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ORIGINAL RESEARCH

A note on a theorem of Ligh and Richou

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Abstract In this note, we generalize the main results of Ligh et al. (Bull Austral Math Soc 16:75–77, 1977), Wei et al. (An Ştiinţ Univ Al I Cuza Iaşi Mat (N.S.) 61:97–100, 2015) and Wei (Bull Malays Math Sci Soc 38:1589–1599, 2015).

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1 Introduction

Throughout this paper, all rings are associative with identity. Let R be a ring, we use $N(R)$ and $Z(R)$ to denote the set of all nilpotent elements and the center, respectively.

In 1977, Ligh and Richou proved that if R is a ring with 1 which satisfies the identities: $(xy)^k = x^k y^k$, $k = n, n+1, n+2$, where n is a positive integer, then R is commutative (see [1]). In 2015, Wei and Fan proved that if R is a ring with 1, $n \geq 1$ and for any $x \in R \backslash N(R)$ and any $y \in R$, $(xy)^k = x^k y^k$, $k = n, n+1, n+2$, then R is commutative (see Theorem 2.7 of [2] and Theorem 1.1 of [3]).

In this note, we generalize the above results as follows.

Theorem 1.1 *Let R be a ring with 1. Suppose that for any $x, y \in R \backslash N(R)$, there exists a nonnegative integer $n = n(x, y)$ which relies on x and y such that $(xy)^k = x^k y^k$, $k = n, n+1, n+2$. Then R is commutative.*

2 Preliminaries

Lemma 2.1 *Let R be a ring with 1. Suppose that for any $x, y \in R$, there exists a nonnegative integer $n = n(x, y)$ which relies on x and y such that $(xy)^k = x^k y^k$, $k = n, n+1, n+2$. Then for any $x, y \in R$, we have $x^n[x, y^n]y = 0$ and $x^{n+1}[x, y^{n+1}]y = 0$.*

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Proof For any $x, y \in R$, by the hypotheses, there exists a nonnegative integer $n = n(x, y)$ which relies on x and y such that $(xy)^k = x^k y^k$, $k = n, n+1, n+2$. Then $x^{n+1}y^{n+1} = (xy)^{n+1} = (xy)^n xy = x^n y^n xy$. Hence $x^n(xy^n - y^n x)y = 0$, i.e., $x^n[x, y^n]y = 0$. Similarly, we have $x^{n+2}y^{n+2} = (xy)^{n+2} = (xy)^{n+1}xy = x^{n+1}y^{n+1}xy$. Hence $x^{n+1}(xy^{n+1} - y^{n+1}x)y = 0$, i.e., $x^{n+1}[x, y^{n+1}]y = 0$. □

Lemma 2.2 *Let R be a ring with 1 and $a, x \in R$. Suppose that there exist nonnegative integers m, n such that $[a, x]x^m = 0$ and $[a, x](1 + x)^n = 0$. Then $[a, x] = 0$.*

Proof There is no loss of generality to assume that $m \geq 1$ and $n \geq 1$. Let $M = \{f$ is a nonnegative integer$|$ $[a, x]x^f = 0\}$. Since $[a, x]x^m = 0$, we see that M is nonempty. Then there exists $m_0 \in M$ such that m_0 is the smallest number of M. Assume that $m_0 \geq 1$ and we work to obtain a contradiction. Since $m_0 \geq 1$ and $[a, x]x^{m_0} = 0$, it is not very difficult to see that $[a, x]\{(1+x)^n x^{m_0-1} - x^{m_0-1}\} = 0$, i.e., $[a, x](1+x)^n x^{m_0-1} - [a, x]x^{m_0-1} = 0$. Recall that $[a, x](1 + x)^n = 0$. Hence $[a, x]x^{m_0-1} = 0$, and thus $m_0 - 1 \in M$. This is a contradiction since m_0 is the smallest number of M. Hence $m_0 = 0$, and thus $[a, x] = 0$. □

Lemma 2.3 *Let R be a ring with 1. Suppose that for any $x, y \in R$, there exists a nonnegative integer $n = n(x, y)$ which relies on x and y such that $(xy)^k = x^k y^k$, $k = n, n + 1, n + 2$. Then $N(R) \subseteq Z(R)$.*

Proof It suffices to prove that for any $a \in N(R)$ and any $y \in R$, $[a, y] = 0$. For any $a \in N(R)$ and any $y \in R$, by the the hypotheses, we have

$$\{(1 + a)y\}^k = (1 + a)^k y^k, k = n, n + 1, n + 2, n = n(1 + a, y); \quad (2.1)$$
$$\{(1 + a)(1 + y)\}^k = (1 + a)^k(1 + y)^k, k = n_1, n_1 + 1, n_1 + 2, n_1 = n_1(1 + a, 1 + y). \quad (2.2)$$

By (2.1) and Lemma 2.1, it follows that

$$(1 + a)^n[1 + a, y^n]y = 0, (1 + a)^{n+1}[1 + a, y^{n+1}]y = 0. \quad (2.3)$$

Since $a \in N(R)$, we see that $1 + a$ is invertible. Hence

$$[a, y^n]y = [1 + a, y^n]y = 0, \quad (2.4)$$
$$[a, y^{n+1}]y = [1 + a, y^{n+1}]y = 0. \quad (2.5)$$

By (2.4) and (2.5), it follows that

$$[a, y]y^{n+1} = [a, y^{n+1}]y - y[a, y^n]y = 0. \quad (2.6)$$

Similarly, by (2.2), we have

$$[a, (1 + y)^{n_1}](1 + y) = 0, [a, (1 + y)^{n_1+1}](1 + y) = 0. \quad (2.7)$$

Similarly, by (2.7), we have

$$[a, y](1 + y)^{n_1+1} = [a, 1 + y](1 + y)^{n_1+1} = 0. \quad (2.8)$$

By (2.6), (2.8) and Lemma 2.2, we see that $[a, y] = 0$. This completes the proof. □

By Lemma 2.3, it is not very difficult to prove the following lemma.

Lemma 2.4 *Let R be a ring with 1. Suppose that for any $x, y \in R$, there exists a nonnegative integer $n = n(x, y)$ which relies on x and y such that $(xy)^k = x^k y^k$, $k = n, n + 1, n + 2$. Then $N(R)$ is an ideal of R.*

Lemma 2.5 *Let R be a ring with 1. Suppose that for any $x, y \in R$, there exist nonnegative integers $m = m(x, y), n = n(x, y)$ which rely on x and y such that $x^m[x, y]y^n = 0$. Then R is commutative.*

Proof It suffices to prove that for any $x, y \in R$, $[x, y] = 0$. For any $x, y \in R$, by the hypotheses, we have

$$x^m[x, y]y^n = 0, m = m(x, y), n = n(x, y); \quad (2.9)$$
$$(1 + x)^{m_1}[x, y]y^{n_1} = (1 + x)^{m_1}[1 + x, y]y^{n_1} = 0, m_1 = m_1(1 + x, y), n_1 = n_1(1 + x, y). \quad (2.10)$$

Let $M = \{f$ is a nonnegative integer$|$ there exists a nonnegative integer h such that $x^f[x, y]y^h = 0\}$. By (2.9), we see that M is nonempty. Then there exists $m_0 \in M$ such that m_0 is the smallest number of M. Since $m_0 \in M$, there exists a nonnegative integer n_0 such that $x^{m_0}[x, y]y^{n_0} = 0$. Assume that $m_0 \geq 1$ and we work to obtain a contradiction. Since $m_0 \geq 1$ and $x^{m_0}[x, y]y^{n_0} = 0$, it is not very difficult to see that

$$\{x^{m_0-1}(1+x)^{m_1} - x^{m_0-1}\}[x, y]y^{n_0+n_1} = 0. \tag{2.11}$$

Combine (2.10) with (2.11), we see that

$$x^{m_0-1}[x, y]y^{n_0+n_1} = 0. \tag{2.12}$$

Hence $m_0 - 1 \in M$. This is a contradiction since m_0 is the smallest number of M. Hence $m_0 = 0$, and thus $[x, y]y^{n_0} = 0$. Since y is an arbitrary element of R, we see that there exists a nonnegative integer n_2 such that $[x, y](1+y)^{n_2} = 0$. By Lemma 2.2, it follows that $[x, y] = 0$. This completes the proof. □

Lemma 2.6 *Let R be a ring with 1. Suppose that for any $x, y \in R$, there exists a nonnegative integer $n = n(x, y)$ which relies on x and y such that $(xy)^k = x^k y^k$, $k = n, n+1, n+2$. Then R is commutative.*

Proof It suffices to prove that for any $x, y \in R$, $[x, y] = 0$. By Lemmas 2.3 and 2.4, it follows that $N(R) \subseteq Z(R)$ and $N(R)$ is an ideal of R. For any $x, y \in R$, by the hypotheses, we have

$$(xy)^k = x^k y^k, k = n, n+1, n+2, n = n(x, y); \tag{2.13}$$

$$\{(1+x)y\}^k = (1+x)^k y^k, k = n_1, n_1+1, n_1+2, n_1 = n_1(1+x, y). \tag{2.14}$$

By Lemma 2.1, it follows that

$$x^n[x, y^n]y = 0, x^{n+1}[x, y^{n+1}]y = 0, \tag{2.15}$$

$$(1+x)^{n_1}[x, y^{n_1}]y = (1+x)^{n_1}[1+x, y^{n_1}]y = 0, \tag{2.16}$$

$$(1+x)^{n_1+1}[x, y^{n_1+1}]y = (1+x)^{n_1+1}[1+x, y^{n_1+1}]y = 0. \tag{2.17}$$

Since $x^{n+1}\{[x, y^n]y + y^n[x, y]\}y = x^{n+1}[x, y^{n+1}]y = 0$, we see that

$$x^{n+1}y^n[x, y]y = 0. \tag{2.18}$$

Similarly, since $(1+x)^{n_1+1}\{[x, y^{n_1}]y + y^{n_1}[x, y]\}y = (1+x)^{n_1+1}[x, y^{n_1+1}]y = 0$, we see that

$$(1+x)^{n_1+1}y^{n_1}[x, y]y = 0. \tag{2.19}$$

By (2.18), we see that

$$y^n[x, y]yx^{n+1} \in N(R). \tag{2.20}$$

By (2.19), we see that

$$y^{n_1}[x, y]y(1+x)^{n_1+1} \in N(R). \tag{2.21}$$

Let $M = \{t$ is a nonnegative integer$|$ there exists a nonnegative integer s such that $y^s[x, y]yx^t \in N(R)\}$. By (2.20), we see that M is nonempty. Then there exists $t_0 \in M$ such that t_0 is the smallest number of M. Since $t_0 \in M$, there exists a nonnegative integer s_0 such that

$$y^{s_0}[x, y]yx^{t_0} \in N(R). \tag{2.22}$$

Assume that $t_0 \geq 1$ and we work to obtain a contradiction. Since $N(R)$ is an ideal of R, by (2.21) and (2.22), we have

$$y^{n_1+s_0}[x, y]y\{x^{t_0-1}(1+x)^{n_1+1} - x^{t_0-1}\} \in N(R), \tag{2.23}$$

$$y^{n_1+s_0}[x, y]yx^{t_0-1}(1+x)^{n_1+1} = y^{n_1+s_0}[x, y]y(1+x)^{n_1+1}x^{t_0-1} \in N(R). \tag{2.24}$$

Combine (2.23) with (2.24), we see that

$$y^{n_1+s_0}[x, y]yx^{t_0-1} \in N(R). \tag{2.25}$$

Hence $t_0 - 1 \in M$. This is a contradiction since t_0 is the smallest number of M. Hence $t_0 = 0$. By (2.22), it follows that $y^{s_0}[x, y]y \in N(R)$, and thus $[x, y]y^{s_0+1} \in N(R)$. Since $N(R) \subseteq Z(R)$, we see that

$$[x, y]y^{s_0+1} \in Z(R). \tag{2.26}$$

By (2.18), it follows that $x^{n+1}y^n[x, y]y^{s_0+1} = 0$. By (2.26), it follows that $x^{n+1}[x, y]y^{n+s_0+1} = 0$. By Lemma 2.5, we see that R is commutative. □

3 Proof of Theorem 1.1

Proof of Theorem 1.1 At first, we work to prove that $[x, y] = 0$ for any $x \in N(R)$ and any $y \in R$.

Assume that $y \in N(R)$. Since $x \in N(R)$, we see that $1+x$ is invertible, in particular, $1+x \in R\backslash N(R)$. Similarly, $1 + y$ is invertible, in particular, $1 + y \in R\backslash N(R)$. Let $a = 1 + x$ and $b = 1 + y$. By the the hypotheses, we have

$$(ab)^k = a^k b^k, k = n, n+1, n+2, n = n(a, b). \tag{3.27}$$

By Lemma 2.1, it follows that $a^n[a, b^n]b = 0, a^{n+1}[a, b^{n+1}]b = 0$. Since a, b are invertible, we see that $[a, b^n] = 0, [a, b^{n+1}] = 0$. Hence $[a, b]b^n = [a, b^{n+1}] - b[a, b^n] = 0$. Since b is invertible, we see that $[a, b] = 0$, i.e., $[1 + x, 1 + y] = 0$. Hence $[x, y] = 0$.

Assume that $y \in R\backslash N(R)$ and $1 + y \in N(R)$. Since $1 + y \in N(R)$, by the above proof, we see that $[x, 1 + y] = 0$, i.e., $[x, y] = 0$.

Assume that neither y nor $1 + y$ is a nilpotent element of R. Since $x \in N(R)$, we see that $1 + x$ is invertible, in particular, $1 + x \in R\backslash N(R)$. By the the hypotheses, we have

$$\{(1+x)y\}^k = (1+x)^k y^k, k = n, n+1, n+2, n = n(1+x, y); \tag{3.28}$$

$$\{(1+x)(1+y)\}^k = (1+x)^k(1+y)^k, k = n_1, n_1+1, n_1+2, n_1 = n_1(1+x, 1+y). \tag{3.29}$$

By (3.28) and Lemma 2.1, it follows that $(1+x)^n[1+x, y^n]y = 0$, $(1+x)^{n+1}[1+x, y^{n+1}]y = 0$. Since $1+x$ is invertible, we see that $[1+x, y^n]y = 0$, $[1+x, y^{n+1}]y = 0$, i.e., $[x, y^n]y = 0$, $[x, y^{n+1}]y = 0$. Hence

$$[x, y]y^{n+1} = [x, y^{n+1}]y - y[x, y^n]y = 0. \tag{3.30}$$

By (3.29) and Lemma 2.1, it follows that

$$(1+x)^{n_1}[1+x, (1+y)^{n_1}](1+y) = 0, (1+x)^{n_1+1}[1+x, (1+y)^{n_1+1}](1+y) = 0. \tag{3.31}$$

Since $1 + x$ is invertible, we see that $[1 + x, (1 + y)^{n_1}](1 + y) = 0$, $[1 + x, (1 + y)^{n_1+1}](1 + y) = 0$. Hence

$$[1+x, 1+y](1+y)^{n_1+1} = [1+x, (1+y)^{n_1+1}](1+y) - (1+y)[1+x, (1+y)^{n_1}](1+y) = 0. \tag{3.32}$$

Hence

$$[x, y](1+y)^{n_1+1} = 0. \tag{3.33}$$

By (3.30), (3.33) and Lemma 2.2, it follows that $[x, y] = 0$.

Now we have proved that for any $x \in N(R)$ and any $y \in R$, $[x, y] = 0$.

For $x, y \in R$, if either x or y is a nilpotent element of R, by the above proof, it follows that $[x, y] = 0$, in particular, $(xy)^k = x^k y^k, k = n, n+1, n+2, n = 1$. If neither x nor y is a nilpotent element of R, by the the hypotheses, there exists a nonnegative integer $n = n(x, y)$ which relies on x and y such that $(xy)^k = x^k y^k, k = n, n+1, n+2$. By Lemma 2.6, it follows that R is commutative. This completes that proof. □

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