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ORIGINAL RESEARCH

Duality and interpolation for symmetric Banach spaces of noncommutative quasi-martingales

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Abstract Let E be a symmetric Banach space with the Fatou property and $1 < p_E \leq q_E < p$. We prove the duality for symmetric Banach space ${}_p\widehat{E}(\mathcal{M})$ which is a kind of noncommutative quasi-martingale space. As its applications, we discuss concrete description of the symmetric Banach space ${}_p\widehat{E}(\mathcal{M})$ as interpolations of quasi-martingale L_p-spaces.

Keywords Symmetric Banach space · Martingale · Hardy space

Mathematics Subject Classification 46L53 · 46L52 · 60G42

1 Introduction

The theory of noncommutative symmetric spaces has been rapidly developed. Many of the noncommutative martingale results have been transferred to the noncommutative symmetric case. Especially, in [1], J. Yong proved Burkholder-Gundy inequalities for symmetric Banach spaces of noncommutative martingales. In [9], T. N. Bekjan proved the duality for conditional Hardy spaces of martingales in noncommutative symmetric Banach spaces.

The quasi-martingales are generalizations of martingales and play important roles in many different areas of mathematics. In [15], we studied duality theorems for L_p-spaces of noncommutative quasi-martingales. In this paper, we will extend the above results to the noncommutative symmetric case. Let E be a symmetric Banach space on [0, 1] with the Fatou property and $1 < p_E \leq q_E < p$. Then

$$({}_p\widehat{E}(\mathcal{M}))^* = {}_{p'}\widehat{E}^{\times}(\mathcal{M}),$$

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where ${}_p\widehat{E}(\mathcal{M})$ and ${}_{p'}\widehat{E}^{\times}(\mathcal{M})$ denote the symmetric Banach spaces of noncommutative quasi-martingales which we refer to the next section for formal definitions. As applications of this result, we obtain the description of the symmetric space ${}_p\widehat{E}(\mathcal{M})$ as interpolations of noncommutative quasi-martingale L_p-spaces.

The organization of the paper is as follows. In Section 2, we give some preliminaries and notations on symmetric Banach spaces, quasi-martingale spaces and interpolations. We prove the main results in Section 3.

2 Preliminaries

Let E be a symmetric Banach space on [0, 1]. The Köthe dual of E is the function space defined by setting:

$$E^{\times} = \{f \in L_0([0,1]) : \int_0^1 |f(t)g(t)|dt < \infty, \forall g \in E\}.$$

When equipped with the norm $\|f\|_{E^{\times}} := \sup\{\int_0^1 |f(t)g(t)|dt : \|g\|_E \le 1\}$, $E^{\times}$ is a symmetric Banach space.

A symmetric Banach space E on [0, 1] is said to have the Fatou property if for every sequence $(x_n)_n$ in E satisfying $0 \le x_n \uparrow$ and $\sup_n \|x_n\|_E < \infty$, the supremum $x = \sup_n x_n$ belongs to E and $\|x_n\|_E \uparrow \|x\|_E$. Note that E has the Fatou property if and only if $E = E^{\times\times}$ isometrically. Examples of symmetric spaces with the Fatou property are separable symmetric spaces and duals of separable symmetric spaces.

For any $s > 0$ we define the dilation operator D_s on $L_0[0, 1]$ by

$$(D_s f)(t) = f(st)\chi_{[0,1]}(st), \quad t \in [0,1].$$

If E is a symmetric Banach space on [0, 1], then D_s is a bounded linear operator. Define the lower and upper Boyd indices of E by

$$p_E := \lim_{s\to\infty} \frac{\log s}{\log \|D_s\|} \quad \text{and} \quad q_E := \lim_{s\to 0^+} \frac{\log s}{\log \|D_s\|},$$

respectively. It is well known that $1 \le p_E \le q_E \le \infty$ and E has non-trivial Boyd indices, whenever $1 < p_E \le q_E < \infty$. We shall need the following duality for Boyd indices:

$$\frac{1}{p_E} + \frac{1}{q_{E^{\times}}} = 1, \quad \frac{1}{q_E} + \frac{1}{p_{E^{\times}}} = 1.$$

Let E be a symmetric Banach space on [0, 1]. For $0 < r < \infty$, we define $E^{(r)}$ and $E_{(r)}$ by

$$E^{(r)} := \{x : |x|^r \in E\}, \quad \|x\|_{E^{(r)}} := \big\||x|^r\big\|_E^{\frac{1}{r}},$$
$$E_{(r)} := \{x : |x|^{\frac{1}{r}} \in E\}, \quad \|x\|_{E_{(r)}} := \big\||x|^{\frac{1}{r}}\big\|_E^{r},$$

respectively. It is clear from the definitions that $E^{(r)}$, $E_{(r)}$ are symmetric and

$$p_{E_{(r)}} = \frac{1}{r}p_E, \quad q_{E_{(r)}} = \frac{1}{r}q_E, \quad p_{E^{(r)}} = rp_E, \quad q_{E^{(r)}} = rq_E.$$

Let E_i be a quasi Banach idea space on [0, 1], $i = 1, 2$. The pointwise product space of E_1 and E_2 is defined as

$$E_1 \odot E_2 = \{x : x = x_1x_2, x_i \in E_i, i = 1,2\}$$

with a functional $\|x\|_{E_1\odot E_2}$ defined by

$$\|x\|_{E_1\odot E_2} = \inf\{\|x\|_{E_1}\|x\|_{E_2} : x = x_1x_2, x_i \in E_i, i = 1,2\}.$$

Note that if E and F are symmetric Banach spaces on [0, 1], then we have the following results (see Theorem 1 in [1]).

(i) If $0 < p < \infty$, then $(E \odot F)^{(p)} = E^{(p)} \odot F^{(p)}$.
(ii) If $1 < p < \infty$, then $(E^{(p)})^{\times} = (E^{\times})^{(p)} \odot L_{p'}[0,1]$.

Let $\mathcal{M}$ be a semi-finite von Neumann algebra with a faithful normal semi-finite trace τ. The set of all τ-measurable operators is denoted by $L_0(\mathcal{M})$. For $x \in L_0(\mathcal{M})$, define its generalized singular number by

$$\mu_t(x) = \inf\{\lambda > 0 : \tau(\chi_{(\lambda,\infty)}(|x|)) \leq t\}, \quad t > 0.$$

For a given symmetric Banach function space E on $[0, 1]$, we define the corresponding noncommutative space by setting:

$$E(\mathcal{M}, \tau) = \{x \in L_0(\mathcal{M}) : \mu_t(x) \in E\}.$$

Equipped with the norm $\|x\|_{E(\mathcal{M},\tau)} := \|\mu_t(x)\|_E$, the space $E(\mathcal{M}, \tau)$ is a Banach space and is referred to as the noncommutative symmetric Banach space associated with $(\mathcal{M}, \tau)$ corresponding to the function space $(E, \|\cdot\|_E)$. Note that if $1 \leq p < \infty$ and $E = L_p([0, 1])$, then $E(\mathcal{M}, \tau) = L_p(\mathcal{M}, \tau)$ is the usual noncommutative L_p-space associated with $(\mathcal{M}, \tau)$.

2.1 Noncommutative quasi-martingales

We first recall the general setup for noncommutative martingales. Let $(\mathcal{M}_n)_{n\geq1}$ be an increasing sequence of von Neumann subalgebras of $\mathcal{M}$ such that the union of the $\mathcal{M}_n$'s is weak*-dense in $\mathcal{M}$. For every $n \geq 1$, the restriction $\tau|_{\mathcal{M}_n}$ of τ to $\mathcal{M}_n$ remains semi-finite, still denoted by τ, and we assume that there exists a trace preserving conditional expectation $\mathcal{E}_n$ from $\mathcal{M}$ onto $\mathcal{M}_n$. In this case, $(\mathcal{M}_n)_{n\geq1}$ is called a filtration of $\mathcal{M}$. Note that $\mathcal{E}_n$ extends to a contractive projection from $L_p(\mathcal{M})$ onto $L_p(\mathcal{M}_n)$ for all $1 \leq p \leq \infty$. A noncommutative $E(\mathcal{M})$-martingale with respect to $(\mathcal{M}_n)_{n\geq1}$ is a sequence $x = (x_n)_{n\geq1}$ such that $x_n \in E(\mathcal{M}_n)$ and $\mathcal{E}_n(x_{n+1}) = x_n$ for any $n \geq 1$. Let $\|x\|_{E(\mathcal{M})} = \sup_{n\geq1} \|x_n\|_{E(\mathcal{M})}$. If $\|x\|_{E(\mathcal{M})} < \infty$, then x is called a bounded $E(\mathcal{M})$-martingale. The martingale difference sequence $dx = (dx_n)_{n\geq1}$ of x is defined by $dx_n = x_n - x_{n-1}$ for $n \geq 1$. Here and in the following, we set $x_0 = 0$ and $\mathcal{E}_0 = \mathcal{E}_1$ for the sake of convenience.

In this paper, we are concerned with the following quasi-martingales in noncommutative symmetric Banach spaces.

Definition 2.1 Let E be a symmetric Banach space on $[0, 1]$ and $1 \leq p \leq \infty$. A noncommutative ${}_pE(\mathcal{M})$-quasi-martingale with respect to $(\mathcal{M}_n)_{n\geq1}$ is a sequence $x = (x_n)_{n\geq1}$ such that $x_n \in E(\mathcal{M}_n)$ for $n \geq 1$ and (with $\mathcal{E}_0 = 0$, $x_0 = 0$)

$$\sum_{n=1}^{\infty} \|\mathcal{E}_{n-1}(dx_n)\|^p_{E(\mathcal{M})} < \infty.$$

Let $y_n = \sum_{k=1}^{n} (dx_k - \mathcal{E}_{k-1}(dx_k))$ for $n \geq 1$. We set

$$\|x\|_{{}_p\widehat{E}(\mathcal{M})} := \sup_n \|y_n\|_{E(\mathcal{M})} + (\sum_{n=1}^{\infty} \|\mathcal{E}_{n-1}(dx_n)\|^p_{E(\mathcal{M})})^{\frac{1}{p}}.$$

If $\|x\|_{{}_p\widehat{E}(\mathcal{M})} < \infty$, then x is called a bounded ${}_pE(\mathcal{M})$-quasi-martingale. The quasi-martingale space ${}_p\widehat{E}(\mathcal{M})$ is defined as the space of all bounded ${}_pE(\mathcal{M})$-quasi-martingales, equipped with the norm $\|\cdot\|_{{}_p\widehat{E}(\mathcal{M})}$. We remark that if $1 \leq q \leq \infty$ and $E = L_q([0, 1])$ then ${}_p\widehat{E}(\mathcal{M}) = {}_p\widehat{L_q}(\mathcal{M})$, where ${}_p\widehat{L_q}(\mathcal{M})$ consists of $x = (x_n)_{n\geq1} \subset L_q(\mathcal{M})$ for which

$$\|x\|_{{}_p\widehat{L_q}(\mathcal{M})} = \sup_n \|y_n\|_{L_q(\mathcal{M})} + (\sum_{n=1}^{\infty} \|\mathcal{E}_{n-1}(dx_n)\|^p_{L_q(\mathcal{M})})^{\frac{1}{p}}.$$

Now we define the noncommutative space ${}_pG_E(\mathcal{M})$ which is used in the proof of our main results.

Definition 2.2 Let E be a symmetric Banach space on [0, 1] and $1 \leq p \leq \infty$. The noncommutative space ${}_pG_E(\mathcal{M})$ is defined as the subspace of $l_p(E(\mathcal{M}))$ consisting of all sequences $dx = (dx_n)_{n\geq 1}$ such that $x = (x_n)_{n\geq 1}$ is a predictable ${}_pE(\mathcal{M})$-quasi-martingale with $x_1 = 0$, and is equipped with the norm

$$\|x\|_{{}_pG_E(\mathcal{M})} = (\sum_{n=1}^{\infty} \|dx_n\|_{E(\mathcal{M})}^p)^{\frac{1}{p}}.$$

Note that if $1 \leq q \leq \infty$ and $E = L_q([0, 1])$ then ${}_pG_E(\mathcal{M}) = {}_pG_q(\mathcal{M})$, where ${}_pG_q(\mathcal{M})$ denotes the space of $x = (x_n)_{n\geq 1} \subset L_q(\mathcal{M})$ for which

$$\|dx\|_{{}_pG_q(\mathcal{M})} = (\sum_{n=1}^{\infty} \|dx_n\|_{L_q(\mathcal{M})}^p)^{\frac{1}{p}}.$$

The following theorem plays an important role in our paper which we call Doob's decomposition.

Theorem 2.3 *(Doob's decomposition) Let E be a symmetric Banach space on [0, 1] and $1 \leq p \leq \infty$. Then each bounded ${}_pE(\mathcal{M})$-quasi-martingale $x = (x_n)_{n\geq 1}$ can be uniquely decomposed as a sum of two sequences $y = (y_n)_{n\geq 1}$ and $z = (z_n)_{n\geq 1}$, where $y = (y_n)_{n\geq 1}$ is a bounded $E(\mathcal{M})$-martingale and $z = (z_n)_{n\geq 1}$ is a predicable ${}_pE(\mathcal{M})$-quasi-martingale with $z_1 = 0$.*

Proof The proof is similar with Lemma 2.2 in [15].

2.2 Interpolations

For a compatible Banach couple (X_0, X_1), we define the K-functional by setting for any $x \in X_0 + X_1$ and $t > 0$,

$$K_t(x; X_0, X_1) = \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} : x = x_0 + x_1, x_0 \in X_0, x_1 \in X_1\}.$$

The interpolation space $(X_0, X_1)_{E,K}$ is defined as the space of all elements $x \in X_0 + X_1$ such that

$$\|x\|_{(X_0,X_1)_{E,K}} := \left\| \frac{K_t(x; X_0, X_1)}{t} \right\|_E < \infty.$$

We may state the following interpolation result which is needed in the sequel (see Theorem 2.2 in [19] and [20]).

Theorem 2.4 *Let E be a symmetric Banach space on* [0, 1] *with the Fatou property and* $1 < p < p_E \leq q_E < q < \infty$. *Then there exists a symmetric Banach space F with nontrivial Boyd indices such that*

$$E(\mathcal{M}) = (L_p(\mathcal{M}), L_q(\mathcal{M}))_{F,K} \ \textit{(with equivalent norms)}. \tag{2.1}$$

Proof By Theorem 2.2 in [19], there is a symmetric Banach function space F on [0, 1] such that $f \in E$ if and only if $t \to K_t(f; L_p[0, 1], L_q[0, 1]) \in F$ and there exist a constant C such that

$$C^{-1}\|t \to K_t(f; L_p[0, 1], L_q[0, 1])\|_F \leq \|f\|_E \leq C\|t \to K_t(f; L_p[0, 1], L_q[0, 1]). \tag{2.2}$$

For any $x \in E(\mathcal{M})$, using the results $K_t(\mu(x); L_p[0, 1], L_q[0, 1]) \approx K_t(x; L_p(\mathcal{M}), L_p(\mathcal{M})$ and $\|\mu(x)\|_E = \|x\|_{E(\mathcal{M})}$, we can extend (2.2) to the noncommutative setting. The proof is completed.

Throughout the paper p' will denote the conjugate index of p. □

3 Main results

Our first result in this section is concerned with the dual space of ${}_p\widehat{E}(\mathcal{M})$ which is the symmetric Banach space of noncommutative quasi-martingales.

Theorem 3.1 *Let E be a symmetric Banach space on $[0, 1]$ with the Fatou property and $1 < p_E \leq q_E < p$. Then*

$$\left({}_p\widehat{E}(\mathcal{M})\right)^* = {}_{p'}\widehat{E}^{\times}(\mathcal{M})$$

isometrically, with associated duality bracket given by

$$\forall x \in {}_p\widehat{E}(\mathcal{M}), \quad \forall u \in {}_{p'}\widehat{E}^{\times}(\mathcal{M}), \quad (x, u) = \tau(\nu y) + \sum_{n=1}^{\infty} \tau(d\omega_n dz_n),$$

where $\mu_n = \nu_n + \omega_n$ and $x_n = y_n + z_n (n \geq 1)$ are the Doob's decomposition of u and x respectively.

For the proof we need the following Lemma.

Lemma 3.2 *Let E be a symmetric Banach space on $[0, 1]$ with the Fatou property and $1 < p_E \leq q_E < p$. Then*

$$E = F \odot E^{\times(\frac{p}{p'})},$$

where $F = (E^{\times(\frac{1}{p'})})^{\times}$ is separable.

Proof From the proof of Lemma 2.1 in [1], we know that $E^{\times(\frac{1}{p'})}$ is reflexive and F is separable. By (ii) of the properties of pointwise product spaces, we have

$$E = (E^{\times})^{\times} = ([E^{\times(\frac{1}{p'})}]^{(p')})^{\times} = ([E^{\times(\frac{1}{p'})}]^{\times})^{(p')} \odot L_p[0, 1] = F^{(p')} \odot L_p[0, 1].$$

Using the equality $L_1[0, 1] = E \odot E^{\times}$ (see Theorem 1.2 in [1]) and (i) of the properties of pointwise product spaces, we obtain that

$$E = F^{(p')} \odot (F \odot E^{\times(\frac{1}{p'})})^{(p)} = F^{(p')} \odot (F^{(p)} \odot E^{\times(\frac{1}{p'})(p)}) = F \odot E^{\times(\frac{p}{p'})}.$$

The proof is completed. □

We also require the following duality result (see Theorem 5.6 in [6]).

Lemma 3.3 *Let E be a symmetric Banach space on $[0, 1]$ with the Fatou property, then $\left(E(\mathcal{M})\right)^* = E^{\times}(\mathcal{M})$ isometrically, with associated duality bracket given by*

$$(x, y) = \tau(xy), \quad x \in E(\mathcal{M}), y \in E^{\times}(\mathcal{M}).$$

Now, we concern the dual space of $l_p(E(\mathcal{M}))$ which is the main ingredient in the proof of Theorem 3.1.

Lemma 3.4 *Let E be a symmetric Banach space on $[0, 1]$ with the Fatou property and $1 < p_E \leq q_E < p$. Then*

$$\left(l_p(E(\mathcal{M}))\right)^* = l_{p'}(E^{\times}(\mathcal{M}))$$

with equivalent norms.

Proof Let $x = (x_n)_{n\geq 1} \in l_p(E(\mathcal{M}))$ and $y = (y_n)_{n\geq 1} \in l_{p'}(E^{\times}(\mathcal{M}))$. Now, we define a linear functional on $l_p(E(\mathcal{M}))$ by

$$l_y(x) = \sum_{n=1}^{\infty} \tau(x_n y_n).$$

Then by Lemma 3.3 and Hölder's inequality,

$$\begin{aligned}\left|\sum_{n=1}^{\infty} \tau(x_n y_n)\right| &\leq \sum_{n=1}^{\infty} |\tau(x_n y_n)| \\ &\leq \sum_{n=1}^{\infty} \|x_n\|_{E(\mathcal{M})} \|y_n\|_{E^{\times}(\mathcal{M})} \\ &\leq (\sum_{n=1}^{\infty} \|x_n\|_{E(\mathcal{M})}^p)^{\frac{1}{p}} (\sum_{n=1}^{\infty} \|y_n\|_{E^{\times}(\mathcal{M})}^{p'})^{\frac{1}{p'}} \\ &= \|x\|_{l_p(E(\mathcal{M}))} \|y\|_{l_{p'}(E^{\times}(\mathcal{M}))}.\end{aligned}$$

Thus the series $\sum_{n=1}^{\infty} \tau(x_n y_n)$ converges absolutely. Therefore, $l_y(x)$ is continuous on $l_p(E(\mathcal{M}))$ and $\|l_y\| \leq \|y\|_{l_{p'}(E^{\times}(\mathcal{M}))}$.

We pass to the converse inclusion. Let $l \in \big(l_p(E(\mathcal{M}))\big)^*$ of norm one. For every $n \geq 1$, set

$$l_n(x_n) = l(\theta), \quad x_n \in E(\mathcal{M}),$$

where $\theta = (\underbrace{0, \ldots, 0, x_n,}_{n} 0, \ldots)$.Then

$$|l_n(x_n)| = |l(\theta)| \leq \|l\| \|\theta\|_{l_p(E(\mathcal{M}))} = \|x_n\|_{E(\mathcal{M})}.$$

This implies that $l_n \in (E(\mathcal{M}))^*$. Since $\big(E(\mathcal{M})\big)^* = E^{\times}(\mathcal{M})$, the representation theorem allows us to find an element $y_n \in E^{\times}(\mathcal{M})$ such that

$$l_n(x_n) = \tau(x_n y_n), \quad x_n \in E(\mathcal{M}).$$

Thus we have that

$$l(x) = \sum_{n=1}^{\infty} l_n(x_n) = \sum_{n=1}^{\infty} \tau(x_n y_n) \tag{3.1}$$

for any finite sequence $x = (x_n)_{n\geq 1} \in l_p(E(\mathcal{M}))$. We must show that $y = (y_n)_{n\geq 1} \in l_{p'}(E^{\times}(\mathcal{M}))$ and is of norm ≤ 1. Now, fix an n. Note that for any $k \leq n$

$$\|y_k\|_{E^{\times}(\mathcal{M})}^{p'} = \|(|y_k|^{p'})^{\frac{1}{p'}}\|_{E^{\times}(\mathcal{M})}^{p'} = \||y_k|^{p'}\|_{E^{\times(\frac{1}{p'})}(\mathcal{M})} = \sup\{\tau(a_k|y_k|^{p'}) : a_k \in F(\mathcal{M}), \|a_k\|_{F(\mathcal{M})} \leq 1\},$$

where $\big(F(\mathcal{M})\big)^* = E^{\times(\frac{1}{p'})}(\mathcal{M})$. Thus for an arbitrarily given $\varepsilon > 0$, there exists $a_k^{\varepsilon} \in F(\mathcal{M})$ and $\|a_k^{\varepsilon}\|_{F(\mathcal{M})} \leq 1$ such that

$$\|y_k\|_{E^{\times}(\mathcal{M})}^{p'} \leq \tau(a_k^{\varepsilon}|y_k|^{p'}) + \frac{\varepsilon}{2^k}. \tag{3.2}$$

Set $z_k = \frac{1}{\gamma_n} a_k^{\varepsilon} |y_k|^{p'-2} y^*$, where $\gamma_n = (\sum_{k=1}^{n} \|y_k\|_{E^{\times}(\mathcal{M})}^{p'})^{\frac{1}{p}}$. Then noting that $|y_k|^{p'-2}y^* \in E^{\times(\frac{p}{p'})}$ and by Lemma 3.2, we get $z_k \in E(\mathcal{M})$ and

$$\|z_k\|_{E(\mathcal{M})} \leq \frac{1}{\gamma_n} \|a_k^{\varepsilon}\|_{F(\mathcal{M})} \||y_k|^{p'-2}y^*\|_{E^{\times(\frac{p}{p'})}} \leq \frac{1}{\gamma_n} \|y_k\|_{E^{\times}(\mathcal{M})}^{p'-1}. \tag{3.3}$$

Thus we have that

$$\left(\sum_{k=1}^{n}\|z_k\|_{E(\mathcal{M})}^{p}\right)^{\frac{1}{p}} \le \frac{1}{\gamma_n}(\sum_{k=1}^{n}\|y_k\|_{E^\times(\mathcal{M})}^{((p'-1)p)})^{\frac{1}{p}} = \frac{1}{\gamma_n}\left(\sum_{k=1}^{n}\|y_k\|_{E^\times(\mathcal{M})}^{p'}\right)^{\frac{1}{p}} = 1.$$

Let $z^{(n)} = (z_1, \dots, z_n, 0, \dots)$. Then $z^{(n)} \in l_p(E(\mathcal{M}))$ and $\|z^{(n)}\|_{l_p(E(\mathcal{M}))} \le 1$. Using (3.1) and (3.2), we obtain that

$$\begin{aligned} l(z^{(n)}) &= \sum_{k=1}^{n}\tau(z_k y_k) \\ &= \frac{1}{\gamma_n}\sum_{k=1}^{n}\tau(a_k^{\varepsilon}|y_k|^{p'}) \\ &\ge \frac{1}{\gamma_n}\sum_{k=1}^{n}(\|y_k\|_{E^\times(\mathcal{M})}^{p'} - \frac{\varepsilon}{2^k}) \\ &\ge (\sum_{k=1}^{n}\|y_k\|_{E^\times(\mathcal{M})}^{p'})^{\frac{1}{p'}} - \frac{1}{\gamma_n}\varepsilon. \end{aligned}$$

Thus we have that

$$\left(\sum_{k=1}^{n}\|y_k\|_{E^\times(\mathcal{M})}^{p'})^{\frac{1}{p'}} \le \frac{1}{\gamma_n}\varepsilon + l(z^{(n)}\right) \le 1.$$

It follows that

$$\left(\sum_{n=1}^{\infty}\|y_n\|_{E^\times(\mathcal{M})}^{p'}\right)^{\frac{1}{p'}} \le 1$$

as $n \to \infty$ which implies

$$y \in l_{p'}(E^\times(\mathcal{M})) \ \text{ and } \ \|y\|_{l_{p'}(E^\times(\mathcal{M}))} \le 1.$$

For any $x = (x_n)_{n\ge 1} \in l_p(E(\mathcal{M}))$, let $x^{(n)} = (x_1, \dots, x_n, 0, \dots)$ $(n \ge 1)$. Then

$$\|x - x^{(n)}\|_{l_p(E(\mathcal{M}))} \to 0 \ \ (n \to \infty).$$

Using (3.1), we have

$$l(x) = \lim_{n\to\infty} l(x^{(n)}) = \lim_{n\to\infty}\sum_{i=1}^{n}\tau(x_i y_i) = \sum_{i=1}^{\infty}\tau(x_i y_i).$$

The proof is completed. □

The proof of Theorem 3.1 Let $\mu = (\mu_n)_{n\ge 1} \in {}_{p'}\widehat{E}^\times(\mathcal{M})$ and $x = (x_n)_{n\ge 1} \in {}_p\widehat{E}(\mathcal{M})$. Let $\mu_n = \nu_n + \omega_n$ and $x_n = y_n + z_n (n \ge 1)$ be the Doob's decomposition of μ and x respectively. Then $y = (y_n)_{n\ge 1}$ is a bounded $E(\mathcal{M})$-martingale and $\nu = (\nu_n)_{n\ge 1}$ is a bounded $E^\times(\mathcal{M})$-martingale. Thus there exist $y_\infty \in E(\mathcal{M})$ and $\nu_\infty \in E^\times(\mathcal{M})$ such that $y_n \xrightarrow{E(\mathcal{M})} y_\infty, \ \nu_n \xrightarrow{E^\times(\mathcal{M})} \nu_\infty$.

Now we define a linear functional on ${}_p\widehat{E}(\mathcal{M})$ by

$$l_\mu(x) = \tau(\nu_\infty y_\infty) + \sum_{n=1}^{\infty}\tau(d\omega_n dz_n).$$

Then by Lemma 3.3 and Hölder's inequality,

$$\begin{aligned}|l_\mu(x)| &\le \|\nu_\infty\|_{E^\times(\mathcal{M})}\|y_\infty\|_{E(\mathcal{M})} + \sum_{n=1}^{\infty}\|d\omega_n\|_{E^\times(\mathcal{M})}\|dz_n\|_{E(\mathcal{M})}\\ &\le \|\nu_\infty\|_{E^\times(\mathcal{M})}\|y_\infty\|_{E(\mathcal{M})} + \left(\sum_{n=1}^{\infty}\|d\omega_n\|^{p'}_{E^\times(\mathcal{M})}\right)^{\frac{1}{p'}}\left(\sum_{n=1}^{\infty}\|dz_n\|^{p}_{E(\mathcal{M})}\right)^{\frac{1}{p}}\\ &\le \left(\|\nu_\infty\|_{E^\times(\mathcal{M})} + \left(\sum_{n=1}^{\infty}\|d\omega_n\|^{p'}_{E^\times(\mathcal{M})}\right)^{\frac{1}{p'}}\right)\left(\|y_\infty\|_{E(\mathcal{M})} + \left(\sum_{n=1}^{\infty}\|dz_n\|^{p}_{E(\mathcal{M})}\right)^{\frac{1}{p}}\right)\\ &= \|\mu\|_{{}_{p'}\widehat{E}^\times(\mathcal{M})}\|x\|_{{}_{p}\widehat{E}(\mathcal{M})}.\end{aligned}$$

Thus $l_\mu(x)$ is continuous on ${}_p\widehat{E}(\mathcal{M})$ and $\|l_\mu\| \le \|\mu\|_{{}_{p'}\widehat{E}^\times(\mathcal{M})}$.

We pass to the converse inclusion. Let $l \in \left({}_p\widehat{E}(\mathcal{M})\right)^*$ of norm one. Let l_1 be the restriction of l on $E(\mathcal{M})$. Noting that $\left(E(\mathcal{M})\right)^* = E^\times(\mathcal{M})$, there exists $\nu \in E^\times(\mathcal{M})$ and $\|\nu\|_{E^\times(\mathcal{M})} \le 1$ such that

$$l_1(a) = \tau(a\nu), \quad a \in E(\mathcal{M}). \tag{3.4}$$

On the other hand, define a functional on ${}_pG_E(\mathcal{M})$ by

$$l_2(db) = l(b), \quad db = (db_n)_{n\ge1} \in {}_pG_E(\mathcal{M}).$$

Then $|l_2(db)| \le \|l\|\|b\|_{{}_p\widehat{E}(\mathcal{M})} = \|db\|_{{}_pG_E(\mathcal{M})}$. Thus we have that l_2 is a continuous linear functional on ${}_pG_E(\mathcal{M})$ and $\|l_2\| \le 1$. Recall that ${}_pG_E(\mathcal{M})$ is the closed subspace of $l_p(E(\mathcal{M}))$. By the Hahn-Banach theorem, l_2 extends to a norm one functional $\widetilde{l}_2$ on $l_p(E(\mathcal{M}))$. Consequently, by Lemma 3.4, $\widetilde{l}_2$ is given by a norm one element $\omega' = (\omega'_n)_{n\ge1}$ of $l_{p'}(E^\times(\mathcal{M}))$. Thus

$$l_2(db) = \sum_{n=1}^{\infty}\tau(d\omega'_n db_n) \;\; db = (db_n)_{n\ge1} \in {}_pG_E(\mathcal{M}). \tag{3.5}$$

Set $\omega_1 = 0$ and $\omega_n = \sum_{k=1}^{n}\mathcal{E}_{k-1}(\omega'_k)(n \ge 2)$. For any $db = (db_n)_{n\ge1} \in {}_pG_E(\mathcal{M})$, noting that $db = (db_n)_{n\ge1}$ is predicable, it follows from (3.5) that

$$l_2(db) = \sum_{n=1}^{\infty}\tau(\mathcal{E}_{n-1}(\omega'_n db_n)) = \sum_{n=1}^{\infty}\tau(db_n\mathcal{E}_{n-1}(\omega'_n)) = \sum_{n=1}^{\infty}\tau(d\omega_n db_n). \tag{3.6}$$

It is easy to see that $\omega = (\omega_n)_{n\ge1}$ is predicable with $\omega_1 = 0$ and

$$\left(\sum_{n=1}^{\infty}\|d\omega_n\|^{p'}_{E^\times(\mathcal{M})}\right)^{\frac{1}{p'}} = \left(\sum_{n=1}^{\infty}\|\mathcal{E}_{n-1}(\omega'_n)\|^{p'}_{E^\times(\mathcal{M})}\right)^{\frac{1}{p'}} \le \left(\sum_{n=1}^{\infty}\|\omega'_n\|^{p'}_{E^\times(\mathcal{M})}\right)^{\frac{1}{p'}} = 1.$$

Set $\mu_n = \nu_n + \omega_n(n \ge 1)$, where $\nu_n = \mathcal{E}_n(\nu)(n \ge 1)$. Then $\mu = (\mu_n)_{n\ge1} \in {}_{p'}\widehat{E}^\times(\mathcal{M})$ and

$$\|\mu\|_{{}_{p'}\widehat{E}^\times(\mathcal{M})} = \|\nu\|_{E^\times(\mathcal{M})} + \left(\sum_{n=1}^{\infty}\|d\omega_n\|^{p'}_{E^\times(\mathcal{M})}\right)^{\frac{1}{p'}} \le 2.$$

For any $x = (x_n)_{n\ge1} \in {}_p\widehat{E}(\mathcal{M})$, let $x_n = y_n + z_n(n \ge 1)$ be its Doob's decomposition. Noting that $y = (y_n)_{n\ge1}$ is a bounded $E(\mathcal{M})$-martingale and $dz = (dz_n)_{n\ge1} \in {}_pG_E(\mathcal{M})$, it follows from (3.4) and (3.6) that

$$l(x) = l_1(y) + l_2(dz) = \tau(y_\infty\upsilon_\infty) + \sum_{n=1}^{\infty}\tau(d\omega_n dz_n),$$

where y_∞ is the limit of $(y_n)_{n\ge1}$ in $E(\mathcal{M})$. The proof is completed. □

As applications of Theorem 3.1, we shall consider the symmetric space ${}_p\widehat{E}(\mathcal{M})$ as interpolations of noncommutative quasi-martingale L_p-spaces, which is a generalization of Theorem 2.4.

Theorem 3.5 *Let E be a symmetric Banach space on [0, 1] with the Fatou property and $1 < p < p_E \le q_E < q < \infty$. Then there exists a symmetric Banach space F with nontrivial Boyd indices such that*

$${}_s\widehat{E}(\mathcal{M}) = ({}_s\widehat{L}_p(\mathcal{M}), {}_s\widehat{L}_q(\mathcal{M}))_{F,K},$$

where $1 < s < p_E$.

For the proof of Theorem 3.5, we need the following lemmas (see [2]).

Lemma 3.6 *Let E be a symmetric Banach space on [0, 1] with the Fatou property, and let (X_1, X_2) be a compatible Banach couple. Then*

$$(X_1, X_2)^*_{F,K} = (X_2^*, X_1^*)_{F^\times,K}.$$

Lemma 3.7 *Let E be a symmetric Banach space on [0, 1] with the Fatou property and $1 < p < p_E \le q_E < q < \infty$. Then there exists a symmetric Banach space F with nontrivial Boyd indices such that*

$$l_s(E(\mathcal{M})) = \big(l_s(L_p(\mathcal{M})), l_s(L_q(\mathcal{M}))\big)_{F,K},$$

where $1 < s < p_E$.

Proof Let $x = (x_n)_{n\ge1} \in l_s(E(\mathcal{M}))$. Then by Theorem 2.4, there exists a symmetric Banach space F with nontrivial Boyd indices such that $x_n \in (L_p(\mathcal{M}), L_q(\mathcal{M}))_{F,K}$ for any $n \ge 1$. For any $a, b > 0$, $\alpha_s(a^s + b^s) \le (a+b)^s \le \beta_s(a^s + b^s)$ for some constants α_s, β_s depending only on s. Using this fact, it is easy to show that

$$\begin{aligned}&\big(K_t(x; l_s(L_p(\mathcal{M})), l_s(L_q(\mathcal{M})))\big)^s\\&\le \beta_s \sum_{n=1}^{\infty} \inf_{x_n = x_n^0 + x_n^1} (\|x_n^0\|^s_{L_p(\mathcal{M})} + t^s \|x_n^1\|^s_{L_q(\mathcal{M})})\\&\le \frac{\beta_s}{\alpha_s} \sum_{n=1}^{\infty} \big(K_t(x_n; L_p(\mathcal{M}), L_q(\mathcal{M}))\big)^s.\end{aligned}$$

Noting that $F_{(s)}$ is a quasi-Banach space, we have that

$$\begin{aligned}&\|x\|^s_{(l_s(L_p(\mathcal{M})), l_s(L_q(\mathcal{M})))_{F,K}}\\&= \left\| \frac{K_t(x; l_s(L_p(\mathcal{M})), l_s(L_q(\mathcal{M})))^s}{t^s} \right\|_{F_{(s)}}\\&\le C_s \sum_{n=1}^{\infty} \left\| \frac{K_t(x_n; L_p(\mathcal{M}), L_q(\mathcal{M}))^s}{t^s} \right\|_{F_{(s)}}\\&= C_s \sum_{n=1}^{\infty} \left\| \frac{K_t(x_n; L_p(\mathcal{M}), L_q(\mathcal{M}))}{t} \right\|^s_F,\end{aligned}$$

where C_s is a constant depending on s. This means that

$$\|x\|_{\left(l_s(L_p(\mathcal{M})), l_s(L_q(\mathcal{M}))\right)_{F,K}} \le C_s \|x\|_{l_s(E(\mathcal{M}))}$$

and

$$l_s(E(\mathcal{M})) \subset \big(l_s(L_p(\mathcal{M})), l_s(L_q(\mathcal{M}))\big)_{F,K}. \tag{3.7}$$

Similarly, we have that $l_{s'}(E^\times(\mathcal{M})) \subset \big(l_{s'}(L_{q'}(\mathcal{M})), l_{s'}(L_{p'}(\mathcal{M}))\big)_{F^\times,K}$. It follows that

$$(l_{s'}(E^\times(\mathcal{M})))^* \supset \big((l_{s'}(L_{q'}(\mathcal{M})), l_{s'}(L_{p'}(\mathcal{M})))\big)^*_{F^\times,K}.$$

Observe that $p_{E^\times} \le q_{E^\times} \le s'$. Thus by Lemma 3.4 and Lemma 3.6, we have that

$$l_s(E(\mathcal{M})) \supset \big(l_s(L_p(\mathcal{M})), l_s(L_q(\mathcal{M}))\big)_{F,K}. \tag{3.8}$$

Putting (3.7) and (3.8) together, we obtain that

$$l_s(E(\mathcal{M})) = \big(l_s(L_p(\mathcal{M})), l_s(L_q(\mathcal{M}))\big)_{F,K}.$$

The proof is completed.

The following is an interpolation result on the space ${}_pG_E(\mathcal{M})$.

Lemma 3.8 *Let E be a symmetric Banach space on [0, 1] with the Fatou property and $1 < p < p_E \le q_E < q < \infty$. Then there exists a symmetric Banach space F with nontrivial Boyd indices such that*

$$ {}_sG_E(\mathcal{M}) = \big({}_sG_p(\mathcal{M}), {}_sG_q(\mathcal{M})\big)_{F,K},$$

where $1 < s < p_E$.

Proof Note that ${}_sG_p(\mathcal{M})$ consists of quasi-martingale difference sequences in $l_s(L_p(\mathcal{M}))$. So ${}_sG_p(\mathcal{M})$ is 1-complemented in $l_s(L_p(\mathcal{M}))$ via the projection

$$P : \begin{cases} l_s(L_p(\mathcal{M})) \to {}_sG_p(\mathcal{M}); \\ (a_n)_{n\ge1} \to (\mathcal{E}_{n-1}(a_n))_{n\ge1}. \end{cases}$$

It follows that for any $x \in \big({}_sG_p(\mathcal{M}), {}_sG_q(\mathcal{M})\big)_{F,K}$,

$$K_t(x; {}_sG_p(\mathcal{M}), {}_sG_q(\mathcal{M})) = K_t(x; l_s(L_p(\mathcal{M})), l_s(L_q(\mathcal{M})))\ \ t > 0.$$

Thus

$$\|x\|_{\left({}_sG_p(\mathcal{M}), {}_sG_q(\mathcal{M})\right)_{F,K}} = \|x\|_{\left(l_s(L_p(\mathcal{M})), l_s(L_q(\mathcal{M}))\right)_{F,K}}.$$

Therefore, using Lemma 3.6, we have finished the proof of the theorem.

Proof of Theorem 3.5 Let $x \in ({}_s\widehat{L}_p(\mathcal{M}), {}_s\widehat{L}_q(\mathcal{M}))_{F,K}$ and $x = x^0 + x^1$ be a decomposition of x where $x^0 \in {}_s\widehat{L}_p(\mathcal{M})$ and $x^1 \in {}_s\widehat{L}_q(\mathcal{M})$. Let $x_n^k = y_n^k + z_n^k$ $(n \ge 1)$ be the Doob's decomposition of x^k $(k = 0, 1)$. Then we have that $y^0 \in L_p(\mathcal{M})$, $y^1 \in L_q(\mathcal{M})$ and $dz^0 \in {}_sG_p(\mathcal{M})$, $dz^1 \in {}_sG_q(\mathcal{M})$. Set $y = y^0 + y^1$ and $z = z^0 + z^1$. Then

$$\begin{aligned} &K_t(y; L_p(\mathcal{M}), L_q(\mathcal{M})) + K_t(dz; {}_sG_p(\mathcal{M}), {}_sG_q(\mathcal{M})) \\ &\quad \le \|y^0\|_{L_p(\mathcal{M})} + t\|y^1\|_{L_q(\mathcal{M})} + \|dz^0\|_{{}_sG_p(\mathcal{M}))} + t\|dz^1\|_{{}_sG_q(\mathcal{M})} \\ &= \|x^0\|_{{}_s\widehat{L}_p(\mathcal{M})} + t\|x^1\|_{{}_s\widehat{L}_q(\mathcal{M})}. \end{aligned}$$

Thus we get that

$$K_t(y; L_p(\mathcal{M}), L_q(\mathcal{M})) + K_t(dz; {}_sG_p(\mathcal{M}), {}_sG_q(\mathcal{M})) \le K_t(x; {}_s\widehat{L}_p(\mathcal{M}), {}_s\widehat{L}_q(\mathcal{M})),$$

where the infimum runs over all decomposition of x. Using the equality $\|x\|_{(X_0,X_1)_{F,K}} = \|\frac{K_t(x;X_0,X_1)}{t}\|_F$, we have that

$$\|y\|_{(L_p(\mathcal{M}),L_q(\mathcal{M}))_{F,K}} + \|dz\|_{({}_sG_p(\mathcal{M}),{}_sG_q(\mathcal{M}))_{F,K}} \le 2\|x\|_{({}_s\widehat{L}_p(\mathcal{M}),{}_s\widehat{L}_q(\mathcal{M}))_{F,K}}.$$

By Lemma 2.4 and Lemma 3.8, we get that

$$\|y\|_{E(\mathcal{M})} + \|dz\|_{{}_sG_E(\mathcal{M})} \le 2\|x\|_{({}_s\widehat{L}_p(\mathcal{M}),{}_s\widehat{L}_q(\mathcal{M}))_{F,K}}$$

which implies that $\|x\|_{{}_s\widehat{E}(\mathcal{M})} \le 2\|x\|_{({}_s\widehat{L}_p(\mathcal{M}),{}_s\widehat{L}_q(\mathcal{M}))_{F,K}}$ and

$${}_s\widehat{E}(\mathcal{M}) \supset ({}_s\widehat{L}_p(\mathcal{M}), {}_s\widehat{L}_q(\mathcal{M}))_{F,K}.$$

By Theorem 3.1 and Lemma 3.6, we have that

$${}_s\widehat{E}(\mathcal{M}) = ({}_{s'}\widehat{E^{\times}}(\mathcal{M}))^* \subset \big(({}_{s'}\widehat{L}_{q'}(\mathcal{M}), {}_{s'}\widehat{L}_{p'}(\mathcal{M}))_{F^{\times},K}\big)^* = ({}_s\widehat{L}_p(\mathcal{M}), {}_s\widehat{L}_q(\mathcal{M}))_{F,K}.$$

Therefore,

$${}_s\widehat{E}(\mathcal{M}) = ({}_s\widehat{L}_p(\mathcal{M}), {}_s\widehat{L}_q(\mathcal{M}))_{F,K}.$$

The proof is completed.

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