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ORIGINAL RESEARCH

Isolated toughness and path-factor uniform graphs (II)

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Abstract A spanning subgraph F of G is called a path-factor if each component of F is a path. A $P_{\geq k}$-factor of G means a path-factor such that each component is a path with at least k vertices, where $k \geq 2$ is an integer. A graph G is called a $P_{\geq k}$-factor covered graph if for each $e \in E(G)$, G has a $P_{\geq k}$-factor covering e. A graph G is called a $P_{\geq k}$-factor uniform graph if for any two different edges $e_1, e_2 \in E(G)$, G has a $P_{\geq k}$-factor covering e_1 and avoiding e_2. In other word, a graph G is called a $P_{\geq k}$-factor uniform graph if for any $e \in E(G)$, the graph $G - e$ is a $P_{\geq k}$-factor covered graph. In this article, we demonstrate that (i) an $(r+3)$-edge-connected graph G is a $P_{\geq 2}$-factor uniform graph if its isolated toughness $I(G) > \frac{r+3}{2r+3}$, where r is a nonnegative integer; (ii) an $(r+3)$-edge-connected graph G is a $P_{\geq 3}$-factor uniform graph if its isolated toughness $I(G) > \frac{3r+6}{2r+3}$, where r is a nonnegative integer. Furthermore, we claim that these conditions on isolated toughness and edge-connectivity in our main results are best possible in some sense.

Keywords Graph · Isolated toughness · Edge-connectivity · Path-factor · Path-factor uniform graph.

Mathematics Subject Classification 05C70 · 05C38

1 Introduction

The graphs discussed here are finite, undirected and simple. Let G be a graph. The vertex set and the edge set of G are denoted by $V(G)$ and $E(G)$, respectively. Denote by $d_G(v)$ the degree of a vertex v in G. Let $i(G)$ and $\omega(G)$ denote the number of isolated vertices and connected components in G, respectively. For any $X \subseteq V(G)$, $G[X]$ is the subgraph of G induced on X, and $G - X$ denotes the subgraph $G[V(G) \setminus X]$. For any $E' \subseteq E(G)$, $G - E'$ denotes the subgraph derived from G by removing E'. A vertex subset X of G is called an independent set if $G[X]$ has no edges. The path and the complete graph with n vertices are denoted by P_n and K_n, respectively. We use $G_1 \vee G_2$ to denote the join of two disjoint graphs G_1 and G_2, and $G_1 \cup G_2$ to denote the union of two disjoint graphs G_1 and G_2.

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Yang, Ma and Liu [14] introduced the notion of isolated toughness, which is defined by

$$I(G) = \min\{\frac{|X|}{i(G-X)} : X \subseteq V(G), i(G-X) \geq 2\}$$

if G is not a complete graph; otherwise, $I(G) = +\infty$.

A spanning subgraph F of G is called a path-factor if each component of F is a path. A $P_{\geq k}$-factor of G means a path-factor such that each component is a path with at least k vertices, where $k \geq 2$ is an integer.

Johnson et al. [7] studied the existence of path-factors in graphs. Egawa and Furuya [3] posed some sufficient conditions for a graph to admit a path-factor. Asratian and Casselgren [2] derived a sufficient condition for the existence of path-factors in graphs. Ando et al. [1] verified that a claw-free graph with minimum degree at least k admits a $P_{\geq k+1}$-factor. Kano, Lee and Suzuki [10] proved that every connected cubic bipartite graph with at least 8 vertices admits a $P_{\geq 8}$-factor. Johansson [6] gave an El-Zahár type condition ensuring path-factors in graphs. Kano, Lu and Yu [11] claimed that a graph G with $i(G-X) \leq \frac{2}{3}|X|$ for any $X \subseteq V(G)$ admits a $P_{\geq 3}$-factor. Zhou [17–19], Zhou, Bian and Pan [20], Zhou, Sun and Liu [23], Zhou, Wu and Bian [24], Zhou, Wu and Xu [25],and Gao, Wang and Chen [5] presented some sufficient conditions for graphs admitting $P_{\geq 3}$-factors with given properties. Some other results on graph factors see [13,16,21]. Las Vergnas [12] showed a criterion for a graph with a $P_{\geq 2}$-factor.

Theorem 1 *([12]). A graph G has a $P_{\geq 2}$-factor if and only if*

$$i(G-X) \leq 2|X|$$

for any $X \subseteq V(G)$.

A graph R is called a factor-critical graph if for any $v \in V(R)$, $R - v$ has a 1-factor. A graph H is called a sun if $H = K_1$, $H = K_2$ or H is the corona of a factor-critical graph R with at least three vertices, namely, H is derived from R by adding a new vertex $u = u(v)$ together with a new edge vu for each $v \in V(R)$. A sun with at least six vertices is called a big sun. The number of sun components of G is denoted by $sun(G)$. Kaneko [8] presented a characterization of a graph with a $P_{\geq 3}$-factor, and Kano, Katona and Király [9] gave a shorter proof.

Theorem 2 *([8,9]). A graph G has a $P_{\geq 3}$-factor if and only if*

$$sun(G-X) \leq 2|X|$$

for any $X \subseteq V(G)$.

A graph G is called a $P_{\geq k}$-factor covered graph if for each $e \in E(G)$, G has a $P_{\geq k}$-factor covering e, which was first defined by Zhang and Zhou [15]. Furthermore, they put forward two characterizations for a graph to a $P_{\geq 2}$-factor covered graph and $P_{\geq 3}$-factor covered graph, which are stated as follows.

Theorem 3 *([15]). A connected graph G is a $P_{\geq 2}$-factor covered graph if and only if*

$$i(G-X) \leq 2|X| - \varepsilon_1(X)$$

for any $X \subseteq V(G)$, where $\varepsilon_1(X)$ is defined by

$$\varepsilon_1(X) = \begin{cases} 2, & if\ X\ is\ not\ an\ independent\ set; \\ 1, & if\ X\ is\ a\ nonempty\ independent\ set,\ and\ G-X\ admits \\ & \quad a\ nontrivial\ component; \\ 0, & otherwise. \end{cases}$$

Theorem 4 *([15]). A connected graph G is a $P_{\geq 3}$-factor covered graph if and only if*

$$sun(G-X) \leq 2|X| - \varepsilon_2(X)$$

for any subset X of $V(G)$, where $\varepsilon_2(X)$ is defined by

$$\varepsilon_2(X) = \begin{cases} 2, & if\ X\ is\ not\ an\ independent\ set; \\ 1, & if\ X\ is\ a\ nonempty\ independent\ set,\ and\ G-X\ admits \\ & \quad a\ non-sun\ component; \\ 0, & otherwise. \end{cases}$$

A graph G is called a $P_{\geq k}$-factor uniform graph if for any two different edges $e_1, e_2 \in E(G)$, G has a $P_{\geq k}$-factor covering e_1 and avoiding e_2. In other word, a graph G is called a $P_{\geq k}$-factor uniform graph if for any $e \in E(G)$, the graph $G - e$ is a $P_{\geq k}$-factor covered graph. Gao and Wang [4] derived a result on the existence of $P_{\geq 3}$-factor uniform graphs. Zhou, Sun and Liu [22] posed two isolated toughness conditions for a graph to be a $P_{\geq 2}$-factor uniform graph and $P_{\geq 3}$-factor uniform graph, which are stated as follows.

Theorem 5 *([22]). A 3-edge-connected graph G is a $P_{\geq 2}$-factor uniform graph if its isolated toughness $I(G) > 1$.*

Theorem 6 *([22]). A 3-edge-connected graph G is a $P_{\geq 3}$-factor uniform graph if its isolated toughness $I(G) > 2$.*

In Theorems 5 and 6, the conditions on $I(G)$ are sharp. However, it is natural to expect that we can weaken the condition on $I(G)$ if we replace the assumption that G is 3-edge-connected by a stronger assumption. Along this line, we derive the following results.

Theorem 7 *Let r be a nonnegative integer. An $(r+3)$-edge-connected graph G is a $P_{\geq 2}$-factor uniform graph if its isolated toughness $I(G) > \frac{r+3}{2r+3}$.*

Theorem 8 *Let r be a nonnegative integer. An $(r+3)$-edge-connected graph G is a $P_{\geq 3}$-factor uniform graph if its isolated toughness $I(G) > \frac{3r+6}{2r+3}$.*

2 The proof of Theorem 7

Proof of Theorem 7 Theorem 7 holds obviously for a complete graph. Hence, we may assume that G is not a complete graph.

We proceed by contradiction. Assume that there exists an edge $e = uv$ in G such that $G' = G - e$ is not a $P_{\geq 2}$-factor covered graph. It follows from Theorem 3 that

$$i(G' - X) \geq 2|X| - \varepsilon_1(X) + 1 \tag{1}$$

for some vertex subset X of G'.

The following proof will be divided into three cases by the value of $|X|$.

Case 1. $0 \leq |X| \leq r + 1$.

According to (1) and $\varepsilon_1(X) \leq |X|$, we obtain

$$i(G' - X) \geq 2|X| - \varepsilon_1(X) + 1 \geq |X| + 1,$$

which implies that $G' - X$ has at least one isolated vertex w, and so $d_{G'-X}(w) = 0$. Thus, we have

$$d_G(w) \leq d_{G'}(w) + 1 \leq d_{G'-X}(w) + |X| + 1 = |X| + 1 \leq r + 2,$$

which contradicts that G is $(r+3)$-edge-connected.

Case 2. $|X| = r + 2$.

In terms of (1) and $\varepsilon_1(X) \leq 2$, we admit

$$i(G' - X) \geq 2|X| - \varepsilon_1(X) + 1 \geq 2|X| - 1. \tag{2}$$

Note that $i(G - X) \geq i(G' - X) - 2$. Combining this with (2), we get

$$i(G - X) \geq i(G' - X) - 2 \geq 2|X| - 3 = 2(r+2) - 3 = 2r + 1,$$

which implies that there exists $w \in V(G - X)$ with $d_{G-X}(w) = 0$. Since G is $(r+3)$-edge-connected, we have

$$r + 3 \leq d_G(w) \leq d_{G-X}(w) + |X| = |X| = r + 2,$$

which is a contradiction.

Case 3. $|X| \geq r + 3$.

Note that $i(G-X) \geq i(G'-X)-2$. It follows from (1) and $\varepsilon_1(X) \leq 2$ that

$$i(G-X) \geq i(G'-X)-2 \geq 2|X|-\varepsilon_1(X)+1-2 \geq 2|X|-3 \geq 2r+3. \tag{3}$$

In light of (3) and the definition of $I(G)$, we derive

$$\begin{aligned} I(G) &\leq \frac{|X|}{i(G-X)} \leq \frac{|X|}{2|X|-3} = \frac{1}{2} + \frac{3}{4|X|-6} \\ &\leq \frac{1}{2} + \frac{3}{4(r+3)-6} = \frac{1}{2} + \frac{3}{4r+6} \\ &= \frac{r+3}{2r+3}, \end{aligned}$$

which contradicts $I(G) > \frac{r+3}{2r+3}$. The proof of Theorem 7 is complete. □

Remark 1 We now show that $I(G) > \frac{r+3}{2r+3}$ in Theorem 7 cannot be weakened to $I(G) \geq \frac{r+3}{2r+3}$.

We construct a graph $G = K_{r+3} \vee ((2r+3)K_1) \cup K_2$, where r is a nonnegative integer. Then $I(G) = \frac{|V(K_{r+3})|}{i(G-V(K_{r+3}))} = \frac{r+3}{2r+3}$ and G is $(r+3)$-edge-connected. Write $G' = G - e$ for $e \in E(K_2)$. Let $X = V(K_{r+3})$. Then $|X| = r+3$ and $\varepsilon_1(X) = 2$. Thus, we admit

$$i(G'-X) = 2r+5 > 2r+4 = 2|X| - \varepsilon_1(X).$$

In light of Theorem 3, G' is not a $P_{\geq 2}$-factor covered graph, and so G is not a $P_{\geq 2}$-factor uniform graph.

Remark 2 The condition that G is $(r+3)$-edge-connected in Theorem 7 cannot be replaced by G being $(r+2)$-edge-connected.

To show this, we construct a graph $G = K_{r+2} \vee ((2r+1)K_1 \cup K_2)$, where $r \geq 1$ is an integer. Then G is $(r+2)$-edge-connected and $I(G) = \frac{|V(K_{r+2})|}{i(G-V(K_{r+2}))} = \frac{r+2}{2r+1} > \frac{r+3}{2r+3}$. Let $G' = G - e$ for $e \in E(K_2)$ and $X = V(K_{r+2})$. Then $\varepsilon_1(X) = 2$, and so

$$i(G'-X) = 2r+3 > 2r+2 = 2|X| - \varepsilon_1(X).$$

According to Theorem 3, G' is not a $P_{\geq 2}$-factor covered graph, and so G is not a $P_{\geq 2}$-factor uniform graph.

3 The proof of Theorem 8

Proof of Theorem 8 Theorem 8 is true for a complete graph. Therefore, we may assume that G is not a complete graph.

Theorem 8 holds for $r = 0$ by Theorem 6. In what follows, we may assume $r \geq 1$. We proceed by contradiction. Assume that there exists an edge $e = uv$ in G such that $G' = G - e$ is not a $P_{\geq 3}$-factor covered graph. Then by Theorem 4 we obtain

$$sun(G'-X) \geq 2|X| - \varepsilon_2(X) + 1 \tag{4}$$

for some vertex subset X of G'.

Claim 1. $|X| \geq r+2$.

Proof If $0 \leq |X| \leq r$, then it follows from (4) and $\varepsilon_2(X) \leq |X|$ that

$$sun(G'-X) \geq 2|X| - \varepsilon_2(X) + 1 \geq |X| + 1 \geq 1,$$

which implies that there exists $w \in V(G'-X)$ such that $d_{G'-X}(w) \leq 1$. Thus, we derive

$$d_G(w) \leq d_{G'}(w) + 1 \leq d_{G'-X}(w) + |X| + 1 \leq |X| + 2 \leq r+2,$$

which contradicts that G is $(r+3)$-edge-connected.

If $|X| = r+1$, the by (4), $\varepsilon_2(X) \leq 2$ and $r \geq 1$, we have

$$sun(G-X) \geq sun(G'-X) - 2 \geq 2|X| - \varepsilon_2(X) + 1 - 2 \geq 2|X| - 3 = 2r - 1 \geq 1,$$

which implies that there exists $w \in V(G-X)$ with $d_{G-X}(w) \leq 1$, and so

$$d_G(w) \leq d_{G-X}(w) + |X| \leq |X| + 1 = r + 2,$$

which contradicts that G is $(r+3)$-edge-connected. Hence, $|X| \geq r+2$. This completes the proof of Claim 1. □

Assume that $G'-X$ admits a isolated vertices, b K_2's and c big sun components $H_1, H_2, \cdots, H_c$, where $|V(H_i)| \geq 6$ for $1 \leq i \leq c$. In terms of (4), $\varepsilon_2(X) \leq 2$ and Claim 1, we infer

$$sun(G'-X) = a + b + c \geq 2|X| - \varepsilon_2(X) + 1 \geq 2|X| - 1 = 2r + 3. \tag{5}$$

Write R_i for the factor-critical graph of H_i, $i = 1, 2, \cdots, c$. Set $D_i = V(R_i)$ and $D = \bigcup_{i=1}^{c} D_i$. Obviously, $i(H_i - D_i) = |D_i| = \frac{|V(H_i)|}{2} \geq 3$ and $|D| \geq 3c$. Select one vertex from each K_2 component of $G'-X$, and denote the set of such vertices by Q. We denote by W the union of all non-sun components of $G'-X$.

Note that $i(G'-X) - 2 \leq i(G-X) \leq i(G'-X)$. The following proof will be divided into three cases.

Case 1. $i(G-X) = i(G'-X)$.

Clearly, $u, v \notin V(aK_1)$ (otherwise, $i(G-X) < i(G'-X)$, a contradiction).

Subcase 1.1. $u \in V(W)$.

In this subcase, we get $i(G-X-Q-D-u) \geq a + b + |D|$, and so

$$I(G) \leq \frac{|X \cup Q \cup D \cup \{u\}|}{i(G-X-Q-D-u)} \leq \frac{|X| + b + |D| + 1}{a + b + |D|}.$$

Combining this with $I(G) > \frac{3r+6}{2r+3}$, we obtain

$$\frac{3r+6}{2r+3} < I(G) \leq \frac{|X| + b + |D| + 1}{a + b + |D|},$$

which implies

$$0 > (3r+6)a + (r+3)b + (r+3)|D| - (2r+3)|X| - (2r+3). \tag{6}$$

It follows from (5), (6), $|D| \geq 3c$ and Claim 1 that

$$\begin{aligned}
0 &> (3r+6)a + (r+3)b + (r+3)|D| - (2r+3)|X| - (2r+3) \\
&\geq (r+3)a + (r+3)b + (r+3)c - (2r+3)|X| - (2r+3) \\
&= (r+3)(a+b+c) - (2r+3)|X| - (2r+3) \\
&\geq (r+3)(2|X|-1) - (2r+3)|X| - (2r+3) \\
&= 3|X| - (3r+6) = 3(r+2) - (3r+6) = 0,
\end{aligned}$$

a contradiction.

Subcase 1.2. $u \notin V(W)$.

In this subcase, $u \in V(bK_2)$, or $u \in D_i$ $(1 \leq i \leq c)$, or $u \in V(H_i) \setminus D_i$ $(1 \leq i \leq c)$.

Claim 2. $I(G) \leq \frac{|X|+b+|D|}{a+b+|D|}$.

Proof If $u \in V(bK_2)$, then we choose such set Q with $u \in Q$. Thus, we get $i(G-X-Q-D) = a + b + |D|$, and so

$$I(G) \leq \frac{|X \cup Q \cup D|}{i(G-X-Q-D)} = \frac{|X| + b + |D|}{a + b + |D|}.$$

If $u \in D_i$ $(1 \leq i \leq c)$, then we admit $i(G-X-Q-D) = a + b + |D|$, and so

$$I(G) \leq \frac{|X \cup Q \cup D|}{i(G-X-Q-D)} = \frac{|X| + b + |D|}{a + b + |D|}.$$

If $u \in V(H_i) \setminus D_i$ $(1 \le i \le c)$, then there exists $w \in D_i$ with $uw \in E(H_i)$. Thus, we derive $i(G - X - Q - u - (D \setminus \{w\})) = a + b + |D|$, and so

$$I(G) \le \frac{|X \cup Q \cup \{u\} \cup (D \setminus \{w\})|}{i(G - X - Q - u - (D \setminus \{w\}))} = \frac{|X| + b + |D|}{a + b + |D|}.$$

We finish the proof of Claim 2.

According to Claim 2 and $I(G) > \frac{3r+6}{2r+3}$, we acquire

$$\frac{3r+6}{2r+3} < I(G) \le \frac{|X| + b + |D|}{a + b + |D|},$$

namely,

$$0 > (3r+6)a + (r+3)b + (r+3)|D| - (2r+3)|X|. \tag{7}$$

It follows from (5), (7), $|D| \ge 3c$ and Claim 1 that

$$\begin{aligned} 0 &> (3r+6)a + (r+3)b + (r+3)|D| - (2r+3)|X| \\ &\ge (r+3)(a+b+c) - (2r+3)|X| \\ &\ge (r+3)(2|X|-1) - (2r+3)|X| \\ &= 3|X| - (r+3) = 3(r+2) - (r+3) = 2r+3 > 0, \end{aligned}$$

which is a contradiction.

Case 2. $i(G - X) = i(G' - X) - 1$.

In this case, $u \in V(aK_1)$ and $v \notin V(aK_1)$, or $u \notin V(aK_1)$ and $v \in V(aK_1)$. Without loss of generality, let $u \in V(aK_1)$ and $v \notin V(aK_1)$. Obviously, $a \ge 1$.

Claim 3. $I(G) \le \frac{|X|+b+|D|+1}{a+b+|D|}$.

Proof The proof is similar to that of Claim 2 by discussing $v \in V(bK_2)$, $v \in D_i$, $v \in V(H_i) \setminus D_i$ or $v \in V(W)$. Claim 3 is verified. □

According to Claim 3 and $I(G) > \frac{3r+6}{2r+3}$, we have

$$\frac{3r+6}{2r+3} < I(G) \le \frac{|X| + b + |D| + 1}{a + b + |D|},$$

that is,

$$(2r+3)|X| > (3r+6)a + (r+3)b + (r+3)|D| - (2r+3). \tag{8}$$

In terms of (5), (8), $a \ge 1$ and $|D| \ge 3c$, we deduce

$$\begin{aligned} (2r+3)|X| &> (3r+6)a + (r+3)b + (r+3)|D| - (2r+3) \\ &\ge (3r+6)a + (r+3)b + (r+3)c - (2r+3) \\ &= (r+3)(a+b+c) + (2r+3)(a-1) \\ &\ge (r+3)(2|X|-1), \end{aligned}$$

which implies $|X| < \frac{r+3}{3}$, which contradicts Claim 1.

Case 3. $i(G - X) = i(G' - X) - 2$.

In this case, $u, v \in V(aK_1)$, and so $a \ge 2$. Thus, we possess $i(G - X - Q - v - D) = a + b + |D| - 1$, and so

$$I(G) \le \frac{|X \cup Q \cup \{v\} \cup D|}{i(G - X - Q - v - D)} = \frac{|X| + b + |D| + 1}{a + b + |D| - 1}. \tag{9}$$

In light of (9) and $I(G) > \frac{3r+6}{2r+3}$, we derive

$$\frac{3r+6}{2r+3} < I(G) \le \frac{|X| + b + |D| + 1}{a + b + |D| - 1},$$

which implies

$$(2r + 3)|X| > (3r + 6)a + (r + 3)b + (r + 3)|D| - 5r - 9. \tag{10}$$

It follows from (5), (10), $a \geq 2$ and $|D| \geq 3c$ that

$$\begin{aligned}(2r + 3)|X| &> (3r + 6)a + (r + 3)b + (r + 3)|D| - 5r - 9\\ &\geq (3r + 6)a + (r + 3)b + (r + 3)c - 5r - 9\\ &= (r + 3)(a + b + c) + (2r + 3)a - 5r - 9\\ &\geq (r + 3)(2|X| - 1) + 2(2r + 3) - 5r - 9\\ &= 2(r + 3)|X| - 2r - 6,\end{aligned}$$

which implies $|X| < \frac{2r+6}{3}$, which contradicts Claim 1 by $r \geq 1$. Theorem 8 is verified. □

Remark 3 In what follows, we claim that $I(G) > \frac{3r+6}{2r+3}$ in Theorem 8 cannot be replaced by $I(G) > \frac{3r+5}{2r+3}$.

To explain this, we construct a graph $G = K_{r+3} \vee ((2r + 4)K_2)$, where r is a nonnegative integer. We select one vertex from each K_2 component of $G - V(K_{r+3})$, and denote the set of such vertices by Y. Then $\frac{3r+6}{2r+3} > I(G) = \frac{|V(K_{r+3}) \cup Y|}{i(G-(V(K_{r+3}) \cup Y))} = \frac{3r+7}{2r+4} > \frac{3r+5}{2r+3}$ and G is $(r + 4)$-edge-connected. Let $G' = G - e$ for $e \in E((2r + 4)K_2)$. Let $X = V(K_{r+3})$. Then $|X| = r + 3$ and $\varepsilon_2(X) = 2$. Thus, we obtain

$$sun(G' - X) = 2r + 5 > 2r + 4 = 2|X| - \varepsilon_2(X).$$

In view of Theorem 4, G' is not a $P_{\geq 3}$-factor covered graph, and so G is not a $P_{\geq 3}$-factor uniform graph.

Remark 4 We now claim that 3-edge-connected in Theorem 8 is best possible in some sense.

To show this, we construct a graph $G = K_{r+1} \vee ((2r)K_2)$, where r is an integer with $1 \leq r \leq 2$. We select one vertex from each K_2 component of $G - V(K_{r+1})$, and denote the set of such vertices by Y. Then G is 2-edge-connected and $I(G) = \frac{|V(K_{r+1}) \cup Y|}{i(G-(V(K_{r+1}) \cup Y))} = \frac{3r+1}{2r} > \frac{3r+6}{2r+3}$. Write $G' = G - e$ for $e \in E((2r)K_2)$ and $X = V(K_{r+1})$. Then $|X| = r + 1$ and $\varepsilon_2(X) = 2$. Hence, we derive

$$sun(G' - X) = 2r + 1 > 2r = 2|X| - \varepsilon_2(X).$$

By virtue of Theorem 4, G' is not a $P_{\geq 3}$-factor covered graph, and so G is not a $P_{\geq 3}$-factor uniform graph.

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