Indian J Pure Appl Math (2023) 54:789–796
https://doi.org/10.1007/s13226-022-00297-8

ORIGINAL RESEARCH

Cohomology bounds and Chern class inequalities for stable sheaves on a smooth projective variety

Tohru Nakashima

Received: 19 May 2022 / Accepted: 19 July 2022 / Published online: 12 August 2022

Abstract We give effective upper bounds for dimensions of the $(n-1)$-th cohomology groups of μ-semistable torsion-free sheaves on a smooth projective variety of dimension n defined over an algebraically closed fieled of characteristic zero. As a corollary to this result, we obtain bounds for the dimension of the moduli space of μ-stable vector bundles. We also prove Bogomolov-Gieseker type inequalities for the fourth Chern classes $c_4(E)$ of μ-semistable vector bundles E on a smooth projective fourfold.

Keywords Bogomolov-Gieseker type inequality · μ-semistable sheaves · Moduli spaces

Mathematics Subject Classification 14J60 · 14F05 · 14J32

1 Introduction

Let X be a smooth projective variety defined over an algebraically closed field of characteristic 0 and let H be an ample line bundle on X. The classical Bogomolov-Gieseker inequality states that $\Delta_H(E) = (2rc_2(E) - (r-1)c_1(E)^2) \cdot H^{n-2} \geq 0$ for any torsion-free sheaf E of rank r and Chern classes $c_i(E)$ on X which is μ-semistable with respect to H. Recently, some conjectures for the third Chern character $\mathrm{ch}_3(E)$ of μ-stable sheaves E on a threefold have been proposed ([1], [2]). We gave explicit bounds for the cohomology groups for μ-semistable sheaves E on a threefold and applied them to obtain inequalities for $\mathrm{ch}_3(E)$ in [6], [7]. On the other hand, it seems that no Bogomolov-Gieseker type inequality has been known for the top Chern class $c_n(E)$ for μ-semistable sheaves E on a variety of dimension $n \geq 4$.

In this note we give an effective bound for the dimension of the cohomology group $\mathrm{Ext}^1(E, E_1)$ for μ-semistable torsion-free sheaves E and E_1 on a smooth projective variety of dimension $n \geq 3$. As in [7], we prove this by reducing the problem to the three dimensional case using the restriction theorem due to A.Langer ([4], [5]) and a vanishing theorem of H.Sun ([8]). As a corollary, we obtain upper bounds for the dimension of the moduli of μ-stable vector bundles. We also obtain explicit upper bounds for $c_4(E)$ of μ-semistable bundles E in terms of r, $c_i(E)$ $(1 \leq i \leq 3)$, $c_i(X)$ and H on a smooth projective fourfold.

Communicated by Indranil Biswas.

T. Nakashima (✉)
Department of Mathematics, Physics and Computer Science, Japan Women's University, Mejirodai, Bunkyoku, Tokyo 112-8681, Japan
E-mail: nakashima@fc.jwu.ac.jp

2 Notations and Preliminaries

In what follows all varieties will be assumed to be defined over an algebraically closed field of characteristic 0. Let X be a smooth projective variety of dimension $n \geq 3$ and let H be an ample line bundle on X. Let K_X denote the canonical bundle of X and let $A^i(X)$ denote the codimension i Chow group of X. For a torsion-free sheaf E on X, the *slope* $\mu_H(E)$ is defined to be the following number

$$\mu_H(E) := \frac{c_1(E) \cdot H^{n-1}}{\operatorname{rk} E}.$$

A torsion-free sheaf E on X is said to be μ*-stable*(resp.μ*-semistable*) with respect to H (or simply H*-(semi)stable*) if, for any coherent subsheaf $F \subset E$ with $0 < \operatorname{rk} F < \operatorname{rk} E$, we have $\mu_H(F) < \mu_H(E)$ (resp.$\mu_H(F) \leq \mu_H(E)$).

The *discriminant* $\Delta(E) \in A^2(X)$ of E is defined as follows.

$$\Delta(E) = 2rc_2(E) - (r-1)c_1(E)^2.$$

We set $\Delta_H(E) := \Delta(E) \cdot H^{n-2}$. We recall the following results concerning the restriction of μ-(semi)stable sheaves to divisors ([5]).

Proposition 1 *Let X be a smooth projective variety X of dimension $n \geq 2$ and let H be a very ample line bundle on X. Let E be an H-semistable torsion-free sheaf of rank $r \geq 2$ on X. Let a be an integer with*

$$\binom{a+n}{n} > \frac{1}{2}\left(\max\{\frac{r^2-1}{4}, 1\}H^n + 1\right)\Delta_H(E) + 1.$$

Then, for general $D \in |aH|$, the restriction $E_{|D}$ is an H_D-semistable torsion-free sheaf.

We also need the following vanishing result due to H.Sun which has been proved by techniques of tilt stability ([8, Corollary 1.9]).

Proposition 2 *Let X be a smooth projective variety X of dimension $n \geq 2$ and H an ample line bundle on X. Let E be an H-semistable torsion-free sheaf of rank $r \geq 2$ and Chern classes $c_i(E) = c_i$ on X. Let*

$$\begin{aligned}\overline{\Delta}_H(E) :&= (c_1(E) \cdot H^{n-1})^2 - 2H^n r \operatorname{ch}_2(E) \cdot H^{n-2} \\ &= (c_1(E) \cdot H^{n-1})^2 + H^n(\Delta_H(E) - c_1(E)^2 \cdot H^{n-2}).\end{aligned}$$

Then, for any integer l with

$$l > \frac{\overline{\Delta}_H(E)}{H^n} - \frac{\mu_H(E)}{H^n},$$

we have

$$H^{n-1}(X, E(K_X + lH)) = 0.$$

Let X be a smooth projective variety of dimension $n \geq 2$. For a coherent sheaf E of rank r and Chern classes c_i and a very ample line bundle H on X, we define the following numbers depending only on r, $c_i(E)$ ($i = 1, \ldots, n$) and H.

$$\begin{aligned}a(E, H) &:= \min\{a \in \mathbb{N} \,|\, \binom{a+n}{n} > \frac{1}{2}\left(\max\{\frac{r^2-1}{4}, 1\}H^n + 1\right)\Delta_H(E) + 1\}, \\ c(E, H) &:= \lfloor \frac{\overline{\Delta}_H(E)}{H^n} - \frac{\mu_H(E)}{H^n} \rfloor + 1.\end{aligned}$$

Here, for a real number x, $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. Let

$$a_0(E, H) := \lfloor \{\frac{n!}{2}(\max\{\frac{r^2-1}{4}, 1\}H^n + 1)\Delta_H(E) + 1\}^{\frac{1}{n}} \rfloor + 1.$$

We see that $a_0 := a_0(E, H) \geq a(E, H)$ since

$$\binom{a_0+n}{n} > \frac{(a_0+1)^n}{n!} > \frac{a_0^n}{n!} > \frac{1}{2}(\max\{\frac{r^2-1}{4}, 1\}H^n + 1)\Delta_H(E) + 1.$$

Let X be a smooth projective threefold and let H be a very ample line bundle on X. Let E_1 be an H-semistable vector bundle of rank r_1 on X. In the rest of this section, we recall the upper bound of $\dim \mathrm{Ext}^1(E, E_1)$ obtained in [7] for H-semistable torsion-free sheaves E on X. We notice that we gave a bound of different type for sheaves on a Calabi-Yau threefold in [6].

For an H-semistable torsion-free sheaf E on X of rank $r \geq 2$ and Chern classes $c_i(E) = c_i$, let $\chi(E)$ denote the Euler characteristic of E. Then Riemann-Roch formula yields

$$\begin{aligned}\chi(E) = &\frac{1}{6}(c_1(E)^3 - 3c_1(E)\cdot c_2(E) + 3c_3(E)) + \frac{1}{12}c_1(E)\cdot(K_X^2 + c_2(X)) \\ &+ r\chi(\mathcal{O}_X).\end{aligned}$$

For a line bundle on L on X, we set $\alpha(E, L) := \chi(E \otimes L) - \chi(E)$. Then we have (cf. [7]):

$$\begin{aligned}\alpha(E, L) = L \cdot \Big(&\frac{L^2}{6} + \frac{(2c_1(E) - rK_X)\cdot L}{4} + \frac{c_1(E)\cdot(c_1(E) - K_X)}{2} \\ &- c_2(E) + \frac{r(K_X^2 + c_2(X))}{12}\Big).\end{aligned}$$

We set $E' = E \otimes E_1^{\vee}$ and $l = \max\{a(E', H), c(E', H)\}$. We divide into the following six cases.

$$\begin{aligned}
&Case\ 1\text{-}1: && (K_X + lH)\cdot H^2 < 0 && \text{and} && \mu_H(E') \geq -(K_X + lH)\cdot H^2\\
&Case\ 1\text{-}2: && (K_X + lH)\cdot H^2 < 0 && \text{and} && 0 < \mu_H(E') < -(K_X + lH)\cdot H^2\\
&Case\ 1\text{-}3: && (K_X + lH)\cdot H^2 < 0 && \text{and} && \mu_H(E') \leq 0\\
&Case\ 2\text{-}1: && (K_X + lH)\cdot H^2 \geq 0 && \text{and} && \mu_H(E') < -(K_X + lH)\cdot H^2\\
&Case\ 2\text{-}2: && (K_X + lH)\cdot H^2 \geq 0 && \text{and} && -(K_X + lH)\cdot H^2 \leq \mu_H(E') \leq 0\\
&Case\ 2\text{-}3: && (K_X + lH)\cdot H^2 \geq 0 && \text{and} && \mu_H(E') > 0.
\end{aligned}$$

Then we have the following bound for $\dim \mathrm{Ext}^1(E, E_1)$([7, Theorem 3.3]).

Theorem 3 *Let X, H, E and E_1 be as above and let $B_i := B_i(E', H, l)$. Then we have* $\dim \mathrm{Ext}^1(E, E_1) \leq B$ *where*

$$B = \begin{cases} B_1 + B_2 + B_3 & \text{in Case 2-2}\\ B_1 + B_3 & \text{in Case 1-1 and Case 2-3}\\ B_2 + B_3 & \text{in Case 1-3 and Case 2-1}\\ B_3 & \text{in Case 1-2.}\end{cases}$$

Here B_j are defined as follows.

$$B_1(E, H, l) = rH^3\Big(\frac{\frac{\mu_H(E) + (K_X + lH)\cdot H^2}{H^3}}{2} + f(r) + 2\Big),$$

$$B_2(E, H, l) = rH^3\Big(\frac{-\frac{\mu_H(E)}{H^3}}{2} + f(r) + 2\Big),$$

$$\begin{aligned}B_3(E, H, l) &= -\alpha(E(K_X), lH)\\
&= -lH \cdot \Big(\frac{l^2H^2}{6} + \frac{l(2c_1(E(K_X)) - rK_X)\cdot H}{4}\\
&\quad + \frac{c_1(E(K_X))\cdot(c_1(E(K_X)) - K_X)}{2} - c_2(E(K_X))\\
&\quad + \frac{r(K_X^2 + c_2(X))}{12}\Big).\end{aligned}$$

3 Effective bounds in dimension $n \geq 3$

We shall adopt the notations introduced in the previous section. The purpose of this section is to give effective bounds for several invariants of μ-semistable sheaves on smooth projective variety. Let X be a smooth projective variety of dimension $n \geq 3$ and let H be a very ample line bundle on X. For integers $1 \leq i \leq n-2$ and $l_1, l_2, \ldots, l_i$, we denote by $Y_i \in |l_1 H \cap \cdots \cap l_{i-1} H|$ a general smooth complete intersections of divisors $l_1 H, l_2 H, \ldots, l_i H$. Let $l_1(E, H) := \max\{a_0(E, H), c(E, H)\}$ and for $2 \leq i \leq n-2$, define

$$l_i = l_i(E, H) := \max\{a_0(E_{|Y_{i-1}}, H_{Y_{i-1}}), c(E_{|Y_{i-1}}, H_{Y_{i-1}})\}.$$

For $1 \leq j \leq 3$, let $C_j = C_j(E, H) := B_j(E_{|Y}, H_Y, l_{n-2})$ for general smooth threefold $Y = Y_{n-3} \in |l_1 H \cap \cdots \cap l_{n-3} H|$.

Proposition 4 *Let l_i and C_j be as above. Then*

1. *For each $1 \leq i \leq n-2$, l_i depends only on r, $c_1(E)$, $c_2(E)$ and H.*
2. *For $1 \leq j \leq 3$, C_j depends only on r, $c_1(E)$, $c_2(E)$, $c_1(X)$, $c_2(X)$ and H.*

Proof For any integer $l > 0$ and general smooth $Y \in |lH|$, let $\iota : Y \hookrightarrow X$ denote the inclusion. Then we have

$$\Delta_{H_Y}(E_{|Y}) = l\Delta_H(E),$$
$$\overline{\Delta}_{H_Y}(E_{|Y}) = l^2\overline{\Delta}_H(E).$$

Hence we obtain

$$a_0(E_{|Y}, H_Y) = \lfloor\Big\{\frac{(n-1)!l}{2}(\max\{\frac{r^2-1}{4}, 1\}H^n + 1)\Delta_H(E) + 1\Big\}^{\frac{1}{n-1}}\rfloor + 1,$$
$$c(E_{|Y}, H_Y) = \lfloor\frac{l\overline{\Delta}_H(E)}{H^n} - \frac{\mu_H(E)}{H^n}\rfloor + 1.$$

By induction, the claim (1) follows immediately.

We notice that there exists the following exact sequence of tangent bundles on Y:

$$0 \to T_Y \to \iota^* T_X \to N_{Y/X} \to 0$$

where $N_{Y/X}$ is the normal bundle of Y in X. Hence the total Chern class of T_Y is given by

$$c(T_Y) = c(\iota^* T_X)/c(N_{Y/X})$$

where

$$c(N_{Y/X}) = \prod_{i=1}^{n-3}(1 + l_i H_Y).$$

Hence the claim (2) follows. □

Let E_1 be an H-semistable vector bundle of rank r_1 on X. We are interested in estimating $\dim \mathrm{Ext}^1(E, E_1)$ from above for any H-semistable torsion-free sheaf E on X. Let $E' = E \otimes E_1^\vee$. Then E' is H-semistable by [3, Theorem 3.1.4]. Let $l_i := l_i(E', H)$ for $1 \leq i \leq n-2$, $l := \sum_{i=1}^{n-2} l_i$ and $C_j := C_j(E', H)$ for $1 \leq j \leq 3$. As in the case of threefolds, we consider the following six cases.

$$\begin{aligned}
&Case\ 1\text{-}1: && (K_X + lH)\cdot H^{n-1} < 0 && \text{and} && \mu_H(E') \geq -(K_X + lH)\cdot H^{n-1}\\
&Case\ 1\text{-}2: && (K_X + lH)\cdot H^{n-1} < 0 && \text{and} && 0 < \mu_H(E') < -(K_X + lH)\cdot H^{n-1}\\
&Case\ 1\text{-}3: && (K_X + lH)\cdot H^{n-1} < 0 && \text{and} && \mu_H(E') \leq 0\\
&Case\ 2\text{-}1: && (K_X + lH)\cdot H^{n-1} \geq 0 && \text{and} && \mu_H(E') < -(K_X + lH)\cdot H^{n-1}\\
&Case\ 2\text{-}2: && (K_X + lH)\cdot H^{n-1} \geq 0 && \text{and} && -(K_X + lH)\cdot H^{n-1} \leq \mu_H(E') \leq 0\\
&Case\ 2\text{-}3: && (K_X + lH)\cdot H^{n-1} \geq 0 && \text{and} && \mu_H(E') > 0.
\end{aligned}$$

Theorem 5 *Let X, H, E and E_1, l_i and C_i be as above. Then we have* $\dim \operatorname{Ext}^1(E, E_1) \le C$ *where*

$$C = \begin{cases} C_1 + C_2 + C_3 & \text{in Case 2-2} \\ C_1 + C_3 & \text{in Case 1-1 and Case 2-3} \\ C_2 + C_3 & \text{in Case 1-3 and Case 2-1} \\ C_3 & \text{in Case 1-2} \end{cases}$$

Proof For any $1 \le i \le n-3$ and general smooth $Y_{i+1} \in |l_{i+1} H_{Y_i}|$, we have the exact sequence on Y_i:

$$0 \to E_{|Y_i}(-l_{i+1}H_{Y_i}) \to E_{|Y_i} \to E_{|Y_{i+1}} \to 0.$$

By tensoring the above sequence with $E_1^\vee(K_{Y_i} + l_{i+1}H)$, we obtain the exact sequence

$$0 \to E'(K_{Y_i}) \to E'(K_{Y_i} + l_{i+1}H) \to E'(K_{Y_i} + l_{i+1}H)_{|Y_{i+1}} \to 0.$$

By Proposition 1, $E'_{|Y_i}$ is an H_{Y_i}-semistable sheaf on Y_i. Hence Proposition 2 yields $H^{n-i-1}(E'(K_{Y_i}+l_{i+1}H)) = 0$. Then we obtain the surjection

$$H^{n-i-2}(E'(K_X + l_{i+1}H)_{|Y_i}) \to H^{n-i-1}(E'(K_{Y_i})).$$

Therefore we have $h^{n-i-1}(E'(K_{Y_i})) \le h^{n-i-2}(E'(K_{Y_i} + l_{i+1}H_{Y_i})_{|Y_{i+1}})$. Since Serre duality yields

$$H^{n-i-1}(E'(K_{Y_i})) \cong \operatorname{Ext}^1(E_{|Y_i}, E_{1|Y_i})^\vee,$$
$$H^{n-i-2}(E'(K_{Y_i} + l_{i+1}H_{Y_i})_{|Y_{i+1}}) \cong \operatorname{Ext}^1(E_{|Y_{i+1}}, E_{1|Y_{i+1}})^\vee,$$

we obtain $\dim \operatorname{Ext}^1(E_{|Y_i}, E_{1|Y_i}) \le \dim \operatorname{Ext}^1(E_{|Y_{i+1}}, E_{1|Y_{i+1}})$ for all $1 \le i \le n-3$. It follows that $\dim \operatorname{Ext}^1(E, E_1) \le \dim \operatorname{Ext}^1(E_{|Y}, E_{1|Y})$ for general smooth $Y = Y_{n-3} \in |l_1 H \cap \cdots \cap l_{n-3}H|$. We have

$$(K_Y + l_{n-2}H_Y) \cdot H_Y^2 = l'(K_X + lH) \cdot H^{n-1},$$
$$\mu_{H_Y}(E'_{|Y}) = l' \mu_H(E')$$

where $l = \sum_{i=1}^{n-2} l_i$ and $l' = \prod_{i=1}^{n-3} l_i$. Hence, applying Theorem 3 to the threefold Y and the H_Y-semistable sheaf $E_{|Y}$, we obtain the claim for $\dim \operatorname{Ext}^1(E, E_1)$. □

We notice that the constant C in the theorem above depends only on r, c_i, $c_i(X)$ and H and not on the choice of E, Y.

Corollary 6 *Let X be a smooth projective variety of dimension $n \ge 3$ and let H be a very ample line bundle on X. Let $l_i = l_i(E(-K_X), H)$ and $m_i = l_i(E^\vee, H)$ for $1 \le i \le n-2$.*

1. *For any H-semistable torsion-free sheaf E on X of rank $r \ge 2$, $c_i(E) = c_i$, we have $h^{n-1}(E) \le \sum_{j=1}^3 C_j$ where $C_j = C_j(E(-K_X), H)$.*
2. *For any H-semistable vector bundle E on X of rank $r \ge 2$, $c_i(E) = c_i$, we have $h^1(E) \le \sum_{j=1}^3 D_j$ where $D_j = C_j(E^\vee, H)$.*

Proof By Serre duality, we have $H^{n-1}(E) = \operatorname{Ext}^{n-1}(\mathcal{O}_X, E) \cong \operatorname{Ext}^1(E, K_X)^\vee$. Hence, applying Theorem 5 to the sheaves E, $E_1 = K_X$, we obtain (1). If E is a vector bundle, then $H^1(E) \cong \operatorname{Ext}^1(E^\vee, \mathcal{O}_X)$. Hence we apply Theorem 3 to $E^\vee$ and $E_1 = \mathcal{O}_X$ and obtain (2). □

Let X be a smooth projective variety of dimension $n \ge 2$ and let H be a very ample line bundle on X. For a coherent sheaf E on X, the *Mukai vector* $v(E)$ of E is the element of the rational cohomology ring $H^*(X, \mathbb{Q}) := \oplus_{i=0}^4 H^{2i}(X, \mathbb{Q})$ defined as follows.

$$v(E) := \operatorname{ch}(E) \cdot \sqrt{\operatorname{td}(X)}$$

where $\operatorname{td}(X)$ denotes the Todd class of X. For given $v \in H^*(X, \mathbb{Q})$, let $\mathcal{M}(v)$ denote the moduli space of μ-stable torsion-free sheaves with Mukai vector v. Let $\mathcal{M}(v)_0 \subset \mathcal{M}(v)$ denote the open subscheme of μ-stable locally free sheaves. Let E_1 be a μ-stable rigid vector bundle on X. We define the *Brill-Noether locus* $\mathcal{M}(v)_{i,j}$ *of type* (i, j) as follows.

$$\mathcal{M}(v)_{i,j} := \{E \in \mathcal{M}(v) \mid i = \dim \operatorname{Hom}(E_1, E) \text{ and } j = \dim \operatorname{Ext}^1(E, E_1)\}.$$

We are interested in the *higher dimensional Brill-Noether problem* concerning the existence of these loci. Theorem 5 yields the following

Corollary 7 *Let X be a smooth projective variety of dimension $n \geq 3$ and let H be a very ample line bundle on X. Then $\mathcal{M}(v)_{i,j}$ is empty if $i \geq 0$ and $j > C$ where C is the constant in Theorem 5.*

In general, we have the following inequality ([3, Corollary 4.5.2])

$$\dim \mathcal{M}_{[E]}(v) \leq \dim \operatorname{Ext}^1(E, E).$$

We notice that effective bounds for $\dim \mathcal{M}_{[E]}(v)$ have been investigated for sheaves on a threefold in [7]. Applying Theorem 5 to $E_1 = E$, we obtain the following result in dimension $n \geq 3$.

Proposition 8 *Let X be a smooth projective variety of dimension $n \geq 3$ and let H be a very ample line bundle on X. For a μ-stable vector bundle $E \in \mathcal{M}(v)_0$ on X, let $l_i := l_i(\mathcal{E}nd E, H)$ $(1 \leq i \leq n-2)$ and let $C_j := C_j(\mathcal{E}nd E, H)$. Then we have $\dim \mathcal{M}_{[E]}(v) \leq \sum_{j=1}^{3} C_j$.*

4 Chern class inequalities on a fourfold

In this section we obtain an upper bound for the fourth Chern class $c_4(E)$ of μ-semistable bundles on a smooth projective fourfold. First, we recall the following Riemann-Roch formula for sheaves on a fourfold.

Lemma 9 *Let X be a smooth projective fourfold. Let E be a coherent sheaf of rank r with Chern classes c_i on X. Then*

$$\begin{aligned}\chi(E) &= \frac{1}{24}(c_1(E)^4 - 4c_1(E)^2 \cdot c_2(E) + 4c_1(E) \cdot c_3(E) + 2c_2(E)^2 - 4c_4(E)) \\ &\quad - \frac{1}{12}(c_1(E)^3 - 3c_1(E) \cdot c_2(E) + 3c_3(E)) \cdot K_X \\ &\quad + \frac{1}{24}(c_1(E)^2 - 2c_2(E)) \cdot (K_X^2 + c_2(X)) - \frac{1}{24}c_1(E) \cdot K_X \cdot c_2(X) + r\chi(\mathcal{O}_X).\end{aligned}$$

We make explicit the constants $l_i(E, H)$ and $c_j(E, H)$ introduced in the previous section for sheaves E on a fourfold.

Lemma 10 *Let X be a smooth projective fourfold and let H be a very ample line bundle on X. Let E be a torsion-free sheaf on X. Let $l_i := l_i(E, H)$ for $i = 1, 2$ and $C_j := C_j(E, H)$ for $1 \leq j \leq 3$. Then*

$$\begin{aligned}l_1 &= \max\{\lfloor \{12(\max\{\frac{r^2-1}{4}, 1\}H^4 + 1)\Delta_H(E)\}^{\frac{1}{4}} \rfloor, \lfloor \frac{\overline{\Delta}_H(E)}{H^4} - \frac{\mu_H(E)}{H^4} \rfloor\} + 1, \\ l_2 &= \max\{\lfloor \{3(\max\{\frac{r^4-1}{4}, 1\}l_1 H^4 + 1)l_1 \Delta_H(E)\}^{\frac{1}{3}} \rfloor, \lfloor \frac{l_1 \overline{\Delta}_H(E)}{H^4} - \frac{\mu_H(E)}{H^4} \rfloor\} + 1\end{aligned}$$

and

$$\begin{aligned}C_1 &= r l_1 H^4 \left(\frac{\frac{\mu_H(E) + (K_X + (l_1 + l_2)H) \cdot H^3}{H^4} + f(r) + 2}{2} \right), \\ C_2 &= r l_1 H^4 \left(\frac{-\frac{\mu_H(E)}{H^4} + f(r) + 2}{2} \right), \\ C_3 &= -l_1 l_2 H^2 \cdot \Big(\frac{l_2^2 H^2}{6} + \frac{l_2\{2c_1(E(K_X + l_1 H)) - r(K_X + l_1 H)\} \cdot H}{4} \\ &\quad + \frac{c_1(E(K_X + l_1 H)) \cdot \{c_1(E(K_X + l_1 H)) - (K_X + l_1 H)\}}{2} \\ &\quad - c_2(E(K_X + l_1 H)) + \frac{r\{(K_X + l_1 H)^2 + c_2(X) + l_1(K_X + l_1 H) \cdot H\}}{12} \Big).\end{aligned}$$

Proof Let $\iota : Y \hookrightarrow X$ denote the inclusion map. Since we have $K_Y = (K_X + l_1 H)_{|Y}$ and $c_2(Y) = \iota^*(c_2(X) + l_1(K_X + l_1)H) \cdot H)$, the claim follows immediately. □

Now we apply Theorem 5 to obtain a bound for the fourth Chern class of μ-semistable bundles on a fourfold.

Theorem 11 *Let X be a smooth projective fourfold and let H be a very ample line bundle on X. Let E be an H-semistable vector bundle E on X of rank $r \geq 2$, $c_i(E) = c_i$. Let $l_i = l_i(E(-K_X), H)$ and $m_i = l_i(E^\vee, H)$ for $i = 1, 2$. We define*

$$\begin{aligned} F = &\frac{1}{4}(c_1(E)^4 - 4c_1(E)^2 \cdot c_2(E) + 4c_1(E) \cdot c_3(E) + 2c_2(E)^2) \\ &- \frac{1}{2}(c_1(E)^3 - 3c_1(E) \cdot c_2(E) + 3c_3(E)) \cdot K_X \\ &+ \frac{1}{4}(c_1(E)^2 - 2c_2(E)) \cdot (K_X^2 + c_2(X)) - \frac{1}{4}c_1(E) \cdot K_X \cdot c_2(X) + 6r\chi(\mathcal{O}_X), \end{aligned}$$

$$\begin{aligned} C = &rl_1H^4\left\{\binom{\frac{\mu_H(E(-K_X)) + (K_X + (l_1+l_2)H) \cdot H^3}{H^4} + f(r) + 2}{2} + \binom{-\frac{\mu_H(E(-K_X))}{H^4} + f(r) + 2}{2}\right\} \\ &- l_1l_2H^2 \cdot \Big(\frac{l_2^2H^2}{6} + \frac{l_2\{2c_1(E(l_1H)) - r(K_X + l_1H)\} \cdot H}{4} \\ &+ \frac{c_1(E(l_1H) \cdot \{c_1(E(l_1H) - (K_X + l_1H)\}}{2} \\ &- c_2(E(l_1H) + \frac{r\{(K_X + l_1H)^2 + c_2(X) + l_1(K_X + l_1H) \cdot H\}}{12}\Big) \end{aligned}$$

and

$$\begin{aligned} D = &rm_1H^4\left\{\binom{\frac{\mu_H(E^\vee) + (K_X + (m_1+m_2)H) \cdot H^3}{H^4} + f(r) + 2}{2} + \binom{-\frac{\mu_H(E^\vee)}{H^4} + f(r) + 2}{2}\right\} \\ &- m_1m_2H^2 \cdot \Big(\frac{m_2^2H^2}{6} + \frac{m_2\{2c_1(E^\vee(K_X + m_1H)) - r(K_X + m_1H)\} \cdot H}{4} \\ &+ \frac{c_1(E^\vee(K_X + m_1H)) \cdot \{c_1(E^\vee(K_X + m_1H)) - (K_X + m_1H)\}}{2} \\ &- c_2(E^\vee(K_X + m_1H)) + \frac{r\{(K_X + m_1H)^2 + c_2(X) + m_1(K_X + m_1H) \cdot H\}}{12}\Big). \end{aligned}$$

Then we have $c_4(E) \leq F + 6(C + D)$.

Proof By Corollary 6, we have $h^3(E) \leq C := \sum_{j=1}^3 C_j$ and $h^1(E) \leq D := \sum_{j=1}^3 D_j$ where $C_j = C_j(E(-K_X), H)$, $D_j = C_j(E^\vee, H)$. Let $F = 6\chi(E) + c_4(E)$. This yields $\chi(E) \geq -(C + D)$ and hence

$$\begin{aligned} c_4(E) &= F - 6\chi(E) \\ &\leq F + 6(C + D). \end{aligned}$$

Therefore the claim follows from Lemma 9 and Lemma 10. □

We obtain the following bound in the case of abelian fourfolds.

Corollary 12 *Let X be an abelian fourfold and let H be a very ample line bundle on X. Let E be an H-semistable vector bundle on X of rank $r \geq 2$ on X. Let $l_i = l_i(E, H)$ and $m_i = l_i(E^\vee, H)$. Then we have*

$c_4(E) \le F + 6(C+D)$ *where*

$$F = \frac{1}{4}(c_1(E)^4 - 4c_1(E)^2c_2(E) + 4c_1(E)c_3(E) + 2c_2(E)^2),$$

$$C = rl_1H^4\left\{\binom{\frac{\mu_H(E)+(l_1+l_2)H^4}{H^4} + f(r) + 2}{2} + \binom{-\frac{\mu_H(E)}{H^4} + f(r) + 2}{2}\right\}$$

$$- l_1l_2H^2 \cdot \Big(\frac{l_2^2H^2}{6} + \frac{l_2\{2c_1(E(l_1H)) - rl_1H\} \cdot H}{4}$$

$$+ \frac{c_1(E(l_1H)) \cdot \{c_1(E(l_1H)) - l_1^2H\}}{2} - c_2(E(l_1H)) + \frac{rl_1^2H^2}{6}\Big),$$

$$D = rm_1H^4\left\{\binom{\frac{-\mu_H(E)+(m_1+m_2)H)\cdot H^3}{H^4} + f(r) + 2}{2} + \binom{\frac{\mu_H(E)}{H^4} + f(r) + 2}{2}\right\}$$

$$- m_1m_2H^2 \cdot \Big(\frac{m_2^2H^2}{6} - \frac{m_2\{2c_1(E^\vee(m_1H)) - rm_1H\} \cdot H}{4}$$

$$+ \frac{c_1(E^\vee(m_1H)) \cdot \{c_1(E^\vee(m_1H)) - m_1^2H\}}{2} - c_2(E^\vee(m_1H)) + \frac{rm_1^2H^2\}}{6}\Big).$$

We notice that there cannot exist an analogous upper bound for $c_4(E)$ for H-semistable *torsion-free sheaves* E on a smooth projective fourfold X. Indeed, the following result holds in arbitrary dimension.

Proposition 13 *Let X be a smooth projective variety of dimension $n \ge 2$ and H an ample line bundle on X. Assume that n is even (resp. odd). Then there does not exist an upper (resp. lower) bound for $c_n(E)$ for H-semistable torsion-free sheaves E on X in terms of r, c_i for $1 \le i \le n-1$, H and $c_i(X)$.*

Proof Let E be an H-semistable torsion-free sheaf on X of rank r, $c_i(E) = c_i$ and any point $p \in X$, let E_p denote the kernel of the natural evaluation map $E \to \mathcal{O}_p$. Then E_p is an H-semistable torsion-free sheaf of rank r, $c_i(E_p) = c_i$ for $1 \le i \le n-1$ and $c_n(E_p) = c_n(E) - (-1)^{n+1}(n-1)!$. Therefore the claim follows by choosing arbitrarily many points of X. □

Acknowledgements The author was supported by JSPS KAKENHI Grant Number JP20K03544.

References

1. A.Bayer, E.Macrì and P.Stellari, The space of stability conditions on Abelian threefolds and some Calabi-Yau threefolds, *Invent. Math.* **206** (2016), 869–933
2. A.Bayer, E.Macrì and Y.Toda, Bridgeland stability conditions on threefolds I: Bogomolov-Gieseker inequalities, *J. Algebraic Geom.* **23** (2014), 117–163
3. D.Huybrechts and M.Lehn, *The Geometry of Moduli Spaces of Sheaves, Second ed.*, Cambridge University Press, 2010
4. A.Langer, Moduli spaces of sheaves and principal G-bundles, *Proc. of Sympos. in Pure Math.* **80**(2010), 273–308
5. A.Langer, A note on restriction theorems for semistable sheaves, *Math. Res. Lett.* **17**(2010), 823–832
6. T.Nakashima, Higher Brill-Noether loci and Bogomolov-Gieseker inequality, *Archiv Math..* **109** (2017), 335–340
7. T.Nakashima, Effective bounds for semistable sheaves on a threefold, *J. Geom. and Phys.* **140**(2019), 271-279
8. H.Sun, Tilt-stability, vanishing theorems and Bogomolov-Gieseker type inequalities, *Adv. in Math* **347**(2019), 677–707