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ORIGINAL RESEARCH

Residue of some Eisenstein series

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Abstract The real analytic Eisenstein series is a special function that has been studied classically. Its generalization to the case of many variables has been studied extensively. Moreover, the analytic properties of certain Eisenstein series on the Siegel modular groups have also been investigated. The purpose of this study is to provide concrete forms of the residue of $E_0^{(m)}(z,s)$ at $s = m/2$.

Keywords Eisenstein series · Siegel modular groups

Mathematics Subject Classification 11F46 · 11F30

1 Introduction

The real analytic Eisenstein series is a special function that has been studied classically. It is used in the representation theory of $SL(2,\mathbb{R})$, and in analytic number theory (e.g., cf. [4]). Its generalization to the case of many variables was initiated by Siegel and later studied more extensively by Langlands [5] and Shimura [10]. Let

$$E_k^{(m)}(z,s) = \det(y)^s \sum_{\{c,d\}} \det(cz+d)^{-k} \, |\det(cz+d)|^{-2s}$$

be the Eisenstein series of degree m (for a precise definition, see § 3.1).

Shimura [10] studied the analytic properties of the Eisenstein series, including this type. He reveals the holomorphy of $E_k^{(m)}(z,s)$ in s at $s = 0$ by analyzing the Fourier coefficients. The Fourier coefficient essentially consists of two parts. One is the confluent hypergeometric function, and the other is the Siegel series. Therefore, the analytic properties of Fourier coefficients, and the Eisenstein series results in the study of the analytic properties of these two parts. In [9], Shimura established the analytic theory of confluent hypergeometric functions on tube domains and then applied them to analysis of Eisenstein series. The results of holomorphy of $E_k^{(m)}(z,s)$ studied and extended by Weissauer [11].

In Shimura's paper [10], apart from the holomorphy, the residue of Eisenstein series is mentioned. His statement is as follows:

The residue of the Eisenstein series $E_{(m-1)/2}^{(m)}(z,s)$ at $s = 1$ can be expressed as the product of π^{-m} and a holomorphic modular form of weight $(m-1)/2$, with rational Fourier coefficients.

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(It is known that the holomorphic modular form stated above is a (rational) constant multiple of Eisenstein series $E^{(m)}_{(m-1)/2}(z,0)$.)

Other than his work, few papers mention concrete forms of the residue for the Eisenstein series, except for the classical work by Kaufhold [3] (see § 5.2).

This study aims to provide concrete forms of residue $E^{(m)}_0(z,s)$ at $s=m/2$. Our results strongly depend on Mizumoto's work [7], especially his work on the Fourier expansion of $E^{(m)}_k(z,s)$, which is a refinement of Maass's result.

Theorem

$$\begin{aligned}&\operatorname*{Res}_{s=m/2} E^{(m)}_0(z,s)\\ &= \mathbb{A}^{(m)}(y)+\mathbb{B}^{(m)}(y)\sum_{h\in\Lambda^{(1)}_m}\sigma_0(\mathrm{cont}(h))\eta_m(2y,\pi h;m/2,m/2)\boldsymbol{e}(\sigma(hx)),\end{aligned}$$

where

$$\begin{aligned}\mathbb{A}^{(m)}(y) &= \frac{1}{2}\alpha_m(y,m/2)\cdot C^{(m)}_{m-1}(y)+\frac{1}{8}v(m-1)det(2y)^{-(m-1)/2}\alpha'_m(y,m/2)\\ &\quad+\frac{1}{8}\beta'_m(y,m/2),\\ \mathbb{B}^{(m)}(y) &= 2^{m-2}\pi^{m\kappa(m)}det(y)^{m/2}\Gamma_m(m/2)^{-1}\zeta(m)^{-1}\\ &\quad\cdot\prod_{j=1}^{m-2}\zeta(m-j)\prod_{j=1}^{m-1}\zeta(2m-2j)^{-1}.\end{aligned}$$

Here, $C^{(m)}_{m-1}$ is the constant term of the completed Koecher–Maass zeta function $\xi^{(m)}_{m-1}(2y,s)$ at $s=m/2$ (see (4.3)), and $\alpha_m(y,s)$ and $\beta_m(y,s)$ are defined in (4.2) and (4.7), respectively, which are essentially products of the gamma functions and zeta functions. The set $\Lambda^{(1)}_m$ is the set of half-integral matrices of size m and rank 1, and for an element $h\in\Lambda^{(1)}_m$, $\mathrm{cont}(h)$ is defined by $\mathrm{cont}(h):=\max\{\ell\in\mathbb{N}\mid \ell^{-1}h\in\Lambda^{(1)}_m\}$, and $\sigma_0(a)=\sum_{0<d|a}1$ (the number of the positive divisors of $a\in\mathbb{N}$).

In the degree 2 case, the constant $C^{(2)}_1(y)$ can be calculated explicitly from the first Kronecker limit formula.

Corollary

$$\begin{aligned}\operatorname*{Res}_{s=1} E^{(2)}_0(z,s) &= \frac{18}{\pi^2\sqrt{det(y)}}\left(\frac{1}{2}\gamma+\frac{1}{2}\log\frac{v'}{4\pi}-\log|\eta(W_y)|^2\right)\\ &\quad+\frac{36\,det(y)}{\pi^2}\sum_{h\in\Lambda^{(1)}_2}\sigma_0(cont(h))\eta_2(2y,\pi h;1,1)\boldsymbol{e}(\sigma(hx)).\end{aligned}\tag{1.1}$$

(for the notation, see § 5.1.1.) In [8], the author provided a formula for $E^{(2)}_2(z,0)$ (Siegel Eisenstein series of degrees 2 and 2):

$$\begin{aligned}E^{(2)}_2(z,0) =&1-\frac{18}{\pi^2\sqrt{det(y)}}\left(1+\frac{1}{2}\gamma+\frac{1}{2}\log\frac{v'}{4\pi}-\log|\eta(W_y)|^2\right)\\ &-\frac{72}{\pi^3}\sum_{\substack{0\neq h\in\Lambda_2\\ discr(h)=\square}}\varepsilon_h\sigma_0(cont(h))\eta_2(2y,\pi h;2,0)\boldsymbol{e}(\sigma(hx))\\ &+288\sum_{0\neq h\in\Lambda_2}\sum_{d|cont(h)}d\,H\left(\frac{|discr(h)|}{d^2}\right)\boldsymbol{e}(\sigma(hz)).\end{aligned}\tag{1.2}$$

It is interesting that the same term appears in each Fourier coefficient in (1.1) and (1.2).

2 Notation

1° For an $m \times n$ matrix a, we write it as $a^{(m,n)}$, and as $a^{(m)}$ if $m = n$, ${}^t a$ denotes the transpose of a, and a_{ij} denotes the (i, j)-entry of a. For a matrix a, we write $\sigma(a)$ as the trace of a. If the right-hand side is defined as the identity matrix (resp. zero matrix) of size m and is denoted by 1_m (resp. 0_m). For a commutative ring R with 1, we denote $R^{(m,n)}$ by the R-module of all $m \times n$ matrices with entries from R. We set $R^{(m)} := R^{(m,m)}$ and $R^m := R^{(1,m)}$.

2° We put

- $\kappa(\nu) = \frac{\nu+1}{2}$ for $\nu \in \mathbb{Z}_{\geqq 0}$.
- $\boldsymbol{e}(z) = \exp(2\pi i z)$ for $z \in \mathbb{C}$.
- $H_m := \{ z \in \mathbb{C}^{(m)} \mid {}^t z = z,\ \mathrm{Im}(z) > 0 \}$: upper half space.
- $V_m = \{ x \in \mathbb{R}^{(m)} \mid {}^t x = x \}$.
- $V_m(\mathbb{C}) = V_m \otimes_{\mathbb{R}} \mathbb{C}$.
- $P_m := \{ x \in V_m \mid x > 0 \}$.
- $V_m(p, q, r)$: subset of V_m consisting of the elements with p positive,
- q negative, r zero eigenvalues.

3° The function $\Gamma_m(s)$ is defined by

$$\Gamma_m(s) = \pi^{\frac{m(m-1)}{4}} \prod_{\nu=0}^{m-1} \Gamma\left(s - \frac{\nu}{2}\right)$$

for $m > 0$, and $\Gamma_0(s) := 1$. 4° The set of symmetric half-integral matrices of size m is denoted by Λ_m. We set

$$\Lambda_m^{(\nu)} := \{ h \in \Lambda_m \mid \mathrm{rank}(h) = \nu \}.$$

For $\nu \in \mathbb{Z}$ with $1 \leqq \nu \leqq m$,

$$\mathbb{Z}_{\mathrm{prim}}^{(m,\nu)} = \{ a \in \mathbb{Z}^{(m,\nu)} \mid a \text{ is primitive} \}.$$

5° Throughout the paper, we understand that the product (resp. sum) over an empty set is equal to 1 (resp. 0).

3 Preliminary

3.1 Eisenstein series

For $m \in \mathbb{Z}_{>0}$ and $k \in 2\mathbb{Z}_{\geq 0}$, let

$$E_k^{(m)}(z, s) = \det(y)^s \sum_{\{c,d\}} \det(cz + d)^{-k} \left|\det(cz + d)\right|^{-2s} \tag{3.1}$$

be the Eisenstein series for $\Gamma_m = Sp_m(\mathbb{Z})$ (Siegel modular groups of degrees m). Here, $z = x + iy$ is a variable on H_m, s is a complex variable, and $\{c, d\}$ runs over a complete system of representatives $\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)$ of $\left\{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \in \Gamma_m\right\} \backslash \Gamma_m$. The right-hand side of (3.1) converges absolutely, locally, and uniformly on the

$$\{ (z, s) \in H_m \times \mathbb{C} \mid \mathrm{Re}(s) > (m + 1 - k)/2 \}.$$

As is well known, the Eisenstein series $E_k^{(m)}(Z, s)$ has a meromorphic continuation to the whole s-plane (Langlands [5], Mizumoto [7]).

3.2 Confluent hypergeometric functions

Shimura studied the confluent hypergeometric functions on the tube domains ([9]) and applied his results to develop the theory of the Eisenstein series ([10]). In this section, we summarize some results on the confluent hypergeometric functions that will be used later. For $g \in P_m$, $h \in V_m$, and $(\alpha, \beta) \in \mathbb{C}^2$,

$$\xi_m(g, h; \alpha, \beta) = \int_{V_m} \boldsymbol{e}^{-\sigma(hx)} \det(x + ig)^{-\alpha} \det(x - ig)^{-\beta} dx, \tag{3.2}$$

with $dx = \prod_{i \leq j} dx_{ij}$, which is convergent for $\mathrm{Re}(\alpha + \beta) > m$;

$$\eta_m(g, h; \alpha, \beta) = \int_{\substack{V_m \\ x \pm h > 0}} e^{-\sigma(gx)} \det(x + h)^{\alpha - \kappa(m)} \det(x - h)^{\beta - \kappa(m)} dx, \tag{3.3}$$

which is convergent for $\mathrm{Re}(\alpha) > \kappa(m) - 1$, $\mathrm{Re}(\beta) > m$.

We also use

$$\eta_m^*(g, h; \alpha, \beta) = \det(g)^{\alpha + \beta - \kappa(m)} \eta_m(g, h; \alpha, \beta),$$

which satisfies the property

$$\eta_m^*(g[a], h[{}^t a^{-1}]; \alpha, \beta) = \eta_m^*(g, h; \alpha, \beta) \quad \text{for all} \quad a \in GL_m(\mathbb{R}).$$

By [9, eqn.(1.29)],

$$\xi_m(g, h; \alpha, \beta) = i^{m(\beta - \alpha)} \cdot 2^m \pi^{m\kappa(m)} \Gamma_n(\alpha)^{-1} \Gamma_n(\beta)^{-1} \eta_m(2g, \pi h; \alpha, \beta). \tag{3.4}$$

for $\mathrm{Re}(\alpha) > \kappa(m) - 1$, $\mathrm{Re}(\beta) > m$. When $h = 0_m$, the following identity holds:

Proposition 3.1 *(Shimura [9, eqn.(1.31)]) If* $\mathrm{Re}(\alpha + \beta) > 2\kappa(m) - 1$*, then*

$$\begin{aligned} \xi_m(g, 0_m; \alpha, \beta) = {} & i^{m\beta - m\alpha} \cdot 2^{m(1 - \kappa(m))} (2\pi)^{m\kappa(m)} \\ & \cdot \Gamma_m(\alpha)^{-1} \Gamma_m(\beta)^{-1} \Gamma_m(\alpha + \beta - \kappa(m)) \\ & \cdot det(2g)^{\kappa(m) - \alpha - \beta}. \end{aligned} \tag{3.5}$$

For $g \in P_m$, $h \in V_m(p, q, r)$ with $p + q + r = m$, we put

$$\begin{aligned} \delta_+(hg) &:= \text{the product of all positive eigenvalues of } g^{\frac{1}{2}} h g^{\frac{1}{2}}, \\ \delta_-(hg) &:= \delta_+((-h)g). \end{aligned}$$

We then put

$$\begin{aligned} \omega_m(g, h; \alpha, \beta) :=& 2^{-p\alpha - q\beta} \Gamma_p\left(\beta - (m-p)/2\right)^{-1} \Gamma_q\left(\alpha - (m-q)/2\right)^{-1} \\ & \cdot \Gamma_r\left(\alpha + \beta - \kappa(m)\right)^{-1} \\ & \cdot \delta_+(hg)^{\kappa(m) - \alpha - q/4} \delta_-(hg)^{\kappa(m) - \beta - p/4} \eta_n^*(g, h; \alpha, \beta), \end{aligned} \tag{3.6}$$

One of the main results in [9] is as follows:

Theorem 3.2 *(Shimura [9, Theorem 4.2])Function* ω_m *can be continued as a holomorphic function in* (α, β) *to the whole* $\mathbb{C}^2$ *and satisfies*

$$\omega_m(g, h; \alpha, \beta) = \omega_m\left(g, h; \kappa(m) + (r/2) - \beta, \kappa(m) + (r/2) - \alpha\right).$$

3.3 Fourier expansion

For $m \in \mathbb{Z}_{>0}$ and $k \in 2\mathbb{Z}_{\geqq 0}$, let s be a complex variable, where $\mathrm{Re}(s) > \kappa(m)$, and let $z = x + iy$ be a variable on H_m with $x \in V_m$. and $y \in P_m$. Maass ([6]) provided a formula for the Fourier expansion of the Eisenstein series $E_k^{(m)}(z, s)$:

$$E_k^{(m)}(z,s) = \det(y)^s + \det(y)^s \sum_{\nu=1}^{m} \sum_{h \in \Lambda_\nu} \sum_{q \in \mathbb{Z}_{\mathrm{prim}}^{(m,\nu)}/GL_\nu(\mathbb{Z})} S_\nu(h, 2s+k)\xi_\nu(y[q], h; s+k, s)\boldsymbol{e}(\sigma(h[{}^t q]x)), \tag{3.7}$$

where

$$S_\nu(h,s) = \sum_{r \in V_\nu \cap \mathbb{Q}^{(\nu)} \bmod 1} n(r)^{-s}\boldsymbol{e}(\sigma(hr)) \tag{3.8}$$

is the singular series (Siegel series), where $n(r)$ is the product of the reduced positive denominators of the elementary divisors of r, and ξ_ν is the confluent hypergeometric function defined in (3.2).

From [7, Lemma 1.1], we have

Lemma 3.3 *For $\nu \in \mathbb{Z}_{>0}$, each $h \in \Lambda_\nu$ of* rank $\lambda > 0$ *(that is, $h \in \Lambda_\nu^{(\lambda)}$) is expressed uniquely as*

$$h = h_0[{}^t w]$$

with $h_0 \in \Lambda_\lambda^{(\lambda)}$ and $w \in \mathbb{Z}_{\mathrm{prim}}^{(\nu,\lambda)}/GL_\lambda(\mathbb{Z})$.

Mizumoto provided a reduced formula for ξ_ν ([7, Lemma 1.4]):

Proposition 3.4 *Let $h = h_0[{}^t w]$ be, as in the above lemma. Suppose that* $\mathrm{Re}(s) > \nu$. *Then, in (3.7), we have*

$$\begin{aligned}&\xi_\nu(y[q], h; s+k, s)\\ &= (-1)^{k\nu/2} 2^\nu \pi^{\nu\kappa(\nu)+\lambda(\nu-\lambda)/2} \cdot \Gamma_{\nu-\lambda}(2s+k-\kappa(\nu))\Gamma_\nu(s)^{-1}\Gamma_\nu(s+k)^{-1}\\ &\cdot \det(2y[q])^{\kappa(\nu)-k-2s}\eta_\lambda^*(2y[qw], \pi h_0; s+k+(\lambda-\nu)/2, s+(\lambda-\nu)/2).\end{aligned} \tag{3.9}$$

Let $m,\ \lambda \in \mathbb{Z}$ with $m \geqq \lambda \geqq 1$. We define the subgroup $\Delta_\lambda^{(m)}$ of $GL_m(\mathbb{Z})$ by

$$\Delta_\lambda^{(m)} := \left\{ \begin{pmatrix} * & * \\ 0^{(m-\lambda,\lambda)} & * \end{pmatrix} \in GL_m(\mathbb{Z}) \right\}.$$

For $r \in \mathbb{Z}_{\mathrm{prim}}^{(m,\lambda)}$, u_r is an element of $GL_m(\mathbb{Z})$ corresponding to r under a bijection

$$\begin{array}{ccc} \mathbb{Z}_{\mathrm{prim}}^{(m,\lambda)}/GL_\lambda(\mathbb{Z}) & \longleftrightarrow & GL_m(\mathbb{Z})/\Delta_\lambda^{(m)} \\ r & \longmapsto & u_r \end{array}$$

which is determined up to the right action of $\Delta_\lambda^{(m)}$.

For $y \in P_m$, we write the Jacobi decomposition of $y[u_r]$ as

$$y[u_r] = \mathrm{diag}(y[r], g(y, u_r)) \begin{bmatrix} 1_\lambda & b \\ 0 & 1_{m-\lambda} \end{bmatrix}.$$

Explicitly, we place $u_r = (r\, r_1)$ and then

$$g(y, u_r) = y[r_1] - (y[r])^{-1}[{}^t r y r_1]. \tag{3.10}$$

Next, we provide a definition of Koecher–Maass zeta functions. For $1 \leqq \nu \leqq m$ and $g \in P_m$, we define

$$\zeta_\nu^{(m)}(g, s) := \sum_{a \in \mathbb{Z}_{\mathrm{prim}}^{(m,\nu)}/GL_\nu(\mathbb{Z})} \det(g[a])^{-s} \tag{3.11}$$

which is convergent for $\mathrm{Re}(s) > m/2$. By definition,

$$\zeta_m^{(m)}(g, s) = \det(g)^{-s}.$$

For later purposes, we put

$$\zeta_0^{(m)}(*, s) := 1 \qquad \text{for all} \quad m \in \mathbb{Z}_{\geqq 0}.$$

Mizumoto's refinement of Maass' expression is as follows:

Theorem 3.5 *(Mizumoto [7, Theorem 1.8]) For $m \in \mathbb{Z}_{>0}$, $k \in 2\mathbb{Z}_{\geqq 0}$, and $\mathrm{Re}(s) > m$, the Eisenstein series $E_k^{(m)}(z, s)$ has the following expression:*

$$E_k^{(m)}(z, s) = \sum_{\nu=0}^{m} \sum_{\lambda=0}^{\nu} F_{k,\nu,\lambda}^{(m)}(z, s) \tag{3.12}$$

where

$$\begin{aligned} &F_{k,\nu,0}^{(m)}(z, s) \\ &= (-1)^{k\nu/2} 2^{\nu} \pi^{\nu\kappa(\nu)} \Gamma_\nu(2s + k - \kappa(\nu)) \Gamma_\nu(s)^{-1} \Gamma_\nu(s + k)^{-1} \\ &\cdot S_\nu(0_\nu, 2s + k) \det(y)^s \, \zeta_\nu^{(m)}(2y, 2s + k - \kappa(\nu)), \end{aligned} \tag{3.13}$$

for $0 \leqq \nu \leqq m$, and

$$F_{k,\nu,\lambda}^{(m)}(z, s) = \sum_{h \in \Lambda_\lambda^{(\lambda)}} \sum_{r \in \mathbb{Z}_{\mathrm{prim}}^{(m,\lambda)}/GL_\lambda(\mathbb{Z})} b_{k,\nu,\lambda}^{(m)}(h[{}^t r], y, s) \boldsymbol{e}(\sigma(h[{}^t r]x)) \tag{3.14}$$

for $1 \leqq \lambda \leqq \nu \leqq m$ with

$$\begin{aligned} &b_{k,\nu,\lambda}^{(m)}(h[{}^t r], y, s) \\ &:= (-1)^{k\nu/2} 2^{\nu} \pi^{\nu\kappa(\nu) + \lambda(\nu-\lambda)/2} \Gamma_{\nu-\lambda}(2s + k - \kappa(\nu)) \Gamma_\nu(s)^{-1} \Gamma_\nu(s + k)^{-1} \\ &\cdot S_\nu(\mathrm{diag}(h, 0_{\nu-\lambda}), 2s + k) \det(y)^s \det(2y[r])^{\kappa(\nu) - k - 2s} \\ &\cdot \eta_\lambda^*(2y[r], \pi h; s + k + (\lambda - \nu)/2, s + (\lambda - \nu)/2) \\ &\cdot \zeta_{\nu-\lambda}^{(m-\lambda)}(2g(y, u_r), 2s + k - \kappa(\nu)). \end{aligned} \tag{3.15}$$

Here, $\zeta_\nu^{(m)}(g, s)$ for $0 \leqq \nu \leqq m$ is the Koecher–Maass zeta function defined in (3.11), and $g(y, u_r)$ is defined by (3.10). Matrix $h[{}^t r]$ runs over the set $\Lambda_m^{(\lambda)}$ exactly once if h runs over $\Lambda_\lambda^{(\lambda)}$ and r runs over a complete set of representatives of $\mathbb{Z}_{\mathrm{prim}}^{(m,\lambda)}/GL_\lambda(\mathbb{Z})$.

3.4 Siegel series

In this section, we summarize the results of the Siegel series $S_\nu(h, s)$ that appear in the Fourier expansions (3.7) and (3.15).

For $h \in \Lambda_\lambda^{(\lambda)}$, we set

$$d(h) := (-1)^{[\lambda/2]} 2^{-\delta((\lambda-1)/2)} \det(2h)$$

where

$$\delta(x) := \begin{cases} 1 & x \in \mathbb{Z}, \\ 0 & x \notin \mathbb{Z} \end{cases}$$

for $x \in \mathbb{Q}$. By [7, eqn.(5.1)],

$$
\begin{aligned}
S_\nu(\mathrm{diag}(h, 0_{\nu-\lambda}), s) = \zeta(s+\lambda-\nu)\zeta(s)^{-1} \\
&\cdot \prod_{j=1}^{\nu-\lambda} (\zeta(2s-\nu-j)\,\zeta(2s-2j)^{-1}) \\
&\cdot S_\lambda(h, s-\nu+\lambda) \qquad (3.16)
\end{aligned}
$$

and

$$
S_\lambda(h, s) = \sum_{d \in A(h)} (\det(d))^{\lambda+1-2s} \widehat{S}_\lambda(h[d^{-1}], s),
$$

$$
\widehat{S}_\lambda(h, s) = \zeta(s)^{-1} \prod_{j=1}^{[\lambda/2]} \zeta(2s-2j)^{-1} L\left(s-\lambda/2, \left(\frac{d(h)}{*}\right)\right)^{\delta(\lambda/2)} \prod_p a_p(h, s)
$$

where $L\left(s, \left(\frac{d(h)}{*}\right)\right)$ is Dirichlet L-function associated to the quadratic character $\left(\frac{d(h)}{*}\right)$, the product of p runs over the prime divisors of $d(h)$,

$$
A(h) := GL_\lambda(\mathbb{Z})\backslash\{\, d \in \mathbb{Z}^{(\lambda)} \mid \det(d) \neq 0 \ \text{ and } \ h[d^{-1}] \in \Lambda_\lambda \,\},
$$

and from [2], we have

$$
a_p(h, s) =
\begin{cases}
\prod_{j=1}^{r/2} (1 - p^{2j-1+\lambda-2s}) & (\lambda, r) \equiv (1, 0) \pmod 2, \\
(1 + \lambda_p(h) p^{(\lambda+r)/2-s}) \prod_{j=1}^{(r-1)/2} (1 - p^{2j-1+\lambda-2s}) & (\lambda, r) \equiv (1, 1) \pmod 2, \\
\prod_{j=1}^{(r-1)/2} (1 - p^{2j+\lambda-2s}) & (\lambda, r) \equiv (0, 1) \pmod 2, \\
(1 + \lambda_p(h) p^{(\lambda+r)/2-s}) \prod_{j=1}^{r/2-1} (1 - p^{2j+\lambda-2s}) & (\lambda, r) \equiv (0, 0) \pmod 2.
\end{cases}
$$

Here, $r := r(p)$ is the maximal number, which is the condition $h[u] \equiv \begin{pmatrix} h^* & 0 \\ 0 & 0_r \end{pmatrix} \pmod p$ for some $u \in \mathbb{Z}^{(\lambda)}$ and $\lambda_p(h) := \left(\frac{d(h^*)}{p}\right)$.

Remark 3.6 (1) We understand that $S_0(*, s) = 1$. Therefore, from (3.16), we obtain the following. Formula for $S_\nu(0_\nu, s)$:

$$
\begin{aligned}
S_\nu(0_\nu, s) &= \zeta(s-\nu)\zeta(s)^{-1} \prod_{j=1}^{\nu} (\zeta(2s-\nu-j)\,\zeta(2s-2j)^{-1}) \\
&= \zeta(s-\nu)\zeta(s)^{-1} \prod_{j=1}^{[\nu/2]} (\zeta(2s-2\nu-1+2j)\,\zeta(2s-2j)^{-1}). \qquad (3.17)
\end{aligned}
$$

(2) In the following discussion, the concrete form of $a_p(h, s)$ is not needed, only its holomorphy in s.

3.5 Koecher–Maass zeta function

Analytic properties of the Koecher–Maass zeta function $\zeta_\nu^{(m)}(g, s)$ are important for the analysis of the Fourier coefficient $F_{k,\nu,\lambda}^{(m)}(z, s)$. In this section, we recall Arakawa's results for the Koecher–Maass zeta function.

For $1 \leqq \nu \leqq m$ and $g \in P_m$, we define the completed Koecher–Maass zeta function by

$$
\xi_\nu^{(m)}(g, s) := \prod_{i=0}^{\nu-1} \xi(2s-i)\,\zeta_\nu^{(m)}(g, s) \qquad (3.18)
$$

where

$$\xi(s) := \pi^{-s/2}\Gamma(s/2)\,\zeta(s)$$

and we understand

$$\xi_0^{(m)}(g,s) := 1.$$

The following result is due to Arakawa, which plays an important role in our investigation.

Proposition 3.7 *(Arakawa [1])*

(1) Suppose $m \geqq 2\nu - 1$. *The function* $\xi_\nu^{(m)}(g,s)$ *has simple poles at* $s = 0, \frac{1}{2}, \cdots, \frac{\nu-1}{2}$ *and* $s = \frac{m-\nu+1}{2}, \cdots, \frac{m}{2}$. *For* $0 \leqq \mu \leqq \nu - 1$, *the residues of* $\xi_\nu^{(m)}(g,s)$ *at* $s = \frac{\mu}{2}$ *and* $s = \frac{m-\mu}{2}$ *are given by*

$$\operatorname*{Res}_{s=\mu/2} \xi_\nu^{(m)}(g,s) = -\frac{1}{2}v(\nu-\mu)\xi_\mu^{(m)}(g,\tfrac{\nu}{2}), \tag{3.19}$$

$$\operatorname*{Res}_{s=(m-\mu)/2} \xi_\nu^{(m)}(g,s) = \frac{1}{2}v(\nu-\mu)\det(g)^{-\frac{\nu}{2}}\xi_\mu^{(m)}(g^{-1},\tfrac{\nu}{2}), \tag{3.20}$$

where

$$v(\nu) = \begin{cases} \displaystyle\prod_{i=2}^{\nu} \xi(i) & (\nu \geqq 2), \\ 1 & (\nu = 1). \end{cases}$$

(2) Suppose $\nu \leqq m \leqq 2\nu - 2$. *The function* $\xi_\nu^{(m)}(g,s)$ *has poles at* $s = 0, \frac{1}{2}, \cdots, \frac{m}{2}$ *of which* $s = 0, \frac{1}{2}, \cdots, \frac{m-\nu}{2}$ *and* $s = \frac{\nu}{2}, \frac{\nu+1}{2}, \cdots, \frac{m}{2}$ *are simples poles. The poles at* $s = \frac{m-\nu+1}{2}, \frac{m-\nu+2}{2}, \cdots, \frac{\nu-1}{2}$ *are double poles. For* $0 \leqq \mu \leqq m - \nu$, *the residues of* $\xi_\nu^{(m)}(g,s)$ *at* $s = \frac{\mu}{2}$ *and* $s = \frac{m-\mu}{2}$ *are given by* (3.19) and (3.20), *respectively.*

Remark 3.8 When $m = 2$ and $\nu = 1$, the function $\zeta_1^{(2)}(g,s)$ appears as a simple factor of Epstein's zeta function for g. Therefore, the residue and constant term at $s = 1$ is explicitly expressed by the Kronecker limit formula (see § 5.1.1).

4 Residue of Eisenstein series

In the rest of this paper, we assume that $m \geqq 2$. In this section, we provide an explicit formula for

$$\operatorname*{Res}_{s=m/2} E_0^{(m)}(z,s)$$

which is the main result of this paper.

4.1 Fourier coefficient of $E_0^{(m)}(z,s)$

We recall the Fourier expansion

$$E_0^{(m)}(z,s) = \sum_{\nu=0}^{m}\sum_{\lambda=0}^{\nu} F_{0,\nu,\lambda}^{(m)}(z,s)$$

in Theorem 3.5 and study the analytic property, particularly the singularity of $F_{0,\nu,\lambda}^{(m)}(z,s)$ and $b_{0,\nu,\lambda}^{(m)}(*,y,s)$. For this purpose, we use the results introduced in § 3 and consider them dividing into several cases. 1° $(\nu,\lambda) = (0,0)$:

$$F_{0,0,0}^{(m)}(z,s) = \det(y)^s.$$

2° $(\nu,\lambda)=(\nu,0),(0<\nu<m)$:

$$F^{(m)}_{0,\nu,0}(z,s)=c_{\nu,0}(s)\Gamma_\nu(s)^{-2}\zeta(2s-\nu)\zeta(2s)^{-1}\cdot\prod_{j=1}^{\nu}\zeta(4s-2j)^{-1}\,\xi^{(m)}_\nu(2y,2s-\kappa(\nu)),$$

where $c_{\nu,0}(s)$ is a holomorphic function in s. The equality is the result of (3.13) and (3.17), and we rewrote $\zeta^{(m)}_\nu$ with the completed Koecher-Maass zeta function $\xi^{(m)}_\nu$ in (3.18). 3° $(\nu,\lambda)=(m,0)$:

$$F^{(m)}_{0,m,0}(z,s)=c_{m,0}(s)\Gamma_m(2s-\kappa(m))\Gamma_m(s)^{-2}\zeta(2s-m)\zeta(2s)^{-1}\cdot\prod_{j=1}^{m}(\zeta(4s-m-j)\zeta(4s-2j)^{-1}),$$

where $c_{m,0}(s)$ is a holomorphic function in s. The above expression is also the result of (3.13) and (3.17). In this case, the factor $\zeta^{(m-\lambda)}_{\nu-\lambda}(2y,2s-\kappa(\nu))$ is just $\zeta^{(m)}_m(2y,2s-\kappa(m))=\det(2y)^{-2s+\kappa(m)}$ which is holomorphic in s. 4° $(\nu,\lambda)=(\nu,\nu),(0<\nu\leqq m)$:

$$b^{(m)}_{0,\nu,\nu}(*,y,s)=c_{\nu,\nu}(s)\Gamma_\nu(s)^{-2}\zeta(2s)^{-1}\prod_{j=1}^{[\nu/2]}\zeta(4s-2j)^{-1}\cdot\eta_\nu(*,*;s,s),$$

where $c_{\nu,\nu}(s)$ is a holomorphic function in s. 5° $(\nu,\lambda),(0<\lambda<\nu<m)$:

$$\begin{aligned}b^{(m)}_{0,\nu,\lambda}(*,y,s)&=c_{\nu,\lambda}(s)\Gamma_\nu(s)^{-2}\zeta(2s)^{-1}\prod_{j=1}^{\nu-\lambda}\zeta(4s-2j)^{-1}\\&\cdot\prod_{j=1}^{[\lambda/2]}\zeta(4s-2\nu+2\lambda-2j)^{-1}\\&\cdot\eta_\lambda(*,*;s+(\lambda-\nu)/2,s+(\lambda-\nu)/2)\cdot\xi^{(m-\lambda)}_{\nu-\lambda}(*,2s-\kappa(\nu)),\end{aligned}$$

for a holomorphic function $c_{\nu,\lambda}(s)$. 6° $(\nu,\lambda)=(m,\lambda),(0<\lambda<m)$:

$$\begin{aligned}b^{(m)}_{0,m,\lambda}(*,y,s)&=c_{m,\lambda}(s)\Gamma_{m-\lambda}(2s-\kappa(m))\Gamma_m(s)^{-2}\\&\cdot\prod_{j=1}^{m-\lambda}(\zeta(4s-m-j)\zeta(4s-2j)^{-1})\prod_{j=1}^{[\lambda/2]}\zeta(4s-2m+2\lambda-2j)^{-1}\\&\cdot\eta_\lambda(*,*;s+(\lambda-m)/2,s+(\lambda-m)/2),\end{aligned}$$

where $c_{\nu,\lambda}(s)$ are holomorphic function in s.

4.2 Analytic property of Fourier coefficients

We investigate the analytic property of $F^{(m)}_{0,\nu,\lambda}(z,s)$ and $b^{(m)}_{0,\nu,\lambda}(*,y,s)$ at $s=m/2$, based on the description in § 4.1.

Proposition 4.1 *Functions $F^{(m)}_{0,\nu,\lambda}(z,s)$ and $b^{(m)}_{0,\nu,\lambda}(*,y,s)$ are holomorphic in s at $s=m/2$, except for the following three cases:*

$$\text{(i) } \nu=m-1,\ \lambda=0\qquad\text{(ii) } \nu=m,\ \lambda=0\qquad\text{(iii) } \nu=m,\ \lambda=1.$$

Proof We use the expressions $1^\circ - 6^\circ$ given in the previous section.

The holomorphy for the case 1° is trivial.

First, we consider the case 5°. The Γ-factor $\Gamma_\nu(s)^{-2}$, and ζ-factors $\zeta(2s)^{-1}, \cdots$ are all holomorphic at $s = m/2$. From Theorem 3.2 and (3.6), the holomorphy of $\eta_\lambda(*, *; s + (\lambda - \nu)/2, s + (\lambda - \nu)/2)$ is reduced to that of

$$\Gamma_p(s + (\lambda - \nu)/2 - q/2) \quad \Gamma_q(s + (\lambda - \nu)/2 - p/2) \qquad (p + q = \lambda),$$

and are both holomorphic at $s = m/2$. (Note that the factor $\Gamma_r(*)^{-1}$ does not appear in η_λ.)

Function $\xi^{(m-\lambda)}_{\nu-\lambda}(*, 2s-\kappa(\nu))$ is holomorphic at $s = m/2$ because $\mathrm{Re}(2s-\kappa(\nu)) > (m-\lambda)/2$. Consequently, the functions in the case 5° are holomorphic at $s = m/2$.

By a similar argument, we observe that the functions in the case of 4° are holomorphic at $s = m/2$. The cases we must consider are the cases of 2° and 6°.

In the case of 2°, only $F^{(m)}_{0,m-1,0}(z, s)$ and $F^{(m)}_{0,m,0}(z, s)$ are non-holomorphic at $s = m/2$ because $F^{(m)}_{0,m-1,0}(z, s)$ has a factor $\zeta(2s - m + 1)\xi^{(m)}_{m-1}(*, 2s - m/2)$, and $F^{(m)}_{0,m,0}(z, s)$ have the factors $\Gamma(2s - m)\zeta(4s - 2m + 1)$, respectively. In fact, the factors above have double poles at $s = m/2$.

In the case 6°, only the function $b^{(m)}_{0,m,1}(*, y, s)$ is nonholomorphic at $s = m/2$, because it contains the factor $\zeta(4s - 2m + 1)$, which has a simple pole at $s = m/2$.

These facts complete the proof. □

Remark 4.2 The explicit formulas for $F^{(m)}_{0,m-1,0}(z, s)$, $F^{(m)}_{0,m,0}(z, s)$, and $F^{(m)}_{0,m,1}(z, s)$ will be given in the next sections.

Here, we arrange functions $F^{(m)}_{0,\nu,\lambda}$ as follows:

$$\begin{array}{llllll}
F^{(m)}_{0,0,0} & & & & & \\
F^{(m)}_{0,1,0} & F^{(m)}_{0,1,1} & & & & \\
\vdots & \vdots & \ddots & & & \\
F^{(m)}_{0,m-2,0} & F^{(m)}_{0,m-2,1} & \cdots & \ddots & & \\
\boldsymbol{F^{(m)}_{0,m-1,0}} & F^{(m)}_{0,m-1,1} & F^{(m)}_{0,m-1,2} & \cdots & F^{(m)}_{0,m-1,m-1} & \\
\boldsymbol{F^{(m)}_{0,m,0}} & \boldsymbol{F^{(m)}_{0,m,1}} & F^{(m)}_{0,m,2} & \cdots & F^{(m)}_{0,m,m-1} & F^{(m)}_{0,m,m}
\end{array}$$

The proposition asserts that only functions $F^{(m)}_{0,\nu,\lambda}$ printed in bold are non-holomorphic at $s = m/2$.

4.3 Residue of the constant term

We investigate the analytic property of the constant term

$$\sum_{\nu=0}^{m} F^{(m)}_{0,\nu,0}(z, s)$$

at $s = m/2$. More specifically, we show that the constant term has a simple pole at $s = m/2$ and calculate the residue. By Proposition 4.1, it is sufficient to investigate only $F^{(m)}_{0,m-1,0}(z, s)$ and $F^{(m)}_{0,m,0}(z, s)$ as far as considering the residue.

Analysis of $\boldsymbol{F^{(m)}_{0,m-1,0}(z, s)}$:

From the definition of $F^{(m)}_{0,\nu,0}(z, s)$ (see (3.13)), we have

$$F^{(m)}_{0,m-1,0}(z, s) = 2^{m-1}\pi^{2(m-1)s}\det(y)^s\Gamma_{m-1}(s)^{-2}\zeta(2s - m + 1)\zeta(2s)^{-2} \cdot \prod_{j=1}^{m-1} \zeta(4s - 2j)^{-1} \cdot \xi^{(m)}_{m-1}(2y, 2s - m/2). \tag{4.1}$$

(We rewrote (3.13) with the completed Koecher–Maass zeta function $\xi^{(m)}_{m-1}$.)

We separate $F^{(m)}_{0,m-1,0}(z,s)$ into holomorphic and non-holomorphic parts. We define the function $\alpha_m(y,s)$ by

$$F^{(m)}_{0,m-1,0}(z,s) = \zeta(2s-m+1)\xi^{(m)}_{m-1}(2y, 2s-m/2)\cdot\alpha_m(y,s).$$

Explicitly,

$$\alpha_m(y,s) := 2^{m-1}\pi^{2(m-1)s}\det(y)^s\Gamma_{m-1}(s)^{-2}\zeta(2s)^{-1}\prod_{j=1}^{m-1}\zeta(4s-2j)^{-1}. \tag{4.2}$$

Functions $\zeta(2s-m+1)$ and $\xi^{(m)}_{m-1}(2y,2s-m/2)$ have a simple pole at $s=m/2$ (for $\xi^{(m)}_{m-1}$, see Proposition 3.7), and $\alpha_m(y,s)$ is holomorphic at $s=m/2$. These facts imply that $F^{(m)}_{0,m-1,0}(z,s)$ has a double pole at $s=m/2$.

We set

$$F^{(m)}_{0,m-1,0}(z,s) = \sum_{l=-2}^{\infty} A^{(m)}_l(y)(s-m/2)^l \quad \text{(Laurent expansion at } s=m/2)$$

and calculate $A^{(m)}_{-2}(y)$ and $A^{(m)}_{-1}(y)$.

As a preparation, we investigate the analytic behavior of $\xi^{(m)}_{m-1}(2y,2s-m/2)$ at $s=m/2$. We consider the completed Koecher–Maass zeta function $\xi^{(m)}_{m-1}(2y,s)$. According to Arakawa's result (Proposition 3.7), this function has a simple pole with residue

$$\operatorname*{Res}_{s=m/2}\xi^{(m)}_{m-1}(2y,s) = \frac{1}{2}v(m-1)\det(2y)^{-(m-1)/2}.$$

From this, we see that

$$\operatorname*{Res}_{s=m/2}\xi^{(m)}_{m-1}(2y,2s-m/2) = \frac{1}{4}v(m-1)\det(2y)^{-(m-1)/2}. \tag{4.3}$$

Definition 4.3 Define a constant $C^{(m)}_{m-1}(y)$ by

$$C^{(m)}_{m-1}(y) := \lim_{s\to m/2}\left(\xi^{(m)}_{m-1}(2y,s) - \operatorname*{Res}_{s=m/2}\xi^{(m)}_{m-1}(2y,s)(s-m/2)^{-1}\right).$$

That is, $C^{(m)}_{m-1}(y)$ is the constant term of the Laurent expansion of $\xi^{(m)}_{m-1}(2y,s)$ at $s=m/2$.

Remark 4.4 (1) It should be noted that the constant $C^{(m)}_{m-1}(y)$ is defined from $\xi^{(m)}_{m-1}(2y,s)$ not $\xi^{(m)}_{m-1}(y,s)$, and the constant term of the Laurent expansion of $\xi^{(m)}_{m-1}(2y,2s-m/2)$ at $s=m/2$ is equal to that of $\xi^{(m)}_{m-1}(2y,s)$.
(2) In the case $m=2$, the constant $C^{(2)}_1(y)$ is explicitly expressed by the Kronecker limit formula (see § 5.1.1).

Proposition 4.5 *Explicit forms of $A^{(m)}_{-2}(y)$ and $A^{(m)}_{-1}(y)$ are given as follows:*

$$A^{(m)}_{-2}(y) = \frac{1}{8}v(m-1)\det(2y)^{-(m-1)/2}\alpha_m(y,m/2), \tag{4.4}$$

$$A^{(m)}_{-1}(y) = \alpha_m(y,m/2)\left(\frac{1}{2}C^{(m)}_{m-1}(y) + \frac{\gamma}{4}v(m-1)\det(2y)^{-(m-1)/2}\right) + \frac{1}{8}v(m-1)\det(2y)^{-(m-1)/2}\cdot\alpha'_m(y,m/2), \tag{4.5}$$

where γ is the Euler constant and $\alpha'_m(y,m/2) = \frac{d}{ds}\alpha_m(y,s)\big|_{s=m/2}$.

Proof The formulas are derived from the expression

$$\begin{aligned}\zeta(2s-m+1)\xi^{(m)}_{m-1}(2y,2s-m/2)\\ &= \frac{1}{8}\,\upsilon(m-1)\det(2y)^{-(m-1)/2}(s-m/2)^{-2}\\ &+\left(\frac{1}{2}\,C^{(m)}_{m-1}(y)+\frac{\gamma}{4}\,\upsilon(m-1)\det(2y)^{-(m-1)/2}\right)(s-m/2)^{-1}\\ &+\text{(a holomorphic function at } s=m/2).\end{aligned}$$

□

Analysis of $F^{(m)}_{0,m,0}(z,s)$: By (3.13) and (3.17),

$$\begin{aligned}F^{(m)}_{0,m,0}(z,s) &= 2^{-2ms+m(m+3)/2}\pi^{m(m+1)/2}\det(y)^{-s+(m+1)/2}\Gamma_m(2s-\kappa(m))\\ &\cdot\Gamma_m(s)^{-2}\zeta(2s-m)\zeta(2s)^{-1}\cdot\prod_{j=1}^{m}(\zeta(4s-m-j)\zeta(4s-2j)^{-1}).\end{aligned}\tag{4.6}$$

Similar to that in case $F^{(m)}_{0,m,-1,0}(z,s)$, we define the function $\beta_m(y,s)$ as $\alpha_m(y,s)$:

$$F^{(m)}_{0,m,0}(z,s)=\Gamma(2s-m)\zeta(4s-2m+1)\,\beta_m(y,s).$$

Explicitly,

$$\begin{aligned}\beta_m(y,s) :&= 2^{-2ms+m(m+3)/2}\pi^{m(m+3)/4}\det(y)^{-s+(m+1)/2}\\ &\cdot\prod_{j=0}^{m-2}\Gamma(2s-(m+1+j)/2)\prod_{j=0}^{m-1}\Gamma(s-j/2)^{-2}\\ &\cdot\zeta(2s-1)\zeta(2s)^{-1}\cdot\prod_{j=1}^{m-2}\zeta(4s-m-j)\cdot\prod_{j=1}^{m-1}\zeta(4s-2j)^{-1}.\end{aligned}\tag{4.7}$$

Functions $\Gamma(2s-m)$ and $\zeta(4s-2m+1)$ have a simple pole at $s=m/2$, respectively, and $\beta_m(y,s)$ is holomorphic at $s=m/2$. Consequently, we observe that $F^{(m)}_{0,m,0}(z,s)$, has a double pole at $s=m/2$, as in the previous case.

We set

$$F^{(m)}_{0,m,0}(z,s)=\sum_{l=-2}^{\infty}B^{(m)}_l(y)(s-m/2)^l$$

and calculate $B^{(m)}_{-2}(y)$ and $B^{(m)}_{-1}(y)$.

Proposition 4.6 *The explicit forms of $B^{(m)}_{-2}(y)$ and $B^{(m)}_{-1}(y)$ are as follows:*

$$B^{(m)}_{-2}(y)=\frac{1}{8}\,\beta_m(y,m/2),\tag{4.8}$$

$$B^{(m)}_{-1}(y)=\frac{\gamma}{4}\,\beta_m(y,m/2)+\frac{1}{8}\,\beta'_m(y,m/2)\tag{4.9}$$

where γ is the Euler constant and $\beta'_m(y,m/2)=\frac{d}{ds}\beta_m(y,s)\big|_{s=m/2}$.

Proof Function $\Gamma(2s-m)\zeta(4s-2m+1)$ has the Laurent expansion as

$$\begin{aligned}&\Gamma(2s-m)\zeta(4s-2m+1)\\ &=\frac{1}{8}\,(s-m/2)^{-2}+\frac{\gamma}{4}\,(s-m/2)^{-1}+\text{(a holomorphic function at } s=m/2).\end{aligned}$$

The formulas for $B^{(m)}_{-2}(y)$ and $B^{(m)}_{-1}(y)$ are obtained from this expression. □

An important point is the following relationship between $A_{-2}^{(m)}(y)$ and $B_{-2}^{(m)}(y)$.

Proposition 4.7 *The following identity holds.*

$$A_{-2}^{(m)}(y) = -B_{-2}^{(m)}(y). \tag{4.10}$$

Proof A direct calculation shows that

$$A_{-2}^{(m)}(y) = 2^{(-m^2+4m-8)/2}\pi^{(m^3+3m-2)/4}\det(2y)^{1/2}\zeta(m)^{-1} \cdot \prod_{i=2}^{m-1}\Gamma(i/2)\prod_{j=0}^{m-2}\Gamma((m-j)/2)^{-2}\prod_{i=2}^{m-1}\zeta(i)\prod_{j=1}^{m-1}\zeta(2m-2j)^{-1}. \tag{4.11}$$

Meanwhile,

$$B_{-2}^{(m)}(y) = -2^{(-m^2+4m-8)/2}\pi^{(m^3+3m)/4}\det(2y)^{1/2}\zeta(m)^{-1} \cdot \prod_{j=0}^{m-2}\Gamma((m-1-j)/2)\prod_{j=0}^{m-1}\Gamma((m-j)/2)^{-2} \cdot \prod_{j=1}^{m-2}\zeta(m-j)\prod_{j=1}^{m-1}\zeta(2m-2j)^{-1}. \tag{4.12}$$

Noting that

$$\prod_{j=0}^{m-2}\Gamma((m-1-j)/2) = \Gamma(1/2)\cdot\prod_{i=2}^{m-1}\Gamma(i/2),$$

we conclude that $A_{-2}^{(m)}(y) = -B_{-2}^{(m)}(y)$. □

From this proposition, we observe that the singularity of function

$$F_{0,m-1,0}^{(m)}(z,s) + F_{0,m,0}^{(m)}(z,s)$$

at $s = m/2$ is a simple pole.

Theorem 4.8 *The residue of the constant term is as follows:*

$$\operatorname*{Res}_{s=m/2}\sum_{\nu=0}^{m}F_{0,\nu,0}^{(m)}(z,s) = \operatorname*{Res}_{s=m/2}(F_{0,m-1,0}^{(m)}(z,s) + F_{0,m,0}^{(m)}(z,s)) = \frac{1}{2}\alpha_m(y,m/2)\cdot C_{m-1}^{(m)}(y) + \frac{1}{8}v(m-1)det(2y)^{-(m-1)/2}\alpha_m'(y,m/2) + \frac{1}{8}\beta_m'(y,m/2) \tag{4.13}$$

where the notation is as above.

Proof We have

$$\operatorname*{Res}_{s=m/2}(F_{0,m-1,0}^{(m)}(z,s) + F_{0,m,0}^{(m)}(z,s)) = A_{-1}^{(m)}(y) + B_{-1}^{(m)}(y) = \frac{1}{2}\alpha_m(y,m/2)\cdot C_{m-1}^{(m)}(y) + \frac{\gamma}{4}v(m-1)\det(2y)^{-(m-1)/2}\alpha_m(y,m/2) + \frac{1}{8}v(m-1)\det(2y)^{-(m-1)/2}\alpha_m'(y,m/2) + \frac{1}{8}\beta_m'(y,m/2) + \frac{\gamma}{4}\beta_m(y,m/2).$$

By the identity (4.10), the sum of the second and fifth terms in the last formula is equal to zero. This implies (4.13). □

4.4 Calculation of $F^{(m)}_{0,m,1}(z,s)$

We have one more non-holomorphic term, that is, $F^{(m)}_{0,m,1}(z,s)$.

Proposition 4.9 *Function $F^{(m)}_{0,m,1}(z,s)$ has the following expression:*

$$\begin{aligned}&F^{(m)}_{0,m,1}(z,s)\\&=2^m\pi^{m\kappa(m)}det(y)^s\Gamma_m(s)^{-2}\zeta(2s)^{-1}\prod_{j=1}^{m-1}(\zeta(4s-m-j)\cdot\zeta(4s-2j)^{-1})\\&\cdot\sum_{h\in\Lambda_m^{(1)}}\sigma_{m-2s}(cont(h))\cdot\eta_m(2y,\pi h;s,s)\boldsymbol{e}(\sigma(hx)),\end{aligned}\tag{4.14}$$

where $\sigma_s(a)=\sum_{0<d|a}d^s$, and for $0\neq h\in\Lambda_m$, $\mathrm{cont}(h):=\max\{l\in\mathbb{N}\mid l^{-1}h\in\Lambda_m\}$.

Proof By definition (3.15),

$$\begin{aligned}&F^{(m)}_{0,m,1}(z,s)\\&=2^m\pi^{m\kappa(m)+(m-1)/2}\Gamma_{m-1}(2s-\kappa(m))\Gamma_m(s)^{-2}\det(y)^s\det(2y)^{\kappa(m)-2s}\\&\cdot\sum_{0\neq h\in\mathbb{Z}}\sum_{w\in\mathbb{Z}^{(m,1)}_{\text{prim}}/\{\pm1\}}S_m(\mathrm{diag}(h,0_{m-1}),2s)\\&\qquad\cdot(2y[w])^{2s-m}\eta_1(2y[w],\pi h;s-(m-1)/2,s-(m-1)/2)\boldsymbol{e}(\sigma(h[{}^tw]x).\end{aligned}\tag{4.15}$$

It follows from (3.16) that

$$\begin{aligned}&S_m(\mathrm{diag}(h,0_{m-1}),2s)\\&=\zeta(2s-m+1)\zeta(2s)^{-1}\prod_{j=1}^{m-1}(\zeta(4s-m-j)\,\zeta(4s-2j)^{-1})S_1(h,2s-m+1)\\&=\sigma_{m-2s}(h)\zeta(2s)^{-1}\prod_{j=1}^{m-1}(\zeta(4s-m-j)\,\zeta(4s-2j)^{-1}).\end{aligned}\tag{4.16}$$

In the above, we used the formula

$$S_1(h,s)=\zeta(s)^{-1}\sigma_{1-s}(h)$$

for $h\in\mathbb{Z}_{>0}$ (e.g. cf. [3, eqn.(8)]).

From Proposition 3.4, function η_1 is expressed as

$$\begin{aligned}&\eta_1(2y[w],\pi h;s-(m-1)/2,s-(m-1)/2)\\&=\pi^{-(m-1)/2}\Gamma_{m-1}(2s-\kappa(m))^{-1}\det(2y)^{2s-\kappa(m)}(2y[w])^{-2s+m}\\&\cdot\eta_m(2y,\pi h[{}^tw],s,s).\end{aligned}\tag{4.17}$$

Substituting (4.16) and (4.17) into (4.15), we obtain the equality (4.14). □

From the above proposition, we obtain the following result:

Theorem 4.10 *Function $F^{(m)}_{0,m,1}(z,s)$ has a simple pole at $s=m/2$.*

$$\begin{aligned}\mathop{\mathrm{Res}}_{s=m/2}F^{(m)}_{0,m,1}(z,s)&=2^{m-2}\pi^{m\kappa(m)}det(y)^{m/2}\Gamma_m(m/2)^{-1}\zeta(m)^{-1}\\&\cdot\prod_{j=1}^{m-2}\zeta(m-j)\prod_{j=1}^{m-1}\zeta(2m-2j)^{-1}\\&\cdot\sum_{h\in\Lambda_m^{(1)}}\sigma_0(\mathrm{cont}(h))\eta_m(2y,\pi h;m/2,m/2)\boldsymbol{e}(\sigma(hx)).\end{aligned}\tag{4.18}$$

Proof In the expression (4.14) of $F^{(m)}_{0,m,1}(z,s)$, only the last factor $\zeta(4s-2m+1)$ in the product $\prod_{j=1}^{m-1}\zeta(4s-m-j)$ has a simple pole at $s=m/2$ with residue $1/4$. From this fact, we obtain (4.18). □

4.5 Conclusion

We summarize our results in the previous sections.

The following is a main result of this study.

Theorem 4.11

$$\begin{aligned}&\operatorname*{Res}_{s=m/2} E_0^{(m)}(z,s)\\ &= \mathbb{A}^{(m)}(y)+\mathbb{B}^{(m)}(y)\sum_{h\in\Lambda_m^{(1)}}\sigma_0(\mathrm{cont}(h))\eta_m(2y,\pi h;m/2,m/2)\boldsymbol{e}(\sigma(hx)),\end{aligned}$$

where

$$\begin{aligned}\mathbb{A}^{(m)}(y)&=\frac{1}{2}\alpha_m(y,m/2)\cdot C^{(m)}_{m-1}(y)+\frac{1}{8}v(m-1)det(2y)^{-(m-1)/2}\alpha'_m(y,m/2)\\ &\quad+\frac{1}{8}\beta'_m(y,m/2),\\ \mathbb{B}^{(m)}(y)&=2^{m-2}\pi^{m\kappa(m)}det(y)^{m/2}\Gamma_m(m/2)^{-1}\zeta(m)^{-1}\\ &\quad\cdot\prod_{j=1}^{m-2}\zeta(m-j)\prod_{j=1}^{m-1}\zeta(2m-2j)^{-1}.\end{aligned}$$

5 Remarks

5.1 Low degree cases

In this section, we provide more explicit formulas for $\mathrm{Res}_{s=m/2}E_0^{(m)}(z,s)$, $m=2,\ 3$. We used the notations in the previous sections as they are.

5.1.1 Case $\boldsymbol{m=2}$

In this case, the constant $C_1^{(2)}(y)$ appearing in the term $\mathbb{A}^{(2)}(y)$ can be expressed more explicitly because we can apply the Kronecker limit formula. For $g\in P_2$, we consider the Epstein zeta function

$$\zeta_g(s):=\sum_{\mathbf{0}\neq a\in\mathbb{Z}^{(2,1)}/\{\pm1\}}g[a]^{-s},\qquad \mathrm{Re}(s)>1.$$

The first Kronecker limit formula asserts that $\zeta_g(s)$ has the following expression:

$$\zeta_g(s)=\frac{1}{2}(4\det(g))^{-s/2}\left[\frac{2\pi}{s-1}+4\pi\beta(g)+O(s-1)\right], \tag{5.1}$$

$$\beta(g)=\gamma+\frac{1}{2}\log\frac{v'}{2\sqrt{\det(g)}}-\log|\eta(W_g)|^2. \tag{5.2}$$

Here, for $g=\begin{pmatrix}v'&w\\w&v\end{pmatrix}\in P_2$,

$$W_g:=\frac{w+i\sqrt{\det(g)}}{v'}\in H_1,$$

and $\eta(z)$ is the Dedekind eta function:

$$\eta(z)=\boldsymbol{e}(z/24)\prod_{n=1}^{\infty}(1-\boldsymbol{e}^n(z)),\qquad z\in H_1.$$

The relationship between the complete zeta function $\xi_1^{(2)}(g,s)$ and $\zeta_g(s)$ is expressed as follows:

$$\xi_1^{(2)}(g,s)=\xi(2s)\zeta_1^{(2)}(g,s)=\pi^{-s}\Gamma(s)\zeta_g(s).$$

Therefore, the constant $C_1^{(2)}(y)$, which is the constant term of $\xi_1^{(2)}(2y,s)$, can be expressed as

$$C_1^{(2)}(y)=\frac{1}{2}(4\det(y))^{-1/2}\left(\gamma+\log\frac{v'}{8\pi}-\log(\det(y))-2\log|\eta(W_y)|^2\right).$$

Concerning $\alpha_m(y,m/2),\alpha_m'(y,m/2),\cdots$ appearing in $\mathbb{A}^{(m)}(y)$, we can calculate them explicitly as

$$\begin{aligned}
&\alpha_2(y,1)=2\pi^2\det(y)\,\zeta(2)^{-2},\\
&\alpha_2'(y,1)=2\pi^2\det(y)\,\zeta(2)^{-2}(2\log\pi+\log(\det(y))+2\gamma-6\zeta'(2)\,\zeta(2)^{-1}),\\
&\beta_2(y,1)=-\pi^2\det(y)^{1/2}\,\zeta(2)^{-2},\\
&\beta_2'(y,1)=2\pi^2\det(y)^{1/2}\,\zeta(2)^{-2}\left(2\log 2+\tfrac{1}{2}\log(\det(y))-\gamma+2\zeta'(0)+3\zeta'(2)\zeta(2)^{-1}\right).
\end{aligned}$$

From these formulas,

$$\begin{aligned}
\mathbb{A}^{(2)}(y)&=\operatorname*{Res}_{s=1}(F_{0,1,0}^{(2)}(z,s)+F_{0,2,0}^{(2)}(z,s))\\
&=\frac{1}{2}\alpha_2(y,1)\cdot C_1^{(2)}(y)+\frac{1}{8}v(1)\cdot\det(2y)^{-1/2}\alpha_2'(y,1)+\frac{1}{8}\beta_2'(y,1)\\
&=\frac{18}{\pi^2\sqrt{\det(y)}}\left(\frac{1}{2}\gamma+\frac{1}{2}\log\frac{v'}{4\pi}-\log|\eta(W_y)|^2\right).
\end{aligned}$$

Combining with $\mathrm{Res}_{s=1}F_{0,2,1}^{(2)}(z,s)$, we obtain

Proposition 5.1

$$\begin{aligned}
\operatorname*{Res}_{s=1}E_0^{(2)}(z,s)=&\frac{18}{\pi^2\sqrt{det(y)}}\left(\frac{1}{2}\gamma+\frac{1}{2}\log\frac{v'}{4\pi}-\log|\eta(W_y)|^2\right)\\
&+\frac{36det(y)}{\pi^2}\sum_{h\in\Lambda_2^{(1)}}\sigma_0(cont(h))\eta_2(2y,\pi h;1,1)\boldsymbol{e}(\sigma(hx)).
\end{aligned}\tag{5.3}$$

Remark 5.2 In [8], the author provided a formula for $E_2^{(2)}(z,0)$ (Siegel Eisenstein series of degree 2 and weight 2):

$$\begin{aligned}
E_2^{(2)}(z,0)=&1-\frac{18}{\pi^2\sqrt{\det(y)}}\left(1+\frac{1}{2}\gamma+\frac{1}{2}\log\frac{v'}{4\pi}-\log|\eta(W_y)|^2\right)\\
&-\frac{72}{\pi^3}\sum_{\substack{0\neq h\in\Lambda_2\\ \mathrm{discr}(h)=\square}}\varepsilon_h\sigma_0(\mathrm{cont}(h))\eta_2(2y,\pi h;2,0)\boldsymbol{e}(\sigma(hx))\\
&+288\sum_{0\neq h\in\Lambda_2}\sum_{d|\mathrm{cont}(h)}d\,H\left(\frac{|\mathrm{discr}(h)|}{d^2}\right)\boldsymbol{e}(\sigma(hz)).
\end{aligned}\tag{5.4}$$

Here, $H(N)$ is the Kronecker–Hurwitz class number and $\varepsilon_h=1/2$ if $\mathrm{rank}(h)=1$; $=1$ if $\mathrm{rank}(h)=2$. It is interesting that the same term appears in each Fourier coefficient in (5.3) and (5.4).

5.1.2 *Case* $m = 3$

From Theorem 4.11, we can write

$$\operatorname*{Res}_{s=3/2} E_0^{(3)}(z,s)$$
$$= \mathbb{A}^{(3)}(y) + \mathbb{B}^{(3)}(y) \sum_{h \in \Lambda_3^{(1)}} \sigma_0(\mathrm{cont}(h))\eta_3(2y, \pi h; 3/2, 3/2)\boldsymbol{e}(\sigma(hx)).$$

The quantities $\mathbb{A}^{(3)}(y)$ and $\mathbb{B}^{(3)}(y)$ are given as follows:

$$\mathbb{A}^{(3)}(y)$$
$$= 2^3\pi^4 \det(y)^{3/2}\zeta(2)^{-1}\zeta(3)^{-1}\zeta(4)^{-1} \cdot C_2^{(3)}(y) + 2^{-2}\pi^3\det(y)^{1/2}\zeta(3)^{-1}\zeta(4)^{-1}$$
$$\cdot (-2\Gamma'(1) - 4\zeta'(2)\zeta(2)^{-1} + 4\zeta'(0) + 2\log(\det(y)) + 4\log\pi + 6\log 2), \qquad \mathbb{B}^{(3)}(y)$$
$$= 2^2\pi^{7/2}\det(y)^{3/2}\zeta(3)^{-1}\zeta(4)^{-1}.$$

Remark 5.3 In the above formulas, we may substitute

$$\zeta(2) = \pi^2/6, \quad \zeta(4) = \pi^4/90, \quad \zeta'(0) = (-\log 2\pi)/2, \quad \Gamma'(1) = -\gamma.$$

5.2 Residue at the other point

The residue we considered above was to $s = m/2$, and it is represented as a Fourier series. The case $\mathrm{Res}_{s=(m+1)/2} E_0^{(m)}(z,s)$ is easier than in the above case. In fact, it becomes a constant, explicitly

$$\operatorname*{Res}_{s=(m+1)/2} E_0^{(m)}(z,s)$$
$$= \operatorname*{Res}_{s=(m+1)/2} \xi(2s-m)\xi(2s)^{-1} \prod_{j=1}^{[m/2]} (\xi(4s-2m-1+2j)\,\xi(4s-2j)^{-1}).$$

Remark 5.4 Kaufhold [3] noted that the residue of

$$\Phi_0(s) := E_0^{(2)}(z, s/2)$$

at $s = 3$ is $90\pi^{-2}$. This is a special case of the above formula because

$$\operatorname*{Res}_{s=3/2} E_0^{(2)}(z,s) = \operatorname*{Res}_{s=3/2} \xi(2s-2)\xi(2s)^{-1}\xi(4s-3)\xi(4s-2)^{-1} = \frac{45}{\pi^2}.$$

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