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ORIGINAL RESEARCH

Ill-posedness for the Burgers equation in Sobolev spaces

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Abstract In this paper, we consider the Cauchy problem for the Burgers equation in the line. We shall prove that this problem is ill-posed in the Sobolev space $H^s(\mathbb{R})$ with $1 \le s < \frac{3}{2}$ in the sense of "norm inflation" by constructing an explicit example of initial data.

Keywords Burgers equation · Ill-posedness

Mathematics Subject Classification 35Q35 · 35B30

1 Introduction

1.1 The Concept of Well-posedness

We say that the Cauchy problem

$$\begin{cases} \partial_t f = F(f), \\ f(0,x) = f_0(x). \end{cases} \tag{1.1}$$

is locally well-posed in a Banach space X if the following three conditions hold

1. (Local existence) For any initial data $u_0 \in X$, there exists a short time $T = T(u_0) > 0$ and a solution $\mathbf{S}_t(u_0) \in \mathcal{C}([0,T), X)$ to the Cauchy problem (1.1);
2. (Uniqueness) This solution $\mathbf{S}_t(u_0)$ is unique in the space $\mathcal{C}([0,T), X)$;
3. (Continuous Dependence) The data-to-solution map $u_0 \mapsto \mathbf{S}_t(u_0)$ is continuous in the following sense: for any $T_1 < T$ and $\varepsilon > 0$, there exists $\delta > 0$, such that if $\|u_0 - \widetilde{u}_0\|_X \le \delta$, then $\mathbf{S}_t(\widetilde{u}_0)$ exists up to T_1 and

$$\|\mathbf{S}_t(u_0) - \mathbf{S}_t(\widetilde{u}_0)\|_{\mathcal{C}([0,T),X)} \le \varepsilon.$$

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The problem is said to be ill-posed in X if it is not well-posed in the above sense. Based on the definition of well-posedness, at least three types of ill-posedness were studied in the literature: nonexistence, non-uniqueness, and discontinuous dependence on the data. In this paper we are interested in discontinuity with respect to the data.

1.2 The Burgers equation

The Burgers equation with fractional dissipation is written as

$$\begin{cases} \partial_t u + uu_x + \Lambda^\gamma u = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.2}$$

where $\gamma \in [0, 2]$ and the fractional power operator Λ^γ is defined by Fourier multiplier with the symbol $|\xi|^\gamma$

$$\Lambda^\gamma u(x) = \mathcal{F}^{-1}\big(|\xi|^\gamma \mathcal{F}u(\xi)\big).$$

The Burgers equation (1.2) with $\gamma = 0$ and $\gamma = 2$ has received an extensive amount of attention since the studies by Burgers in the 1940s. If $\gamma = 0$, the equation is perhaps the most basic example of a PDE evolution leading to shocks. If $\gamma = 2$, it provides an accessible model for studying the interaction between nonlinear and dissipative phenomena. Kiselev et al. [8] gave a complete study for general $\gamma \in [0, 2]$ for the periodic case. In particular, for the case $\gamma = 1$, they proved the global well-posedness of the equation in the critical Hilbert space $H^{\frac{1}{2}}(\mathbb{T})$ by using the method of modulus of continuity. Subsequently, Miao-Wu [12] proved the global well-posedness of the critical Burgers equation in critical Besov spaces $B^{1/p}_{p,1}(\mathbb{R})$ with $p \in [1, \infty)$ with the help of Fourier localization technique and the method of modulus of continuity. For more results on the fractional Burgers equation and dispersive perturbations of Burgers equations, we refer the readers to see [1,3,9–11] and the references therein. We should mention that Molinet et al. [11] proved that the Cauchy problem for a class of dispersive perturbations of Burgers equations is locally well-posed in $H^s(\mathbb{R})$.

In this paper, we focus on the well-posedness problem of the following Burgers equation as the most simple quasilinear symmetric hyperbolic equation.

$$\begin{cases} \partial_t u + uu_x = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.3}$$

Kato [6] demonstrated that the flow map cannot be of Hölder continuous type. Roughly speaking, (1.3) can be viewed as the simplest in the family of partial differential equations modeling the Euler and Navier-Stokes equation nonlinearity. The local well-posedness of the Burgers equation (1.3) for data in $H^s(\mathbb{R})$ with any $s > 3/2$ can be proved by combining the Sobolev embedding $H^{s-1}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ and the classical energy estimate

$$\|u\|_{H^s} \le \|u_0\|_{H^s} \exp\left(C\int_0^t \|u_x\|_{L^\infty} \mathrm{d}\tau\right).$$

Also, in [13] it is obtained that the solution map is continuous dependence while not uniformly continuous dependence on initial data for the Burgers equation (1.3) in the same space $H^s(\mathbb{R})$ with $s > 3/2$. For the endpoint case, Linares et al. in [10] proved that the Cauchy problem for (1.3) is ill-posed in $H^{3/2}(\mathbb{R})$, where the key point is that the available local well-posedness theory in $H^s(\mathbb{R})$ with any $s > 3/2$ have be used, for more details see Remark 1.6. Using the idea developed in [10], Guo et al. in [4] proved the ill-posedness for the Camassa-Holm equation in the critical Sobolev space $H^{3/2}(\mathbb{R})$ and even in the Besov space $B^{1+1/p}_{p,r}(\mathbb{R})$ with $r > 1$.

In this paper, we shall prove that the Cauchy problem for the Burgers equation (1.3) in the Sobolev space $H^s(\mathbb{R})$ with $1 \le s < \frac{3}{2}$ is ill-posed in the sense of "norm inflation" by constructing an explicit example of initial data.

1.3 Main Result

Now let us state our main ill-posedness result of this paper.

Theorem 1.1 *Let $1 \le s < \frac{3}{2}$. For any $\delta > 0$, there exists initial data satisfying*

$$\|u_0\|_{H^s} \le \delta,$$

such that a solution $u(t) \in \mathcal{C}([0, T_0]; H^s)$ of the Cauchy problem (1.3) satisfies

$$\|u(T_0)\|_{H^s} \ge \frac{1}{\delta} \quad \text{for some} \quad 0 < T_0 < \delta.$$

Remark 1.1 Theorem 1.1 indicates that the Burgers equation is ill-posed in $H^s(\mathbb{R})$ with $1 \le s < \frac{3}{2}$ by exhibiting a strong discontinuity with respect to the initial data known as a norm inflation.

Strategies to Proof. We shall outline the main ideas in the proof of Theorem 1.1.

- Firstly, we construct an explicit example for initial data u_0, where the norm $\|u_0\|_{H^s}$ is sufficiently small while $\|u_0'\|_{L^\infty}$ can be large enough.
- Secondly, we express the solution to the Burgers equation (1.3) by exploring fully the properties of the flow map and give the explicit blow-up time T^*.
- Lastly, we mainly observe that the transport term does cause growth of the L^2-norm of u_x as t tends to T^*. Precisely speaking, we estimate the L^2-norm of u_x over $(-\psi(t, q_0), \psi(t, q_0))$ and obtain that its lower bound can be arbitrarily large as t tends to T^*.

The structure of the paper. In Section 2 we provide several key Lemmas. In Section 3 we present the proof of Theorem 1.1.

Let us complete this section with some notations we shall use throughout this paper.

Notations. The notation $A \le a \wedge b$ means that $A \le a$ and $A \le b$. $a \approx b$ means $C^{-1}b \le a \le Cb$ for some positive harmless constants C. Given a Banach space X, we denote its norm by $\|\cdot\|_X$. For $I \subset \mathbb{R}$, we denote by $\mathcal{C}(I; X)$ the set of continuous functions on I with values in X. For all $f \in \mathcal{S}'$, the Fourier transform $\widehat{f}$ is defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)\mathrm{d}x \quad \text{for any } \xi \in \mathbb{R}.$$

We denote $J^s := (1 - \partial_x^2)^{\frac{s}{2}}$. For $s \in \mathbb{R}$, the nonhomogeneous Sobolev space is defined by

$$\|f\|_{H^s}^2 = \|(1 - \partial_x^2)^{\frac{s}{2}} f\|_{L^2}^2 = \int_{\mathbb{R}} (1 + |\xi|^2)^s |\widehat{f}(\xi)|^2 \mathrm{d}\xi.$$

2 Preliminary

In the section, we make some preparations for the proof of the main theorem.

2.1 Key Example for Initial Data

Firstly, we construct an explicit example as follows. Set

$$u_0(x) := p_0\big(e^{-|x+q_0|} - e^{-|x-q_0|}\big),$$

where two positive numbers p_0 and $q_0 \in (0, 1)$ will be fixed later.

It is easy to check that $u_0(x)$ is an odd function. Furthermore, we can deduce that the following result holds:

Lemma 2.1 *For every $q_0 \in (0, 1)$ and $s \in (\frac{1}{2}, \frac{3}{2})$, there exists $C = C_s > 0$ such that*

$$C^{-1} p_0 q_0^{3/2-s} \le \|u_0\|_{H^s} \le C p_0 q_0^{3/2-s}. \tag{2.4}$$

Proof The proof essentially follows that of Lemma 3.1 in [2] or Proposition 1 in [5]. For the sake of readability, we sketch the proof here. Defining the function

$$f(x) := e^{-|x+q_0|} - e^{-|x-q_0|},$$

then using the fact $\widehat{e^{-|x|}}(\xi) = 2/\left(1+\xi^2\right)$, we have

$$\widehat{f}(\xi) = \frac{2(e^{iq_0\xi} - e^{-iq_0\xi})}{1+\xi^2} = \frac{4i\sin(q_0\xi)}{1+\xi^2}.$$

Using the definition of the H^s-norm and the change of variable setup $y = q_0\xi$, we have

$$\begin{aligned}
\|f\|^2_{H^s(\mathbb{R})} &= 16\int_{\mathbb{R}} \left(1+\xi^2\right)^{s-2}\sin^2(q_0\xi)\mathrm{d}\xi \\
&\geq 32\left(1+\frac{\pi^2}{q_0^2}\right)^{s-2}\int_0^{\pi/q_0}\sin^2(q_0\xi)\mathrm{d}\xi \\
&= 32\left(1+\frac{\pi^2}{q_0^2}\right)^{s-2}\cdot\frac{1}{q_0}\int_0^{\pi}\sin^2 y\mathrm{d}y \\
&\geq 16\pi\left(1+\pi^2\right)^{-3/2} q_0^{3-2s},
\end{aligned}$$

where we have used $s \in \left(\frac{1}{2}, \frac{3}{2}\right)$ and $q_0 \in (0, 1)$. This proves the lower bound.

To get the upper bound, we split the domain of integration as

$$\begin{aligned}
\|f\|^2_{H^s(\mathbb{R})} &= 32\int_0^{\infty}\left(1+\xi^2\right)^{s-2}\sin^2(q_0\xi)\mathrm{d}\xi \\
&= 32\left(\int_0^{1/q_0} + \int_{1/q_0}^{\infty}\right)\left(1+\xi^2\right)^{s-2}\sin^2(q_0\xi)\mathrm{d}\xi \\
&=: 32(I_1 + I_2).
\end{aligned}$$

Due to the simple fact $\sin^2(q_0\xi) \leq |q_0\xi|^2 \wedge 1$, we have

$$\begin{aligned}
I_1 &\leq 32q_0^2\int_0^{1/q_0}\xi^{2s-2}\mathrm{d}\xi \leq \left(\frac{32}{2s-1}\right)q_0^{3-2s}, \\
I_2 &\leq 32\int_{1/q_0}^{\infty}\xi^{2s-4}\mathrm{d}\xi \leq \left(\frac{32}{3-2s}\right)q_0^{3-2s},
\end{aligned}$$

which completes the proof of Lemma 2.1.

2.2 Existence and Blow-up criterion

Lemma 2.2 *For every $s \in [1, \frac{3}{2})$, there exists a solution $u \in \mathcal{C}([0, T^*); H^s) \cap L^\infty([0, T^*); \mathrm{Lip})$ for the Burgers equation (1.3), where $T^* < \infty$ is the maximal time for initial data u_0. Furthermore, we have*

$$\lim_{t\uparrow T^*}\left(\|u(t)\|_{H^s} + \|\partial_x u(t)\|_{L^\infty}\right) = +\infty \quad \Leftrightarrow \quad \lim_{t\uparrow T^*}\|\partial_x u(t)\|_{L^\infty} = +\infty.$$

Proof Easy computations give that

$$u_0'(x) = \begin{cases} -p_0(e^{-q_0} - e^{q_0})e^x, & \text{if } x \in (-\infty, -q_0), \\ -p_0e^{-q_0}(e^x + e^{-x}), & \text{if } x \in (-q_0, q_0), \\ -p_0(e^{-q_0} - e^{q_0})e^{-x}, & \text{if } x \in (q_0, +\infty), \end{cases} \tag{2.5}$$

from which and Lemma 2.1, we get that $u_0 \in H^s \cap \mathrm{Lip}$.

Applying J^s to (1.3) and taking the L^2 inner product of the resulting equality with $J^s u$, then using the following commutator estimate (see [7])

$$\|[J^s, f]g\|_{L^2} \leq C\big(\|\partial_x f\|_{L^\infty}\|g\|_{H^{s-1}} + \|f\|_{H^s}\|g\|_{L^\infty}\big), \quad s > 0,$$

we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{H^s}^2 = \frac{1}{2}\int_{\mathbb{R}} \partial_x u |J^s u|^2 \mathrm{d}x - \int_{\mathbb{R}} [J^s, u]\partial_x u \cdot J^s u \mathrm{d}x \leq C\|\partial_x u\|_{L^\infty}\|u\|_{H^s}^2,$$

which implies that

$$\|u(t)\|_{H^s} \leq \|u_0\|_{H^s} \exp\left(C\int_0^t \|\partial_x u(\tau)\|_{L^\infty}\mathrm{d}\tau\right).$$

Applying ∂_x to (1.3) and taking the inner product of the resulting equality with $|\partial_x u|^{p-2}\partial_x u$ with $p \geq 2$, we obtain

$$\|\partial_x u\|_{L^p}^{p-1}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x u\|_{L^p} = \frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x u\|_{L^p}^p = \frac{p-1}{p}\int_{\mathbb{R}} \partial_x u |\partial_x u|^p \mathrm{d}x \leq \|\partial_x u\|_{L^\infty}\|\partial_x u\|_{L^p}^p,$$

which reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x u\|_{L^p} \leq \|\partial_x u\|_{L^\infty}\|\partial_x u\|_{L^p}.$$

Using Gronwall's inequality and letting $p = \infty$, we get

$$\|\partial_x u(t)\|_{L^\infty} \leq \|\partial_x u_0\|_{L^\infty} \exp\left(C\int_0^t \|\partial_x u(\tau)\|_{L^\infty}\mathrm{d}\tau\right).$$

This is enough to complete the proof of Lemma 2.2.

2.3 The Equation Along the Flow

Given a Lipschitz velocity field u, we may solve the following ODE to find the flow induced by u:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\psi(t,x) = u(t,\psi(t,x)), \\ \psi(0,x) = x, \end{cases} \tag{2.6}$$

which is equivalent to the integral form

$$\psi(t,x) = x + \int_0^t u(\tau, \psi(\tau,x))\mathrm{d}\tau.$$

Furthermore, we get from (1.3) that

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t,\psi(t,x)) = u_t(t,\psi(t,x)) + u_x(t,\psi(t,x))\frac{\mathrm{d}}{\mathrm{d}t}\psi(t,x) = 0,$$

which means that

$$u(t,\psi(t,x)) = u_0(x), \quad \text{namely,} \quad u(t,x) = u_0(\psi^{-1}(t,x)). \tag{2.7}$$

Thus we can give the explicit expression of the flow as

$$\psi(t,x) = x + tu_0(x). \tag{2.8}$$

Let $y = \psi(t,x)$, then we have

$$\psi^{-1}(t,y) = y - tu_0(\psi^{-1}(t,y)) = y - tu(t,y). \tag{2.9}$$

Differentiating (1.3) with respect to space variable x, we find

$$u_{tx} + uu_{xx} + (u_x)^2 = 0.$$

Combining the above and (2.6), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}u_x(t,\psi(t,x)) &= u_{tx}(t,\psi(t,x)) + u_{xx}(t,\psi(t,x))\frac{\mathrm{d}}{\mathrm{d}t}\psi(t,x),\\ &= u_{tx}(t,\psi(t,x)) + u_{xx}(t,\psi(t,x))u(t,\psi(t,x))\\ &= -(u_x)^2(t,\psi(t,x)),\end{aligned}$$

which reduces to

$$u_x(t,\psi(t,x)) = \frac{1}{t+\frac{1}{u_0'(x)}}. \tag{2.10}$$

We should mention that the above can also be deduced from (2.7) and (2.8).

According to the definition of u_0, we can deduce that the maximal existence time of the solution to (1.3) is

$$T^* = -\frac{1}{\inf\limits_{x\in\mathbb{R}} u_0'(x)} = -\frac{1}{u_0'(q_0^-)} \in \left(\frac{1}{2p_0}, \frac{1}{p_0}\right).$$

Because the velocity field is Lipschitz, then we get that for $t \in [0, T^*)$

$$\psi_x(t,x) = \exp\left(\int_0^t u_x(\tau,\psi(\tau,x))\mathrm{d}\tau\right) > 0.$$

This shows that $\psi(t,\cdot)$ is an increasing diffeomorphism over $\mathbb{R}$, that is, for all $x, y \in \mathbb{R}$, there holds that $\psi(t,x) < \psi(t,y)$ if $x < y$.

3 Proof of Main Theorem

In this section, we prove Theorem 1.1.

Proof of Theorem 1.1 By the definition of $u_0'(x)$ and (2.10), we know that $u_x(t,\psi(t,x))$ is continuous in $[0,T^*)\times(-q_0,q_0)$. We should emphasize that $u_x(t,x)$ is discontinuous in $[0,T^*)\times\mathbb{R}$, but we can claim that $u_x(t,x)$ is continuous in $[0,T^*)\times(-\psi(t,q_0),\psi(t,q_0))$. In fact, for any $x,y\in(-\psi(t,q_0),\psi(t,q_0))$, we have $\psi^{-1}(t,x),\psi^{-1}(t,y)\in(-q_0,q_0)$. Moreover, we deduce from (2.9) that for $t\in[0,T^*)$

$$\begin{aligned}|\psi^{-1}(t,x)-\psi^{-1}(t,y)| &\le |\partial_x\psi^{-1}(t,x)||x-y|\\ &\le |x-y|\big(1+T^*\|u_x\|_{L_t^\infty(L^\infty)}\big).\end{aligned} \tag{3.11}$$

Also, it follows from (2.9) and (1.3) that

$$\begin{aligned}|\psi^{-1}(t,x)-\psi^{-1}(s,x)| &= |tu(t,x)-su(s,x)|\\ &\le \|u_0\|_{L^\infty}|t-s| + T^*\left|\int_s^t \|\partial_t u(\tau,\cdot)\|_{L^\infty}\mathrm{d}\tau\right|\\ &\le C\big(1+T^*\|u_x\|_{L_t^\infty(L^\infty)}\big)|t-s|.\end{aligned} \tag{3.12}$$

Thus, we obtain from (2.5) and (3.11)-(3.12) for $s,t\in[0,T^*)$ and $x,y\in(-\psi(t,q_0),\psi(t,q_0))$

$$\begin{aligned}|u_x(t,x)-u_x(s,y)| &\le |u_x(t,x)-u_x(t,y)| + |u_x(t,y)-u_x(s,y)|\\ &\le |u_x(t,\psi(t,\psi^{-1}(t,x))) - u_x(t,\psi(t,\psi^{-1}(t,y)))|\\ &\quad + |u_x(t,\psi(t,\psi^{-1}(t,y))) - u_x(s,\psi(s,\psi^{-1}(s,y)))|\\ &\to 0 \quad \text{as} \quad (t,x)\to(s,y).\end{aligned}$$

By the Burgers equation $\partial_t u = -uu_x$, we can deduce that $\partial_t u(t,x)$ is continuous in $[0,T^*)\times(-\psi(t,q_0),\psi(t,q_0))$. That is $u(t,x)\in\mathcal{C}^1([0,T^*)\times(-\psi(t,q_0),\psi(t,q_0)))$. Furthermore, one has

$$u_{xx}(t,\psi(t,x)) = \frac{u_0''(x)}{\big(1+tu_0'(x)\big)^3}.$$

The similar argument shows that $u_x(t,x) \in \mathcal{C}^1([0,T^*) \times (-\psi(t,q_0), \psi(t,q_0)))$.

For notational convenience we now set

$$\widetilde{m}(t) := u_x(t, \psi(t,0)) = \frac{1}{t + \frac{1}{u_0'(0)}}.$$

Thus, we have

$$u_x(t, \psi(t,x)) \le \widetilde{m}(t) \quad \text{for all } x \in (-q_0, q_0). \tag{3.13}$$

Set $w(t,x) := u_x(t,x)$, then we obtain from (1.3)

$$\partial_t w + \partial_x(uw) = 0,$$

which implies that for $(t,x) \in [0,T^*) \times (-\psi(t,q_0), \psi(t,q_0))$

$$\partial_t(w^2) + \partial_x(uw^2) + \partial_x u w^2 = 0. \tag{3.14}$$

Integrating (3.14) with respect to space variable x over $[-\psi(t,q_0), \psi(t,q_0)]$, we have

$$\int_{|x|\le\psi(t,q_0)} \partial_t(w^2)\mathrm{d}x + \int_{|x|\le\psi(t,q_0)} \partial_x(uw^2)\mathrm{d}x + \int_{|x|\le\psi(t,q_0)} \partial_x u w^2 \mathrm{d}x = 0. \tag{3.15}$$

As $u_0(x)$ is odd, the solution of Burgers equation satisfies $u(t,x) = -u(t,-x)$, which tells us that $w(t,x) = w(t,-x)$. Thus we have

$$\int_{|x|\le\psi(t,q_0)} \partial_t(w^2)\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_{|x|\le\psi(t,q_0)} w^2\mathrm{d}x - 2u(t,\psi(t,q_0))w^2(t,\psi^-(t,q_0)), \tag{3.16}$$

and

$$\int_{|x|\le\psi(t,q_0)} \partial_x(uw^2)\mathrm{d}x = 2u(t,\psi(t,q_0))w^2(t,\psi^-(t,q_0)). \tag{3.17}$$

Inserting (3.16) and (3.17) into (3.15) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{|x|\le\psi(t,q_0)} w^2\mathrm{d}x + \int_{|x|\le\psi(t,q_0)} \partial_x u(t,x) w^2\mathrm{d}x = 0. \tag{3.18}$$

To simplify notation let

$$A(t) := \int_{|x|\le\psi(t,q_0)} w^2(t,x)\mathrm{d}x \quad \text{for} \quad t \in [0,T^*),$$

combining (3.13), then (3.18) reduces to

$$A'(t) = \int_{|x|\le\psi(t,q_0)} -u_x(t,x)w^2\mathrm{d}x \ge -\widetilde{m}(t)A(t).$$

Solving the above differential inequality gives us that

$$A(t) \ge A_0 \exp\left(\int_0^t -\widetilde{m}(\tau)\mathrm{d}\tau\right) = A_0 \cdot \frac{\widetilde{m}(t)}{\widetilde{m}(0)},$$

which implies

$$\|w\|_{L^2} \ge A_0^{\frac{1}{2}} \cdot \sqrt{\frac{\widetilde{m}(t)}{u_0'(0)}}. \tag{3.19}$$

Notice that

$$A_0 = \int_{|x|\le q_0} \left(u_0'(x)\right)^2\mathrm{d}x \approx p_0^2 q_0,$$

and

$$\lim_{t\uparrow T^*}\widetilde{m}(t)=\frac{1}{-\frac{1}{u_0'(q_0^-)}+\frac{1}{u_0'(0)}}=\frac{u_0'(q_0^-)u_0'(0)}{u_0'(q_0^-)-u_0'(0)},$$

combining the above and (3.19) yields

$$\begin{aligned}\lim_{t\uparrow T^*}\|u(t)\|_{H^1}\geq \lim_{t\uparrow T^*}\|w\|_{L^2}&\\ &\geq Cp_0\sqrt{q_0}\sqrt{\frac{u_0'(q_0^-)}{u_0'(q_0^-)-u_0'(0)}}\\ &\geq C\frac{p_0\sqrt{q_0}}{1-e^{-q_0}}\\ &\approx Cp_0q_0^{-\frac{1}{2}},\end{aligned}$$

where we have used that

$$u_0'(0)=-2p_0e^{-q_0}\quad\text{and}\quad u_0'(q_0^-)=-p_0(e^{-2q_0}+1)$$

and in the last step used

$$\frac{q_0}{2}\leq 1-e^{-q_0}\leq q_0\quad\text{for}\quad q_0\in(0,1).$$

By Lemma 2.1, one has

$$\|u_0\|_{H^s}\leq c_1p_0q_0^{\frac{3}{2}-s}\leq\delta\quad\text{and}\quad T^*\leq\frac{1}{p_0}\leq\delta,$$

but

$$\lim_{t\uparrow T^*}\|u(t)\|_{H^1}\geq c_2p_0q_0^{-\frac{1}{2}}\geq\frac{1}{\delta^2},$$

if some large p_0 and small q_0 is chosen. In fact, we can take p_0 and q_0 such that

$$p_0\geq\frac{1}{\delta}\quad\text{and}\quad q_0\leq\big(\delta/(c_1p_0)\big)^{2/(3-2s)}\wedge c_2^2p_0^2\delta^4.$$

Hence, we can choose $T_0\in[0,T^*)$ such that

$$\|u(T_0)\|_{H^s}\geq C\|u(T_0)\|_{H^1}\geq\frac{1}{\delta}.$$

This completes the proof of Theorem 1.1.

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Declarations

Data availability Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Conflict of interest The authors declare that they have no conflict of interest.

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