Indian J Pure Appl Math (2024) 55:508–512
https://doi.org/10.1007/s13226-023-00383-5

ORIGINAL RESEARCH

Normalized solutions to the Kirchhoff equation with L^2-subcritical or critical nonlinearities in $\mathbb{R}^2$

Jiaqing Hu · **Anmin Mao**

Received: 8 September 2022 / Accepted: 2 February 2023 / Published online: 14 February 2023

Abstract In this paper, we study the existence of normalized solutions to the Kirchhoff equation with L^2-subcritical or critical nonlinearities

$$-\left(a + b\int_{\mathbb{R}^2} |\nabla u|^2 dx\right)\Delta u = \lambda u + |u|^{p-2}u + \mu |u|^{q-2}u \ \text{ in } \mathbb{R}^2,$$

where $a, b, \mu > 0$, $2 < q < p \leq 6$. By minimizing methods and the concentration compactness principle, we prove the existence and nonexistence of normalized solutions when (p, q) belongs to a certain domain in $\mathbb{R}^2$, and discuss how μ affects the existence of normalized solutions. Our main results may be illustrated by the red areas and green areas shown in Figure 1. Our results partially extend the results of Soave (J. Differ. Equ. 2020).

Keywords Normalized solution · Kirchhoff equation · Minimizing method

Mathematics Subject Classification 35J60 · 35A15 · 35B38

1 Introduction

In this paper, we study the existence of normalized solutions to the Kirchhoff equation with L^2-subcritical or critical nonlinearities

$$-\left(a + b\int_{\mathbb{R}^2} |\nabla u|^2 dx\right)\Delta u = \lambda u + |u|^{p-2}u + \mu |u|^{q-2}u \ \text{ in } \mathbb{R}^2, \tag{1.1}$$

where $a, b, \mu > 0$, $2 < q < p \leq 6$. The problem (1.1) is related to the equation

$$-\left(a + b\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)\Delta u = f(x, u) \tag{1.2}$$

proposed by Kirchhoff as an extension of the classical D'Alembert's wave equations. Mathematically, the problem (1.2) is often referred to be nonlocal as the appearance of the term $b\int_{\mathbb{R}^N} |\nabla u|^2 dx$ implies that (1.2) is no longer a pointwise identity. This phenomenon causes some mathematical difficulties, which make the study of (1.2) particularly interesting.

Communicated by K Sandeep.

Supported by the NSFC(12171014, ZR2020MA005, ZR2021MA096)

J. Hu · A. Mao (✉)
School of Mathematical Sciences, Qufu Normal University, Shandong 273165, People's Republic of China E-mail: maoam@163.com

J. Hu
E-mail: hujiaqing6013@163.com

After the pioneering work of Lions [7], the Kirchhoff type problem (1.2) began to receive much attention and many important results were established, see [1,3,9,14]. When $f(x,u)=\lambda u+g(x,u)$, to find solutions for problem (1.2), a possible choice is to consider fixed λ, or even with an additional external and fixed potential V [4,6]. Another possible choice is to search for solutions to (1.2) having prescribed L^2-norm, see [5,12,13]. In this case, (1.2) can be viewed as an eigenvalue problem by taking λ as amn unknown Lagrange multiplier, and solved by studying some constrained variational problems. Inspired by the works of [10,12], we consider the following minimization problem

$$m(c):=\inf_{S_c} E_\mu(u), \tag{1.3}$$

where $E_\mu(u)=\frac{a}{2}|\nabla u|_2^2+\frac{b}{4}|\nabla u|_2^4-\frac{1}{p}|u|_p^p-\frac{\mu}{q}|u|_q^q$, $S_c=\left\{u\in H^1(\mathbb{R}^2): |u|_2^2=c\right\}$, and $|\cdot|_r$ denotes the standard norm in $L^r(\mathbb{R}^2)$. If $u\in S_c$ is a minimizer of problem (1.3), then there exists $\lambda_c\in\mathbb{R}$ such that $E_\mu{}'(u)=\lambda_c u$, namely, $u\in S_c$ is a solution of (1.1) for some λ_c. It is known that $p=6$ is a L^2-critical exponent, that is, E_μ is bounded from below on S_c if $2<p<6$, while E_μ is not bounded from below on S_c if $p>6$.

When $\mu=0$, Ye [12] considered problem (1.3) with L^2-subcritical case, and proved the existence of minimizers to (1.3). Later, Zeng and Zhang [13] gave a new proof for the results of [12] by using technical energy estimates, and showed that the minimizer of (1.3), if exists, is unique and must be a scaling of Q, where Q is the unique radially symmetric positive solution of the following equation

$$-\Delta u+\tfrac{2}{p-2}u-\tfrac{2}{p-2}|u|^{p-2}u=0 \text{ in } \mathbb{R}^2. \tag{1.4}$$

When $\mu\neq 0$, Li, Luo and Yang [5] considered the existence of multiple solutions to problem (1.1) with the L^2-subcritical and supercritical case in $\mathbb{R}^3$ by constructing a suitable Palais-Smale sequence and Ekeland's variational principle, and obtained the existence of ground state solution in the L^2-supercritical case by using minimax methods. However, to the best of our knowledge, the existence of normalized solutions to problem (1.1) with L^2-subcritical or critical case in $\mathbb{R}^2$ is still unknown.

Motivated by the above works, in this paper we consider that the existence of normalized solutions to problem (1.1) with L^2-subcritical or critical case, namely, $2<q<4$ and $2<p\le 6$ in $\mathbb{R}^2$. In addition, we discuss how μ affects the existence of normalized solutions to problem (1.1).

We recall that Q is an optimizer of the following sharp Gagliardo-Nirenberg [11] inequality

$$|u|_p^p\le \frac{p}{2|Q|_2^{p-2}}|\nabla u|_2^{p-2}|u|_2^2,\quad \forall u\in H^1(\mathbb{R}^2).$$

Moreover, combining the Pohozaev identity, Q satisfies $|\nabla Q|_2^2=|Q|_2^2=\frac{2}{p}|Q|_p^p$. Let

$$c^*:=\frac{b|Q|_2^4}{2},\quad c_p^*(a,b):=2|Q|_2^{p-2}\left(\frac{a}{6-p}\right)^{\frac{6-p}{2}}\left(\frac{b}{2(p-4)}\right)^{\frac{p-4}{2}},$$

our first result is the nonexistence of minimizers to problem (1.3).

Theorem 1.1 *Assume that $2<q<p\le 6$ and $c,\mu>0$. Then problem (1.3) has no minimizer if one of the following conditions holds.*

(i) $q=4$, $p=6$, *either* $c<c^*$ *and* $\mu c\le a|Q|_2^2$, *or* $c=c^*$ *and* $\mu c<a|Q|_2^2$.

(ii) $q=4<p<6$, $\mu c<a|Q|_2^2$, *and* $c<c_p^*(a_0,b)$, *where* $a_0=\frac{a|Q|_2^2-\mu c}{|Q|_2^2}>0$.

(iii) $4<q<6=p$, $c<c^*$, *and* $\mu c<c_q^*(a,b_0)$, *where* $b_0=\frac{b|Q|_2^4-2c}{|Q|_2^4}>0$.

Remark 1.1 (a) Theorem 1.1 may be illustrated by the red area shown in Figure 1. (b) When $\mu=0$, problem (1.3) also has no minimizer under the conditions (i), (ii) and (iii) (see [13]). Theorem 1.1 shows that the nonexistence of minimizers to problem (1.3) still holds for small μ, which partially extends the result of Zeng and Zhang [13]. (c) Theorem 1.1 shows that the nonexistence of minimizers to problem (1.3). In fact, we may proceed as in Theorem 1.2 of [12] to prove that E_μ has no constraint critical point on S_c under the conditions (i) by Gagliardo-Nirenberg inequality and Pohozaev identity. For the cases (ii) and (iii), we can further constrain μ and c by Young's inequality such that E_μ has no constraint critical point on S_c.

Let $c_p^*=+\infty$ if $2<p<6$, $c_p^*=c^*$ if $p=6$, we have the following result.

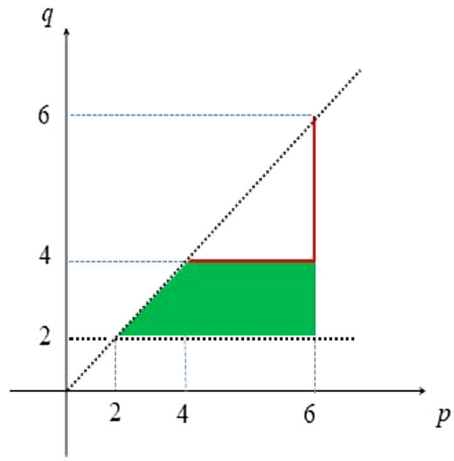

Fig. 1 The nonexistence (red area) and existence (green area) of constraint minimizers

Theorem 1.2 *Assume that* $2 < q < 4$, $2 < p \le 6$, $q < p$ *and* $\mu > 0$. *Then the problem (1.3) has a minimizer for every* $c \in (0, c_p^*)$.

Remark 1.2 (a) The results in Theorem 1.2 may be illustrated by the green area shown in Figure 1. (b) Theorem 1.2 shows the existence of normalized solutions to problem (1.1) with $2 < q < 4$ and $2 < p \le 6$, which partially extends existence result of Soave [10] in case where $b = 0$ and $2 < q < 4 \le p < +\infty$.

Remark 1.3 If $4 \le p \le 6$, Ye [12] showed that the nonexistence of minimizers to (1.3) with $\mu = 0$ for small $c > 0$. But Theorem 1.2 shows that $\mu > 0$ will affect the existence of minimizers to problem (1.3) for $c \in (0, c_p^*)$ in the case where $2 < q < 4$ and $q < p$. The reason is that under the conditions of Theorem 1.2, we have $m(c) < 0$, which guarantees that $m(c)$ satisfies the strict sub-additivity inequalities in Lemma 2.4. This is the key to prove the existence of normalized solutions.

To prove Theorem 1.2, since that every minimizing sequence for (1.3) is bounded, we only need to exclude the two possibilities of vanishing and dichotomy, namely, $u_n \rightharpoonup 0$ in $H^1(\mathbb{R}^2)$ and $u_n \rightharpoonup u \ne 0$ in $H^1(\mathbb{R}^2)$, $0 < |u|_2^2 < c$. By vanishing lemma of Lions, the former can be avoided. The latter can be excluded by the strict inequality $m(c) < m(\alpha) + m(c - \alpha)$ for every $0 < \alpha < c$.

Regarding the notation, in this paper, we use $\to$ and $\rightharpoonup$ to denote the strong and weak convergence in the related function space respectively. $:=$ and $=:$ denote definitions. The rest of this paper is organized as follows. In Sect. 2, we present some preliminary results for Theorems 1.1 and 1.2. In Sect. 3, we prove Theorems 1.1 and 1.2.

2 Preliminary results for Theorems 1.1 and 1.2

Lemma 2.1 *Assume* $c, \mu > 0$, *then* E_μ *is coercive on* S_c *if one of the following conditions holds.*

(i) $2 < q < p < 6$.
(ii) $2 < q < 6 = p$, $c < c^*$.
(iii) $q = 4$, $p = 6$, $c = c^*$ *and* $\mu c < a|Q|_2^2$.

Especially, if the condition (iii) holds, then $E_\mu(u) > 0$ *for every* $u \in S_c$.

Proof For every $u \in S_c$, we have $E_\mu(u) \ge \frac{a}{2}|\nabla u|_2^2 + \frac{b}{4}|\nabla u|_2^4 - \frac{c}{2|Q|_2^{p-2}}|\nabla u|_2^{p-2} - \frac{\mu c}{2|Q|_2^{q-2}}|\nabla u|_2^{q-2}$. It is easy to check that $E_\mu(u) \to +\infty$ as $|\nabla u|_2 \to +\infty$. □

Lemma 2.2 *(1) Assume that the conditions of Lemma 2.1 hold, then* $m(c)$ *is well defined and* $m(c) \le 0$.
(2) Assume that $2 < q < 4$, $2 < p \le 6$, $q < p$ *and* $\mu > 0$, *then* $m(c) < 0$ *for every* $c \in (0, c_p^*)$.

Proof (1) By Lemma 2.1, E_μ is bounded from below on S_c. Let $u_t(x) = t^{\frac{1}{2}} u\left(t^{\frac{1}{2}} x\right)$, then $u_t \in S_c$, and

$$E_\mu(u_t) = \frac{a}{2}t|\nabla u|_2^2 + \frac{b}{4}t^2|\nabla u|_2^4 - \frac{1}{p}t^{\frac{p-2}{2}}|u|_p^p - \frac{\mu}{q}t^{\frac{q-2}{2}}|u|_q^q \to 0 \text{ as } t \to 0. \text{ Thus } m(c) \le 0.$$

(2) Let $u_s(x) = c^{\frac{1}{2}} \frac{s^{\frac{1}{2}} Q\left(s^{\frac{1}{2}} x\right)}{|Q|_2}$ for $s > 0$, then $u_s \in S_c$, and

$$E_\mu(u_s) = \frac{a}{2}cs + \frac{b}{4}c^2 s^2 - \frac{c^{\frac{p}{2}}}{2|Q|_2^{p-2}} s^{\frac{p-2}{2}} - \frac{\mu c^{\frac{q}{2}} |Q|_q^q}{q|Q|_2^q} s^{\frac{q-2}{2}}.$$

It is easy to see that there exists $s_0 > 0$ such that $E_\mu(u_{s_0}) < 0$ when $2 < q < 4$, $2 < p \le 6$ and $c \in (0, c_p^*)$. □

Remark 2.1 By Lemma 2.1, we know that if $q = 4$, $p = 6$, $c = c^*$ and $\mu c < a|Q|_2^2$, then $E_\mu(u) > 0$ for every $u \in S_c$, so $m(c)$ has no minimizer.

Lemma 2.3 *Assume that* $2 < q < 4$, $2 < p \le 6$, $q < p$ *and* $\mu > 0$. *Then* $m(c)$ *is continuous on* $(0, c_p^*)$.

Proof The proof is similar to that of Theorem 2.1 in [2], so we omit it.

Lemma 2.4 *Assume that* $2 < q < 4$, $2 < p \le 6$, $q < p$ *and* $\mu > 0$. *Then for every* $c \in (0, c_p^*)$, $m(c) < m(\alpha) + m(c - \alpha)$ *when* $0 < \alpha < c$.

Proof By Lemma 2.2, $m(c) < 0$ for every $c \in (0, c_p^*)$. Let $\{u_n\} \subset S_c$ be a minimizing sequence for $m(c)$, that is, $E_\mu(u_n) \to m(c)$. Since E_μ is coercive on S_c, then $\{u_n\}$ is bounded in $H^1(\mathbb{R}^2)$. We claim that there exists a constant $k_1 > 0$ such that $|\nabla u_n|_2^2 \ge k_1$. In fact, if $|\nabla u_n|_2 \to 0$ then

$$E_\mu(u_n) \ge \frac{a}{2}|\nabla u_n|_2^2 + \frac{b}{4}|\nabla u_n|_2^4 - \frac{c}{2|Q|_2^{p-2}}|\nabla u_n|_2^{p-2} - \frac{\mu c}{2|Q|_2^{q-2}}|\nabla u_n|_2^{q-2} \to 0.$$

Since $E_\mu(u_n) \le \frac{a}{2}|\nabla u_n|_2^2 + \frac{b}{4}|\nabla u_n|_2^4 \to 0$, then $E_\mu(u_n) \to 0$, a contradiction.

For every $u \in S_c$, let $u^\theta(x) = u\left(\theta^{-\frac{1}{2}} x\right)$ for $\theta > 0$, then $u^\theta \in S_{\theta c}$, so $u_n^\theta \in S_{\theta c}$. Let $\theta > 1$ such that $\theta c < c_p^*$, then

$$\begin{aligned} m(\theta c) \le E_\mu(u_n^\theta) &= \frac{a}{2}|\nabla u_n|_2^2 + \frac{b}{4}|\nabla u_n|_2^4 - \frac{1}{p}\theta |u_n|_p^p - \frac{\mu}{q}\theta |u_n|_q^q \\ &= \theta E_\mu(u_n) + \frac{a}{2}(1-\theta)|\nabla u_n|_2^2 + \frac{b}{4}(1-\theta)|\nabla u_n|_2^4 \\ &\le \theta E_\mu(u_n) + \frac{a}{2}(1-\theta)k_1 + \frac{b}{4}(1-\theta)k_1^2. \end{aligned}$$

Since the second term on the right is negative, we have $m(\theta c) < \theta m(c)$, and from this the thesis follows as in Lemma II.1 of [8]. □

3 Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1 It is enough to show that $E_\mu(u) > 0$ for every $u \in S_c$. Since $m(c) \le 0$, then $m(c)$ has no minimizer.

(i) If $q = 4$, $p = 6$, and either $c < c^*$, $\mu c \le a|Q|_2^2$ or $c = c^*$, $\mu c < a|Q|_2^2$, then by Gagliardo-Nirenberg inequality, we have $E_\mu(u) \ge \frac{1}{2}\left(a - \frac{\mu c}{|Q|_2^2}\right)|\nabla u|_2^2 + \frac{1}{2}\left(\frac{b}{2} - \frac{c}{|Q|_2^4}\right)|\nabla u|_2^4 > 0$.

(ii) If $q = 4 < p < 6$ and $\mu c < a|Q|_2^2$, then

$$E_\mu(u) \ge \frac{1}{2}\left(a - \frac{\mu c}{|Q|_2^2}\right)|\nabla u|_2^2 + \frac{b}{4}|\nabla u|_2^4 - \frac{c}{2|Q|_2^{p-2}}|\nabla u|_2^{p-2}.$$

Let $|\nabla u|_2^2 = t$, $\alpha = \frac{6-p}{2}$, $\beta = \frac{p-4}{2}$. By Young's inequality, we deduce that

$$\frac{a_0}{2}t + \frac{b}{4}t^2 = \alpha \cdot \frac{a_0}{2\alpha}t + \beta \cdot \frac{b}{4\beta}t^2 \ge \left(\frac{a_0}{2\alpha}\right)^\alpha \left(\frac{b}{4\beta}\right)^\beta t^{\alpha+2\beta} = \left(\frac{a_0}{2\alpha}\right)^\alpha \left(\frac{b}{4\beta}\right)^\beta t^{\frac{p-2}{2}},$$

where the equality holds if and only if $t = \frac{2(p-4)a_0}{(6-p)b}$. Thus $E_\mu(u) \ge \frac{c_p^*(a_0,b)-c}{2|Q|_2^{p-2}}|\nabla u|_2^{p-2} > 0$ for every $u \in S_c$.

(iii) If $4 < q < 6 = p$, $c < c^*$ and $\mu c < c_q^*(a, b_0)$, then for every $u \in S_c$, by Young's inequality

$$E_\mu(u) \geq \frac{a}{2}|\nabla u|_2^2 + \frac{1}{4}\left(b - \frac{2c}{|Q|_2^4}\right)|\nabla u|_2^4 - \frac{\mu c}{2|Q|_2^{q-2}}|\nabla u|_2^{q-2} \geq \frac{c_q^*(a,b_0)-\mu c}{2|Q|_2^{q-2}}|\nabla u|_2^{q-2} > 0.$$

Proof of Theorem 1.2. Let $\{u_n\} \subset S_c$ be a minimizing sequence for $m(c)$, namely, $E_\mu(u_n) \to m(c)$ and $|u_n|_2^2 = c$. Since E_μ is coercive on S_c, then $\{u_n\}$ is bounded in $H^1(\mathbb{R}^2)$. Then

$$\delta := \underline{\lim}_{n\to\infty} \sup_{y\in\mathbb{R}^2} \int_{B_1(y)} |u_n|^2 dx > 0.$$

Otherwise, by vanishing lemma of Lions, $u_n \to 0$ in $L^r(\mathbb{R}^2)$ for $2 < r < +\infty$, and then $0 \leq \underline{\lim}_{n\to\infty}\left(\frac{a}{2}|\nabla u_n|_2^2 + \frac{b}{4}|\nabla u_n|_2^4\right) = \lim_{n\to\infty} E_\mu(u_n) = m(c) < 0$, a contradiction. Therefore, there exists a sequence $\{y_n\} \subset \mathbb{R}^2$ such that $\int_{B_1(y_n)} |u_n|^2 dx > \frac{\delta}{2} > 0$. Let $v_n = u_n(x + y_n)$, and then

$$\int_{B_1(0)} |v_n|^2 dx > \frac{\delta}{2}. \tag{3.1}$$

Moreover $\{v_n\} \subset S_c$ also a bounded minimizing sequence for $m(c)$, and then we may assume that $v_n \rightharpoonup v_0$ in $H^1(\mathbb{R}^2)$, $v_n \to v_0$ in $L^r_{loc}(\mathbb{R}^2)$ for $r \in [1, +\infty)$, $v_n(x) \to v_0(x)$ a.e. in $\mathbb{R}^2$, which and (3.1) imply that $v_0 \neq 0$. Thus $\alpha := |v_0|_2^2 \in (0, c]$.

We now prove that $\alpha = c$. Suppose that $\alpha < c$, then $c = |v_n|_2^2 = |v_0|_2^2 + |v_n - v_0|_2^2 + o(1)$. Combining Brezis-Lieb Lemma and Lemma 2.3, we see that

$$m(c) = \lim_{n\to\infty} E_\mu(v_n) = E_\mu(v_0) + \lim_{n\to\infty} E_\mu(v_n - v_0) \geq m(\alpha) + m(c - \alpha),$$

which contradicts Lemma 2.4. So $\alpha = c$, namely, $v_0 \in S_c$. Since $m(c) \leq E_\mu(v_0) \leq \underline{\lim}_{n\to\infty} E_\mu(v_n) = m(c)$, then $v_0 \in S_c$ is a minimizer of $m(c)$. □

References

1. C. Alves, G. Figueiredo, On multiplicity and concentration of positive solutions for a class of quasilinear problems with critical exponential growth in $\mathbb{R}^N$, J. Differ. Equ., 246 (2009) 1288-1311.
2. J. Bellazzini, G. Siciliano, Scaling properties of functionals and existence of constrained minimizers, J. Funct. Anal., 261 (2011) 2486-2507.
3. G. Figueiredo, Existence of a positive solution for a Kirchhoff problem type with critical growth via truncation argument, J. Math. Anal. Appl., 401 (2013) 706-713.
4. X. He, W. Zou, Existence and concentration behavior of positive solutions for a Kirchhoff equation in $\mathbb{R}^3$, J. Differ. Equ., 252 (2012) 1813-1834.
5. G. Li, X. Luo, T. Yang, Normalized solutions to a class of Kirchhoff equations with Sobolev critical exponent. arXiv:2103.08106
6. G. Li, H. Ye, Existence of positive ground state solutions for the nonlinear Kirchhoff type equations in $\mathbb{R}^3$, J. Differ. Equ., 257 (2014) 566-600.
7. J. Lions, On some questions in boundary value problems of mathematical physics, North-Holland Math. Stud., 30 (1978) 284-346.
8. P. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case, part 1, Ann. Inst. Henri Poincaré, Anal. Non Linéaire, 1 (1984) 109-145.
9. A. Mao, Z. Zhang, Sign-changing and multiple solutions of Kirchhoff type problems without the P.S. condition, Nonlinear Anal., 70 (2009) 1275-1287.
10. N. Soave, Normalized ground states for the NLS equation with combined nonlinearities, J. Differ. Equ., 269 (2020) 6941-6987.
11. M. Weinstein, Nonlinear Schrödinger equations and sharp interpolations estimates, Commun. Math. Phys., 87 (1983) 567-576.
12. H. Ye, The sharp existence of constrained minimizers for a class of nonlinear Kirchhoff equations, Math. Methods Appl. Sci., 38 (2015) 2663-2679.
13. X. Zeng, Y. Zhang, Existence and uniqueness of normalized solutions for the Kirchhoff equation, Appl. Math. Lett., 74 (2017) 52-59.
14. Z. Zhang, K. Perera, Sign changing solutions of Kirchhoff type problems via invariant sets of descent flow, J. Math. Anal. Appl., 317 (2006) 456-463.