1,j} : 1 \leq j \leq k\}$–factor, where $k \geq 2$ be an integer. We refer the readers to [18] for more details on graph factors.

Motivated by [17], it is natural and interesting to ask whether or not there is a spectral condition to guarantee the existence of $\{P_2, C_n : n \geq 3\}$–factor in a graph? Note that every even cycle can be decomposed into a combination of P_2. Then the above problem has another statement as follows: whether or not there is a spectral condition to guarantee the existence of $\{P_2, C_{2i+1} : i \geq 1\}$–factor in a graph?

Inspired by the ideas from Miao and Li [16] and using the typical spectral techniques, we provide some sufficient conditions to ensure a graph G contains a $\{P_2, C_{2i+1} : i \geq 1\}$–factor in terms of the size and the spectral radius of G, respectively.

Theorem 1.1 *Let G be a graph of order n. Then we have*

(i) for $n \in \{3, 4, 10\}$ or $n \geq 12$, if $|E(G)| > \binom{n-2}{2} + 2$, then G has a $\{P_2, C_{2i+1} : i \geq 1\}$–factor;
(ii) for $n \in \{5, 7, 9, 11\}$, if $|E(G)| > \frac{(3n-1)(n-1)}{8}$, then G has a $\{P_2, C_{2i+1} : i \geq 1\}$–factor;
(iii) for $n \in \{6, 8\}$, if $|E(G)| > \frac{3n(n-2)}{8}$, then G has a $\{P_2, C_{2i+1} : i \geq 1\}$–factor.

Theorem 1.2 *Let G be a graph of order n. Then we have*

(i) for $n \in \{3, 4, 8\}$ or $n \geq 10$, if $\rho(G) > \theta(n)$, then G has a $\{P_2, C_{2i+1} : i \geq 1\}$–factor, where $\theta(n)$ is the largest root of $x^3 - (n-4)x^2 - (n-1)x + 2(n-4) = 0$;
(ii) for $n \in \{5, 7, 9\}$, if $\rho(G) > \frac{n-3+\sqrt{5n^2-6n+5}}{4}$, then G has a $\{P_2, C_{2i+1} : i \geq 1\}$–factor;
(iii) for $n = 6$, if $\rho(G) > \frac{1+\sqrt{33}}{2}$, then G has a $\{P_2, C_{2i+1} : i \geq 1\}$–factor.

Recently, Miao and Li [16] provided a sufficient condition to ensure a graph G has a $\{K_{1,j} : 1 \leq j \leq k\}$–factor in terms of $\rho(G)$. As a continuance of their work, we further deduce a sharp lower bound on $q(G)$ to ensure that G has a $\{K_{1,j} : 1 \leq j \leq k\}$–factor.

Theorem 1.3 *Let $k \geq 2$ be an integer and G be a graph of order $n \geq 2k + 12$. If*

$$q(G) \geq q(K_1 \vee (K_{n-k-2} \cup (k+1)K_1)),$$

then G has a $\{K_{1,j} : 1 \leq j \leq k\}$–factor unless $K_1 \vee (K_{n-k-2} \cup (k+1)K_1)$.

For a graph G of order n, the *Laplacian eigenratio* of G is defined as $\frac{\mu_2(G)}{\mu_n(G)}$, and it has attracted great concern about the relations between the Laplacian eigenratio and other graph properties [10,11]. For example, it is known that for a graph G of order n, if n is even and $\frac{\mu_2(G)}{\mu_n(G)} \geq \frac{1}{2}$, then G has a perfect matching [5]. Furthermore, we also present the following Laplacian eigenratio conditions for the existence of $\{P_2, C_{2i+1} : i \geq 1\}$–factor, $P_{\geq 3}$–factor and $\{K_{1,j} : 1 \leq j \leq k\}$–factor in a graph, respectively.

Theorem 1.4 *Let G be a graph of order n. If $\frac{\mu_2(G)}{\mu_n(G)} \geq \frac{1}{2}$, then G has a $\{P_2, C_{2i+1} : i \geq 1\}$–factor.*

Theorem 1.5 *Let G be a graph of order n. If $\frac{\mu_2(G)}{\mu_n(G)} \geq \frac{1}{3}$, then G has a $P_{\geq 3}$–factor.*

Theorem 1.6 *Let $k \geq 2$ be an integer and G be a graph of order n. If $\frac{\mu_2(G)}{\mu_n(G)} \geq \frac{1}{k+1}$, then G has a $\{K_{1,j} : 1 \leq j \leq k\}$–factor.*

For $S \subseteq V(G)$, let $c(G-S)$ and $i(G-S)$ denote the numbers of connected components and isolated vertices in $G-S$, respectively. Recall that the toughness $t(G)$ of a connected graph G is defined as

$$t(G) = \min\left\{\frac{|S|}{c(G-S)} : \ S \subseteq V(G),\ \ c(G-S) \geq 2\right\}.$$

By convention, a complete graph has infinite toughness. This parameter was introduced by Chvátal [6] in 1973 and is closely related to many graph properties, including Hamiltonicity, pancyclicity and spanning trees, see [3]. Very recently, Gu and Haemers [11] derived a lower bound of $t(G)$ by building the relationship between the toughness of G and its Laplacian eigenvalues as follows: $t(G) \geq \frac{\mu_2(G)}{\mu_n(G)-\mu_2(G)}$ for any connected graph G of order n. This implies that $t(G) \geq \frac{\mu_2(G)}{\mu_n(G)-\mu_2(G)} \geq 1$ when $\frac{\mu_2(G)}{\mu_n(G)} \geq \frac{1}{2}$. Thus, by Theorem 1.4, we have the following result immediately.

Corollary 1.7 *Let G be a graph of order n. If $t(G) \geq 1$, then G has a $\{P_2,\ C_{2i+1} : i \geq 1\}$–factor.*

Remark 1.1 Recall that Guan, Ma and Shi [12] showed that for a graph G, if $t(G) \geq 1$, then G has a $\{P_2, C_{2i+1} : i \geq 2\}$–factor. So Theorem 1.4 can be viewed as a generalization of their result.

Similarly, we have the following results for the existence of $P_{\geq 3}$–factor and $\{K_{1,j} : 1 \leq j \leq k\}$–factor in a graph, respectively.

Corollary 1.8 *Let G be a graph of order n. If $t(G) \geq \frac{1}{2}$, then G has a $P_{\geq 3}$–factor.*

Remark 1.2 Recall that Zhou *et al.* [21] showed that for a graph G, if $t(G) \geq \frac{2}{3}$, then G has a $\mathcal{P}_{\geq 3}$–factor. So Corollary 1.8 can be viewed as a slight improvement of the result due to Zhou et.al.

Corollary 1.9 *Let $k \geq 2$ be an integer and G be a graph of order n. If $t(G) \geq \frac{1}{k}$, then G has a $\{K_{1,j} : 1 \leq j \leq k\}$–factor.*

A *matching M* in a graph is a set of pairwise non-adjacent edges. A *perfect matching* is a matching which matches all vertices of a graph. Let $f : E(G) \to \{0, 1, \ldots, k\}$ be an assignment such that the sum of weights of edges incident with any vertex is at most k, *i.e.*, $\sum_{e\sim v} f(e) \leq k$ for any vertex $v \in V(G)$. A k*--matching* is a subgraph induced by the edges with weight among $1, \ldots, k$ such that $\sum_{e\sim v} f(e) \leq k$. The sum of all weights, *i.e.*, $\sum_{e\in E(G)} f(e)$, is called the *size* of a k-matching f. A k–matching is *perfect* if $\sum_{e\sim v} f(e) = k$ for every vertex $v \in V(G)$. Clearly, a k–matching is perfect if and only if its size is $k|V(G)|/2$. For $k = 1$, the perfect 1–matching is also known as the the perfect matching. For $k = 2$, Tutte [17] showed that a connected graph G has a perfect 2–matching is equivalent to G has a $\{P_2,\ C_{2i+1} : i \geq 1\}$–factor. Moreover, for $k \geq 4$ be an even integer, it was shown by Lu and Wang [15] that a graph G contains a perfect k–matching if and only if G contains a perfect 2–matching. Thus, by Theorems 1.1, 1.2 and 1.4 and Corollary 1.7, we then have the following conditions for a graph G has a perfect k–matching.

Corollary 1.10 *Let $k \geq 2$ be an even integer and G be a graph of order n. Then we have*

(i) for $n \in \{3, 4, 10\}$ or $n \geq 12$, if $|E(G)| > \binom{n-2}{2} + 2$, then G has a perfect k–matching;
(ii) for $n \in \{5, 7, 9, 11\}$, if $|E(G)| > \frac{(3n-1)(n-1)}{8}$, then G has a perfect k–matching;
(iii) for $n \in \{6, 8\}$, if $|E(G)| > \frac{3n(n-2)}{8}$, then G has a perfect k–matching.

Corollary 1.11 *Let $k \geq 2$ be an even integer and G be a graph of order n. Then we have*

(i) for $n \in \{3, 4, 8\}$ or $n \geq 10$, if $\rho(G) > \theta(n)$, then G has a perfect k–matching, where $\theta(n)$ is the largest root of $x^3 - (n-4)x^2 - (n-1)x + 2(n-4) = 0$;
(ii) for $n \in \{5, 7, 9\}$, if $\rho(G) > \frac{n-3+\sqrt{5n^2-6n+5}}{4}$, then G has a perfect k–matching;
(iii) for $n = 6$, if $\rho(G) > \frac{1+\sqrt{33}}{2}$, then G has a perfect k–matching.

Corollary 1.12 *Let $k \geq 2$ be an even integer and G be a graph of order n. If $\frac{\mu_2(G)}{\mu_n(G)} \geq \frac{1}{2}$, then G has a perfect k–matching.*

Corollary 1.13 *Let $k \geq 2$ be an even integer and G be a graph of order n. If $t(G) \geq 1$, then G has a perfect k–matching.*

The remainder of the paper is organized as follows. In Section 2, we present some preliminary results, which will be used in the subsequent section. In Section 3, we will give the proofs of Theorems 1.1, 1.2 and 1.3, respectively. In section 4, we present the proofs of Theorems 1.4, 1.5 and 1.6, respectively. In the last section we construct extremal graph to show the bound obtained in Theorems 1.1 and 1.2 are best possible.

2 Preliminary

In this section, we present some preliminary results and lemmas which are useful.

Let $\phi_M(x) := det(xI - M)$ be the characteristic polynomial of a square matrix M, where I is the identity matrix, whose order is the same as that of M. Consider an $n \times n$ real symmetric matrix

$$M = \begin{pmatrix} M_{1,1} & M_{1,2} & \cdots & M_{1,m} \\ M_{2,1} & M_{2,2} & \cdots & M_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ M_{m,1} & M_{m,2} & \cdots & M_{m,m} \end{pmatrix},$$

whose rows and columns are partitioned according to a partitioning $X_1, X_2, \ldots, X_m$ of $\{1, 2, \ldots, n\}$. The *quotient matrix* $\mathcal{B}$ of the matrix M is the $m \times m$ matrix whose entries are the average row sums of the blocks $M_{i,j}$ of M. The partition is *equitable* if each block $M_{i,j}$ of M has constant row (and column) sum.

Lemma 2.1 *([19]) Let M be a square matrix with an equitable partition π and let M_π be the corresponding quotient matrix. Then every eigenvalue of M_π is an eigenvalue of M. Furthermore, if M is nonnegative and M_π is irreducible, then the largest eigenvalues of M and M_π are equal.*

Lemma 2.2 *([17]) A graph G has a $\{P_2, C_{2i+1} : i \geq 1\}$–factor if and only if $i(G-S) \leq |S|$ for any $S \subseteq V(G)$.*

Lemma 2.3 *([1]) For any integer $k \geq 2$, a graph G has a $\{K_{1,j} : 1 \leq j \leq k\}$–factor if and only if $i(G-S) \leq k|S|$ for any $S \subseteq V(G)$.*

In order to prove the main results for $\rho(G)$ and $q(G)$ simultaneously, we introduce the matrix $A_a(G) = aD(G) + A(G)$ and denote by $\rho_a(G)$ the largest eigenvalue of $A_a(G)$, where $a \geq 0$. Clearly, $A_0(G) = A(G)$ (resp. $A_1(G) = Q(G)$) and $\rho_0(G) = \rho(G)$ (resp. $\rho_1(G) = q(G)$).

Lemma 2.4 *([2]) If H is a spanning subgraph of a graph G, then $\rho_a(H) \leq \rho_a(G)$, with equality if and only if $G \cong H$; Moreover, if H is a proper subgraph of G, then $\rho_a(H) < \rho_a(G)$.*

Lemma 2.5 *([7]) Let G be a graph of order n with m edges. Then*

$$q(G) \leq \frac{2m}{n-1} + n - 2.$$

3 Proofs of Theorems 1.1, 1.2 and 1.3

We now give the proofs of Theorems 1.1, 1.2 and 1.3, respectively.

Proof of Theorem 1.1: Suppose to the contrary that G has no $\{P_2, C_{2i+1} : i \geq 1\}$–factor. Then Lemma 2.2 implies that there exists a non-empty subset $S \subseteq V(G)$ satisfying $i(G-S) \geqslant |S| + 1$. We choose such a connected graph G of order n so that its size is as large as possible. According to the choice of G, we see that the induced subgraph $G[S]$ and each connected component of $G - S$ are complete graphs, respectively, and $G \cong G[S] \vee (G - S)$.

First, we claim that there is at most one non-trivial connected component in $G - S$. Otherwise, we can add edges among all nontrivial connected components to get a bigger non-trivial connected component, which is a contradiction to the choice of G. For convenience, let $i(G-S) = i$ and $|S| = s$. We now consider the following two possible cases.

Case 1 $G - S$ has only one non-trivial connected component, say G_1.

In this case, let $|V(G_1)| = n_1 \geq 2$. We are to show $i = s + 1$. If $i \geq s + 2$, let H_1 be a new graph obtained from G by joining each vertex of G_1 with one vertex in $I(G-S)$ by an edge, where $I(G-S)$ is a set of isolated vertices in $G - S$. Then we have $|E(H_1)| = |E(G)| + n_1 > |E(G)|$ and $i(H_1 - S) \geq s + 1$, a contradiction to the choice of G. Hence $i \leq s + 1$. Recall that $i \geq s + 1$. Therefore, we have $i = s + 1$ and $G = K_s \vee (K_{n-2s-1} \cup (s+1)K_1)$.

Bear in mind that $n = s + s + 1 + n_1 \geq 2s + 3 \geq 5$ and $|E(G)| = s(s+1) + \binom{n-s-1}{2}$. By a directed calculation, we have

$$\begin{aligned}\binom{n-2}{2} + 2 - |E(G)| &= \frac{1}{2}(s-1)(2n-3s-8)\\ &\geq \frac{1}{2}(s-1)(4s+6-3s-8)\\ &= \frac{1}{2}(s-1)(s-2) \geq 0.\end{aligned}$$

Thus, $|E(G)| \leq \binom{n-2}{2}+2$ for $n \geq 5$. By a direct calculation, we have $\binom{n-2}{2}+2 < \frac{(3n-1)(n-1)}{8}$ for $n \in \{5, 7, 9, 11\}$ and $\binom{n-2}{2} + 2 < \frac{3n(n-2)}{8}$ for $n \in \{6, 8\}$, a contradiction.

Case 2 $G - S$ has no non-trivial connected component.

In this case, we are to prove $i \leq s + 2$. If $i \geq s + 3$, let H_2 be a new graph obtained from G by adding an edge in $I(G - S)$. Clearly, $i(H_2 - S) \geq s + 1$, $H_2 - S$ has exactly one non-trivial connected component and $|E(G)| < |E(H_2)|$, contradicting to the choice of G. Bear in mind that $i \geq s+1$, it suffices to consider $i = s+1$ (i.e., $n = 2s + 1$) and $i = s + 2$ (i.e., $n = 2s + 2$).

For $i = s + 1$, we have $G \cong K_s \vee (s + 1)K_1$. Therefore, $n = 2s + 1$ and $|E(G)| = s(s + 1) + \binom{s}{2}$. By a directed calculation, we have

$$\binom{n-2}{2} + 2 - |E(G)| = \binom{2s-1}{2} + 2 - s(s+1) - \binom{s}{2} = \frac{1}{2}(s-1)(s-6).$$

Thus, $|E(G)| \leq \binom{n-2}{2} + 2$ for $s = 1$ or $s \geq 6$, which is a contradiction for $n = 3$ or $n \geq 13$. For $s \in \{2, 3, 4, 5\}$ (or $n \in \{5, 7, 9, 11\}$), we have

$$|E(G)| = s(s+1) + \binom{s}{2} = \frac{3s^2+s}{2} = \frac{3(n-1)^2+2(n-1)}{8} = \frac{(3n-1)(n-1)}{8},$$

a contradiction.

For $i = s + 2$, we have $G \cong K_s \vee (s + 2)K_1$. Therefore, $n = 2s + 2$ and $|E(G)| = s(s + 2) + \binom{s}{2}$. By a directed calculation, we have

$$\binom{n-2}{2} + 2 - |E(G)| = \binom{2s}{2} + 2 - s(s+2) - \binom{s}{2} = \frac{1}{2}(s-1)(s-4).$$

Thus, $|E(G)| \leq \binom{n-2}{2} + 2$ for $s = 1$ or $s \geq 4$, which is a contradiction for $n = 4$ or $n \geq 10$. For $s \in \{2, 3\}$ (or $n \in \{6, 8\}$), we have

$$|E(G)| = s(s+2) + \binom{s}{2} = \frac{3s^2+3s}{2} = \frac{3}{2} \cdot \frac{n-1}{2} \cdot \left(\frac{n-1}{2}+1\right) = \frac{3n(n-2)}{8},$$

a contradiction.

In view of Cases 1 and 2, the proof of Theorem 1.1 is completed. □

Proof of Theorem 1.2: Suppose to the contrary that G has no $\{P_2, C_{2i+1} : i \geq 1\}$–factor. Then Lemma 2.1 implies that there exists a non-empty subset $S \subseteq V(G)$ satisfying $i(G - S) \geq |S| + 1$. We choose such a connected graph G of order n so that its adjacency spectral radius is as large as possible. By Lemma 2.4 and the choice of G, it follows that the induced subgraph $G[S]$ and each connected component of $G - S$ are complete graphs, respectively, and $G \cong G[S] \vee (G - S)$.

For convenience, let $i(G - S) = i$ and $|S| = s$. We claim that there exists at most one non–trivial connected component in $G - S$. Otherwise, we can add edges among all non–trivial connected components to get a non–trivial connected component of larger size, which gives a contradiction (based on Lemma 2.4). Let $\phi(x) = x^3 - (n-4)x^2 - (n-1)x + 2(n-4)$ be a real function in x and $\theta(n)$ be the largest root of $\phi(x) = 0$. We now consider the following two possible cases.

Case 1 $G - S$ has just one non–trivial connected component, say G_1.

In this case, let $|V(G_1)| = n_1 \geq 2$. We are to show $i = s+1$. If $i \geq s+2$, let H_1 be a new graph obtained from G by joining each vertex of G_1 with one vertex in $I(G-S)$ by an edge. Then $i(H_1 - S) \geq s+1$ and G is a proper spanning subgraph of H_1. By Lemma 2.4, we have $\rho(G) < \rho(H_1)$, a contradiction to the choice of G. Therefore $i \leq s+1$. Recall that $i \geq s+1$. Hence $i = s+1$ ($n = 2s+1+n_1 \geq 2s+3$) and $G = K_s \vee (K_{n-2s-1} \cup (s+1)K_1)$. We now consider the partition $V(G) = V(K_s) \cup V((s+1)K_1) \cup V(K_{n-2s-1})$. Then the corresponding quotient matrix of $A(G)$ is

$$B_1 = \begin{pmatrix} s-1 & s+1 & n-2s-1 \\ s & 0 & 0 \\ s & 0 & n-2s-2 \end{pmatrix},$$

and the characteristic polynomial of B_1 is

$$\begin{aligned}\phi_{B_1}(x) &= x^3 - (n-s-3)x^2 - \left(s^2+n-2\right)x - s^3 + ns^2 - s^3 - 3s^2 + ns - s^2 - 2s \\ &= x^3 - (n-s-3)x^2 - \left(s^2+n-2\right)x - 2s^3 + (n-4)s^2 + (n-2)s.\end{aligned}$$

Note that the partition $V(G) = V(K_s) \cup V((s+1)K_1) \cup V(K_{n-2s-1})$ is equitable. Then Lemma 2.1 implies that the largest root θ_1 of $\phi_{B_1}(x) = 0$ is $\rho(G)$. In order to prove $\theta(n) \geq \rho(G)$, it suffices to show $\phi(\theta_1) < 0$. Note that

$$\begin{aligned}\phi(\theta_1) &= \phi(\theta_1) - \phi_{B_1}(\theta_1) \\ &= (s-1)(-\theta_1^2 + (s+1)\theta_1 + 2s^2 + s - ns + 1 - n + 4s + 4 - n + s + 3) \\ &= (s-1)(-\theta_1^2 + (s+1)\theta_1 + 2s^2 - (n-6)s - 2n + 8) \\ &\leq (s-1)(-\theta_1^2 + (s+1)\theta_1 + 2 - s) \\ &= (s-1)(-\theta_1(\theta_1 - s - 1) + 2 - s)\end{aligned}$$

and K_{s+2} is a proper subgraph of $K_s \vee (K_{n-2s-1} \cup (s+1)K_1)$. Then by Lemma 2.4, we have $\rho(G) = \theta_1 > s+1$. It follows that

$$\begin{aligned}\phi(\theta_1) &= \phi(\theta_1) - \phi_{B_1}(\theta_1) \\ &\leq (s-1)(-\theta_1(\theta_1 - s - 1) + 2 - s) \\ &< (s-1)(-\theta_1(s+1-s-1) + 2 - s) \\ &= (s-1)(2-s) \leq 0.\end{aligned}$$

Thus, $\phi(\theta_1) = \phi(\theta_1) - \phi_{B_1}(\theta_1) < 0$ for $s \geq 1$, which implies that $\rho(G) = \theta_1 < \theta(n)$, a contradiction for $n \notin \{5, 6, 7, 9\}$. By a direct calculation, we have $\theta(5) \approx 2.34 < \frac{5-3+\sqrt{5\times5^2-6\times5+5}}{4} = 3$, $\theta(6) \approx 3.18 < \frac{1+\sqrt{33}}{2}$, $\theta(7) \approx 4.11 < \frac{7-3+\sqrt{5\times7^2-6\times7+5}}{4} = 1+\sqrt{33}$, $\theta(9) \approx 6.05 < \frac{9-3+\sqrt{5\times9^2-6\times9+5}}{4} = \frac{3+\sqrt{89}}{2}$.

Case 2 $G-S$ has no non–trivial connected component.

If $i \geq s+3$, let H_2 be a graph obtained form G by adding an edge in $I(G-S)$. Then $i(H_2 - S) \geq s+1$ and $H_2 - S$ has one non–trivial connected component. From Case 1, we have $\rho(G) \leq \theta(n)$, contradicting to the choice of G. Thus, we only consider $i = s+1$ (*i.e.*, $n = 2s+1$) and $i = s+2$ (*i.e.*, $n = 2s+2$).

For $i = s+1$, we have $G \cong K_s \vee (s+1)K_1$ and $n = 2s+1$. One may consider the partition $V(K_s \vee (s+1)K_1) = V(K_s) \cup V((s+1)K_1)$. Its corresponding quotient matrix of $A(K_s \vee (ks+1)K_1)$ is

$$B_2 = \begin{pmatrix} s-1 & s+1 \\ s & 0 \end{pmatrix}.$$

And the characteristic polynomial of B_2 is

$$\phi_{B_2}(x) = x^2 - (s-1)x - s(s+1).$$

Note that the partition $V(K_s \vee (s+1)K_1) = V(K_s) \cup V((s+1)K_1)$ is equitable. Then Lemma 2.1 implies that $\rho(K_s \vee (s+1)K_1)$ is the largest root θ_2 of $\phi_{B_2}(x) = 0$. As $\phi_{B_2}(x) = 0$ is a quadratic equation with respect to x, by the root formula, we have

$$\rho\left(K_s \vee (s+1)K_1\right) = \theta_2 = \frac{s-1+\sqrt{5s^2+2s+1}}{2}.$$

If $s = 1$, then we have $n = 3$, $\phi_{B_2}(x) = x^2 - 2$ and $\phi(x) = (x+1)(x^2-1)$. That is $\theta_2 = \sqrt{2} = \theta(3)$, a contradiction to the condition. If $s \in \{2, 3, 4\}$ (or $n \in \{5, 7, 9\}$), then $\theta_2 = \frac{s-1+\sqrt{5s^2+2s+1}}{2} = \frac{n-3+\sqrt{5n^2-6n+5}}{4}$, a contradiction. Next, we consider $s \geq 5$. Note that

$$\begin{aligned}\phi(\theta_2) = \phi(\theta_2) - \theta_2\phi_{B_2}(\theta_2) &= -(s-2)\theta_2^2 + (s^2-s)\theta_2 + 4s - 6 \\ &= -\theta_2[(s-2)\theta_2 - s^2 + s] + 4s - 6,\end{aligned}$$

and

$$\theta_2 = \frac{s-1+\sqrt{5s^2+2s+1}}{2} = \frac{s-1+\sqrt{(2s+1)^2+s^2-2s}}{2} > \frac{3s}{2}.$$

We then have

$$\begin{aligned}\phi(\theta_2) &= \phi(\theta_2) - \theta_2\phi_{B_2}(\theta_2) \\ &= -\theta_2[(s-2)\theta_2 - s^2 + s] + 4s - 6 \\ &< -\theta_2[(s-2)\cdot\frac{3s}{2} - s^2 + s] + 4s - 6 \\ &= -\theta_2(\frac{s^2}{2} - 2s) + 4s - 6 \\ &< -\frac{3s^3}{4} + 3s^2 + 4s - 6.\end{aligned}$$

Let $f(s) = -\frac{3s^3}{4} + 3s^2 + 4s - 6$. Note that $f'(s) = -\frac{9}{4}s^2 + 6s + 4 < 0$ when $s \geq 5$. Hence $f(s)$ is a monotonically decreasing functions with respect to s. Therefore,

$$\begin{aligned}\phi(\theta_2) = \phi(\theta_2) - \theta_2\phi_{B_2}(\theta_2) &< -\frac{3s^3}{4} + 3s^2 + 4s - 6 \\ &\leq f(5) = -\frac{19}{4} < 0.\end{aligned}$$

Thus, $\phi(\theta_2) < 0$ for $s \geq 5$ and $\rho(G) = \theta_2 < \theta(n)$, a contradiction.

For $i = s + 2$. In this subcase one has $G = K_s \vee (s+2)K_1$ and $n = 2s + 2$. We consider the partition $V(G) = V(K_s) \cup V((s+2)K_1)$. Then the corresponding quotient matrix of $A(G)$ is

$$B_3 = \begin{pmatrix} s-1 & s+2 \\ s & 0 \end{pmatrix},$$

and its characteristic polynomial is

$$\phi_{B_3}(x) = x^2 - (s-1)x - s(s+2).$$

Note that the partition $V(G) = V(K_s) \cup V((s+2)K_1)$ is equitable. Then Lemma 2.1 implies that the largest root θ_3 of $\phi_{B_3}(x) = 0$ is $\rho(G)$. As $\phi_{B_3}(x) = 0$ is a quadratic equation with respect to x, we may easily obtain that

$$\theta_3 = \rho(G) = \frac{s-1+\sqrt{5s^2+6s+1}}{2}.$$

If $s = 1$, then $n = 4$, we have $\phi_{B_3}(x) = x^2 - 3$ and $\phi(x) = x(x^2-3)$, which implies that $\theta_3 = \sqrt{3} = \theta(4)$, a contradiction to the condition. If $s = 2$ (or $n = 6$), then $\theta_3 = \frac{1+\sqrt{33}}{2}$, a contradiction. Next, we consider $s \geq 3$.

Bear in mind that $n = 2s + 2$ and so

$$\phi(x) = x^3 - (2s-2)x^2 - (2s+1)x - 4(s-1).$$

In what follows, it suffices to prove $\phi(\theta_3) < 0$. Note that

$$\begin{aligned}\phi(\theta_3) &= \phi(\theta_3) - \theta_3\phi_{B_3}(\theta_3) \\ &= -\theta_3^2(s-1) + \theta_3(s^2-1) + 4(s-1) \\ &= -(s-1)(\theta_3^2 - (s+1)\theta_3 - 4),\end{aligned}$$

and

$$\theta_3 = \frac{s-1+\sqrt{5s^2+6s+1}}{2} = \frac{s-1+\sqrt{(s+5)^2+4(s^2-s-6)}}{2} > s+2.$$

Then we have

$$\begin{aligned}\phi(\theta_3) &= \phi(\theta_3) - \theta_3\phi_{B_3}(\theta_3)\\ &= -(s-1)(\theta_3^2-(s+1)\theta_3-4)\\ &< -(s-1)[(s+2)^2-(s+1)(s+2)-4]\\ &= -(s-1)(s-2) < 0.\end{aligned}$$

Thus, $\phi(\theta_3) < 0$ for $s \geq 3$ and $\rho(G) = \theta_3 < \theta(n)$, a contradiction.

In view of Cases 1 and 2, the proof of Theorem 1.2 is completed. □

Proof of Theorem 1.3: Assume that G has no $\{K_{1,j} : 1 \leq j \leq k\}$–factor. Then Lemma 2.3 implies that there exists some nonempty subset $S \subseteq V(G)$ such that $i(G-S) \geq k|S|+1$. Let $|S| = s$. Then G is a spanning subgraph of $G_1 = K_s \vee (K_{n-(k+1)s-1} \cup (ks+1)K_1)$. Hence Lemma 2.4 implies that

$$q(G) \leq q(G_1), \tag{1}$$

with equality if and only if $G \cong G_1$.

If $s = 1$, then $G_1 \cong K_1 \vee (K_{n-k-2} \cup (k+1)K_1)$. Combining this with (1), we conclude that

$$q(G) \leq q(K_1 \vee (K_{n-k-2} \cup (k+1)K_1)),$$

where the equality holds if and only if $G \cong K_1 \vee (K_{n-k-2} \cup (k+1)K_1)$.

For $s \geq 2$, note that Lemma 2.5 implies that

$$\begin{aligned}q(G_1) &\leq \frac{2m(G_1)}{n-1} + n - 2\\ &= \frac{(n-ks-1)(n-ks-2)+2s(ks+1)}{n-1} + n - 2\\ &= \frac{(k^2+2k)s^2-(2kn-3k-2)s+2n^2-6n+4}{n-1}.\end{aligned} \tag{2}$$

Let $f(s) = (k^2+2k)s^2-(2kn-3k-2)s+2n^2-6n+4$. Since $n \geq (k+1)s+1$, then $2 \leq s \leq \frac{n-1}{k+1}$. By a simple calculation, we have

$$\begin{aligned}f(2) - f\left(\frac{n-1}{k+1}\right) &= (k^2+2k)\left(2^2-\left(\frac{n-1}{k+1}\right)^2\right) - (2kn-3k-2)\left(2-\frac{n-1}{k+1}\right)\\ &= \frac{(n-2k-3)(-2k^3+(n-8)k^2-7k-2)}{(k+1)^2}\\ &\geq \frac{9(k-2)(4k+1)}{(k+1)^2} \geq 0,\end{aligned} \tag{3}$$

where the inequality follows from the fact that $n \geq 2k+12$ and $k \geq 2$. This implies that, for $2 \leq s \leq \frac{n-1}{k+1}$, the maximum value of $f(s)$ is attained at $s = 2$. This together with (2) and (3) imply that

$$\begin{aligned}q(G_1) \leq \frac{f(2)}{n-1} &= \frac{2n^2-(4k+6)n+4k^2+14k+8}{n-1}\\ &= 2(n-k-2) - \frac{2(n-2k-6)k-4}{n-1}\\ &\leq 2(n-k-2) - \frac{12k-4}{n-1}\\ &< 2(n-k-2),\end{aligned} \tag{4}$$

where the penultimate inequality follows from $n \geq 2k+12$. Note that K_{n-k-1} is a proper subgraph of $K_1 \vee (K_{n-k-2} \cup (k+1)K_1)$. Then we have

$$q(K_1 \vee (K_{n-k-2} \cup (k+1)K_1)) > q(K_{n-k-1}) = 2(n-k-2).$$

Combining this with (1) and (4), we have

$$q(G) \leq q(G_1) < q(K_1 \vee (K_{n-k-2} \cup (k+1)K_1)).$$

Concluding the above results, we obtain

$$q(G) \leq q(K_1 \vee (K_{n-k-2} \cup (k+1)K_1)),$$

where the equality holds if and only if $G \cong K_1 \vee (K_{n-k-2} \cup (k+1)K_1)$. Let $S = \{u\}$ denote the unique vertex of degree $n-1$ in $K_1 \vee (K_{n-k-2} \cup (k+1)K_1)$. Then $i(K_1 \vee (K_{n-k-2} \cup (k+1)K_1) - S) \geq k+1 > |S|$ since u is adjacent to $k+1$ pendant vertices. Thus by Lemma 2.3, we have $K_1 \vee (K_{n-k-2} \cup (k+1)K_1)$ has no $\{K_{1,j} : 1 \leq j \leq k\}$–factor, and so the result follows. □

4 Proofs of Theorems 1.4, 1.5 and 1.6

In this section, we will present the following Laplacian eigenvalue conditions for the existence of $\{P_2, C_{2i+1} : i \geq 1\}$–factor, $P_{\geq 3}$–factor and $\{K_{1,j} : 1 \leq j \leq k\}$–factor, respectively. In order to prove our results, the following lemma is needed.

Lemma 4.1 *([10]) Let G be a graph of order n. Suppose that $S \subseteq V(G)$ such that $G - S$ is disconnected. Let X and Y be disjoint vertex subsets of $G - S$ such that $X \cup Y = V(G) - S$ with $|X| \leq |Y|$. Then*

$$|X| \leq \frac{\mu_n - \mu_2}{2\mu_n} n,$$

and

$$|S| \geq \frac{2\mu_2}{\mu_n - \mu_2} |X|.$$

Now we are in a position to give the proofs of our results. Our strategy for proving the following results is employed the similar argument as that was used in [10].

Proof of Theorem 1.4: Suppose to the contrary that G has no a $\{P_2, C_{2i+1} : i \geq 1\}$–factor. Then Lemma 2.2 implies that there exists a nonempty subset $S \subseteq V(G)$ such that

$$i(G-S) \geq |S| + 1 \geq 2.$$

Let $v_1, v_2, \ldots, v_c$ be the isolated vertices of $G - S$, where $c = i(G-S)$. Defined $X = \cup_{1 \leq i \leq \lfloor c/2 \rfloor} v_i$ and $Y = V(G) - S - X$. Then $\frac{c-1}{2} \leq |X| \leq |Y|$. By Lemma 4.1, we have

$$|S| \geq \frac{2\mu_2}{\mu_n - \mu_2} \cdot \frac{c-1}{2},$$

with equality holding only when $|X| = |Y| = \frac{c-1}{2}$.

However, note that $2\mu_2 \geq \mu_n$ and $X \geq \frac{c}{2} > \frac{c-1}{2}$ when c is even. Then we have

$$|S| \geq \frac{2\mu_2}{\mu_n - \mu_2} \cdot \frac{c}{2} \geq c = i(G-S).$$

When c is odd, by the definitions of X and Y, we have $|X| < |Y|$. Thus the equality never hold. Then

$$|S| > \frac{2\mu_2}{\mu_n - \mu_2} \cdot \frac{c-1}{2} \geq c - 1 = i(G-S) - 1,$$

a contradiction. □

Let $sun(G-S)$ denote the number of sun components in $G-S$. It is proved in [13] that every graph G admits a $P_{\geq 3}$–factor if and only if $sun(G-X) \leq 2|X|$ for every $X \subseteq V(G)$. Then we have the following result.

Proof of Theorem 1.5: Suppose to the contrary that G has no a $P_{\geq 3}$–factor. Then there exists a nonempty subset $S \subseteq V(G)$ such that

$$sun(G-S) \geq 2|S| + 1 \geq 3.$$

Let $V_1, V_2, \ldots, V_c$ be the sun components of $G-S$, where $c = sun(G-S)$. Without loss of generality, suppose that $|V_1| \leq |V_2| \leq \cdots \leq |V_c|$. Defined $X = \cup_{1 \leq i \leq \lfloor c/2 \rfloor} V_i$ and $Y = V(G) - S - X$. Then $\frac{c-1}{2} \leq |X| \leq |Y|$. By Lemma 4.1, we have

$$|S| \geq \frac{2\mu_2}{\mu_n - \mu_2} \cdot \frac{c-1}{2},$$

with equality holding only when $|X| = |Y| = \frac{c-1}{2}$.

However, note that $3\mu_2 \geq \mu_n$ and $X \geq \frac{c}{2} > \frac{c-1}{2}$ when c is even. Then

$$|S| \geq \frac{2\mu_2}{\mu_n - \mu_2} \cdot \frac{c}{2} \geq \frac{c}{2} = \frac{1}{2} \cdot sun(G-S).$$

When c is odd, by the definitions of X and Y, we have $|X| < |Y|$. Thus the equality never hold. Then

$$|S| > \frac{2\mu_2}{\mu_n - \mu_2} \cdot \frac{c-1}{2} \geq \frac{c-1}{2} = \frac{1}{2} \cdot (sun(G-S) - 1),$$

a contradiction. □

Proof of Theorem 1.6: Suppose to the contrary that G has no a $\{K_{1,j} : 1 \leq j \leq k\}$–factor. Then Lemma 2.3 implies that there exists a nonempty subset $S \subseteq V(G)$ such that

$$i(G-S) \geq k|S| + 1 \geq 3.$$

Let $v_1, v_2, \ldots, v_c$ be the isolated vertices of $G-S$, where $c = i(G-S)$. Defined $X = \cup_{1 \leq i \leq \lfloor c/2 \rfloor} v_i$ and $Y = V(G) - S - X$. Then $\frac{c-1}{2} \leq |X| \leq |Y|$. By Lemma 4.1, we have

$$|S| \geq \frac{2\mu_2}{\mu_n - \mu_2} \cdot \frac{c-1}{2},$$

with equality holding only when $|X| = |Y| = \frac{c-1}{2}$.

However, note that $(k+1)\mu_2 \geq \mu_n$ and $X \geq \frac{c}{2} > \frac{c-1}{2}$ when c is even. Then we have

$$|S| \geq \frac{2\mu_2}{\mu_n - \mu_2} \cdot \frac{c}{2} \geq \frac{c}{k} = \frac{1}{k} \cdot i(G-S).$$

When c is odd, by the definitions of X and Y, we have $|X| < |Y|$. Thus the equality never hold. Then

$$|S| > \frac{2\mu_2}{\mu_n - \mu_2} \cdot \frac{c-1}{2} \geq \frac{c-1}{k} = \frac{1}{k} \cdot (i(G-S) - 1),$$

a contradiction. □

5 Extremal graphs

In this section, we construct the extremal graphs to show that the bounds established in Theorems 1.1 and 1.2 are best possible, respectively.

Note that for a positive integer n, we have

(i) $|E(K_1 \vee (K_{n-3} \cup 2K_1))| = \binom{n-2}{2} + 2$ for $n \in \{3, 4, 10\}$ or $n \geq 12$;
(ii) $|E\left(K_{\frac{n-1}{2}} \vee \frac{n+1}{2}K_1\right)| = \frac{3(n-1)(n-1)}{8}$ for $n \in \{5, 7, 9, 11\}$;
(iii) $|E\left(K_{\frac{n-2}{2}} \vee \frac{n+2}{2}K_1\right)| = \frac{3n(n-2)}{8}$ for $n \in \{6, 8\}$.

Clearly, the graphs $K_1 \vee (K_{n-3} \cup 2K_1)$, $K_{\frac{n-1}{2}} \vee \frac{n+1}{2}K_1$ and $K_{\frac{n-2}{2}} \vee \frac{n+2}{2}K_1$ have no $\{P_2,\ C_{2i+1} : i \geq 1\}$–factor, respectively. This shows that the bound in Theorem 1.1 is best possible.

Theorem 5.1 *Let n be a positive integer and $\theta(n)$ be the largest root of $x^3 - (n-4)x^2 - (n-1)x + 2(n-4) = 0$. Then we have*

(i) $\rho(K_1 \vee (K_{n-3} \cup 2K_1)) = \theta(n)$ *for* $n \in \{3, 4, 8\}$ *or* $n \geq 10$*;*
(ii) $\rho\left(K_{\frac{n-1}{2}} \vee \frac{n+1}{2}K_1\right) = \frac{n-3+\sqrt{5n^2-6n+5}}{4}$ *for* $n \in \{5, 7, 9\}$*;*
(iii) $\rho(P_2 \vee 4K_1) = \frac{1+\sqrt{33}}{2}$ *for* $n = 6$.

Proof For $n \in \{3, 4, 8\}$ or $n \geq 10$, we consider the partition $V(K_1 \vee (K_{n-3} \cup 2K_1)) = V(K_1) \cup V(2K_1) \cup V(K_{n-3})$. The quotient matrix of $A(K_1 \vee (K_{n-3} \cup 2K_1))$ corresponding to the above partition is

$$B_1 = \begin{pmatrix} 0 & 2 & n-3 \\ 1 & 0 & 0 \\ 1 & 0 & n-4 \end{pmatrix}.$$

Then we have

$$\phi_{B_1}(x) = x^3 - (n-4)x^2 - (n-1)x + 2(n-4).$$

Note that the partition is equitable. Then the spectral radius of graph $K_1 \vee (K_{n-3} \cup 2K_1)$ is the largest root of $x^3 - (n-4)x^2 - (n-1)x + 2(n-4) = 0$. It follows that $\rho(K_1 \vee (K_{n-3} \cup 2K_1)) = \theta(n)$.

For $n \in \{5, 7, 9\}$, we consider the partition $V(K_{\frac{n-1}{2}} \vee \frac{n+1}{2}K_1) = V(K_{\frac{n-1}{2}}) \cup V(\frac{n+1}{2}K_1)$. Then the corresponding quotient matrix of $A(K_{\frac{n-1}{2}} \vee \frac{n+1}{2}K_1)$ is

$$B_2 = \begin{pmatrix} \frac{n-3}{2} & \frac{n+1}{2} \\ \frac{n-1}{2} & 0 \end{pmatrix}.$$

Hence we have

$$\phi_{B_2}(x) = x^2 - \frac{n-3}{2}x - \frac{n^2-1}{4}.$$

Note that the spectral radius of graph $K_{\frac{n-1}{2}} \vee \frac{n+1}{2}K_1$ is the largest root of $x^2 - \frac{n-3}{2}x - \frac{n^2-1}{4} = 0$ since the above partition is equitable. It follows that $\rho\left(K_{\frac{n-1}{2}} \vee \frac{n+1}{2}K_1\right) = \frac{n-3+\sqrt{5n^2-6n+5}}{4}$.

For $n = 6$, consider the partition $V(K_2 \vee 4K_1) = V(K_2) \cup V(4K_1)$. The quotient matrix of $A(K_2 \vee 4K_1)$ corresponding to the above partition equals

$$B_3 = \begin{pmatrix} 1 & 4 \\ 2 & 0 \end{pmatrix}.$$

Hence we have $\phi_{B_3}(x) = x^2 - x - 8$. Note that the spectral radius of graph $K_2 \vee 4K_1$ is the largest root of $x^2 - x - 8 = 0$ since the above partition is equitable. It follows that $\rho(K_2 \vee 4K_1) = \frac{1+\sqrt{33}}{2}$. This completes the proof. □

In view of Theorem 5.1, we know that the bound obtained in Theorem 1.2 is best possible.

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References

1. A. Amahashi, M. Kano, On factors with given components, *Discrete Math.*, **42**(1982), 1-6.
2. R. Bapat, *Graphs and matrices*, London, Springer, 2010.
3. D. Bauer, H. Broersma, E. Schmeichel, Toughness in graphs-a survey, *Graphs Combin.*, **22**(2006), 1-35.
4. J. Bondy, U. Murty, *Graph Theory with Applications*, London, Macmillan, 1976.
5. A.E. Brouwer, W.H. Haemers, *Spectra of Graphs*, Universitext, New York: Springer, 2012.
6. V. Chvátal, Tough graphs and Hamiltonian circuits, *Discrete Math.*, **5**(1973), 215-228.
7. K. Das, Maximizing the sum of the squares of the degrees of a graph, *Discrete Math.*, **285**(2004), 57-66.
8. M. Fiedler, Algebraic connectivity of graphs, *Czechoslovak Math. J.*, **23**(1973), 298-305.
9. W. Gao, W. Wang, Remarks on Component Factors, *J. Oper. Res. Soc. China* (2021). https://doi.org/10.1007/s40305-021-00357-6.
10. X. Gu, M. Liu, A tight lower bound on the matching number of graphs via Laplacian eigenvalues, *European J. Combin.*, **101**(2022), 103468.
11. X. Gu, W. Haemers, Graph toughness from Laplacian eigenvalues, *Algebraic Combin.,* **5**(2022), 53-61.
12. X. Guan, T. Ma, C. Shi, Tight toughness, isolated toughness and binding number bounds for the $\{P_2, C_n\}$–factors, *J. Oper. Res. Soc. China*, 2023, https://doi.org/10.1007/s40305-023-00485-1.
13. A. Kaneko, A necessary and sufficient condition for the existence of a path factor every component of which is a path of length at least two, *J. Combin. Theory Ser. B*, **88**(2003), 195-218.
14. S. Li, S. Miao, Characterizing $\mathcal{P}_{\geq 2}$-factor and $\mathcal{P}_{\geq 2}$-factor covered graphs with respect to the size or the spectral radius, *Discrete Math.*, **344**(2021), 112588.
15. H. Lu, W. Wang, On Perfect k-Matchings, *Graphs Combin.*, **30**(2014), 229-235.
16. S. Miao, S. Li, Characterizing star factors via the size, the spectral radius or the distance spectral radius of graphs, *Discrete Appl. Math.*, **326**(2023), 17-32.
17. W. Tutte, The 1-factors of oriented graphs, *Proc. Am. Math. Soc.*, **4**(1953), 922-931.
18. Q. Yu, G. Liu, *Graph factors and matching extensions*, Springer, 2010.
19. L. You, M. Yang, W. So, *et al.*, On the spectrum of an equitable quotient matrix and its application, *Linear Algebra Appl.*, **577**(2019), 21-40.
20. W. Zhang, The spectral radius and $\mathcal{P}_{\geq \ell}$-factors of graphs involving minimum degree, *Graphs Combin.*, **38**(2022), Article number: 176.
21. S. Zhou, J. Wu, T. Zhang, The existence of $\mathcal{P}_{\geq 3}$-factor covered graphs, *Discuss. Math. Graph Theory*, **37**(2017), 1055-1065.