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ORIGINAL RESEARCH

Identities involving generalized derivations act as Jordan homomorphisms

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Abstract Let R be a prime ring with char(R) is not equal to 2 and $\pi(\omega_1, \ldots, \omega_n)$ be a noncentral multilinear polynomial over the extended centroid C of R. If F_1, F_2 and F_3 are generalized derivations on R such that $F_1(F_3(\xi^2)) = F_2(\xi)F_3(\xi)$ for all $\xi = \pi(\omega_1, \ldots, \omega_n)$, $\omega_1, \ldots, \omega_n \in R$, then we describe all possible forms of generalized derivations F_1, F_2 and F_3.

Keywords Differential polynomial identity · Prime ring · Multilinear polynomial · Generalized derivation · Utumi quotient ring

Mathematics Subject Classification 16N60 · 16W25

1 Introduction

Let R be an associative ring. A mapping $\zeta : R \to R$ is said to be a derivation on R if

$$\zeta(u_1 + u_2) = \zeta(u_1) + \zeta(u_2), \zeta(u_1u_2) = \zeta(u_1)u_2 + u_1\zeta(u_2),$$

for all $u_1, u_2 \in R$. The commutator of u_1 and u_2 is denoted by $[u_1, u_2] = u_1u_2 - u_2u_1$ for $u_1, u_2 \in R$, which is called the Lie commutator of u_1 and u_2. For fix $a \in R$, define a mapping $g_a : R \to R$ by $g_a(u) = [a, u]$ for all $u \in R$. We can easily prove that g_a is a derivation on R and usually it is called an inner derivation on R. A mapping $\zeta : R \to R$ is said to be a generalized derivation if there exists a derivation g on R such that

$$\zeta(u_1 + u_2) = \zeta(u_1) + \zeta(u_2), \zeta(u_1u_2) = \zeta(u_1)u_2 + u_1g(u_2),$$

for all $u_1, u_2 \in R$ (for more details see [4]). Let s_1, s_2 be fixed elements in R and $\zeta_{(s_1,s_2)} : R \to R$ be a mapping defined by $\zeta_{(s_1,s_2)}(u) = s_1u + us_2$ for all $u \in R$. Here, we can easily prove that $\zeta_{(s_1,s_2)}$ is a generalized derivation on R and it is called a generalized inner derivation on R.

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A mapping $\zeta : R \to R$ is called homomorphism and anti-homomorphism if

$$\zeta(u_1 + u_2) = \zeta(u_1) + \zeta(u_2), \zeta(u_1 u_2) = \zeta(u_1)\zeta(u_2)$$

and

$$\zeta(u_1 + u_2) = \zeta(u_1) + \zeta(u_2), \zeta(u_1 u_2) = \zeta(u_2)\zeta(u_1),$$

for all $u_1, u_2 \in R$ respectively. A mapping $\zeta : R \to R$ is called Jordan homomorphism if

$$\zeta(u_1 + u_2) = \zeta(u_1) + \zeta(u_2), \zeta(u^2) = \zeta(u)^2,$$

for all $u, u_1, u_2 \in R$. We notice that every homomorphism and anti-homomorphism is a Jordan homomorphism but the converse is not true in general. The following example justify our observation.

Example 1.1 Suppose that $*$ is an involution on a ring R and $S = R \oplus R$ is a ring with the properties $t_1 z t_2 = 0$, for all $t_1, t_2 \in R$, where $z \in Z(R)$. We define a function $\zeta : S \to S$ such that $\zeta(t_1, t_2) = (zt_1, t_2^*)$, for all $t_1, t_2 \in R$. It is clear that ζ is a Jordan homomorphism on S but not a homomorphism on S.

In 1956, Herstein [20] proved that every Jordan homomorphism from a ring R onto a prime ring R' with $\mathrm{char}(R) \neq 2, 3$ is either a homomorphism or anti-homomorphism. Further, in 1957 Smiley [28] extended the Herstein's result [20] and proved that the statement of the Herstein's result is still true without taking the characteristic is not equal to 3.

In the literature, several mathematicians describe the structure of prime ring R with the additive mappings which acts as a homomorphism or anti-homomorphism or Jordan homomorphism on Lie ideals, Jordan ideals or some appropriate subsets of R. In this line of investigation, Bell and Kappe [6] proved the first result in the context of derivation. More precisely, they proved that there is no nonzero derivation on prime ring R which acts as a homomorphism or anti-homomorphism on nonzero right ideal of R. Further, above [6] result was extended by Wang and You [33] to Lie ideal case under suitable restriction.

Recently, generalized derivation which behaves as a Jordan homomorphism, Lie homomorphism, homomorphism, anti-homomorphism were discuss in [11,12,16–18,29–31], where further references can be found.

On the other hand, Posner [27] gave a remarkable results concerning centralizing mapping on prime ring R. More specifically, Posner [27] proved that if d is a nonzero centralizing derivation on prime ring R, then R must be commutative. Further, Posner's [27] result was extended by Brešar [5]. Later on many mathematicians studied the structure of prime rings as well as structure of additive mappings which behaves as a commuting or centralizing mappings. In this line of investigation, the readers are refer to ([1,2,29,32], where further references can be found).

Carini et al. in [8] considered a noncommutative prime ring R of characteristic different from 2 with Utumi quotient ring U and extended centroid C, $\pi(\omega_1, \ldots, \omega_n)$ a multilinear polynomial over C which is not an identity for R, F and G two nonzero generalized derivations of R. If $F(\xi)G(\xi) = 0$ for all $\xi \in \pi(R) = \{\pi(\omega_1, \ldots, \omega_n) \mid \omega_i \in R\}$, then they gave the complete possible forms of F and G.

Throughout the following U denotes the Utumi quotient ring of prime ring R, C denotes the center of U which is called the extended centroid of R and $\pi(\omega_1, \ldots, \omega_n)$ is noncentral multilinear polynomial over C. More details about U and C readers can found in [3] and [9].

Motivated by above cited results, our result is the following.

2 Results

Theorem 2.1 *Suppose that R is a prime ring of characteristic not equal to 2 and G, F, H are three generalized derivations on R. If*

$$G(H(\xi^2)) = F(\xi)H(\xi)$$

for all $\xi = \pi(\omega_1, \ldots, \omega_n)$, $\omega_1, \ldots, \omega_n \in R$, then one of the following holds:

(i) $H = 0$;
(ii) *there exist $\sigma \in C$, $w_1 \in U$ such that $G(r) = F(r) = w_1 r$ and $H(r) = \sigma r$ for all $r \in R$;*
(iii) *there exist $w_1, w_2, w_3 \in U$, $\sigma \in C$ such that $H(r) = r w_1$, $F(r) = w_2 r$ and $G(r) = \sigma r + w_2 r + r w_3$ with $w_1 w_3 = -\sigma w_1$;*

(iv) *there exist* $w_1, w_2, w_3 \in U$ *such that* $H(r) = w_1 r$, $F(r) = r w_2$ *and* $G(r) = w_3 r$ *for all* $r \in R$ *with* $w_3 w_1 = w_2 w_1 = \alpha \in C$;

(v) $\pi(\omega_1, \ldots, \omega_n)^2$ *is central valued on* R *and one of the following holds:*

(a) *there exist* $0 \neq \sigma \in C$, $w_1, w_2 \in U$ *such that* $H(r) = \sigma r$, $F(r) = w_1 r$ *and* $G(r) = [w_2, r] + r w_1$ *for all* $r \in R$;

(b) *there exist* $w_1, w_2, w_3, w_4 \in U$, $\alpha \in C$ *such that* $G(r) = w_3 r + r w_4$ *and either* $H(r) = r w_1$, $F(r) = w_2 r$ *with* $w_3 w_1 + w_1 w_4 = w_2 w_1$ *or* $H(r) = w_1 r$, $F(r) = r w_2$ *for all* $r \in R$ *with* $w_3 w_1 + w_1 w_4 = w_2 w_1 = \alpha \in C$;

(c) *there exist* $w_1, w_2, w_3, w_4 \in U$ *such that* $G(r) = w_3 r + r w_4$, $H(r) = w_1 r + r w_2$, $F = 0$ *with* $w_3(w_1 + w_2) + (w_1 + w_2) w_4 = 0$.

(vi) *there exist* $w_1, w_2, w_3, w_4 \in U$ *such that* $G(r) = w_3 r + r w_4$, $H(r) = w_1 r + r w_2$, $F = 0$ *and* R *satisfies* s_4;

(vii) $G = 0$ *and* $F = 0$;

(viii) *there exist* $\sigma_1, \sigma_2, \sigma_3 \in C$, $w_1, w_2, w_3, w_4 \in U$ *such that* $G(r) = w_3 r + r w_4$, $H(r) = w_1 r + r w_2$, $F = 0$ *with* $w_4 + \sigma_3 w_2 = \sigma_1$, $w_3 - \sigma_3 w_1 = \sigma_2$ *and* $w_3 w_1 + \sigma_1 w_1 = -(w_2 w_4 + \sigma_2 w_2) \in C$.

In particular for $G = I$, the identity mapping and $F = H$ in the Theorem 2.1, we get the following result.

Corollary 2.2 *Suppose that* R *is a prime ring of characteristic not equal to* 2 *and* H *is a generalized derivation on* R. *If* $H(\xi^2) = H(\xi)^2$ *for all* $\xi = \pi(\omega_1, \ldots, \omega_n)$, $\omega_1, \ldots, \omega_n \in R$, *then either* $H = 0$ *or* $H(t) = t$ *for all* $t \in R$.

Corollary 2.3 *([8], Main theorem) By taking* $G = 0$ *in Theorem 2.1, we get the Carini, De Filippis and Gsudo result.*

In particular for $H = I$, the identity mapping in Theorem 2.1, we get the following.

Corollary 2.4 *Suppose that* R *is a prime ring of characteristic not equal to* 2 *and* G, F *are generalized derivations on* R. *If* $G(\xi^2) = F(\xi)\xi$ *for all* $\xi = \pi(\omega_1, \ldots, \omega_n)$, $\omega_1, \ldots, \omega_n \in R$, *then one of the following holds:*

(i) $G = 0$ *and* $F = 0$;

(ii) *there exists* $w_1 \in U$ *with* $F(t) = G(t) = w_1 t$ *for all* $t \in R$;

(iii) *there exist* $w_1, w_2 \in U$ *with* $F(t) = w_1 t$, $G(t) = [w_2, t] + t w_1$ *for all* $t \in R$ *and* $\pi(\omega_1, \ldots, \omega_n)^2$ *is central valued on* R;

(iv) *there exist* $w_1, w_2 \in U$ *with* $G(t) = w_1 t + t w_2$, $F = 0$ *and* R *satisfies* s_4.

Since difference of two generalized derivations is a generalized derivation on ring R. By substituting $G = 0$ and $F = F - I$, where I is the identity mapping on R in Theorem 2.1, we have the following.

Corollary 2.5 *Suppose that* R *is a prime ring of characteristic not equal to* 2 *and* $H \neq 0$, F *are generalized derivations on* R. *If*

$$F(\xi)H(\xi) - \xi H(\xi) = 0,$$

for all $\xi = \pi(\omega_1, \ldots, \omega_n)$, $\omega_1, \ldots, \omega_n \in R$, *then one of the following holds:*

(i) $F = I$, *the identity mapping;*

(ii) *there exist* $w_1, w_2 \in U$ *with* $F(t) = t(w_1 + 1)$, $H(t) = w_2 t$ *for all* $t \in R$ *with* $w_1 w_2 = 0$;

(iii) *there exist* $w_1, w_2 \in U$ *with* $F(t) = (w_1 + 1)t$, $H(t) = t w_2$ *for all* $t \in R$ *with* $w_1 w_2 = 0$ *and* $\pi(\omega_1, \ldots, \omega_n)^2$ *is central valued on* R.

3 Preliminaries

Let h and δ be two derivations on R. We denote by $\pi^h(\omega_1, \ldots, \omega_n)$ the polynomials obtained from $\pi(\omega_1, \ldots, \omega_n)$ replacing each coefficients α_σ with $h(\alpha_\sigma)$. Then we have

$$h(\pi(\omega_1, \ldots, \omega_n)) = \pi^h(\omega_1, \ldots, \omega_n) + \sum_i \pi(\omega_1, \ldots, h(\omega_i), \ldots, \omega_n)$$

and

$$h\delta(\pi(\omega_1,\ldots,\omega_n)) = \pi^{h\delta}(\omega_1,\ldots,\omega_n) + \sum_i \pi^h(\omega_1,\ldots,\delta(\omega_i),\ldots,\omega_n) + \sum_i \pi^\delta(\omega_1,\ldots,h(\omega_i),\ldots,\omega_n) + \sum_i \pi(\omega_1,\ldots,h\delta(\omega_i),\ldots,\omega_n) + \sum_{i\neq j} \pi(\omega_1,\ldots,h(\omega_i),\ldots,\delta(\omega_j),\ldots,\omega_n). \quad (1)$$

The following results are frequently used to prove our theorem. Let R be a prime ring and I be a two sided ideal of R.

Remark 3.1 By ([9]), R, I and U satisfy the same generalized polynomial identities with coefficients in U.

Remark 3.2 By ([24]), R, I and U satisfy the same differential identities.

4 G, F and H are inner maps

This section deals the case when G, F and H all are generalized inner derivations. Suppose $G(x) = p_5x + xp_6$, $F(x) = p_1x + xp_2$ and $H(x) = p_3x + xp_4$ for all $x \in R$, where $p_1, p_2, p_3, p_4, p_5, p_6 \in U$. From the hypothesis $G(H(\pi(\xi)^2)) = F(\pi(\xi))H(\pi(\xi))$ we get the expression $p_5\Big(p_3X^2+X^2p_4\Big)+\Big(p_3X^2+X^2p_4\Big)p_6 = \Big(p_1X+Xp_2\Big)\Big(p_3X+Xp_4\Big)$ that is $a_1X^2+a_2X^2a_3+a_4X^2a_5-a_6Xa_4X-Xa_7X-a_6X^2a_3-Xa_8Xa_3+X^2a_9 = 0$ for all $X = \pi(\omega_1,\ldots,\omega_n)$, where $a_1 = p_5p_3, a_2 = p_5, a_3 = p_4, a_4 = p_3, a_5 = p_6, a_6 = p_1, a_7 = p_2p_3, a_8 = p_2, a_9 = p_4p_6$.

Proposition 4.1 *Suppose R is a prime ring with characteristic is not equal to 2. If H, G, F are three generalized inner derivations on R such that*

$$G(H(\xi^2)) = F(\xi)H(\xi),$$

for all $\xi = \pi(\omega_1,\ldots,\omega_n)$, $\omega_1,\ldots,\omega_n \in R$. Then one of the following holds:

- *(i) $H = 0$;*
- *(ii) there exist $\sigma \in C$, $w_1 \in U$ with $G(r) = F(r) = w_1r$ and $H(r) = \sigma r$ for all $r \in R$;*
- *(iii) there exist $w_1, w_2, w_3 \in U$, $\sigma \in C$ with $H(r) = rw_1$, $F(r) = w_2r$ and $G(r) = \sigma r + w_2r + rw_3$ with $w_1w_3 = -\sigma w_1$;*
- *(iv) there exist $w_1, w_2, w_3 \in U$ with $H(r) = w_1r$, $F(r) = rw_2$ and $G(r) = w_3r$ for all $r \in R$ with $w_3w_1 = w_2w_1 = \alpha \in C$;*
- *(v) $\pi(\omega_1,\ldots,\omega_n)^2$ is central valued on R and one of the following holds:*
  - *(a) there exist $0 \neq \sigma \in C$, $w_1, w_2 \in U$ with $H(r) = \sigma r$, $F(r) = w_1r$ and $G(r) = [w_2, r] + rw_1$ for all $r \in R$;*
  - *(b) there exist $w_1, w_2, w_3, w_4 \in U$, $\alpha \in C$ such that $G(r) = w_3r + rw_4$ and either $H(r) = rw_1$, $F(r) = w_2r$ with $w_3w_1 + w_1w_4 = w_2w_1$ or $H(r) = w_1r$, $F(r) = rw_2$ with $w_3w_1 + w_1w_4 = w_2w_1 = \alpha \in C$;*
  - *(c) there exist $w_1, w_2, w_3, w_4 \in U$ such that $G(r) = w_3r + rw_4$, $H(r) = w_1r + rw_2$, $F = 0$ with $w_3(w_1 + w_2) + (w_1 + w_2)w_4 = 0$.*
- *(vi) there exist $w_1, w_2, w_3, w_4 \in U$ such that $G(r) = w_3r + rw_4$, $H(r) = w_1r + rw_2$, $F = 0$ and R satisfies s_4;*
- *(vii) $G = 0$ and $F = 0$;*
- *(viii) there exist $\sigma_1, \sigma_2, \sigma_3 \in C$, $w_1, w_2, w_3, w_4 \in U$ such that $G(r) = w_3r + rw_4$, $H(r) = w_1r + rw_2$, $F = 0$ with $w_4 + \sigma_3w_2 = \sigma_1$, $w_3 - \sigma_3w_1 = \sigma_2$ and $w_3w_1 + \sigma_1w_1 = -(w_2w_4 + \sigma_2w_2) \in C$.*

We need the following results to prove the above proposition.

Lemma 4.2 *[15, Lemma 1] Let $m \geq 2$ and K be an infinite field. Let $A_1, A_2, \ldots, A_n$ are non scalar matrices in $M_m(K)$ then there exists an invertible matrix $N \in M_m(K)$ such that all matrices NA_1N^{-1}, NA_2N^{-1}, ..., NA_nN^{-1} have all nonzero entries.*

Proposition 4.3 *Let $R = M_m(K)$ be the ring of all $m \times m$ matrices over the field K with characteristic not equal to 2 and $m \geq 2$. Let $q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8, q_9 \in R$ such that $q_1X^2 + q_2X^2q_3 + q_4X^2q_5 - q_6Xq_4X - Xq_7X - q_6X^2q_3 - Xq_8Xq_3 + X^2q_9 = 0$ for all $X = \pi(\omega_1, \ldots, \omega_n) \in f(R)$, then one of the following holds:*

(i) q_4, q_3 *are central;*
(ii) q_4, q_8 *are central;*
(iii) q_6, q_3 *are central;*
(iv) q_6, q_8 *are central.*

Proof We shall prove this by contradiction. Suppose that $q_4 \notin Z(R)$ and $q_6 \notin Z(R)$. By the hypothesis, R satisfies the generalized polynomial identity

$$q_1X^2 + q_2X^2q_3 + q_4X^2q_5 - q_6Xq_4X - Xq_7X - q_6X^2q_3 - Xq_8Xq_3 + X^2q_9 = 0, \tag{2}$$

for all $X \in \pi(R^n)$. We have the following cases.

Case-I: If K is infinite, then by Lemma 4.2 there exists a K-automorphism ϕ of $M_m(K)$ such that $q_4' = \phi(q_4)$, $q_6' = \phi(q_6)$ have all nonzero entries. Clearly $q_4', q_6', q_1' = \phi(q_1), q_2' = \phi(q_2), q_3' = \phi(q_3), q_5' = \phi(q_5)$, $q_7' = \phi(q_7), q_8' = \phi(q_8)$ and $q_9' = \phi(q_9)$ must satisfy the relation (2). Now we can replace $q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8, q_9$ with $q_1', q_2', q_3', q_4', q_5', q_6', q_7', q_8', q_9'$ respectively.

Let e_{ij} be the matrix such that (i, j)-entry is 1 and remaining other entries are zero. It is given $X = \pi(\omega_1, \ldots, \omega_n)$ is not central, by [24] (see also [25]), there exist $\omega_1, \ldots, \omega_n \in M_m(K)$ and $\gamma \in K\backslash\{0\}$ such that $X = \pi(\omega_1, \ldots, \omega_n) = \gamma e_{st}$, with $s \neq t$. Moreover, since the set $\{X = \pi(\omega_1, \ldots, \omega_n) : \omega_1, \ldots, \omega_n \in M_m(K)\}$ is invariant under the action of all K-automorphisms of $M_m(K)$, then $i \neq j$ there exist $\omega_1, \ldots, \omega_n \in M_m(K)$ such that $X = \pi(\omega_1, \ldots, \omega_n) = e_{ij}$. Hence by (2) we have

$$-q_6e_{ij}q_4e_{ij} - e_{ij}q_7e_{ij} - e_{ij}q_8e_{ij}q_3 = 0. \tag{3}$$

Left multiplying above relation by e_{ij}, we obtain $e_{ij}q_6e_{ij}q_4e_{ij} = 0$. It implies that $q_{6_{ij}}q_{4_{ij}}e_{ij} = 0$. This implies that either $q_{6_{ij}} = 0$ or $q_{4_{ij}} = 0$, a contradiction and thus we get either q_4 is central or q_6 is central.

In the same fashion, we can show that either q_3 is central or q_8 is central. Combining these two results, we get our conclusions.

Case-II: Let K be a finite. Suppose F is an infinite extension of K. Suppose that $\overline{R} = M_m(F) \cong R \otimes_K F$. Note that $\pi(\omega_1, \ldots, \omega_n)$ is central valued on R if and only if it is central valued on $\overline{R}$. Suppose $\mathsf{Q}(\omega_1, \ldots, \omega_n)$ is the generalized polynomial such that

$$\mathsf{Q}(\omega_1, \ldots, \omega_n) = q_1X^2 + q_2X^2q_3 + q_4X^2q_5 - q_6Xq_4X - Xq_7X - q_6X^2q_3 - Xq_8Xq_3 + X^2q_9$$

is a GPI for R.

Since $\mathsf{Q}(\omega_1, \ldots, \omega_n)$ is a multihomogeneous of multidegree $(2, \ldots, 2)$ in the indeterminates $\omega_1, \ldots, \omega_n$. By complete linearization of $\mathsf{Q}(\omega_1, \ldots, \omega_n)$ we get a multilinear generalized polynomial $\Theta(\omega_1, \ldots, \omega_n, x_1, \ldots, x_n)$ in $2n$ indeterminates, moreover

$$\Theta(\omega_1, \ldots, \omega_n, \omega_1, \ldots, \omega_n) = 2^n\mathsf{Q}(\omega_1, \ldots, \omega_n).$$

It is clear that the multilinear polynomial $\Theta(\omega_1, \ldots, \omega_n, x_1, \ldots, x_n)$ is a generalized polynomial identity for R and $\overline{R}$. By assumption $\text{char}(R) \neq 2$ we obtain $\mathsf{Q}(\omega_1, \ldots, \omega_n) = 0$ for all $\omega_1, \ldots, \omega_n \in \overline{R}$ and then we get a contradiction from Case-I. Thus we get $q_4 \in C$ or $q_6 \in C$.

Similarly, we can prove that either q_3 is central or q_8 is central. □

Lemma 4.4 *Let R be a prime ring of characteristic not equal to 2. If $q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8, q_9 \in R$ such that $q_1X^2 + q_2X^2q_3 + q_4X^2q_5 - q_6Xq_4X - Xq_7X - q_6X^2q_3 - Xq_8Xq_3 + X^2q_9 = 0$ for all $X = \pi(\xi)$, $\xi = (\omega_1, \ldots, \omega_n) \in R^n$. Then one of the following holds:*

(i) q_4, q_3 *are central;*
(ii) q_4, q_8 *are central;*
(iii) q_6, q_3 *are central;*
(iv) q_6, q_8 *are central.*

Proof First, we will show that one of q_4 or q_6 is central. On contrary suppose that both q_4 and q_6 both are not central. By hypothesis, we have

$$\begin{aligned}\mathrm{h}(\omega_1,\ldots,\omega_n) &= q_1\pi(\omega_1,\ldots,\omega_n)^2 + q_2\pi(\omega_1,\ldots,\omega_n)^2q_3\\ &+q_4\pi(\omega_1,\ldots,\omega_n)^2q_5 - q_6\pi(\omega_1,\ldots,\omega_n)q_4\pi(\omega_1,\ldots,\omega_n)\\ &-\pi(\omega_1,\ldots,\omega_n)q_7\pi(\omega_1,\ldots,\omega_n) - q_6\pi(\omega_1,\ldots,\omega_n)^2q_3\\ &-\pi(\omega_1,\ldots,\omega_n)q_8\pi(\omega_1,\ldots,\omega_n)q_3 + \pi(\omega_1,\ldots,\omega_n)^2q_9,\end{aligned}$$

for all $\omega_1,\ldots,\omega_n \in R$. By Remark 3.1, R and U satisfy same generalized polynomial identity (GPI). Then U satisfies $\mathrm{h}(\omega_1,\ldots,\omega_n) = 0$. Let $\mathrm{h}(\omega_1,\ldots,\omega_n)$ be a trivial GPI for U, T be the free product of U and $C\{\omega_1,\ldots,\omega_n\}$, the free C-algebra in non commuting indeterminates $\omega_1,\ldots,\omega_n$ that is $T = U*_C C\{\omega_1,\ldots,\omega_n\}$. Then, $\mathrm{h}(\omega_1,\ldots,\omega_n) = 0$ in $T = U *_C C\{\omega_1,\ldots,\omega_n\}$. The term

$$\begin{aligned}&-q_6\pi(\omega_1,\ldots,\omega_n)q_4\pi(\omega_1,\ldots,\omega_n) - \pi(\omega_1,\ldots,\omega_n)q_7\pi(\omega_1,\ldots,\omega_n)\\ &-\pi(\omega_1,\ldots,\omega_n)q_8\pi(\omega_1,\ldots,\omega_n)q_3\end{aligned}$$

appears nontrivially in $\mathrm{h}(\omega_1,\ldots,\omega_n)$. Since $q_6 \notin C$, then we get

$$q_6\pi(\omega_1,\ldots,\omega_n)q_4\pi(\omega_1,\ldots,\omega_n) = 0_T,$$

gives a contradiction since neither $q_6 \in C$ nor $q_4 \in C$. Thus we get either $q_4 \in C$ or $q_6 \in C$.

Let $q_4 \in C$. Suppose that $q_3 \notin C$ and $q_8 \notin C$. Since $q_4 \in C$, U satisfies

$$\begin{aligned}\mathrm{Q}(\omega_1,\ldots,\omega_n) &= (q_1 - q_6q_4)\pi(\omega_1,\ldots,\omega_n)^2 + q_2\pi(\omega_1,\ldots,\omega_n)^2q_3\\ &-\pi(\omega_1,\ldots,\omega_n)q_7\pi(\omega_1,\ldots,\omega_n) - q_6\pi(\omega_1,\ldots,\omega_n)^2q_3\\ &-\pi(\omega_1,\ldots,\omega_n)q_8\pi(\omega_1,\ldots,\omega_n)q_3 + \pi(\omega_1,\ldots,\omega_n)^2(q_9 + q_4q_5).\end{aligned}$$

This is again a trivial GPI. Then $\mathrm{Q}(\omega_1,\ldots,\omega_n) = 0_T$ and the term

$$\pi(\omega_1,\ldots,\omega_n)q_8\pi(\omega_1,\ldots,\omega_n)q_3$$

appears nontrivially in $\mathrm{Q}(\omega_1,\ldots,\omega_n)$. This gives that either q_3 is central or q_8 is central, a contradiction. Thus we conclude that either $q_4 \in C, q_3 \in C$ or $q_4 \in C, q_8 \in C$.

In the same fashion, we can prove that either $q_6 \in C, q_3 \in C$ or $q_6 \in C, q_8 \in C$.

Suppose $\mathrm{h}(\omega_1,\ldots,\omega_n)$ is a non trivial GPI for U. If C is infinite field, we have $\mathrm{h}(\omega_1,\ldots,\omega_n) = 0$ for all $\omega_1,\ldots,\omega_n \in U \otimes_C \overline{C}$, where $\overline{C}$ denotes the algebraic closure of C. By [13, Theorems 2.5 and 3.5], $U \otimes_C \overline{C}$ and U both are centrally closed and prime, then we replace R by U or $U \otimes_C \overline{C}$ according to C finite or infinite. Then $\mathrm{h}(\omega_1,\ldots,\omega_n) = 0$ for all $\omega_1,\ldots,\omega_n \in R$ and R is centrally closed over C. By Martindale's theorem [26], R is then a primitive ring with nonzero socle $soc(R)$ and with C as its associated division ring. Then, by Jacobson's theorem [21, p.75], R is isomorphic to a dense ring of linear transformations of a vector space V over C.

If the dimension of V is finite, that is, $\dim_C V = m$. Then $R \cong M_m(C)$ (by density of R). By our hypothesis $\pi(\omega_1,\ldots,\omega_n)$ is not central valued on R it implies that R must be noncommutative. Then we can assume that $m \geq 2$. By Proposition 4.3, it implies that either $q_4 \in C, q_3 \in C$ or $q_4 \in C, q_8 \in C$ or $q_6 \in C, q_3 \in C$ or $q_6 \in C, q_8 \in C$.

Next we suppose that V is infinite dimensional over C. By Martindale's theorem [26, Theorem 3], for any $e^2 = e \in soc(R)$ we have $eRe \cong M_t(C)$ with $t = \dim_C Ve$. We shall prove this case by contradiction. Suppose that none of q_4, q_6, q_3, q_8 are not in the center C. Then there exist $u_1, u_2, u_3, u_4 \in soc(R)$ with $[q_4, u_1] \neq 0$, $[q_3, u_3] \neq 0$, $[q_8, u_4] \neq 0$ and $[q_6, u_2] \neq 0$. By Litoff's Theorem [14], there exists an element $e \in soc(R)$ with $e^2 = e$ and $q_4u_1, u_1q_4, q_6u_2, u_2q_6, q_3u_3, u_3q_3, q_8u_4, u_4q_8 \in eRe$. Since R satisfies generalized identity

$$\begin{aligned}e\Big\{&q_1\pi(e\omega_1e,\ldots,e\omega_ne)^2 + q_2\pi(e\omega_1e,\ldots,e\omega_ne)^2q_3\\ &+q_4\pi(e\omega_1e,\ldots,e\omega_ne)^2q_5 - q_6\pi(e\omega_1e,\ldots,e\omega_ne)q_4\pi(e\omega_1e,\ldots,e\omega_ne)\\ &-\pi(e\omega_1e,\ldots,e\omega_ne)q_7\pi(e\omega_1e,\ldots,e\omega_ne) - q_6\pi(e\omega_1e,\ldots,e\omega_ne)^2q_3\\ &-\pi(e\omega_1e,\ldots,e\omega_ne)q_8\pi(e\omega_1e,\ldots,e\omega_ne)q_3 + \pi(e\omega_1e,\ldots,e\omega_ne)^2q_9\Big\}e,\end{aligned}$$

the subring eRe satisfies

$$\begin{aligned}&eq_1e\pi(\omega_1,\ldots,\omega_n)^2+eq_2e\pi(\omega_1,\ldots,\omega_n)^2eq_3e\\&+eq_4e\pi(\omega_1,\ldots,\omega_n)^2eq_5e-eq_6e\pi(\omega_1,\ldots,\omega_n)eq_4e\pi(\omega_1,\ldots,\omega_n)\\&-\pi(\omega_1,\ldots,\omega_n)eq_7e\pi(\omega_1,\ldots,\omega_n)-eq_6e\pi(\omega_1,\ldots,\omega_n)^2eq_3e\\&-\pi(\omega_1,\ldots,\omega_n)eq_8e\pi(\omega_1,\ldots,\omega_n)eq_3e+\pi(\omega_1,\ldots,\omega_n)^2eq_9e.\end{aligned}$$

Then either eq_4e or eq_6e and either eq_3e or eq_8e are central elements of eRe (by the above finite dimensional case). Then we get either $q_4u_1 = eq_4eu_1 = u_1eq_4e = u_1q_4$ or $q_6u_2 = eq_6eu_2 = u_2eq_6e = u_2q_6$ and either $q_3u_3 = eq_3eu_3 = u_3eq_3e = u_3q_3$ or $q_8u_4 = eq_8eu_4 = u_4eq_8e = u_4q_8$, a contradiction. □

Lemma 4.5 *[12, Lemma 2.9] Let R be a prime ring of characteristic not equal to 2, $q_1, q_2, q_3, q_4 \in U$ and $p(\omega_1,\ldots,\omega_n)$ be any polynomial over C which is not identity for R. If $q_1p(\xi) + p(\xi)q_2 + q_3p(\xi)q_4 = 0$ for all $\xi \in R^n$ then one of the following conditions holds:*

(i) $q_2, q_4 \in C$ and $q_1 + q_2 + q_3q_4 = 0$,
(ii) $q_1, q_3 \in C$ and $q_1 + q_2 + q_3q_4 = 0$,
(iii) $q_1 + q_2 + q_3q_4 = 0$ and $p(\omega_1,\ldots,\omega_n)$ is central valued on R.

Lemma 4.6 *Let R be a prime ring of characteristic is not equal to 2 and $q_1, q_2, q_3, q_4 \in R$. If $q_1\pi(\xi)^2 - \pi(\xi)q_2\pi(\xi) + q_3\pi(\xi)^2q_4 = 0$ for all $\xi = (\omega_1,\ldots,\omega_n) \in R^n$, then one of the following holds:*

(i) $q_4 \in C$ and $q_1 + q_3q_4 = q_2 = \alpha \in C$;
(ii) $q_1, q_3 \in C$ and $q_1 + q_3q_4 = q_2 = \alpha \in C$;
(iii) $\pi(\omega_1,\ldots,\omega_n)^2$ is central valued on R and $q_1 + q_3q_4 = q_2 = \alpha \in C$.

Proof By using similar argument as we have used in Lemma 4.4, we get $q_2 \in C$. Then our hypothesis gives that

$$(q_1-q_2)\pi(\xi)^2+q_3\pi(\xi)^2q_4=0$$

for all $\xi = (\omega_1,\ldots,\omega_n) \in R^n$. From Lemma 4.5, we have one of the following.

- $q_4 \in C$ and $q_1 - q_2 + q_3q_4 = 0$, which implies that $q_1 + q_3q_4 = q_2 = \alpha \in C$. In this case, we get $q_2, q_4 \in C$ and $q_1 + q_3q_4 = q_2 = \alpha \in C$ for some $\alpha \in C$, which is conclusion (i);
- $q_1 - q_2 \in C$, $q_3 \in C$ and $q_1 - q_2 + q_3q_4 = 0$. Thus we get $q_1, q_2, q_3 \in C$ and $q_1 + q_3q_4 = q_2 = \alpha \in C$ for some $\alpha \in C$, which is conclusion (ii);
- $\pi(\omega_1,\ldots,\omega_n)^2$ is central valued on R and $q_1 + q_3q_4 = q_2 = \alpha \in C$ for some $\alpha \in C$, which is conclusion (iii).

□

Lemma 4.7 *Let R be a prime ring of characteristic not equal to 2 and $q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8 \in R$. If $q_1\pi(\xi)^2 + q_2\pi(\xi)^2q_3 + q_4\pi(\xi)^2q_5 + \pi(\xi)^2q_6 - \pi(\xi)q_7\pi(\xi) - \pi(\xi)^2q_8 = 0$ for all $\xi = (\omega_1,\ldots,\omega_n) \in R^n$, then $q_7 \in C$.*

Proof By using similar argument as we have used in Lemma 4.4, we get our conclusion. □

The following lemma is a particular case of Lemma 3 of [2].

Lemma 4.8 *Let R be a noncommutative prime ring of characteristic different from 2 and $q_1, q_2, q_3 \in U$. If $\pi(\xi)q_1\pi(\xi) + \pi(\xi)^2q_2 - q_3\pi(\xi)^2 = 0$ for all $\xi = (\omega_1,\ldots,\omega_n) \in R^n$, then one of the following holds:*

(i) $q_2, q_3 \in C$ and $q_3 - q_2 = q_1 = \mu \in C$,
(ii) $\pi(\omega_1,\ldots,\omega_n)^2$ is central valued on R and $q_3 - q_2 = q_1 = \mu \in C$.

Lemma 4.9 *[19, Proposition 2.13] Let R be a prime ring of characteristic is not equal to 2. Suppose there exist $a, b, c, q, u, v \in R$ such that $ax + bxc + qxu + xv = 0$, for all $x \in S = [R, R]$. Then one of the following holds:*

1. *R satisfies s_4;*
2. *$c, q \in Z(R)$ and $a + bc = -(v + uq) \in Z(R)$;*
3. *$c, u, v \in Z(R)$ and $a + bc + qu + v = 0$;*
4. *$a, b, q \in Z(R)$ and $a + bc + qu + v = 0$;*

5. *$b, u \in Z(R)$ and $a + qu = -(v + bc) \in Z(R)$;*
6. *there exist $\lambda, \mu, \eta \in Z(R)$ such that $u + \eta c = \lambda$, $b - \eta q = \mu$ and $a + \lambda q = -(v + \mu c) \in Z(R)$.*

Now we are ready to prove Proposition 4.1.

Proof of Proposition 4.1: By the hypothesis, we have

$$\begin{aligned} &p_5\Big(p_3\pi(\omega_1,\dots,\omega_n)^2 + \pi(\omega_1,\dots,\omega_n)^2 p_4\Big) + \Big(p_3\pi(\omega_1,\dots,\omega_n)^2 \\ &+\pi(\omega_1,\dots,\omega_n)^2 p_4\Big)p_6 = \Big(p_1\pi(\omega_1,\dots,\omega_n) \\ &+\pi(\omega_1,\dots,\omega_n)p_2\Big)\Big(p_3\pi(\omega_1,\dots,\omega_n) + \pi(\omega_1,\dots,\omega_n)p_4\Big). \end{aligned} \tag{4}$$

That is

$$\begin{aligned} &p_5 p_3\pi(\omega_1,\dots,\omega_n)^2 + p_5\pi(\omega_1,\dots,\omega_n)^2 p_4 + p_3\pi(r\omega_1,\dots,\omega_n)^2 p_6 \\ &+\pi(\omega_1,\dots,\omega_n)^2 p_4 p_6 = p_1\pi(\omega_1,\dots,\omega_n)p_3\pi(\omega_1,\dots,\omega_n) \\ &+\pi(\omega_1,\dots,\omega_n)p_2 p_3\pi(\omega_1,\dots,\omega_n) + p_1 f(\omega_1,\dots,\omega_n)^2 p_4 \\ &+\pi(\omega_1,\dots,\omega_n)p_2\pi(\omega_1,\dots,\omega_n)p_4, \end{aligned}$$

for all $\omega_1, \cdots, \omega_n \in R$. By Lemma 4.4 we get one of the following:

1. p_3, p_4 are central;
2. p_3, p_2 are central;
3. p_1, p_4 are central;
4. p_1, p_2 are central.

Case-I: If $p_3, p_4 \in C$, then $H(t) = (p_3 + p_4)t = \lambda t$ for all $t \in R$. If $\lambda = 0$, then we have conclusion (i). Suppose that $\lambda \neq 0$. Then equation (4) reduces to

$$(p_5 - p_1)\pi(\xi)^2 + \pi(\xi)^2 p_6 - \pi(\xi)p_2\pi(\xi) = 0$$

for all $\xi = (\omega_1, \dots, \omega_n) \in R^n$. Then by Lemma 4.8, we have one of the following:

- $p_6 \in C$, $p_5 - p_1 \in C$ and $p_5 - p_1 + p_6 = p_2 = \alpha \in C$, which gives $p_5 + p_6 = p_1 + p_2$. In this case we get $G(t) = (p_5 + p_6)t$ and $F(t) = (p_1 + p_2)t = (p_5 + p_6)t$ for all $t \in R$, which is conclusion (ii);
- $p_5 - p_1 + p_6 = p_2 = \alpha \in C$ and $\pi(\omega_1, \dots, \omega_n)^2$ is central valued on R which gives $p_5 + p_6 = p_1 + p_2$ and $p_2 \in C$. In this case, we get $G(t) = p_5 t + t p_6 = p_5 t + t(p_1 + p_2 - p_5) = [p_5, t] + t(p_1 + p_2)$, $F(t) = (p_1 + p_2)t$ for all $t \in R$ and $\pi(\omega_1, \dots, \omega_n)^2$ is central valued on R, which is conclusion $((v)(a))$.

Case-II: If $p_3, p_2 \in C$, then $H(t) = t(p_3 + p_4)$ and $F(t) = (p_1 + p_2)t$ for all $t \in R$. Then (4) reduces to

$$(p_5 - p_1 - p_2)\pi(\xi)^2(p_3 + p_4) + \pi(\xi)^2(p_3 + p_4)p_6 = 0,$$

for all $\xi = (\omega_1, \dots, \omega_n) \in R^n$. From Lemma 4.5, we have one of the following:

- $(p_3 + p_4)p_6 \in C$, $p_3 + p_4 \in C$ and $(p_3 + p_4)p_6 + (p_5 - p_1 - p_2)(p_3 + p_4) = 0$. In this case, we have $H(t) = (p_3 + p_4)t$, $F(t) = (p_1 + p_2)t$ for all $t \in R$. If $p_3 + p_4 = 0$, then we get conclusion (i). If $p_3 + p_4 \neq 0$, then $p_6 \in C$ and $p_5 + p_6 = p_1 + p_2$. Thus we get $G(t) = (p_5 + p_6)t = (p_1 + p_2)t$ for all $t \in R$, which is conclusion (ii);
- $p_5 - p_1 - p_2 \in C$ and $(p_3 + p_4)p_6 + (p_5 - p_1 - p_2)(p_3 + p_4) = (p_3 + p_4)(p_5 + p_6 - p_1 - p_2) = 0$. Thus we get $p_5 = \lambda + p_1 + p_2$ for some $\lambda \in C$. In this case, we get $H(t) = t(p_3 + p_4)$, $F(t) = (p_1 + p_2)t$ and $G(t) = p_5 t + t p_6 = \lambda t + (p_1 + p_2)t + t p_6$ for all $t \in R$ with $(p_3 + p_4)p_6 = -(p_3 + p_4)\lambda$, which is conclusion (iii);
- $\pi(\omega_1, \dots, \omega_n)^2$ is central valued on R and $F(t) = (p_1 + p_2)t$, $H(t) = t(p_3 + p_4)$ and $G(t) = p_5 t + t p_6$ with $p_5(p_3 + p_4) + (p_3 + p_4)p_6 - (p_1 + p_2)(p_3 + p_4) = 0$, which is conclusion $((v)(b))$.

Case-III: If $p_1, p_4 \in C$, then $H(t) = (p_3 + p_4)t$ and $F(t) = t(p_1 + p_2)$ for all $t \in R$. Then (4) reduces to

$$p_5(p_3 + p_4)\pi(\xi)^2 - \pi(\xi)(p_1 + p_2)(p_3 + p_4)\pi(\xi) + (p_3 + p_4)\pi(\xi)^2 p_6 = 0,$$

for all $\xi = (\omega_1, \dots, \omega_n) \in R^n$. By Lemma 4.6, we have one of the following:

- $p_6 \in C$ and $p_5(p_3+p_4)+(p_3+p_4)p_6=(p_1+p_2)(p_3+p_4)=\alpha \in C$ for some $\alpha \in C$. In this case we have $H(t)=(p_3+p_4)t$, $F(t)=t(p_1+p_2)$ and $G(t)=(p_5+p_6)t$ for all $t \in R$ with $(p_5+p_6)(p_3+p_4)=(p_1+p_2)(p_3+p_4)=\alpha \in C$, which is conclusion (iv);
- $p_3+p_4 \in C$, $p_5(p_3+p_4) \in C$ and $p_5(p_3+p_4)+(p_3+p_4)p_6=(p_1+p_2)(p_3+p_4)=\alpha \in C$ for some $\alpha \in C$. If $p_3+p_4=0$, then we get conclusion (i). Suppose $p_3+p_4 \neq 0$ that is $H \neq 0$. Then $(p_5+p_6)(p_3+p_4)=(p_1+p_2)(p_3+p_4)=\alpha \in C$, which implies that $p_5+p_6=p_1+p_2$, which gives our conclusion (ii);
- $\pi(\omega_1,\ldots,\omega_n)^2$ is central valued on R and $p_5(p_3+p_4)+(p_3+p_4)p_6=(p_1+p_2)(p_3+p_4)=\alpha \in C$ for some $\alpha \in C$, which is conclusion $((v)(b))$.

Case-IV: If $p_1, p_2 \in C$, then $F(t)=(p_1+p_2)t=\alpha t$. Then (4) reduces to

$$p_5p_3\pi(\xi)^2+p_5\pi(\xi)^2p_4+p_3\pi(\xi)^2p_6+\pi(\xi)^2p_4p_6 \\ -\pi(\xi)\alpha p_3\pi(\xi)-\pi(\xi)^2\alpha p_4=0.$$

From Lemma 4.7, we get $\alpha p_3 \in C$.

If $\alpha \neq 0$, then $p_3 \in C$. Thus we have $p_3 \in C$, $p_1, p_2 \in C$, which is a Case-II.

If $\alpha = 0$, then $F=0$. Thus we get

$$p_5p_3\pi(\xi)^2+p_5\pi(\xi)^2p_4+p_3\pi(\xi)^2p_6+\pi(\xi)^2p_4p_6=0, \tag{5}$$

for all $\xi=(\omega_1,\ldots,\omega_n) \in R^n$. If $\pi(\xi)^2$ is central valued on R, then $p_5(p_3+p_4)+(p_3+p_4)p_6=0$, which is conclusion $((v)(c))$.

Now assume that $\pi(\xi)^2$ for all $\xi=(\omega_1,\ldots,\omega_n) \in R^n$ is not central valued on R. Then R must be noncommutative. Suppose R_1 is a subset of R generated by R_2, where $R_2=\{\pi(\omega_1,\ldots,\omega_n)^2 \mid \omega_1,\ldots,\omega_n \in R\}$. Clearly R_1 is an additive subgroup of R and $R_2 \neq \{0\}$, since $\pi(\omega_1,\ldots,\omega_n)^2$ is noncentral valued on R. From equation (5) we get

$$p_5p_3s+p_5sp_4+p_3sp_6+sp_4p_6=0, \tag{6}$$

for all $s \in R_1$. By [10], we get one of the following

- $R_1 \subseteq Z(R)$;
- R satisfies s_4 and char $(R)=2$, except when R_1 contains a noncentral Lie ideal L of R.

Since $\pi(\omega_1,\ldots,\omega_n)^2$ is not central valued on R, the first case can not possible. Since char $(R) \neq 2$, second case also can not occur except R_1 contains a noncentral Lie ideal of R. Thus there exists Lie ideal $L \subseteq R_1$, where $L \nsubseteq Z(R)$. By [7, Lemma 1], there exists a noncentral two sided ideal I of R such that $[I,R] \subseteq L$. Then (6) gives that

$$p_5p_3s+p_5sp_4+p_3sp_6+sp_4p_6=0,$$

for all $s \in [R,R]$. From Lemma 4.9, we get one of the following:

- R satisfies s_4. In this case, we get $F=0$, $G(t)=p_5t+tp_6$, $H(t)=p_3t+tp_4$ for all $t \in R$, which is conclusion (vi);
- $p_4, p_3 \in C$ and $p_5p_3+p_5p_4=-(p_4p_6+p_6p_3) \in C$. In this case, we get $p_1, p_4 \in C$, which is Case-III.
- $p_4, p_6, p_4p_6 \in C$ and $p_5p_3+p_5p_4+p_3p_6+p_4p_6=0$. In this case, we get $p_1 \in C$, $p_4 \in C$, which is Case-III.
- $p_5p_3, p_5, p_3 \in C$ and $p_5p_3+p_5p_4+p_3p_6+p_4p_6=0$. In this case, we get $p_2 \in C$, $p_3 \in C$, which is Case-II.
- $p_5, p_6 \in C$ and $(p_5+p_6)p_3=-(p_5+p_6)p_4 \in C$. If $p_5+p_6=0$, then $G=0$. In this case, we get $G=0$ and $F=0$, which is conclusion (vii). If $p_5+p_6 \neq 0$, then $p_3, p_4 \in C$. Thus we get $p_1, p_4 \in C$, which is Case-III.
- $\mu, \lambda, \eta \in C$ such that $p_6+\eta p_4=\lambda$, $p_5-\eta p_3=\mu$ and $p_5p_3+\lambda p_3=-(p_4p_6+\mu p_4) \in C$. We get $G(t)=p_5t+tp_6$, $H(t)=p_3t+tp_4$, $F=0$, which is conclusion $(viii)$.

□

5 Proof of the Theorem 2.1

In this section, throughout we shall use the following well known results.

Remark 5.1 By (Proposition 2.5.1 [3]), we can uniquely extend every derivation d to a derivation of U.

Remark 5.2 (Kharchenko [22, Theorem 2]) Suppose that R is a prime ring, $\delta \neq 0$ a derivation on R. If I is a nonzero ideal of R satisfies the differential identity

$$\pi(\omega_1, \ldots, \omega_n, \delta(\omega_1), \ldots, \delta(\omega_n)) = 0$$

for any $\omega_1, \ldots, \omega_n \in I$, then either I satisfies the GPI

$$\pi(\omega_1, \ldots, \omega_n, t_1, \ldots, t_n) = 0$$

or δ is U-inner i.e., for some $q \in U$, $\delta(t) = [q, t]$ and I satisfies the GPI

$$\pi(\omega_1, \ldots, \omega_n, [q, \omega_1], \ldots, [q, \omega_n]) = 0.$$

Proof of theorem 2.1: If $H = 0$ then we are done. We assume $H \neq 0$. Then by [23, Theorem 3] there exist $q, b, a \in U$ and derivations d, δ, θ on U with $G(t) = qt + d(t)$, $F(t) = bt + \delta(t)$ and $H(t) = at + \theta(t)$ for all $t \in U$. Then by our hypothesis we have

$$\begin{aligned}
&(qa + d(a))\,\pi(\omega_1, \ldots, \omega_n)^2 + q\theta(\pi(\omega_1, \ldots, \omega_n)^2) + ad(\pi(\omega_1, \ldots, \omega_n)^2)\\
&\quad + d\theta(\pi(\omega_1, \ldots, \omega_n)^2) = b\pi(\omega_1, \ldots, \omega_n)a\pi(\omega_1, \ldots, \omega_n)\\
&\quad + b\pi(\omega_1, \ldots, \omega_n)\theta(\pi(\omega_1, \ldots, \omega_n)) + \delta(\pi(\omega_1, \ldots, \omega_n))a\pi(\omega_1, \ldots, \omega_n)\\
&\quad + \delta(\pi(\omega_1, \ldots, \omega_n))\theta(\pi(\omega_1, \ldots, \omega_n)),
\end{aligned} \tag{7}$$

which can be written as

$$\begin{aligned}
&(qa + d(a))\,\pi(\omega_1, \ldots, \omega_n)^2 + q\theta(\pi(\omega_1, \ldots, \omega_n))\pi(\omega_1, \ldots, \omega_n)\\
&\quad + q\pi(\omega_1, \ldots, \omega_n)\theta(\pi(\omega_1, \ldots, \omega_n)) + ad(\pi(\omega_1, \ldots, \omega_n))\pi(\omega_1, \ldots, \omega_n)\\
&\quad + a\pi(\omega_1, \ldots, \omega_n)d(\pi(\omega_1, \ldots, \omega_n)) + d\theta(\pi(\omega_1, \ldots, \omega_n))\pi(\omega_1, \ldots, \omega_n)\\
&\quad + \theta(\pi(\omega_1, \ldots, \omega_n))d(\pi(\omega_1, \ldots, \omega_n)) + d(\pi(\omega_1, \ldots, \omega_n))\theta(\pi(\omega_1, \ldots, \omega_n))\\
&\quad + \pi(\omega_1, \ldots, \omega_n)d\theta(\pi(\omega_1, \ldots, \omega_n)) = b\pi(\omega_1, \ldots, \omega_n)a\pi(\omega_1, \ldots, \omega_n)\\
&\quad + b\pi(\omega_1, \ldots, \omega_n)\theta(\pi(\omega_1, \ldots, \omega_n)) + \delta(\pi(\omega_1, \ldots, \omega_n))a\pi(\omega_1, \ldots, \omega_n)\\
&\quad + \delta(\pi(\omega_1, \ldots, \omega_n))\theta(\pi(\omega_1, \ldots, \omega_n)).
\end{aligned} \tag{8}$$

If d, δ and θ all are inner derivations then from Proposition 4.1, the result follows. Suppose all δ, d and θ are not inner derivations together. Now we need to study the following. **Case 1:** Let $d(t) = [c, t]$ for all $t \in R$, where $c \in U$ i.e. d is inner. □

Subcase 1a: Let δ be inner, say $\delta(x) = [p, x]$ and θ be an outer derivation. Putting these values in (8) we get

$$\begin{aligned}
&(qa + [c, a])\,\pi(\omega_1, \ldots, \omega_n)^2 + q\theta(\pi(\omega_1, \ldots, \omega_n))\pi(\omega_1, \ldots, \omega_n)\\
&\quad + q\pi(\omega_1, \ldots, \omega_n)\theta(\pi(\omega_1, \ldots, \omega_n)) + a[c, \pi(\omega_1, \ldots, \omega_n)]\pi(\omega_1, \ldots, \omega_n)\\
&\quad + a\pi(\omega_1, \ldots, \omega_n)[c, \pi(\omega_1, \ldots, \omega_n)] + [c, \theta(\pi(\omega_1, \ldots, \omega_n)]\pi(\omega_1, \ldots, \omega_n)\\
&\quad + \theta(\pi(\omega_1, \ldots, \omega_n))[c, \pi(\omega_1, \ldots, \omega_n)] + [c, \pi(\omega_1, \ldots, \omega_n)]\theta(\pi(\omega_1, \ldots, \omega_n))\\
&\quad + \pi(\omega_1, \ldots, \omega_n)[c, \theta(\pi(\omega_1, \ldots, \omega_n)] = b\pi(\omega_1, \ldots, x_n)a\pi(\omega_1, \ldots, \omega_n)\\
&\quad + b\pi(\omega_1, \ldots, \omega_n)\theta(\pi(\omega_1, \ldots, \omega_n)) + [p, \pi(\omega_1, \ldots, \omega_n)]a\pi(\omega_1, \ldots, \omega_n)\\
&\quad + [p, \pi(\omega_1, \ldots, \omega_n)]\theta(\pi(\omega_1, \ldots, \omega_n)).
\end{aligned}$$

Since θ is an outer derivation on U, by replacing the value of $\theta(\pi(\omega_1, \ldots, \omega_n))$ from (1) and by applying Remark 5.2) U satisfies

$$(qa + [c, a])\,\pi(\omega_1, \ldots, \omega_n)^2 + q\pi^{\theta}(\omega_1, \ldots, \omega_n)\pi(\omega_1, \ldots, \omega_n)$$

$$+q\sum_i \pi(\omega_1,\ldots,\eta_i,\ldots,\omega_n)\pi(\omega_1,\ldots,\omega_n)+q\pi(\omega_1,\ldots,\omega_n)\pi^{\theta}(\omega_1,\ldots,\omega_n)$$
$$+q\pi(\omega_1,\ldots,\omega_n)\sum_i \pi(\omega_1,\ldots,\eta_i,\ldots,\omega_n)+a[c,\pi(\omega_1,\ldots,\omega_n)]\pi(\omega_1,\ldots,\omega_n)$$
$$+a\pi(\omega_1,\ldots,\omega_n)[c,\pi(\omega_1,\ldots,\omega_n)]+[c,\pi^{\theta}(\omega_1,\ldots,\omega_n)]\pi(\omega_1,\ldots,\omega_n)$$
$$+[c,\sum_i \pi(\omega_1,\ldots,\eta_i,\ldots,\omega_n)]\pi(\omega_1,\ldots,\omega_n)+\pi^{\theta}(\omega_1,\ldots,\omega_n)[c,\pi(\omega_1,\ldots,\omega_n)]$$
$$+\sum_i \pi(\omega_1,\ldots,\eta_i,\ldots,\omega_n)[c,\pi(\omega_1,\ldots,\omega_n)]$$
$$+[c,\pi(\omega_1,\ldots,\omega_n)]\pi^{\theta}(\omega_1,\ldots,\omega_n))+[c,\pi(\omega_1,\ldots,\omega_n)]\sum_i \pi(\omega_1,\ldots,\eta_i,\ldots,\omega_n)]$$
$$+\pi(\omega_1,\ldots,\omega_n)[c,\pi^{\theta}(\omega_1,\ldots,\omega_n)]+\pi(\omega_1,\ldots,\omega_n)[c,\sum_i \pi(\omega_1,\ldots,\eta_i,\ldots,\omega_n)]$$
$$=b\pi(\omega_1,\ldots,\omega_n)a\pi(\omega_1,\ldots,\omega_n)+b\pi(\omega_1,\ldots,\omega_n)\pi^{\theta}(\omega_1,\ldots,\omega_n))$$
$$+b\pi(\omega_1,\ldots,\omega_n)\sum_i \pi(\omega_1,\ldots,\eta_i,\ldots,\omega_n)+[p,\pi(\omega_1,\ldots,\omega_n)]a\pi(\omega_1,\ldots,\omega_n)$$
$$+[p,\pi(\omega_1,\ldots,\omega_n)]\pi^{\theta}(\omega_1,\ldots,\omega_n)+[p,\pi(\omega_1,\ldots,\omega_n)]\sum_i \pi(\omega_1,\ldots,\eta_i,\ldots,\omega_n),$$

for all $\omega_i,\eta_i\in U$. In particular U satisfies the blended component

$$q\sum_i \pi(\omega_1,\ldots,\eta_i,\ldots,\omega_n)\pi(\omega_1,\ldots,\omega_n)$$
$$+q\pi(\omega_1,\ldots,\omega_n)\sum_i \pi(\omega_1,\ldots,\eta_i,\ldots,\omega_n)$$
$$+\sum_i \pi(\omega_1,\ldots,\eta_i,\ldots,\omega_n)[c,\pi(\omega_1,\ldots,\omega_n)]$$
$$+[c,\sum_i \pi(\omega_1,\ldots,\eta_i,\ldots,\omega_n)]\pi(\omega_1,\ldots,\omega_n)$$
$$+[c,\pi(\omega_1,\ldots,\omega_n)]\sum_i \pi(\omega_1,\ldots,\eta_i,\ldots,\omega_n)$$
$$+\pi(\omega_1,\ldots,\omega_n)[c,\sum_i \pi(\omega_1,\ldots,\eta_i,\ldots,\omega_n)]$$
$$=b\pi(\omega_1,\ldots,\omega_n)\sum_i \pi(\omega_1,\ldots,\eta_i,\ldots,\omega_n)$$
$$+[p,\pi(\omega_1,\ldots,\omega_n)]\sum_i \pi(\omega_1,\ldots,\eta_i,\ldots,\omega_n),$$

for all $\omega_i,\eta_i\in U$, where $\eta_i=\theta(\omega_i)$. By replacing $\eta_1=\omega_1$ and $\eta_i=0$ for $i=2,3,\cdots,n$ we get

$$2qX^2+2X[c,X]+2[c,X]X=bX^2+[p,X]X,$$

where $X=\pi(\omega_1,\ldots,\omega_n)$. We can write it again as

$$(2q+2c)X^2-X^2 2c=\{(b+p)X-Xp\}X.$$

Now result follows from Proposition 4.1 by taking $G(t)=(2q+2c)t-t(2c)$, $F(t)=(b+p)t-tp$ and $H(t)=t$ for all $t\in R$.

Subcase 1b: Let δ be outer and θ be inner say $\theta(t)=[s,t]$ for all $t\in R$ for some $s\in U$. Substituting these values in (7) we get

$$(qa+[c,a])\,\pi(\omega_1,\ldots,\omega_n)^2+q[s,\pi(\omega_1,\ldots,\omega_n)^2]+a[c,\pi(\omega_1,\ldots,\omega_n)^2]$$

$$\begin{aligned}&+[c,[s,\pi(\omega_1,\dots,\omega_n)^2]]=b\pi(\omega_1,\dots,\omega_n)a\pi(\omega_1,\dots,\omega_n)\\&+b\pi(\omega_1,\dots,\omega_n)[s,\pi(\omega_1,\dots,\omega_n)]+\delta(\pi(\omega_1,\dots,\omega_n))a\pi(\omega_1,\dots,\omega_n)\\&+\delta(\pi(\omega_1,\dots,\omega_n))[s,\pi(\omega_1,\dots,\omega_n)].\end{aligned}$$

Since δ is an outer derivation, by Remark 5.2, we replace $\delta(\pi(\omega_1,\dots,\omega_n))$ from (1) in above expression, we get

$$\begin{aligned}&(qa+[c,a])\,\pi(\omega_1,\dots,\omega_n)^2+q[s,\pi(\omega_1,\dots,\omega_n)^2]+a[c,\pi(\omega_1,\dots,\omega_n)^2]\\&+[c,[s,\pi(\omega_1,\dots,\omega_n)^2]]=b\pi(\omega_1,\dots,\omega_n)a\pi(\omega_1,\dots,\omega_n)\\&+b\pi(\omega_1,\dots,\omega_n)[s,\pi(\omega_1,\dots,\omega_n)]+\pi^{\delta}(\omega_1,\dots,\omega_n)a\pi(\omega_1,\dots,\omega_n)\\&+\sum_i\pi(\omega_1,\dots,\sigma_i,\dots,\omega_n)a\pi(\omega_1,\dots,\omega_n)+\pi^{\delta}(\omega_1,\dots,\omega_n)[s,\pi(\omega_1,\dots,\omega_n)]\\&+\sum_i\pi(\omega_1,\dots,\sigma_i,\dots,\omega_n)[s,\pi(\omega_1,\dots,\omega_n)],\end{aligned}$$

for all $\omega_i,\sigma_i\in U$. In particular U satisfies the blended component

$$\sum_i\pi(\omega_1,\dots,\sigma_i,\dots,\omega_n)a\pi(\omega_1,\dots,\omega_n)+\sum_i\pi(\omega_1,\dots,\sigma_i,\dots,\omega_n)[s,\pi(\omega_1,\dots,\omega_n)].$$

For $\sigma_1=\omega_1$ and $\sigma_i=0$ for $i=2,3,\cdots,n$, we get $X^2s=X(a+s)X$, where $X=\pi(\omega_1,\dots,\omega_n)$. By taking $G(t)=ts$, $F(t)=t(a+s)$ and $H(t)=t$ for all $t\in R$, where $s\in U$ in Proposition 4.1, the result follows.

Subcase 1c: Suppose δ and θ both are outer derivations. Now two cases arise

Subcase 1c(i): The set $\{\delta,\theta\}$ is linearly C-independent. The equation (8) is written as

$$\begin{aligned}&(qa+[c,a])\,\pi(\omega_1,\dots,\omega_n)^2+q\theta(\pi(\omega_1,\dots,\omega_n))\pi(\omega_1,\dots,\omega_n)\\&+q\pi(\omega_1,\dots,\omega_n)\theta(\pi(\omega_1,\dots,\omega_n))+a[c,\pi(\omega_1,\dots,\omega_n)]\pi(\omega_1,\dots,\omega_n)\\&+a\pi(\omega_1,\dots,\omega_n)[c,\pi(\omega_1,\dots,\omega_n)]+[c,\theta(\pi(\omega_1,\dots,\omega_n))]\pi(\omega_1,\dots,\omega_n)\\&+\theta(\pi(\omega_1,\dots,\omega_n))[c,\pi(\omega_1,\dots,\omega_n)]+[c,\pi(\omega_1,\dots,\omega_n)]\theta(\pi(\omega_1,\dots,\omega_n))\\&+\pi(\omega_1,\dots,\omega_n)[c,\theta(\pi(\omega_1,\dots,\omega_n))]=b\pi(\omega_1,\dots,\omega_n)a\pi(\omega_1,\dots,\omega_n)\\&+b\pi(\omega_1,\dots,\omega_n)\theta(\pi(\omega_1,\dots,\omega_n))+\delta(\pi(\omega_1,\dots,\omega_n))a\pi(\omega_1,\dots,\omega_n)\\&+\delta(\pi(\omega_1,\dots,\omega_n))\theta(\pi(\omega_1,\dots,\omega_n)).\end{aligned}$$

Since δ and θ are outer derivations, by Remarks 5.2, we can replace $\delta(\pi(\omega_1,\dots,\omega_n))$ and $\theta(\pi(\omega_1,\dots,\omega_n))$ from (1), where $\delta(\omega_i)=\sigma_i$, $\theta(\omega_i)=\eta_i$ in above expression, then U satisfies

$$\sum_i\pi(\omega_1,\dots,\sigma_i,\dots,\omega_n)\sum_i\pi(\omega_1,\dots,\eta_i,\dots,\omega_n).$$

For $\sigma_1=\eta_1=\omega_1$ and $\sigma_i=\eta_i=0$, $i\geq 2$ we have $X^2=0$, where $X=\pi(\omega_1,\dots,\omega_n)$ a contradiction.

Subcase 1c(ii): Suppose the set $\{\delta,\theta\}$ is linearly C-dependent modulo inner derivation on U. Then there are $\lambda',\mu'\in C$ and $p'\in U$ with $\lambda'\delta(t)+\mu'\theta(t)=[p',t]$ for all $t\in R$. In a case either $\lambda'=0$ or $\mu'=0$, we have a contradiction. Hence both λ' and μ' can not be zero. So we can write $\delta(t)=\lambda\theta(t)+[p,t]$ where $\lambda=-\lambda'^{-1}\mu'$ and $p=\lambda'^{-1}p'$. Substituting this value in (8)

$$\begin{aligned}&(qa+[c,a])\,\pi(\omega_1,\dots,\omega_n)^2+q\theta(\pi(\omega_1,\dots,\omega_n))\pi(\omega_1,\dots,\omega_n)\\&+q\pi(\omega_1,\dots,\omega_n)\theta(\pi(\omega_1,\dots,\omega_n))+a[c,\pi(\omega_1,\dots,\omega_n)]\pi(\omega_1,\dots,\omega_n)\\&+a\pi(\omega_1,\dots,\omega_n)[c,\pi(\omega_1,\dots,\omega_n)]+[c,\theta(\pi(\omega_1,\dots,\omega_n)]\pi(\omega_1,\dots,\omega_n)\\&+\theta(\pi(\omega_1,\dots,\omega_n))[c,\pi(\omega_1,\dots,\omega_n)]+[c,\pi(\omega_1,\dots,\omega_n)]\theta(\pi(\omega_1,\dots,\omega_n))\\&+\pi(\omega_1,\dots,\omega_n)[c,\theta(\pi(\omega_1,\dots,\omega_n))]=b\pi(\omega_1,\dots,\omega_n)a\pi(\omega_1,\dots,\omega_n)\\&+b\pi(\omega_1,\dots,\omega_n)\theta(\pi(\omega_1,\dots,\omega_n))+\lambda\theta(\pi(\omega_1,\dots,\omega_n))a\pi(\omega_1,\dots,\omega_n)\\&+[p,\pi(\omega_1,\dots,\omega_n)]a\pi(\omega_1,\dots,\omega_n)+\lambda\theta(\pi(\omega_1,\dots,\omega_n))\theta(\pi(\omega_1,\dots,\omega_n))\\&+[p,\pi(\omega_1,\dots,\omega_n)]\theta(\pi(\omega_1,\dots,\omega_n)).\end{aligned}$$

Since θ is an outer derivation, by application of Remark 5.2, we can replace $\theta(\pi(\omega_1,\ldots,\omega_n))$ from (1), where $\theta(\omega_i)=\eta_i$ in above expression and then U satisfies

$$\begin{aligned}
&q\sum_i \pi(\omega_1,\ldots,\eta_i,\ldots,\omega_n)\pi(\omega_1,\ldots,\omega_n)\\
&\quad+q\pi(\omega_1,\ldots,\omega_n)\sum_i \pi(\omega_1,\ldots,\eta_i,\ldots,\omega_n)\\
&\quad+\Big[c,\sum_i \pi(\omega_1,\ldots,\eta_i,\ldots,\omega_n)\Big]\pi(\omega_1,\ldots,\omega_n)\\
&\quad+\sum_i \pi(\omega_1,\ldots,\eta_i,\ldots,\omega_n)\Big[c,\pi(\omega_1,\ldots,\omega_n)\Big]\\
&\quad+\Big[c,\pi(\omega_1,\ldots,\omega_n)\Big]\sum_i \pi(\omega_1,\ldots,\eta_i,\ldots,\omega_n)\\
&\quad+\pi(\omega_1,\ldots,\omega_n)\Big[c,\sum_i \pi(\omega_1,\ldots,\eta_i,\ldots,\omega_n)\Big]\\
&=b\pi(\omega_1,\ldots,\omega_n)\sum_i \pi(\omega_1,\ldots,\eta_i,\ldots,\omega_n)\\
&\quad+\lambda\sum_i \pi(\omega_1,\ldots,\eta_i,\ldots,\omega_n)a\pi(\omega_1,\ldots,\omega_n)\\
&\quad+\lambda\pi^{\theta}(\omega_1,\ldots,\omega_n)\sum_i \pi(\omega_1,\ldots,\eta_i,\ldots,\omega_n)\\
&\quad+\lambda\sum_i \pi(\omega_1,\ldots,\eta_i,\ldots,\omega_n)\pi^{\theta}(\omega_1,\ldots,x_n)\\
&\quad+\lambda\sum_i \pi(\omega_1,\ldots,\eta_i,\ldots,\omega_n)\sum_i \pi(\omega_1,\ldots,\eta_i,\ldots,\omega_n)\\
&\quad+\Big[p,\pi(\omega_1,\ldots,\omega_n)\Big]\sum_i \pi(\omega_1,\ldots,\eta_i,\ldots,\omega_n),
\end{aligned}$$

for all $\omega_i,\eta_i\in U$. For $\omega_1=0$ in above expression we get $\lambda\pi(\eta_1,\omega_2,\ldots,\omega_n)^2=0$. Since $\lambda\neq 0$ we get $\pi(\eta_1,\omega_2,\ldots,\omega_n)^2=0$, a contradiction.

Case 2: Let $\delta(t)=[p,t]$ for all $t\in R$ for some $p\in U$ i.e. δ be an inner derivation.

Subcase 2a: Suppose $d(t)=[c,t]$, where $c\in U$ and θ is an outer. This case is same as Subcase $1a$.

Subcase 2b: Suppose $\theta(t)=[s,t]$ for all $t\in R$, $s\in U$ and d is an outer. Then expression (7) becomes

$$\begin{aligned}
&(qa+d(a))\,\pi(\omega_1,\ldots,\omega_n)^2+q\Big[s,\pi(\omega_1,\ldots,\omega_n)^2\Big]\\
&\quad+ad(\pi(\omega_1,\ldots,\omega_n))\pi(\omega_1,\ldots,\omega_n)\\
&\quad+a\pi(\omega_1,\ldots,\omega_n)d(\pi(\omega_1,\ldots,\omega_n))+\Big[d(s),\pi(\omega_1,\ldots,\omega_n)^2\Big]\\
&\quad+\Big[s,d(\pi(\omega_1,\ldots,\omega_n))\pi(\omega_1,\ldots\omega_n)\Big]+\Big[s,\pi(\omega_1,\ldots,\omega_n)d(\pi(\omega_1,\ldots,\omega_n))\Big]\\
&=b\pi(\omega_1,\ldots,\omega_n)a\pi(\omega_1,\ldots,\omega_n)+b\pi(\omega_1,\ldots,\omega_n)\Big[s,\pi(\omega_1,\ldots,\omega_n)\Big]\\
&\quad+\Big[p,\pi(\omega_1,\ldots,\omega_n)\Big]a\pi(\omega_1,\ldots,\omega_n)+\Big[p,\pi(\omega_1,\ldots,\omega_n)\Big]\Big[s,\pi(\omega_1,\ldots,\omega_n)\Big].
\end{aligned}$$

Since d is an outer, By Remark 5.2 we can replace the value of $d(\pi(\omega_1,\ldots,\omega_n))$ from (1), where $d(\omega_i)=\nu_i$ in above expression and then U satisfies

$$\begin{aligned}
&a\sum_i \pi(\omega_1,\ldots,\nu_i,\ldots,\omega_n)\pi(\omega_1,\ldots,\omega_n)\\
&+a\pi(\omega_1,\ldots,\omega_n)\sum_i \pi(\omega_1,\ldots,\nu_i,\ldots,\omega_n)
\end{aligned}$$

$$+\Big[s, \sum_i \pi(\omega_1, \ldots, \nu_i, \ldots, \omega_n)\pi(\omega_1, \ldots, \omega_n)\Big]$$
$$+\Big[s, \pi(\omega_1, \ldots, \omega_n)\sum_i \pi(\omega_1, \ldots, \nu_i, \ldots, \omega_n)\Big],$$

for all $\omega_i, \nu_i \in U$. Substituting $\nu_1 = \omega_1$ and $\nu_i = 0$ for $i \geq 2$ in above we get $2aX^2 + 2[s, X^2] = 0$. Then result follows from Proposition 4.1 by taking $G(t) = [2s, t]$, $F(t) = -2at$ and $H(t) = t$ for all $t \in R$.

Subcase 2c: Suppose d and θ both are outer derivation. Then following two cases arise.

Subcase 2c(i): The set $\{d, \theta\}$ is linearly C-independent. Substituting the value of $\delta(t) = [p, t]$ in (8) and by using Remark 5.2 we can replace $d(\pi(\omega_1, \ldots, \omega_n)), \theta(\pi(\omega_1, \ldots, \omega_n))$ and $d\theta(\pi(\omega_1, \ldots, \omega_n))$ from (1) where $d(\omega_i) = \nu_i, \theta(\omega_i) = \eta_i$ and $d\theta(\omega_i) = w_i$, U satisfies the blended component $\sum_i \pi(\omega_1, \ldots, w_i, \ldots, \omega_n)\pi(\omega_1, \ldots, \omega_n) + \pi(\omega_1, \ldots, \omega_n)\sum_i \pi(\omega_1, \ldots, w_i, \ldots, \omega_n)$. By replacing $w_1 = \omega_1$ and $w_i = 0$ for $i = 2, 3, \cdots, n$, we get $2\pi(\omega_1, \ldots, \omega_n)^2 = 0$, which leads to a contradiction.

Subcase 2c(ii): The set $\{d, \theta\}$ is linearly C-dependent. Then there are $\lambda', \mu' \in C$ and $p' \in U$ with $\lambda' d(t) + \mu'\theta(t) = [p', t]$ for all $t \in R$. If either $\lambda' = 0$ or $\mu' = 0$, we get a contradiction. So consider $0 \neq \lambda'$ and $0 \neq \mu'$. Now we write $d(t) = -\lambda'^{-1}\mu'\theta(t) + [\lambda'^{-1}p', t] = \lambda\theta(t) + [p, t]$, where $\lambda = -\lambda'^{-1}\mu'$ and $p = \lambda'^{-1}p'$. Substituting the values of $\delta(t) = [c, t]$ and $d(t) = \lambda\theta(t) + [p, t]$ in expression (8) we get

$$\begin{aligned}
&(qa + \lambda\theta(a) + [p, a])\,\pi(\omega_1, \ldots, \omega_n)^2 + q\theta(\pi(\omega_1, \ldots, \omega_n))\pi(\omega_1, \ldots, \omega_n) \\
&\quad + q\pi(\omega_1, \ldots, \omega_n)\theta(\pi(\omega_1, \ldots, \omega_n)) + a\lambda\theta(f(\omega_1, \ldots, \omega_n))f(\omega_1, \ldots, \omega_n) \\
&\quad + a\Big[p, \pi(\omega_1, \ldots, \omega_n)\Big]\pi(\omega_1, \ldots, \omega_n) + a\pi(\omega_1, \ldots, \omega_n)\lambda\theta(\pi(\omega_1, \ldots, \omega_n)) \\
&\quad + a\pi(\omega_1, \ldots, \omega_n)\Big[p, \pi(\omega_1, \ldots, x_n)\Big] + \lambda\theta^2(\pi(\omega_1, \ldots, \omega_n))\pi(\omega_1, \ldots, \omega_n) \\
&\quad + \Big[p, \theta(\pi(\omega_1, \ldots, \omega_n))\Big]\pi(\omega_1, \ldots, \omega_n) + \theta(\pi(\omega_1, \ldots, \omega_n))\lambda\theta(\pi(\omega_1, \ldots, \omega_n)) \\
&\quad + \theta(\pi(\omega_1, \ldots, \omega_n))\Big[p, \pi(\omega_1, \ldots, \omega_n)\Big] + \lambda\theta(\pi(\omega_1, \ldots, \omega_n))\theta(\pi(\omega_1, \ldots, \omega_n)) \\
&\quad + \Big[p, \pi(\omega_1, \ldots, \omega_n)\Big]\theta(\pi(\omega_1, \ldots, \omega_n)) + \pi(\omega_1, \ldots, \omega_n)\lambda\theta^2(\pi(\omega_1, \ldots, \omega_n)) \\
&\quad + \pi(\omega_1, \ldots, \omega_n)\Big[p, \theta(\pi(\omega_1, \ldots, \omega_n))\Big] = b\pi(\omega_1, \ldots, \omega_n)a\pi(\omega_1, \ldots, \omega_n) \\
&\quad + b\pi(\omega_1, \ldots, \omega_n)\theta(\pi(\omega_1, \ldots, \omega_n)) + \Big[c, \pi(\omega_1, \ldots, \omega_n)\Big]a\pi(\omega_1, \ldots, \omega_n) \\
&\quad + \Big[c, \pi(\omega_1, \ldots, \omega_n)\Big]\theta(\pi(\omega_1, \ldots, \omega_n)).
\end{aligned}$$

Since θ is an outer derivation, by using Remark 5.2 and then U satisfies

$$\begin{aligned}
&\lambda\sum_i \pi(\omega_1, \ldots, w_i, \ldots, \omega_n)\pi(\omega_1, \ldots, \omega_n) \\
&\quad + \lambda\pi(\omega_1, \ldots, \omega_n)\sum_i \pi(\omega_1, \ldots, w_i, \ldots, \omega_n) = 0,
\end{aligned}$$

for all $\omega_i, w_i \in U$, where $\theta^2(\omega_i) = w_i$. By replacing $w_1 = \omega_1$ and $w_i = 0$ for $i = 2, 3, \cdots, n$, we get $2\lambda\pi(\omega_1, \ldots, \omega_n)^2 = 0$ which is a contradiction.

Case 3: Suppose θ is an inner derivation, say $\theta(t) = [s, t]$ for all $t \in R$ for some $s \in U$. Then (8) gives that

$$\begin{aligned}
&(qa + d(a))\,\pi(\omega_1, \ldots, \omega_n)^2 + q\Big[s, \pi(\omega_1, \ldots, \omega_n)\Big]\pi(\omega_1, \ldots, \omega_n) \\
&\quad + q\pi(\omega_1, \ldots, \omega_n)\Big[s, \pi(\omega_1, \ldots, \omega_n)\Big] + ad(\pi(\omega_1, \ldots, \omega_n))\pi(\omega_1, \ldots, \omega_n) \\
&\quad + a\pi(\omega_1, \ldots, \omega_n)d(\pi(\omega_1, \ldots, \omega_n)) + d\Big(\Big[s, \pi(\omega_1, \ldots, \omega_n)\Big]\Big)\pi(\omega_1, \ldots, \omega_n) \\
&\quad + \Big[s, \pi(\omega_1, \ldots, \omega_n)\Big]d(\pi(\omega_1, \ldots, \omega_n)) + d(\pi(\omega_1, \ldots, \omega_n))\Big[s, \pi(\omega_1, \ldots, \omega_n)\Big] \\
&\quad + \pi(\omega_1, \ldots, \omega_n)d\Big(\Big[s, \pi(\omega_1, \ldots, \omega_n)\Big]\Big) = b\pi(\omega_1, \ldots, \omega_n)a\pi(\omega_1, \ldots, \omega_n)
\end{aligned}$$

$$+b\pi(\omega_1,\ldots,\omega_n)\Big[s,\pi(\omega_1,\ldots,\omega_n)\Big]+\delta(\pi(\omega_1,\ldots,\omega_n))a\pi(\omega_1,\ldots,\omega_n)$$
$$+\delta(\pi(\omega_1,\ldots,\omega_n))\Big[s,\pi(\omega_1,\ldots,\omega_n)\Big], \tag{9}$$

for all $\omega_1,\ldots,\omega_n \in U$. Then we have the following subcases.

Subcase 3a: Let $d(t) = [c, t]$ for all $t \in R$, where $c \in U$ and δ be an outer derivation. This case is same as the Subcase $1b$.

Subcase 3b: Let d be an outer derivation and δ be an inner derivation, say $\delta(t) = [p, t]$ for all $t \in R$ for some $p \in U$. This case is same as the Subcase 2b.

Subcase 3c: Suppose d and δ both are outer derivations. Then two cases arise.

Subcase 3c(i): Let the set $\{d, \delta\}$ be linearly C-independent. Since d and δ are outer derivation, we replace $d(\pi(\omega_1,\ldots,\omega_n))$ with and $\delta(\pi(\omega_1,\ldots,\omega_n))$ from (1) where $d(\omega_i) = \nu_i$ and $\delta(\omega_i) = \sigma_i$ in the expression (9) and then U satisfies the blended component

$$\sum_i \pi(\omega_1,\ldots,\sigma_i,\ldots,\omega_n)a\pi(\omega_1,\ldots,\omega_n)$$
$$+\sum_i \pi(\omega_1,\ldots,\sigma_i,\ldots,\omega_n)\Big[s,\pi(\omega_1,\ldots,\omega_n)\Big].$$

By replacing $\sigma_1 = \omega_1$ and $\sigma_i = 0$ for $i = 2, 3, \cdots, n$, we have

$$XaX + X[s, X] = 0,$$

where $X = \pi(\omega_1,\ldots,\omega_n)$. Now result follows from Proposition 4.1 by taking $G(t) = 0$, $F(t) = t$ and $H(t) = at + [s, t]$ for all $t \in R$.

Subcase 3c(ii): Suppose the set $\{d, \delta\}$ is linearly C-dependent. Then there are scalars $\lambda', \mu' \in C$ and $p' \in U$ with $\lambda' d(t) + \mu'\delta(x) = [p', t]$. In a case either $\lambda' = 0$ or $\mu' = 0$, we get a contradiction. Therefore we assume $0 \neq \lambda'$ and $0 \neq \mu'$. We can write $\delta(t) = -\mu'^{-1}\lambda' d(t) + [\mu'^{-1}p', t] = \mu d(t) + [p, t]$ where $\mu = -\mu'^{-1}\lambda'$ and $p = -\mu'^{-1}p'$. Substituting these values in expression (9) and then applying Remark 5.2 U satisfies the blended component

$$q\sum_i \pi(\omega_1,\ldots,\nu_i,\ldots,\omega_n)\pi(\omega_1,\ldots,\omega_n)$$
$$+a\pi(\omega_1,\ldots,\omega_n)\sum_i \pi(\omega_1,\ldots,\nu_i,\ldots,\omega_n)$$
$$+\Big[s,\sum_i \pi(\omega_1,\ldots,\nu_i,\ldots,\omega_n)\Big]\pi(\omega_1,\ldots,\omega_n)$$
$$+\Big[s,\pi(\omega_1,\ldots,\omega_n)\Big]\sum_i \pi(\omega_1,\ldots,\nu_i,\ldots,\omega_n)$$
$$+\sum_i \pi(\omega_1,\ldots,\nu_i,\ldots,\omega_n)\Big[s,\pi(\omega_1,\ldots,\omega_n)\Big]$$
$$+\pi(\omega_1,\ldots,\omega_n)\Big[s,\sum_i \pi(\omega_1,\ldots,\nu_i,\ldots,\omega_n)\Big]$$
$$=\mu\sum_i \pi(\omega_1,\ldots,\nu_i,\ldots,\omega_n)a\pi(\omega_1,\ldots,\omega_n)$$
$$+\mu\sum_i \pi(\omega_1,\ldots,\nu_i,\ldots,\omega_n)\Big[s,\pi(\omega_1,\ldots,\omega_n)\Big],$$

for all $\omega_i, \nu_i \in U$, where $\nu_i = d(\omega_i)$. By replacing $\nu_1 = \omega_1$ and $\nu_i = 0$, $i = 2, 3, \cdots, n$ the above expression becomes

$$2aX^2 + 2[s, X]X + 2X[s, X] = \mu XaX + \mu X[s, X],$$

where $X=\pi(\omega_1,\ldots,\omega_n)$. Last expression again can be written as $(2a+2s)X^2+X^2(\mu s-2s)=\mu X(a+s)X$. Now result follows from Proposition 4.1 by considering $G(t)=(2a+2s)t+t(\mu s-2s)$, $F(t)=t\mu(a+s)$ and $H(t)=t$ for all $t\in R$.

Case 4: We consider all derivations d, δ and θ are outer derivations. Two cases arise.

Subcase 4a: The set $\{d,\delta,\theta\}$ is linearly C-independent. In this case by application of Remark 5.2 and using similar argument in equation (8) as we have used above, U satisfies the blended component

$$\sum_i \pi(\omega_1,\ldots,t_i,\ldots,\omega_n)\pi(\omega_1,\ldots,\omega_n)$$
$$+\pi(\omega_1,\ldots,\omega_n)\sum_i \pi(\omega_1,\ldots,t_i,\ldots,\omega_n)=0,$$

for all $\omega_i,t_i\in U$, where $t_i=d\theta(\omega_i)$. By replacing $t_1=\omega_1$, $t_i=0$ for $i=2,3,\cdots,n$, we get $2\pi(\omega_1,\ldots,\omega_n)^2=0$ which is a contradiction.

Subcase 4b: The set $\{d,\delta,\theta\}$ is linearly dependent modulo inner derivation. So there are scalars $\lambda',\mu',\nu'\in C$ and $p'\in U$ such that $\lambda' d(t)+\mu'\delta(t)+\nu'\theta(t)=[p',t]$ for all $t\in R$. If any two of λ',μ',ν' are zero simultaneously then we get a contradiction. Thus it implies that either one of λ',μ',ν' is zero or none of λ',μ',ν' are zero.

Firstly, if $\lambda'=0$, $\mu'\neq 0$ and $\nu'\neq 0$ then we write $\delta(t)=\nu\theta(t)+[p,t]$ where $\nu=-\mu'^{-1}\nu'$ and $p=\mu'^{-1}p'$. Substituting these values in expression (8) we get

$$\begin{aligned}
&(qa+d(a))\,\pi(\omega_1,\ldots,\omega_n)^2+q\theta(\pi(\omega_1,\ldots,\omega_n))\pi(\omega_1,\ldots,\omega_n)\\
&\quad+q\pi(\omega_1,\ldots,\omega_n)\theta(\pi(\omega_1,\ldots,\omega_n))+ad(\pi(\omega_1,\ldots,\omega_n))\pi(\omega_1,\ldots,\omega_n)\\
&\quad+a\pi(\omega_1,\ldots,\omega_n)d(\pi(\omega_1,\ldots,\omega_n))+d\theta(\pi(\omega_1,\ldots,\omega_n))\pi(\omega_1,\ldots,\omega_n)\\
&\quad+\theta(\pi(\omega_1,\ldots,\omega_n))d(\pi(\omega_1,\ldots,\omega_n))+d(\pi(\omega_1,\ldots,\omega_n))\theta(\pi(\omega_1,\ldots,\omega_n))\\
&\quad+\pi(\omega_1,\ldots,\omega_n)d\theta(\pi(\omega_1,\ldots,\omega_n))=b\pi(\omega_1,\ldots,\omega_n)a\pi(\omega_1,\ldots,\omega_n)\\
&\quad+b\pi(\omega_1,\ldots,\omega_n)\theta(\pi(\omega_1,\ldots,\omega_n))+\nu\theta(\pi(\omega_1,\ldots,\omega_n))a\pi(\omega_1,\ldots,\omega_n)\\
&\quad+[p,\pi(\omega_1,\ldots,\omega_n)]a\pi(\omega_1,\ldots,\omega_n)+\nu\theta(\pi(\omega_1,\ldots,\omega_n))\theta(\pi(\omega_1,\ldots,\omega_n))\\
&\quad+[p,\pi(\omega_1,\ldots,\omega_n)]\theta(\pi(\omega_1,\ldots,\omega_n)).
\end{aligned}$$

If d and θ are linearly C-dependent modulo inner derivation on U. By using similar argument as we have used in Subcase 2c(ii), the result follows.

If d and θ are linearly C-independent modulo inner derivation on U, then by using Remark 5.2, U satisfies

$$\pi(\omega_1,\ldots,\omega_n)\sum_i \pi(\omega_1,\ldots,w_i,\ldots,\omega_n)+\sum_i \pi(\omega_1,\ldots,w_i,\ldots,\omega_n)\pi(\omega_1,\ldots,\omega_n),$$

for all $\omega_i,w_i\in U$, where $d\theta(\omega_i)=w_i$. By replacing $w_1=\omega_1$ and $w_i=0$ for $i=2,3,\cdots,n$ we get $2\pi(\omega_1,\ldots,\omega_n)^2=0$, which leads to a contradiction.

Secondly, if $0\neq\lambda'$, $0\neq\mu'$ and $\nu'\neq 0$ then we write $d(t)=\lambda\theta(t)+[p,t]$ for all $t\in R$, where $\lambda=-\lambda'^{-1}\nu'$ and $p=\lambda'^{-1}p'$. Substituting these values in expression (8) we get

$$\begin{aligned}
&(qa+d(a))\,\pi(\omega_1,\ldots,\omega_n)^2+q\theta(\pi(\omega_1,\ldots\omega_n))\pi(\omega_1,\ldots,\omega_n)\\
&\quad+q\pi(\omega_1,\ldots,\omega_n)\theta(\pi(\omega_1,\ldots,\omega_n))+a\lambda\theta(\pi(\omega_1,\ldots,\omega_n))\pi(\omega_1,\ldots,\omega_n)\\
&\quad+a[p,\pi(\omega_1,\ldots,\omega_n)]\pi(\omega_1,\ldots,\omega_n)+a\pi(\omega_1,\ldots,\omega_n)\lambda\theta(\pi(\omega_1,\ldots,\omega_n))\\
&\quad+a\pi(\omega_1,\ldots,\omega_n)[p,\pi(\omega_1,\ldots,\omega_n)]+\lambda\theta^2(\pi(\omega_1,\ldots,\omega_n))\pi(\omega_1,\ldots,\omega_n)\\
&\quad+[p,\theta(\pi(\omega_1,\ldots,\omega_n))]\pi(\omega_1,\ldots,\omega_n)+\theta(\pi(\omega_1,\ldots,\omega_n))\lambda\theta(\pi(\omega_1,\ldots,\omega_n))\\
&\quad+\theta(\pi(\omega_1,\ldots,\omega_n))[p,\pi(\omega_1,\ldots,\omega_n)]+\lambda\theta(\pi(\omega_1,\ldots,\omega_n))\theta(\pi(\omega_1,\ldots,x_n))\\
&\quad+[p,\pi(\omega_1,\ldots,\omega_n)]\theta(\pi(\omega_1,\ldots,\omega_n))+\pi(\omega_1,\ldots,\omega_n)\lambda\theta^2(\pi(\omega_1,\ldots,\omega_n))\\
&\quad+\pi(\omega_1,\ldots,\omega_n)[p,\theta(\pi(\omega_1,\ldots,\omega_n))]=b\pi(\omega_1,\ldots,\omega_n)a\pi(\omega_1,\ldots,\omega_n)\\
&\quad+b\pi(\omega_1,\ldots,\omega_n)\theta(\pi(\omega_1,\ldots,\omega_n))+\delta(\pi(\omega_1,\ldots,\omega_n))a\pi(\omega_1,\ldots,\omega_n)\\
&\quad+\delta(\pi(\omega_1,\ldots,\omega_n))\theta(\pi(\omega_1,\ldots,\omega_n)).
\end{aligned}$$

If θ and δ are linearly C-dependent modulo inner derivation on U. By using similar argument as we have used in Subcase 1c(i), the result follows

If d and δ are linearly C-independent modulo inner derivation on U. Then by using Remark 5.2, U satisfies

$$\lambda \sum_i \pi(\omega_1, \ldots, w_i, \ldots, \omega_n)\pi(\omega_1, \ldots, \omega_n)$$
$$+\lambda\pi(\omega_1, \ldots, \omega_n) \sum_i \pi(\omega_1, \ldots, w_i, \ldots, \omega_n),$$

for all $\omega_i, w_i \in U$, where $\theta^2(\omega_i) = w_i$. By replacing $w_1 = \omega_1$ and $w_i = 0$ for $i = 2, 3, \cdots, n$, we get $2\lambda\pi(\omega_1, \ldots, \omega_n)^2 = 0$ which is a contradiction.

Thirdly, if $\lambda' \neq 0$, $\mu' \neq 0$ and $\nu' = 0$ then we write $\delta(t) = \mu d(t) + [p, t]$ for all $t \in R$, where $\mu = -\mu'^{-1}\lambda'$ and $p = \mu'^{-1}p'$. Substituting these values in expression (8) we get

$$\begin{aligned}
&(qa + d(a))\,\pi(\omega_1, \ldots, \omega_n)^2 + q\theta(\pi(\omega_1, \ldots, \omega_n))\pi(\omega_1, \ldots, \omega_n)\\
&+q\pi(\omega_1, \ldots, \omega_n)\theta(\pi(\omega_1, \ldots, \omega_n)) + ad(\pi(\omega_1, \ldots, \omega_n))\pi(\omega_1, \ldots, \omega_n)\\
&+a\pi(\omega_1, \ldots, \omega_n)d(\pi(\omega_1, \ldots, \omega_n)) + d\theta(\pi(\omega_1, \ldots, \omega_n))\pi(\omega_1, \ldots, \omega_n)\\
&+\theta(\pi(\omega_1, \ldots, \omega_n))d(\pi(\omega_1, \ldots, \omega_n)) + d(\pi(\omega_1, \ldots, \omega_n))\theta(\pi(\omega_1, \ldots, \omega_n))\\
&+\pi(\omega_1, \ldots, \omega_n)d\theta(\pi(\omega_1, \ldots, \omega_n)) = b\pi(\omega_1, \ldots, \omega_n)a\pi(\omega_1, \ldots, \omega_n)\\
&+b\pi(\omega_1, \ldots, \omega_n)\theta(\pi(\omega_1, \ldots, \omega_n)) + \mu d(\pi(\omega_1, \ldots, \omega_n))a\pi(\omega_1, \ldots, \omega_n)\\
&+[p, \pi(\omega_1, \ldots, \omega_n)]a\pi(\omega_1, \ldots, \omega_n) + \mu d(\pi(\omega_1, \ldots, \omega_n))\theta(\pi(\omega_1, \ldots, \omega_n))\\
&+[p, \pi(\omega_1, \ldots, \omega_n)]\theta(\pi(\omega_1, \ldots, \omega_n)).
\end{aligned}$$

If θ and d are linearly C-dependent modulo inner derivation on U. By using similar argument as we have used in Subcase 2c(ii), the result follows.

If d and θ are linearly C-independent modulo inner derivation on U. Then by using Remark 5.2, U satisfies

$$\sum_i \pi(\omega_1, \ldots, w_i, \ldots, \omega_n)\pi(\omega_1, \ldots, \omega_n)$$
$$+\pi(\omega_1, \ldots, \omega_n) \sum_i \pi(\omega_1, \ldots, w_i, \ldots, \omega_n),$$

where $d\theta(\omega_i) = w_i$. By replacing $w_1 = \omega_1$ and $w_i = 0$ for $i = 2, 3, \cdots, n$ we get $2\pi(\omega_1, \ldots, \omega_n)^2 = 0$, which is a contradiction.

Finally, consider $0 \neq \lambda'$, $0 \neq \mu'$ and $\nu' \neq 0$ then we write $\delta(t) = \lambda d(t) + \mu\theta(t) + [p, t]$ where $\lambda = -\mu'^{-1}\lambda'$, $\mu = -\mu'^{-1}\nu'$ and $p = \mu'^{-1}p'$. Substituting the value of $\delta(t)$ in expression (8) we get

$$\begin{aligned}
&(qa + d(a))\,\pi(\omega_1, \ldots, \omega_n)^2 + q\theta(\pi(\omega_1, \ldots, \omega_n))\pi(\omega_1, \ldots, \omega_n)\\
&+q\pi(\omega_1, \ldots, \omega_n)\theta(\pi(\omega_1, \ldots, \omega_n)) + ad(\pi(\omega_1, \ldots, \omega_n))\pi(\omega_1, \ldots, \omega_n)\\
&+a\pi(\omega_1, \ldots, \omega_n)d(\pi(\omega_1, \ldots, \omega_n)) + d\theta(\pi(\omega_1, \ldots, \omega_n))\pi(\omega_1, \ldots, \omega_n)\\
&+\theta(\pi(\omega_1, \ldots, \omega_n))d(\pi(\omega_1, \ldots, \omega_n)) + d(\pi(\omega_1, \ldots, \omega_n))\theta(\pi(\omega_1, \ldots, \omega_n))\\
&+\pi(\omega_1, \ldots, \omega_n)d\theta(\pi(\omega_1, \ldots, \omega_n)) = b\pi(\omega_1, \ldots, \omega_n)a\pi(\omega_1, \ldots, \omega_n)\\
&+b\pi(\omega_1, \ldots, \omega_n)\theta(\pi(\omega_1, \ldots, \omega_n)) + \lambda d(\pi(\omega_1, \ldots, \omega_n))a\pi(\omega_1, \ldots, \omega_n)\\
&+\mu\theta(\pi(\omega_1, \ldots, \omega_n))a\pi(\omega_1, \ldots, \omega_n) + [p, \pi(\omega_1, \ldots, \omega_n)]a\pi(\omega_1, \ldots, \omega_n)\\
&+\lambda d(\pi(\omega_1, \ldots, \omega_n))\theta(\pi(\omega_1, \ldots, \omega_n)) + \mu\theta(\pi(\omega_1, \ldots, \omega_n))\theta(\pi(\omega_1, \ldots, \omega_n))\\
&+[p, \pi(\omega_1, \ldots, \omega_n)]\theta(\pi(\omega_1, \ldots, \omega_n)).
\end{aligned}$$

If d and θ are linearly C-independent then it is similar to the Subcase 2c(i) and hence we get a contradiction.

If d and θ are linearly C-dependent then it is similar to the Subcase 2c(ii) and hence we get a contradiction.

Statements and Declarations

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