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ORIGINAL RESEARCH

# Quantitative Theorems for a Rich Class of Novel Miheşan-type Approximation Operators Incorporating the Boas-Buck Polynomials

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**Abstract** In this work, new summation-integral approximation operators based on a versatile generalization of the classic Szász-Mirakjan type operators, and incorporating the Boas-Buck polynomials are considered. We show how the proposed operators can get reduced to a multitude of operators involving classic approximation operators studied over past many decades. Wex nomenclate the individual cases hybrid generalizations of Bernstein, Baskakov, Lupaş and Szász-Mirakjan operators, each incorporating the Boas-Buck, Brenke, Sheffer and Appell polynomials. Indispensable properties of the proposed operators based on first and second order modulus of continuity are derived. Approximation on weighted space is also considered. In addition, quantitative Voronovskaja-type theorems have very recently been acknowledged as valuable properties for approximating functions. These form a noteworthy part of the present work.

**Keywords** Weighted modulus of continuity · Boas-Buck polynomials · Steklov mean

## 1 Introduction

Approximation of functions is of vital significance in engineering mathematics, and also as a mathematical field in its own right. The classic Szász-Mirakjan approximation operators [37] are defined, for $g : [0, \infty) \to \mathbb{R}$ for which the series is convergent, as:

$$S_n(g; y) = \frac{1}{e^{ny}} \sum_{k=0}^{\infty} \frac{(ny)^k}{k!} g\left(\frac{k}{n}\right). \tag{1}$$

In 2008, Miheşan [32] obtained the following generalized Szász-Mirakjan operators $\mathfrak{M}_n^{(\rho)}$ for $\rho \in \mathbb{R}$, $\rho + ny > 0$, by applying Gamma transform to the Szász-Mirakjan operators, which are given by:

$$\mathfrak{M}_n^{(\rho)}(g; y) = \sum_{k=0}^{\infty} \mathfrak{r}_{n,k}^{(\rho)}(y) g\left(\frac{k}{n}\right), \quad y \in [0, \infty) \tag{2}$$

where

$$\mathfrak{r}_{n,k}^{(\rho)}(y) = \frac{(\rho)_k}{k!} \frac{\left(\frac{ny}{\rho}\right)^k}{\left(1 + \frac{ny}{\rho}\right)^{\rho+k}} \tag{3}$$

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and $(\rho)_k$ denotes the Pochhammer symbol for rising factorial of $\rho$, given by

$$(\rho)_k = (\rho)(\rho+1)\dots(\rho+k-1), (\rho)_0 = 1. \tag{4}$$

Note that $\rho \neq 0$. The operators $\mathfrak{M}_n^{(\rho)}$ were shown to be guaranteed to converge [24].

*Remark 1* These operators are a landmark work because they reproduce important classical operators, studied over past many decades, in particular cases [27], [20]:

1. If $\rho = -n$, $\mathfrak{M}_n^{(\rho)}$ get reduced to Bernstein operators [14],
2. If $\rho = n$, $\mathfrak{M}_n^{(\rho)}$ get reduced to Baskakov operators [13],
3. If $\rho \to \infty$, $\mathfrak{M}_n^{(\rho)}$ get reduced to Szász-Mirakjan operators [37],
4. If $\rho = ny$, $y > 0$, $\mathfrak{M}_n^{(\rho)}$ get reduced to Lupaş operators [31] [9].

Due to their versatility, the operators $\mathfrak{M}_n^{(\rho)}$ (2) have been examined comprehensively, like [23], [27]. In [24], several indispensable results for (2) have been established. Modifications of Szász-Mirakjan operators have recently been studied in [25], [1], [23], [27], [6].

In terms of fresh developments, approximation operators which reproduce well-known classic operators have gained importance in frontier research, for e.g. [10]. In the same spirit, in [20], [19], it has been demonstrated how the operators based on generalized Szász-Mirakjan operators due to Miheşan (2) (considered in the present work) get reduced to a multitude of well-known operators studied over past many decades.

Summation-integral type operators have also been intensively studied recently, and Voronovskaya type theorems were obtained in quantitative forms. Some recent comprehensive literature on these is [2], [4], [26], [7], [30], [16]

Let us turn our attention to approximation involving the Appell polynomials and their generalizations like Brenke polynomials and Boas-Buck polynomials. Let $U(\mu)$, $V(\mu)$ and $W(\mu)$ be analytic functions of the form

$$U(\mu) = \sum_{j=0}^{\infty} a_j \mu^j, a_0 \neq 0, \tag{5}$$

$$V(\mu) = \sum_{j=0}^{\infty} b_j \mu^j, b_j \neq 0, \tag{6}$$

$$W(\mu) = \sum_{j=0}^{\infty} h_j \mu^j, h_1 \neq 0. \tag{7}$$

Jakimovski and Leviatan's classic work [22] introduced generalized Szász-Mirakjan-Appell operators, given as:

$$P_n(\mu, y) = \frac{e^{-ny}}{U(1)} \sum_{j=0}^{\infty} \mathfrak{H}_j(ny)\mu\left(\frac{k}{n}\right), \tag{8}$$

where $\mathfrak{H}_j(y)$ are the Appell polynomials, defined as $U(\mu)e^{y\mu} = \sum_{j=0}^{\infty} \mathfrak{H}_j(y)\mu^j$ and $U(\mu)$ is as defined before.

Varma in [41] put forth an operator inspired by Szász-Mirakjan operators and incorporating the Brenke polynomials, $\mathfrak{v}_j(y)$:

$$L_n(g; y) := \frac{1}{U(1)V(ny)} \sum_{k=0}^{\infty} \mathfrak{v}_k(ny) g\left(\frac{k}{n}\right), \quad y \geq 0, n \in \mathbb{N} \tag{9}$$

where the Brenke polynomials $\sum_{j=0}^{\infty} \mathfrak{v}_j(y)\mu^j = U(\mu)V(y\mu)$ et al.

In a very recent development [40], Sucu et al. proposed operators formulated from Szász-Mirakjan operators involving the Boas-Buck polynomials:

$$B_n(g;y)=\frac{1}{U(1)V(nyW(1))}\sum_{k=0}^{\infty}p_k(ny)g\left(\frac{k}{n}\right),\quad y>0,n\in\mathbb{N} \tag{10}$$

where the generating relation, $p_j(y)$, is given by

$$U(\mu)V(yW(\mu))=\sum_{j=0}^{\infty}p_j(y)\mu^j \tag{11}$$

with $U$, $V$ and $W$ as specified in (5)-(7).

*Remark 2* For convergence, the operators (10) were assumed to satisfy:

1. $U(1)\neq 0$, $W^{(1)}=1$, $p_j(y)\geq 0$, $j=0,1,2\ldots$,
2. $V:\mathbb{R}\to(0,\infty)$,
3. (11), (5)–(7) converge for $|\mu|<R$, $R>1$.

Similar to the work of [40], Sidharth et al. [38] recently proposed and investigated the properties of a Szász-Mirakjan-Durrmeyer operator involving the Boas-Buck polynomials:

$$\begin{aligned}M_n(g;y)=&\frac{1}{U(1)V(nyW(1))}\sum_{k=1}^{\infty}\frac{p_k(ny)}{\beta(k,n+1)}\int_0^{\infty}\frac{\mu^{k-1}}{(1+\mu)^{n+k+1}}g(\mu)d\mu\\&+\frac{a_0b_0}{U(1)V(nyW(1))}f(0),\end{aligned} \tag{12}$$

where $\beta(k,n+1)$ is the beta function and $y>0, n\in\mathbb{N}$. For additional significant and recent work involving approximation via these polynomial classes, consult [39], [33], [11], [41], [42], [12], [36], [15], [35], [34].

*Remark 3* It is important to note that in special cases, the Boas-Buck polynomials get reduced to well-studied polynomials as follows:

1. In (11), let $W(\mu)=\mu$. We get the Brenke polynomials.
2. In (11), let $V(\mu)=e^{\mu}$. We get the Sheffer polynomials.
3. In (11), let $V(\mu)=e^{\mu}$, $W(\mu)=\mu$. We get the Appell polynomials.

Thus, approximation operators involving the Boas-Buck polynomials form a rich class, and the results derived for these can be easily get reduced to the results for operators based on the aforementioned polynomials as well.

Therefore, motivated by [40], [19], [20], [38], we propose a new summation-integral operator formulated by Durrmeyer-type modification of (2) and the Boas-Buck polynomials. The main contribution of this study is that the proposed operators can reproduce a large number of approximation operators, based on functions studied in past decades. Thus, the properties of the proposed operators can serve as a general results for these special cases, for which these results can be derived with ease. We dedicate a discussion to this later. The remaining paper contains important results for the proposed operator on uniform convergence, Voronovskaja-type theorem, results involving the usual modulus of continuity, and approximation on weighted space. Further, quantitative Voronovskaja-type theorems have very recently been acknowledged as valuable properties for approximating functions [8]. These form a noteworthy part of this work.

## 2 Theoretical Framework

### 2.1 Construction of the Proposed Operator

Let $\gamma>0$, $C_\gamma[0,\infty)$ be the space $\{g\in C[0,\infty):|g(\mu)|\leq M(1+\mu^\gamma)$ for some $M>0\}$ equipped with the norm

$$||g||_\gamma=\sup_{\mu\in[0,\infty)}\frac{|g(\mu)|}{1+\mu^\gamma}.$$

Then, for $g \in C_\gamma[0,\infty)$, we propose the following novel approximation operator:

$$\begin{aligned}\mathfrak{R}_n^{(\rho)}(g;y) =& \frac{n(\rho-1)}{\rho}\frac{1}{U(1)V(nyW(1))}\sum_{k=1}^{\infty}p_k(ny)\int_0^{\infty}\mathfrak{r}_{n,k-1}^{(\rho)}(\mu)g(\mu)d\mu \\ &+\frac{1}{U(1)V(nyW(1))}p_0(ny)g(0). \end{aligned} \tag{13}$$

*Remark 4* The proposed operators (13) are a generalization of the operators in [22], [41], [38]. Further, using Remark 1 and Remark 3, we have the following particular cases for $\mathfrak{R}_n^{(\rho)}$:

1. In (13), let $\rho = -n$. Then (13) reproduce a hybrid generalization of Bernstein operators [14] incorporating the Boas-Buck polynomials.
2. In (13), let $\rho = n$. Then (13) reproduce a hybrid generalization of Baskakov operators [13] incorporating the Boas-Buck polynomials.
3. In (13), let $\rho \to \infty$. Then (13) reproduce a hybrid generalization of Szász-Mirakjan operators [37] incorporating the Boas-Buck polynomials.
4. In (13), let $\rho = ny$. Then (13) reproduce a hybrid generalization of Lupaş operators [31] [9] incorporating the Boas-Buck polynomials.
5. In (13), let $\rho = -n$ and $W(\mu) = \mu$. Then (13) reproduce a hybrid generalization of Bernstein operators [14] incorporating the Brenke polynomials.
6. In (13), let $\rho = n$ and $W(\mu) = \mu$. Then (13) reproduce a hybrid generalization of Baskakov operators [13] incorporating the Brenke polynomials.
7. In (13), let $\rho \to \infty$ and $W(\mu) = \mu$. Then (13) reproduce a hybrid generalization of Szász-Mirakjan operators [37] incorporating the Brenke polynomials.
8. In (13), let $\rho = ny$ and $W(\mu) = \mu$. Then (13) reproduce a hybrid generalization of Lupaş operators [31] [9] incorporating the Brenke polynomials.
9. In (13), let $\rho = -n$ and $V(\mu) = e^{\mu}$. Then (13) reproduce a hybrid generalization of Bernstein operators [14] incorporating the Sheffer polynomials.
10. In (13), let $\rho = n$ and $V(\mu) = e^{\mu}$. Then (13) reproduce a hybrid generalization of Baskakov operators [13] incorporating the Sheffer polynomials.
11. In (13), let $\rho \to \infty$ and $V(\mu) = e^{\mu}$. Then (13) reproduce a hybrid generalization of Szász-Mirakjan operators [37] incorporating the Sheffer polynomials.
12. In (13), let $\rho = ny$ and $V(\mu) = e^{\mu}$. Then (13) reproduce a hybrid generalization of Lupaş operators [31] [9] incorporating the Sheffer polynomials.
13. In (13), let $\rho = -n$ and $V(\mu) = e^{\mu}$, $W(\mu) = \mu$. Then (13) reproduce a hybrid generalization of Bernstein operators [14] incorporating the Appell polynomials.
14. In (13), let $\rho = n$ and $V(\mu) = e^{\mu}$, $W(\mu) = \mu$. Then (13) reproduce a hybrid generalization of Baskakov operators [13] incorporating the Appell polynomials.
15. In (13), let $\rho \to \infty$ and $V(\mu) = e^{\mu}$, $W(\mu) = \mu$. Then (13) reproduce a hybrid generalization of Szász-Mirakjan operators [37] incorporating the Appell polynomials.
16. In (13), let $\rho = ny$ and $V(\mu) = e^{\mu}$, $W(\mu) = \mu$. Then (13) reproduce a hybrid generalization of Lupaş operators [31] [9] incorporating the Appell polynomials.

### 2.2 Some Auxiliary Results

**Lemma 1** *[40], [38] For the Boas-Buck polynomials (11), we have the following useful results :*

1. $$\sum_{j=0}^{\infty}p_j(ny) = U(1)V(nyW(1))$$
2. $$\sum_{j=0}^{\infty}jp_j(ny) = [U^{(1)}(1)V(nyW(1))] + ny[U(1)V^{(1)}(nyW(1))]$$

3. $$\sum_{j=0}^{\infty} j^2 p_j(ny) = [U^{(2)}(1) + U^{(1)}(1)]V(nyW(1)) + [2U^{(1)}(1) + U(1) + U(1)W^{(2)}(1)]V^{(1)}(nyW(1))(ny) + U(1)V^{(2)}(nyW(1))(ny)^2$$

4. $$\sum_{j=0}^{\infty} j^3 p_j(ny) = [4U^{(2)}(1) + U^{(1)}(1)]V(nyW(1)) + [6U^{(1)}(1) + U(1) + 3U(1)W^{(2)}(1) + 3U^{(2)}(1) + 3U^{(1)}(1)W^{(2)}(1) + U(1)W^{(3)}(1)]V^{(1)}(nyW(1))(ny) + [3U(1) + 3U^{(1)}(1) + 3U(1)W^{(2)}(1)]V^{(2)}(nyW(1))(ny)^2 + [U(1)]V^{(3)}(nyW(1))(ny)^3$$

5. $$\sum_{j=0}^{\infty} j^4 p_j(ny) =[13U^{(2)}(1) + U^{(1)}(1) + U^{(4)}(1)]V(nyW(1)) + [4U^{(3)}(1) + 6U^{(2)}(1)W^{(2)}(1) + 4U^{(1)}(1)W^{(3)}(1) + U(1)W^{(4)}(1) + 36U^{(1)}(1) + U(1) + 7U(1)W^{(2)}(1) + 18U^{(2)}(1) + 18U^{(1)}W^{(2)}(1) + 6U(1)W^{(3)}(1) - 22U^{(1)}(1)]V^{(1)}(nyW(1))(ny) + [6U^{(2)}(1) + 12W^{(2)}(1) + U^{(1)}(1) + 4U(1)W^{(3)}(1) + 3U(1)[W^{(2)}(1)]^2 + 7U(1) + 18U^{(1)}(1) + 18U(1)W^{(2)}(1)]V^{(2)}(nyW(1))(ny)^2 + [4U^{(1)}(1) + 6U(1)W^{(2)}(1) + 6U(1)]V^{(3)}(nyW(1))(ny)^3 + [U(1)]V^{(4)}(nyW(1))(ny)^4.$$

**Lemma 2** *For the moments of the form*

$$\int_0^{\infty} \mathfrak{r}_{n,k-1}^{(\rho)}(\mu)\mu^r d\mu, \quad r = 0, 1, \ldots, 4,$$

*where* $\mathfrak{r}_{n,k-1}^{(\rho)}(\mu)$ *is as in* (3)*, we have the following results:*

1. $\int_0^{\infty} \mathfrak{r}_{n,k-1}^{(\rho)}(\mu)d\mu = \frac{\rho}{n(\rho-1)}$
2. $\int_0^{\infty} \mathfrak{r}_{n,k-1}^{(\rho)}(\mu)\mu d\mu = \frac{\rho^2 k}{n^2(\rho-1)(\rho-2)}$
3. $\int_0^{\infty} \mathfrak{r}_{n,k-1}^{(\rho)}(\mu)\mu^2 d\mu = \frac{\rho^3 k(k+1)}{n^3(\rho-1)(\rho-2)(\rho-3)}$
4. $\int_0^{\infty} \mathfrak{r}_{n,k-1}^{(\rho)}(\mu)\mu^3 d\mu = \frac{\rho^4 k(k+1)(k+2)}{n^4(\rho-1)(\rho-2)(\rho-3)(\rho-4)}$
5. $\int_0^{\infty} \mathfrak{r}_{n,k-1}^{(\rho)}(\mu)\mu^4 d\mu = \frac{\rho^5 k(k+1)(k+2)(k+3)}{n^5(\rho-1)(\rho-2)(\rho-3)(\rho-4)(\rho-5)}.$

*Proof* All parts follow a simple and direct computation. Part (3.) will be proved, and other parts follow likewise. Consider

$$\int_0^{\infty} \mathfrak{r}_{n,k-1}^{(\rho)}(\mu)\mu^2 d\mu = \int_0^{\infty} \frac{(\rho)_{k-1}}{(k-1)!} \frac{(\frac{n\mu}{\rho})^{k-1}}{(1+\frac{n\mu}{\rho})^{\rho+k-1}} \mu^2 d\mu = \frac{(\rho)_{k-1}}{(k-1)!} \frac{\rho^2}{n^2} \int_0^{\infty} \frac{(\frac{n\mu}{\rho})^{k+1}}{(1+\frac{n\mu}{\rho})^{\rho+k-1}} d\mu. \tag{14}$$

Let $\mathfrak{l} = \frac{n\mu}{\rho}$, $d\mu = \frac{\rho}{n} d\mathfrak{l}$. Then

$$\int_0^{\infty} \mathfrak{r}_{n,k-1}^{(\rho)}(\mu)\mu^2 d\mu = \frac{(\rho)_{k-1}}{(k-1)!} \frac{\rho^2}{n^2} \frac{\rho}{n} \int_0^{\infty} \frac{(\mathfrak{l})^{k+1}}{(1+\mathfrak{l})^{\rho+k-1}} d\mathfrak{l} = \frac{(\rho)_{k-1}}{(k-1)!} \frac{\rho^3}{n^3}\, \beta(k+2, \rho-3), \tag{15}$$

where $\beta(m, n)$ is the beta function of second kind defined as:

$$\int_0^{\infty} \frac{\mathfrak{z}^{m-1}}{(1+\mathfrak{z})^{m+n}} d\mathfrak{z} = \beta(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}. \tag{16}$$

Then, substituting for $\beta(k+2, \rho-3)$ in (15), some simplification leads to

$$\int_0^{\infty} \mathfrak{r}_{n,k-1}^{(\rho)}(\mu)\mu^2 d\mu = \frac{\rho^3}{n^3}\frac{(k+1)(k)}{(\rho-1)(\rho-2)(\rho-3)}. \tag{17}$$

□

## 2.3 Results on Moments

Using Lemmas 1 and 2, we present the results on moments of (13).

**Lemma 3** *For the operators $\mathfrak{R}_n^{(\rho)}(g; y)$, the moments are given by:*

1. $\mathfrak{R}_n^{(\rho)}(1; y) = 1$
2. $$\mathfrak{R}_n^{(\rho)}\left(\mu^1; y\right) = \frac{\rho}{n(\rho-2)}\left[\frac{V^{(1)}(nyW(1))}{V(nyW(1))}(ny) + \frac{U^{(1)}(1)}{U(1)}\right]$$
3. $$\mathfrak{R}_n^{(\rho)}\left(\mu^2; y\right) = \frac{\rho^2}{n^2(\rho-2)(\rho-3)}\left[\frac{V^{(2)}(nyW(1))}{V(nyW(1))}(ny)^2 + \left(2\frac{U^{(1)}(1)}{U(1)} + W^{(2)}(1) + 2\right)\frac{V^{(1)}(nyW(1))}{V(nyW(1))}(ny) + 2\frac{U^{(1)}(1)}{U(1)} + \frac{U^{(2)}(1)}{U(1)}\right]$$
4. $$\mathfrak{R}_n^{(\rho)}\left(\mu^3; y\right) = \frac{\rho^3}{n^3(\rho-2)(\rho-3)(\rho-4)}\left[\frac{V^{(3)}(nyW(1))}{V(nyW(1))}(ny)^3 + \left(3\frac{U^{(1)}(1)}{U(1)} + 6 + 3W^{(2)}(1)\right)\right. \\ \cdot \frac{V^{(2)}(nyW(1))}{V(nyW(1))}(ny)^2 + \left(12\frac{U^{(1)}(1)}{U(1)} + W^{(2)}(1) + 3\frac{U^{(1)}(1)}{U(1)}W^{(2)}(1) + W^{(3)}(1) + 4\right)\frac{V^{(1)}(nyW(1))}{V(nyW(1))}(ny) \\ \left. + 7\frac{U^{(2)}(1)}{U(1)} + 6\frac{U^{(1)}(1)}{U(1)}\right]$$
5. $$\mathfrak{R}_n^{(\rho)}\left(\mu^4; y\right) = \frac{\rho^4}{n^4(\rho-2)(\rho-3)(\rho-4)(\rho-5)}\left[\frac{V^{(4)}(nyW(1))}{V(nyW(1))}(ny)^4\right. \\ + \left(4\frac{U^{(1)}(1)}{U(1)} + 6W^{(2)}(1) + 12\right)\frac{V^{(3)}(nyW(1))}{V(nyW(1))}(ny)^3 + \left(6\frac{U^{(2)}(1)}{U(1)}\right. \\ + 12\frac{W^{(2)}(1)}{U(1)}U^{(1)}(1) + 21\frac{U^{(1)}(1)}{U(1)} + 3\frac{W^{(2)}(1)}{U(1)} + 4W^{(3)}(1) + 18W^{(2)}(1) \\ \left. + 3[W^{(2)}(1)]^2 + 21\right)\frac{V^{(2)}(nyW(1))}{V(nyW(1))}(ny)^2 + \left(4\frac{U^{(2)}(1)}{U(1)} + 6\frac{U^{(2)}(1)}{U(1)}W^{(1)}(1)\right. \\ + 36\frac{U^{(2)}(1)}{U(1)}W^{(2)}(1) + 42\frac{U^{(1)}(1)}{U(1)} + 4\frac{U^{(1)}(1)}{U(1)}W^{(3)}(1) + 36\frac{U^{(2)}(1)}{U(1)} + W^{(4)}(1) \\ \left. + 12W^{(3)}(1) + 36W^{(2)}(1) + 24\right)\frac{V^{(1)}(nyW(1))}{V(nyW(1))}(ny) + \frac{U^{(4)}(1)}{U(1)} + 48\frac{U^{(2)}(1)}{U(1)} \\ \left. + 13\frac{U^{(1)}(1)}{U(1)} + 11\right].$$

*Proof* In view of the reults in 2, 1, all parts are the result of a direct calculation, thus omitted. □

2.4 Results on Central Moments

**Lemma 4** *Let $\mathfrak{T}_{n,r}^{(\rho)}(y) := \mathfrak{R}_n^{(\rho)}((\mu-y)^r; y)$ nomenclate the $r^{th}$ central moment, $r = 0, 1, \ldots 4$. Using the fact that the operator $\mathfrak{R}_n^{(\rho)}$ is a positive linear operator, from Lemma 3, the following results on central moments can be obtained:*

1. $\mathfrak{T}_{n,0}^{(\rho)}(y) = 1$

2. $$\mathfrak{T}_{n,1}^{(\rho)}(y) = \left[\frac{\rho}{(\rho-2)}\frac{V^{(1)}(nyW(1))}{V(nyW(1))} - 1\right](y) + \frac{\rho}{n(\rho-2)}\frac{U^{(1)}(1)}{U(1)}$$

3. $$\begin{aligned}\mathfrak{T}_{n,2}^{(\rho)}(y) = &\left[\frac{\rho^2}{(\rho-2)(\rho-3)}\frac{V^{(2)}(nyW(1))}{V(nyW(1))} - \frac{2\rho}{(\rho-2)}\frac{V^{(1)}(nyW(1))}{V(nyW(1))} + 1\right](y)^2\\ &+\left[\frac{\rho^2}{n(\rho-2)(\rho-3)}\left(\frac{2U^{(1)}(1)}{U(1)} + W^{(2)}(1) + 2\right)\frac{V^{(1)}(nyW(1))}{V(nyW(1))} - \frac{2\rho}{n(\rho-2)}\frac{U^{(1)}(1)}{U(1)}\right](y)\\ &+\left[\frac{\rho^2}{n^2(\rho-2)(\rho-3)}\left(2\frac{U^{(1)}(1)}{U(1)} + \frac{U^{(2)}(1)}{U(1)}\right)\right]\end{aligned}$$

4. $$\begin{aligned}\mathfrak{T}_{n,4}^{(\rho)}(y) = (y)^4&\left[\frac{\rho^4}{(\rho-2)(\rho-3)(\rho-4)(\rho-5)}\frac{V^{(4)}(nyW(1))}{V(nyW(1))}\right.\\ &-4\frac{\rho^3}{(\rho-2)(\rho-3)(\rho-4)}\frac{V^{(3)}(nyW(1))}{V(nyW(1))}\\ &\left.+6\frac{\rho^2}{(\rho-2)(\rho-3)}\frac{V^{(2)}(nyW(1))}{V(nyW(1))} - 4\frac{\rho}{(\rho-2)}\frac{V^{(1)}(nyW(1))}{V(nyW(1))} + 1\right]\\ &+(y)^3\left[\frac{\rho^4}{n(\rho-2)(\rho-3)(\rho-4)(\rho-5)}\left(4\frac{U^{(1)}(1)}{U(1)} + 6W^{(2)}(1) + 12\right)\frac{V^{(3)}(nyW(1))}{V(nyW(1))}\right.\\ &-4\frac{\rho^3}{n(\rho-2)(\rho-3)(\rho-4)}\left(3\frac{U^{(1)}(1)}{U(1)} + 6 + 3W^{(2)}(1)\right)\frac{V^{(2)}(nyW(1))}{V(nyW(1))}\\ &\left.+6\frac{\rho^2}{n(\rho-2)(\rho-3)}\left(2\frac{U^{(1)}}{U(1)} + W^{(2)}(1) + 2\right)\frac{V^{(1)}(nyW(1))}{V(nyW(1))} - 4\frac{\rho}{n(\rho-2)}\frac{U^{(1)}}{U(1)}\right]\\ &+(y)^2\left[\frac{\rho^4}{n^2(\rho-2)(\rho-3)(\rho-4)(\rho-5)}\left(6\frac{U^{(2)}(1)}{U(1)} + 12\frac{U^{(1)}(1)}{U(1)}W^{(2)}(1) + 21\frac{U^{(1)}(1)}{U(1)}\right.\right.\\ &\left.+3\frac{W^{(2)}(1)}{U(1)} + 4W^{(3)}(1) + 18W^{(2)}(1) + 3[W^{(2)}(1)]^2 + 21\right)\frac{V^{(2)}(nyW(1))}{V(nyW(1))}\\ &-4\frac{\rho^3}{n^2(\rho-2)(\rho-3)(\rho-4)}\left(12\frac{U^{(1)}}{U(1)} + W^{(2)}(1) + 3\frac{U^{(1)}}{U(1)}W^{(2)}(1) + W^{(3)}(1) + 4\right)\\ &\left.\cdot\frac{V^{(1)}(nyW(1))}{V(nyW(1))}\right]\\ &+(y)\left[\frac{\rho^4}{n^3(\rho-2)(\rho-3)(\rho-4)(\rho-5)}\left(4\frac{U^{(2)}(1)}{U(1)} + 6\frac{U^{(2)}(1)}{U(1)}W^{(1)}(1)\right.\right.\\ &+36\frac{U^{(2)}(1)}{U(1)}W^{(2)}(1) + 42\frac{U^{(1)}(1)}{U(1)} + 4\frac{U^{(1)}(1)}{U(1)}W^{(3)}(1) + 36\frac{U^{(2)}(1)}{U(1)} + W^{(4)}(1)\\ &\left.+12W^{(3)}(1) + 36W^{(2)}(1) + 24\right) - 4\frac{\rho^3}{n^3(\rho-2)(\rho-3)(\rho-4)}\end{aligned}$$

$$\cdot\left(7\frac{U^2(1)}{U(1)}+6U^1(1)U(1)\right)\Bigg]$$
$$+\left[\frac{\rho^4}{n^4\,(\rho-2)\,(\rho-3)\,(\rho-4)\,(\rho-5)}\left(\frac{U^{(4)}(1)}{U(1)}+48\frac{U^{(2)}(1)}{U(1)}+13\frac{U^{(1)}(1)}{U(1)}+11\right)\right].$$

*The result for* $\mathfrak{R}_n^{(\rho)}\left((\mu-y)^6;y\right)$, *which is omitted as it is quite complicated and lengthy, will be needed to establish the quantitative Voronovskaja-type theorem later.*

*Remark 5* For the purpose of the results presented in the paper, we make the following assumptions:

1. $$\rho=\rho(n)\to\infty \text{ as } n\to\infty \text{ and } \lim_{n\to\infty}\frac{n}{\rho}=q\in\mathbb{R}$$
2. $$\lim_{\mu\to\infty}\frac{V^{(k)}(\mu)}{V(\mu)}=1,\,k\in\mathbb{N},\,k\geq1$$
3. $$\lim_{n\to\infty}n\left[\frac{\rho}{(\rho-2)}\frac{V^{(1)}(nyW(1))}{V(nyW(1))}-1\right]=m_1(y)$$
4. $$\lim_{n\to\infty}n\left[\frac{\rho^2}{(\rho-2)(\rho-3)}\frac{V^{(2)}(nyW(1))}{V(nyW(1))}-\frac{2\rho}{(\rho-2)}\frac{V^{(1)}(nyW(1))}{V(nyW(1))}+1\right]=m_2(y)$$
5. $$\lim_{n\to\infty}n^2\left[\frac{\rho^4}{n\,(\rho-2)\,(\rho-3)\,(\rho-4)\,(\rho-5)}\left(4\frac{U^{(1)}(1)}{U(1)}+6W^{(2)}(1)+12\right)\frac{V^{(3)}(nyW(1))}{V(nyW(1))}\right.$$
$$-4\frac{\rho^3}{n\,(\rho-2)\,(\rho-3)\,(\rho-4)}\left(3\frac{U^{(1)}(1)}{U(1)}+6+3W^{(2)}(1)\right)\frac{V^{(2)}(nyW(1))}{V(nyW(1))}$$
$$\left.+6\frac{\rho^2}{n\,(\rho-2)\,(\rho-3)}\left(2\frac{U^{(1)}}{U(1)}+W^{(2)}(1)+2\right)\frac{V^{(1)}(nyW(1))}{V(nyW(1))}-4\frac{\rho}{n\,(\rho-2)}\frac{U^{(1)}}{U(1)}\right]=m_3(y)$$
6. $$\lim_{n\to\infty}n^2\left[\frac{\rho^4}{(\rho-2)\,(\rho-3)\,(\rho-4)\,(\rho-5)}\frac{V^{(4)}(nyW(1))}{V(nyW(1))}-4\frac{\rho^3}{(\rho-2)\,(\rho-3)\,(\rho-4)}\frac{V^{(3)}(nyW(1))}{V(nyW(1))}\right.$$
$$\left.+6\frac{\rho^2}{(\rho-2)\,(\rho-3)}\frac{V^{(2)}(nyW(1))}{V(nyW(1))}-4\frac{\rho}{(\rho-2)}\frac{V^{(1)}(nyW(1))}{V(nyW(1))}+1\right]=m_4(y).$$

Under these assumptions, we have

1. $$\lim_{n\to\infty}n\mathfrak{T}_{n,1}^{(\rho)}(y)=m_1(y)y+\frac{U^{(1)}(1)}{U(1)}$$
2. $$\lim_{n\to\infty}n\mathfrak{T}_{n,2}^{(\rho)}(y)=m_2(y)y^2+\left(W^{(2)}(1)+2\right)y=\nu_1(y),$$
   say.
3. $$\lim_{n\to\infty}n^2\mathfrak{T}_{n,4}^{(\rho)}(y)$$
   $$=m_4(y)y^4+m_3(y)y^3+\left(6\frac{U^{(2)}(1)}{U(1)}+14W^{(2)}(1)\right.$$
   $$\left.+3[W^{(2)}(1)]^2-27\frac{U^{(1)}(1)}{U(1)}+\frac{W^{(2)}(1)}{U(1)}+5\right)y^2=\nu_2(y),$$
   say.

## 3 Direct Results

The following can be immediately established due to the foregoing results.

### 3.1 Uniform Convergence

**Theorem 1** *Let $g \in C_\kappa[0,\infty)$ and $\rho = \rho(n) \to \infty$ as $n \to \infty$. Then,*

$$\lim_{n\to\infty} \mathfrak{R}_n^{(\rho)}(g; y) = g(y), \tag{18}$$

*uniformly in each compact subset of* $[0,\infty)$.

*Proof* From lemma 3 and using the assumptions in the Remark 2, as $n \to \infty$, $\mathfrak{R}_n^{(\rho)}(1; y) = 1$, $\mathfrak{R}_n^{(\rho)}(\mu; y) \to y$, $\mathfrak{R}_n^{(\rho)}(\mu^2; y) \to y^2$ uniformly in each compact subset of $[0,\infty)$. Thus, from the Bohman-Korovkin theorem, $\mathfrak{R}_n^{(\rho)}(g; y) \to g(y)$ as $n \to \infty$ for any $g$, uniformly in each compact subset of $[0,\infty)$. □

### 3.2 Voronovskaja-type Theorem

**Theorem 2** *Let $g \in C_\kappa[0,\infty)$ and $\rho = \rho(n) \to \infty$ as $n \to \infty$. If $g^{(2)}$ exists at $y \in [0,\infty)$ and*

$$\lim_{n\to\infty} \frac{n}{\rho(n)} = q \in \mathbb{R},$$

*then the following holds:*

$$\begin{aligned}&\lim_{n\to\infty} n\Big[\mathfrak{R}_n^{(\rho)}(g; y) - g(y)\Big]\\ &= \left[m_1(y)y + \frac{U^{(1)}(1)}{U(1)}\right] g^{(1)}(y) + \Big[m_2(y)y^2 + \Big(W^{(2)}(1) + 2\Big)\, y\Big] \frac{g^{(2)}(y)}{2}.\end{aligned} \tag{19}$$

*Proof* Consider Taylor's expansion in the form

$$g(\mu) = \sum_{r=0}^{2} \frac{1}{r!} g^{(r)}(y)(\mu - y)^r + \xi(\mu, y)(\mu - y)^2, \tag{20}$$

where $\xi(\mu, y)$ is a function such that $\lim_{\mu\to y} \xi(\mu, y) = 0$. Therefore, operating by $\mathfrak{R}_n^{(\rho)}$ on (20),

$$\mathfrak{R}_n^{(\rho)}\,(g(\mu); y) - g(y) = \sum_{r=1}^{2} \frac{1}{r!} g^{(r)}(y)\mathfrak{T}_{n,r}^{(\rho)}(y) + \mathfrak{R}_n^{(\rho)}(\xi(\mu, y)(\mu - y)^2; y). \tag{21}$$

From the Cauchy-Schwarz inequality,

$$n\mathfrak{R}_n^{(\rho)}(\xi(\mu, y)(\mu - y)^2; y) \le \sqrt{\mathfrak{R}_n^{(\rho)}(\xi^2(\mu, y); y)}\sqrt{n^2\mathfrak{T}_{n,4}^{(\rho)}(y)}. \tag{22}$$

By Theorem 1,

$$\lim_{n\to\infty} \mathfrak{R}_n^{(\rho)}(\xi^2(\mu, y); y) = 0. \tag{23}$$

Using Remark 5 and (23) above, in (22), we get:

$$\lim_{n\to\infty} n\mathfrak{R}_n^{(\rho)}(\xi(\mu, y)(\mu - y)^2; y) = 0. \tag{24}$$

Hence, substituting the values of central moments from lemma 4, we get:

$$\begin{aligned}&\lim_{n\to\infty} n\Big[\mathfrak{R}_n^{(\rho)}(g; y) - g(y)\Big]\\ &= \left[m_1(y)y + \frac{U^{(1)}(1)}{U(1)}\right] g^{(1)}(y) + \Big[m_2(y)y^2 + \Big(W^{(2)}(1) + 2\Big)\, y\Big] \frac{g^{(2)}(y)}{2}.\end{aligned} \tag{25}$$

□

### 3.3 Local Approximation Properties

*Preliminaries* Let $\tilde{C}_B[0,\infty)$ depict the space of all real valued, bounded, and uniformly continuous functions $g$ on $[0,\infty)$, with

$$\|g\|_{\tilde{C}_B[0,\infty)} = \sup_{y\in[0,\infty)} |g(y)| \tag{26}$$

being the norm on $\tilde{C}_B[0,\infty)$. The modulus of continuity of $g \in \tilde{C}_B[0,\infty)$ is defined as

$$\mathfrak{w}(g;\mathfrak{d}) = \sup_{y,\mathfrak{a},\mathfrak{b}\geq 0}\ \sup_{|\mathfrak{a}-\mathfrak{b}|\leq \mathfrak{d}} |g(y+\mathfrak{a}) - g(y+\mathfrak{b})|\,,\ \mathfrak{d}\geq 0 \tag{27}$$

and the second order modulus of continuity is defined as

$$\mathfrak{w}_2(g;\mathfrak{d}) = \sup_{y,\mathfrak{a},\mathfrak{b}\geq 0}\ \sup_{|\mathfrak{a}-\mathfrak{b}|\leq \mathfrak{d}} |g(y+2\mathfrak{a}) - 2g(y+\mathfrak{a}+\mathfrak{b}) + g(y+2\mathfrak{b})|\,,\ \mathfrak{d}\geq 0. \tag{28}$$

For $g \in \tilde{C}_B[0,\infty)$, the Steklov mean is defined as [18]:

$$g_{\mathfrak{h}}(y) = \frac{4}{\mathfrak{h}^2}\int_0^{\frac{\mathfrak{h}}{2}}\int_0^{\frac{\mathfrak{h}}{2}} [2g(y+\mathfrak{a}+\mathfrak{b}) - g(y+2(\mathfrak{a}+\mathfrak{b}))]\, d\mathfrak{a}d\mathfrak{b}. \tag{29}$$

The following properties related to the Steklov mean can be observed, [18]:

1. $\|g_{\mathfrak{h}} - g\|_{\tilde{C}_B[0,\infty)} \leq \mathfrak{w}_2(g;\mathfrak{h})$
2. $g_{\mathfrak{h}}^{(1)}, g_{\mathfrak{h}}^{(2)} \in \tilde{C}_B[0,\infty)$; $\|g_{\mathfrak{h}}^{(1)}\|_{\tilde{C}_B[0,\infty)} \leq \frac{5}{\mathfrak{h}}\mathfrak{w}(g;\mathfrak{h})$;
   $\|g_{\mathfrak{h}}^{(2)}\|_{\tilde{C}_B[0,\infty)} \leq \frac{9}{\mathfrak{h}^2}\mathfrak{w}_2(g;\mathfrak{h})$

**Theorem 3** *Let $g \in \tilde{C}_B[0,\infty)$. Then, for every $y\geq 0$, the following inequality holds:*

$$\left|\mathfrak{R}_n^{(\rho)}(g;y) - g(y)\right| \leq 5\mathfrak{w}\left(g;\sqrt{\mathfrak{T}_{n,2}^{(\rho)}(y)}\right) + \frac{13}{2}\mathfrak{w}_2\left(g;\sqrt{\mathfrak{T}_{n,2}^{(\rho)}(y)}\right) \tag{30}$$

*Proof* Using the definition of Steklov mean from (29), the following can be written:

$$\left|\mathfrak{R}_n^{(\rho)}(g;y) - g(y)\right| \leq \mathfrak{R}_n^{(\rho)}(\left|g-g_{\mathfrak{h}}\right|;y) + \left|\mathfrak{R}_n^{(\rho)}(g_{\mathfrak{h}} - g_{\mathfrak{h}}(y);y)\right| + \left|g_{\mathfrak{h}}(y) - g(y)\right| \tag{31}$$

Also, for every $g \in \tilde{C}_B[0,\infty)$, we have

$$\left|\mathfrak{R}_n^{(\rho)}(g;y)\right| \leq \|g\|_{\tilde{C}_B[0,\infty)} \tag{32}$$

Consider the first term in (31). Using (32) and then property (1.) of Steklov mean, we get

$$\begin{aligned}\mathfrak{R}_n^{(\rho)}(\left|g-g_{\mathfrak{h}}\right|;y) &\leq \|\mathfrak{R}_n^{(\rho)}(g-g_{\mathfrak{h}};y)\|_{\tilde{C}_B[0,\infty)}\\ &\leq \|g-g_{\mathfrak{h}}\|_{\tilde{C}_B[0,\infty)}\\ &\leq \mathfrak{w}_2(g;h)\end{aligned} \tag{33}$$

Now, consider the second term in (31). Expanding $g_{\mathfrak{h}}(\mu)$ as a Taylor series upto second derivative term, we get

$$g_{\mathfrak{h}}(\mu) = \sum_{r=0}^{2} g_{\mathfrak{h}}^{(r)}\frac{(\mu-y)^r}{r!} + \xi(\mu,y)((\mu-y)^2) \tag{34}$$

Therefore

$$\left|\mathfrak{R}_n^{(\rho)}(g_{\mathfrak{h}} - g_{\mathfrak{h}}(y); y)\right| \approx \left|\mathfrak{R}_n^{(\rho)}((\mu - y)g_{\mathfrak{h}}^{(1)}(y); y) + \mathfrak{R}_n^{(\rho)}\left(\frac{(\mu - y)^2}{2}g_{\mathfrak{h}}^{(2)}(y); y\right)\right|$$
$$\leq \left|\mathfrak{R}_n^{(\rho)}((\mu - y)g_{\mathfrak{h}}^{(1)}(y); y)\right| + \left|\mathfrak{R}_n^{(\rho)}\left(\frac{(\mu - y)^2}{2}g_{\mathfrak{h}}^{(2)}(y); y\right)\right| \tag{35}$$

By the definition of supremum norm and linearity of $\mathfrak{R}_n^{(\rho)}(g; y)$, we can write:

$$\left|\mathfrak{R}_n^{(\rho)}(g_{\mathfrak{h}}(\mu) - g_{\mathfrak{h}}(y); y)\right| \leq \|g_{\mathfrak{h}}^{(1)}\|_{\tilde{C}_B[0,\infty)}\left|\mathfrak{T}_{n,1}^{(\rho)}(y)\right| + \frac{1}{2}\|g_{\mathfrak{h}}^{(2)}\|_{\tilde{C}_B[0,\infty)}\left|\mathfrak{T}_{n,2}^{(\rho)}(y)\right| \tag{36}$$

Using Cauchy-Schwarz inequality on the first term, we get:

$$\left|\mathfrak{R}_n^{(\rho)}(g_{\mathfrak{h}} - g_{\mathfrak{h}}(y); y)\right| \leq \|g_h^{(1)}\|\sqrt{\mathfrak{T}_{n,2}^{(\rho)}(y)} + \frac{1}{2}\|g_{\mathfrak{h}}^{(2)}\|\mathfrak{T}_{n,2}^{(\rho)}(y) \tag{37}$$

Now, consider the third term in (31). Using property (1.) of Steklov mean:

$$\left|g_{\mathfrak{h}}(y) - g(y)\right| \leq \|g_{\mathfrak{h}} - g\|_{\tilde{C}_B[0,\infty)} \leq \mathfrak{w}_2(g; \mathfrak{h}) \tag{38}$$

Using (33), (37), (38) in (31), and choosing $\mathfrak{h} = \sqrt{\mathfrak{T}_{n,2}^{(\rho)}(y)}$ gives:

$$\left|\mathfrak{R}_n^{(\rho)}(g; y) - g(y)\right| \leq \mathfrak{w}_2\left(g; \sqrt{\mathfrak{T}_{n,2}^{(\rho)}(y)}\right) + \|g_{\mathfrak{h}}^{(1)}\|\sqrt{\mathfrak{T}_{n,2}^{(\rho)}(y)}$$
$$+ \frac{1}{2}\|g_{\mathfrak{h}}^{(2)}\|\mathfrak{T}_{n,2}^{(\rho)}(y) + \mathfrak{w}_2\left(g; \sqrt{\mathfrak{T}_{n,2}^{(\rho)}(y)}\right)$$

Finally, using property (2.) of Steklov mean gives

$$\left|\mathfrak{R}_n^{(\rho)}(g; y) - g(y)\right| \leq 5\mathfrak{w}\left(g; \sqrt{\mathfrak{T}_{n,2}^{(\rho)}(y)}\right) + \frac{13}{2}\mathfrak{w}_2\left(g; \sqrt{\mathfrak{T}_{n,2}^{(\rho)}(y)}\right)$$

□

**Theorem 4** *For any $g \in \tilde{C}_B^1[0, \infty)$ and $y \in \mathbb{R}_+ \bigcup \{0\}$, we have*

$$\left|\mathfrak{R}_n^{(\rho)}(g; y) - g(y)\right| \leq 2\sqrt{\mathfrak{T}_{n,2}^{(\rho)}(y)}\,\mathfrak{w}\left(g^{(1)}; \sqrt{\mathfrak{T}_{n,2}^{(\rho)}(y)}\right)$$

*Proof* Using

$$\int_y^{\mu}\left(g^{(1)}(\mathfrak{a}) - g^{(1)}(y)\right)d\mathfrak{a} = g(\mu) - g(y) - g^{(1)}(y)(\mu - y)$$

we get

$$g(\mu) - g(y) = g^{(1)}(y)(\mu - y) + \int_y^{\mu}\left(g^{(1)}(\mathfrak{a}) - g^{(1)}(y)\right)d\mathfrak{a} \tag{39}$$

Operate $\mathfrak{R}_n^{(\rho)}$ on both sides

$$\mathfrak{R}_n^{(\rho)}((g(\mu) - g(y)); y) = g^{(1)}(y)\mathfrak{T}_{n,1}^{(\rho)}(y) + \mathfrak{R}_n^{(\rho)}\left(\int_y^{\mu}\left(g^{(1)}(\mathfrak{a}) - g^{(1)}(y)\right)d\mathfrak{a}; y\right)$$

from properties of modulus of continuity,

$$|g(\mu) - g(y)| \leq \mathfrak{w}(g; \mathfrak{d})\left(\frac{|\mu - y|}{\mathfrak{d}} + 1\right), \quad \mathfrak{d} > 0$$

in the form

$$\left|g^{(1)}(\mathfrak{a}) - g^{(1)}(y)\right| \leq \mathfrak{w}(g^{(1)}; \mathfrak{d})\left(\frac{|\mathfrak{a} - y|}{\mathfrak{d}} + 1\right), \quad \mathfrak{d} > 0$$

after integrating both sides and simplifying, we obtain:

$$\left|\int_y^{\mu}(g^{(1)}(\mathfrak{a})-g^{(1)}(y))du\right|\leq\mathfrak{w}(g^{(1)};\mathfrak{d})\left(\frac{|\mu-y|^2}{\mathfrak{d}}+|\mu-y|\right)$$

Therefore

$$\begin{aligned}&\left|\mathfrak{R}_n^{(\rho)}(g;y)-g(y)\right|\\&\leq\left|g^{(1)}(y)\right|\left|\mathfrak{T}_{n,1}^{(\rho)}(y)\right|+\mathfrak{R}_n^{(\rho)}\left(\mathfrak{w}(g^{(1)};\mathfrak{d})\left(\frac{|\mu-y|^2}{\mathfrak{d}}+|\mu-y|\right);y\right)\end{aligned}$$

that is,

$$\begin{aligned}&\left|\mathfrak{R}_n^{(\rho)}(g;y)-g(y)\right|\\&\leq\left|g^{(1)}(y)\right|\left|\mathfrak{T}_{n,1}^{(\rho)}(y)\right|+\mathfrak{w}(g^{(1)};\mathfrak{d})\left(\frac{1}{\mathfrak{d}}\mathfrak{R}_n^{(\rho)}(|\mu-y|^2;y)+\mathfrak{R}_n^{(\rho)}(|\mu-y|;y)\right)\end{aligned}$$

Using the Cauchy-Schwarz inequality at this stage,

$$\left|\mathfrak{R}_n^{(\rho)}(g;y)-g(y)\right|\leq\left|g^{(1)}(y)\right|\left|\mathfrak{T}_{n,1}^{(\rho)}(y)\right|+\mathfrak{w}(g^{(1)};\mathfrak{d})\left(\frac{1}{\mathfrak{d}}\sqrt{\mathfrak{T}_{n,2}^{(\rho)}(y)}+1\right)\sqrt{\mathfrak{T}_{n,2}^{(\rho)}(y)}$$

Selecting $\mathfrak{d}=\sqrt{\mathfrak{T}_{n,2}^{(\rho)}(y)}$, and in the limit of large enough $n$ and $\rho$, we get the stated result. □

Let us depict by $\mathcal{H}_\zeta[0,\infty)$ the space of all real valued functions on $[0,\infty)$ which satisfy $|g(y)|\leq A_g\zeta(y)$, where $A_g$ is a positive constant dependent on $g$, and $\zeta(y)=1+y^2$ is a weight function.

Let $C_\zeta[0,\infty)$ depict the space of all continuous functions in $\mathcal{H}_\zeta[0,\infty)$, equipped with the norm

$$||g||_\zeta:=\sup_{y\in[0,\infty)}\frac{|g(y)|}{\zeta(y)} \tag{40}$$

Also, let $C_\zeta^*[0,\infty)$ depict the space of all functions $g\in C_\zeta[0,\infty)$ for which the limit $\lim_{y\to\infty}\frac{|g(y)|}{\zeta(y)}$ exists and is finite.

The usual modulus of continuity of $g$ on $[0,\lambda]$ is defined as

$$\mathfrak{w}_\lambda(g;\mathfrak{d})=\sup_{0\leq|\mu-y|\leq\mathfrak{d}}\ \sup_{y,\mu\in[0,\lambda]}|g(\mu)-g(y)| \tag{41}$$

**Theorem 5** *Let $g\in C_\zeta[0,\infty)$. Then the following result holds:*

$$\left|\mathfrak{R}_n^{(\rho)}(g;y)-g(y)\right|\leq 4A_g(1+y^2)\mathfrak{T}_{n,2}^{(\rho)}(y)+2\mathfrak{w}_{\lambda+1}\left(g;\sqrt{\mathfrak{T}_{n,2}^{(\rho)}(y)}\right)$$

*Proof* Referring to [21], [23], for $y\in[0,\lambda]$ and $t\geq 0$, we have

$$|g(\mu)-g(y)|\leq 4A_g(1+y^2)(\mu-y)^2+\left(1+\frac{|\mu-y|}{\mathfrak{d}}\mathfrak{w}_{\lambda+1}(g;\mathfrak{d})\right),\quad \mathfrak{d}>0 \tag{42}$$

Therefore

$$\left|\mathfrak{R}_n^{(\rho)}(g(\mu);y)-g(y)\right|\leq 4A_g(1+y^2)\mathfrak{T}_{n,2}^{(\rho)}(y)+\mathfrak{w}_{\lambda+1}(g;\mathfrak{d})\left(1+\frac{1}{\mathfrak{d}}\left|\mathfrak{T}_{n,1}^{(\rho)}(y)\right|\right)$$

Using the Cauchy-Schwarz inequality,

$$\left|\mathfrak{R}_n^{(\rho)}(g(\mu);y)-g(y)\right|\leq 4A_g(1+y^2)\mathfrak{T}_{n,2}^{(\rho)}(y)+\mathfrak{w}_{\lambda+1}(g;\mathfrak{d})\left(1+\frac{1}{\mathfrak{d}}\left|\sqrt{\mathfrak{T}_{n,2}^{(\rho)}(y)}\right|\right)$$

Choosing $\mathfrak{d}=\sqrt{\mathfrak{T}_{n,2}^{(\rho)}(y)}$, we get

$$\left|\mathfrak{R}_n^{(\rho)}(g(\mu);y)-g(y)\right|\leq 4A_g(1+y^2)\mathfrak{T}_{n,2}^{(\rho)}(y)+2\mathfrak{w}_{\lambda+1}\left(g;\sqrt{\mathfrak{T}_{n,2}^{(\rho)}(y)}\right)$$

□

## 4 Weighted Approximation Properties

**Theorem 6** *Let $g \in C^*_\zeta[0,\infty)$ and $\rho = \rho(n)$ be such that as $n \to \infty$, $\rho(n) \to \infty$. Then, the following holds:*

$$\lim_{n\to\infty} \left|\left|\mathfrak{R}_n^{(\rho)}(g(\mu)) - g\right|\right|_\zeta = 0. \tag{43}$$

*Proof* To demonstrate this result, it is sufficient to establish the following three relations [17]:

$$\lim_{n\to\infty} \left|\left|\mathfrak{R}_n^{(\rho)}(\mu^r) - y^r\right|\right|_\zeta = 0, r = 0, 1, 2. \tag{44}$$

Because $\mathfrak{R}_n^{(\rho)}(1; y) = 1$ due to Lemma 3, the condition in (44) holds true for $r = 0$.

Using Lemma 3, we get

$$\lim_{n\to\infty} \left|\left|\mathfrak{R}_n^{(\rho)}(\mu) - y\right|\right|_\zeta$$

$$= \lim_{n\to\infty} \left|\left|\left(\frac{\rho}{n(\rho-2)}\left(\frac{V^{(1)}(nyW(1))}{V(nyW(1))}(ny) + \frac{U^{(1)}(1)}{U(1)}\right)\right) - y\right|\right|_\zeta \tag{45}$$

$$= 0. \tag{46}$$

Therefore, $\lim_{n\to\infty} \left|\left|\mathfrak{R}_n^{(\rho)}(\mu) - y\right|\right|_\zeta = 0.$

$$\lim_{n\to\infty} \left|\left|\mathfrak{R}_n^{(\rho)}(\mu^2) - y^2\right|\right|_\zeta$$

also turns out to be equal to 0 under the assumptions in Remark 2, leading to $\lim_{n\to\infty} \left|\left|\mathfrak{R}_n^{(\rho)}(\mu^2) - y^2\right|\right|_\zeta = 0.$ Hence the stated result follows. □

We invoke the definition of the weighted modulus of continuity $\mathfrak{W}(g; \mathfrak{d})$ defined on $[0,\infty)$ (see [43]) as follows:

$$\mathfrak{W}(g; \mathfrak{d}) = \sup_{|\mathfrak{m}|<\mathfrak{d},\, y\in[0,\infty)} \frac{|g(y+\mathfrak{m}) - g(y)|}{(1+\mathfrak{m}^2)(1+y^2)} \text{ for } g \in C_\zeta[0,\infty) \tag{47}$$

**Lemma 5** *[43] Let $g \in C^*_\zeta[0,\infty)$, then the following hold:*

1. *$\mathfrak{W}(g; \mathfrak{d})$ is monotone increasing function in $\mathfrak{d}$;*
2. *$\lim_{\mathfrak{d}\to 0^+} \mathfrak{W}(g; \mathfrak{d}) = 0$;*
3. *for each $\phi \in \mathbb{N}$, $\mathfrak{W}(g; \phi\mathfrak{d}) \le \phi\mathfrak{W}(g; \mathfrak{d})$;*
4. *for each $\vartheta \in [0,\infty)$, $\mathfrak{W}(g; \vartheta\mathfrak{d}) \le (1+\vartheta)\mathfrak{W}(g; \mathfrak{d})$.*

**Theorem 7** *Let $g \in C^*_\zeta[0,\infty)$ and $\rho = \rho(n)$ be such that as $n \to \infty$, $\rho(n) \to \infty$, and*

$$\lim_{n\to\infty} \frac{n}{\rho} = q \in \mathbb{R},$$

*then there exists $\mathfrak{m}_0 \in \mathbb{N}$ and a constant $Q(q) \in \mathbb{R}^+$ that depends on $q$, such that:*

$$\sup_{y\in\mathbb{R}^+} \frac{\left|\mathfrak{R}_n^{(\rho)}(g; y) - g(y)\right|}{(1+y^2)^{5/2}} \le Q(q)\mathfrak{W}\left(g; n^{-1/2}\right), \text{ for } n > \mathfrak{m}_0. \tag{48}$$

*Proof* For $\mu > 0$, $y \in \mathbb{R}^+$, $\mathfrak{d} > 0$, by using the definition of $\mathfrak{W}(g; \mathfrak{d})$ and the associated Lemma 5, we can write

$$\begin{aligned}|g(\mu)-g(y)| &\leq \left(1+(\mu-y)^2\right)(1+y^2)\mathfrak{W}(g;|\mu-y|)\\ &\leq \left(1+y^2\right)\left(1+(\mu-y)^2\right)\left(1+\frac{|\mu-y|}{\mathfrak{d}}\right)\mathfrak{W}(g;\mathfrak{d})\\ &\leq \left(1+y^2\right)\mathfrak{W}(g;\mathfrak{d})\left(1+(\mu-y)^2+\left(1+(\mu-y)^2\right)\frac{|\mu-y|}{\mathfrak{d}}\right)\end{aligned} \tag{49}$$

Because $\mathfrak{R}_n^{(\rho)}$ is a positive linear operator,

$$\begin{aligned}&\left|\mathfrak{R}_n^{(\rho)}(g;y)-g(y)\right|\\ &\leq \left(1+y^2\right)\mathfrak{W}(g;\mathfrak{d})\left\{1+\mathfrak{T}_{n,2}^{(\rho)}(y)+\mathfrak{R}_n^{(\rho)}\left(\left(1+(\mu-y)^2\right)\frac{|\mu-y|}{\mathfrak{d}};y\right)\right\}\end{aligned} \tag{50}$$

Using Cauchy-Schwarz inequality, we write

$$\mathfrak{R}_n^{(\rho)}\left(\left(1+(\mu-y)^2\right)\frac{|\mu-y|}{\mathfrak{d}};y\right)\leq\frac{1}{\mathfrak{d}}\sqrt{\mathfrak{T}_{n,2}^{(\rho)}(y)}+\frac{1}{\mathfrak{d}}\sqrt{\mathfrak{T}_{n,2}^{(\rho)}(y)}\sqrt{\mathfrak{T}_{n,4}^{(\rho)}(y)} \tag{51}$$

Using the Remark 5, it can be said that there is an $\mathfrak{m}_1\in\mathbb{N}$ s.t.

$$\mathfrak{T}_{n,2}^{(\rho)}(y)\leq Q_1(q)\frac{1+y^2}{n},\ \text{for } n>\mathfrak{m}_1 \tag{52}$$

and an $\mathfrak{m}_2\in\mathbb{N}$ s.t.

$$\sqrt{\mathfrak{T}_{n,4}^{(\rho)}(y)}\leq Q_2(q)\frac{\sqrt{1+y^2}}{n},\ \text{for } n>\mathfrak{m}_2 \tag{53}$$

where $Q_1(q), Q_2(q)\in\mathbb{R}^+$ are constants that depends on $q$. Let $\mathfrak{m}_0=max\{\mathfrak{m}_1,\mathfrak{m}_2\}$. Combining the above results from (49)-(53), and with the choice $\mathfrak{d}=n^{-1/2}$, for $y>\mathfrak{m}_0$, we obtain the stated result. □

## 5 Quantitative Voronovskaja-type theorem

In this section, we establish a quantitative Voronovskaja-type theorem for the operators $\mathfrak{R}_n^{(\rho)}$ by using the weighted modulus of continuity, $\mathfrak{W}(g;\mathfrak{d})$ 5. Recent important works in this direction are [34] [5] [11] [29].

**Theorem 8** *Let $g\in C_\zeta^*[0,\infty)$ such that $g^{(1)}$ and $g^{(2)}\in C_\zeta^*[0,\infty)$, and $y\geq 0$. Then, the following holds:*

$$\begin{aligned}&\Bigg|\mathfrak{R}_n^{(\rho)}(g;y)-g(y)\\ &-\left[\left[\frac{\rho}{(\rho-2)}\frac{V^{(1)}(nyW(1))}{V(nyW(1))}-1\right](y)+\frac{\rho}{n(\rho-2)}\frac{U^{(1)}(1)}{U(1)}\right]g^{(1)}(y)\\ &-\Bigg(\left[\frac{\rho^2}{(\rho-2)(\rho-3)}\frac{V^{(2)}(nyW(1))}{V(nyW(1))}-\frac{2\rho}{(\rho-2)}\frac{V^{(1)}(nyW(1))}{V(nyW(1))}+1\right](y)^2\\ &+\Bigg[\frac{\rho^2}{n(\rho-2)(\rho-3)}\left(\frac{2U^{(1)}(1)}{U(1)}+W^{(2)}(1)+2\right)\frac{V^{(1)}(nyW(1))}{V(nyW(1))}\\ &-\frac{2\rho}{n(\rho-2)}\frac{U^{(1)}(1)}{U(1)}\Bigg](y)+\left[\frac{\rho^2}{n^2(\rho-2)(\rho-3)}\left(2\frac{U^{(1)}(1)}{U(1)}+\frac{U^{(2)}(1)}{U(1)}\right)\right]\Bigg)g^{(2)}(y)\Bigg|\\ &\leq 8(1+y^2)O\left(\frac{1}{n}\right)\mathfrak{W}\left(g^{(2)};n^{-1/2}\right)\end{aligned}$$

*Proof* Let $y, \mu \geq 0$. Using Taylor's expansion, we can write

$$g(\mu) = g(y) + g^{(1)}(y)(t-y) + \frac{g^{(2)}(y)}{2!}(\mu - y)^2 + \mathfrak{E}(\mu, y),$$

where $\mathfrak{E}(\mu, y) = \frac{g^{(2)}(\mathfrak{p}) - g^{(2)}(y)}{2!}(\mu - y)^2$ and $\mathfrak{p}$ lies between $\mu$ and $y$. Operating by $\mathfrak{R}_n^{(\rho)}$ gives

$$\left| \mathfrak{R}_n^{(\rho)}(g; y) - g(y) - g^{(1)}(y)\mathfrak{T}_{n,1}^{(\rho)}(y) - g^{(2)}(y)\mathfrak{T}_{n,2}^{(\rho)}(y) \right| \leq \mathfrak{R}_n^{(\rho)}\left(|\mathfrak{E}(\mu, y)|; y\right) \tag{54}$$

Substituting the central moments from Lemma 4 gives

$$\begin{aligned}
&\Bigg| \mathfrak{R}_n^{(\rho)}(g; y) - g(y) - \\
&\quad \left[\left[\frac{\rho}{(\rho-2)} \frac{V^{(1)}(nyW(1))}{V(nyW(1))} - 1\right](y) + \frac{\rho}{n(\rho-2)} \frac{U^{(1)}(1)}{U(1)}\right] g^{(1)}(y) \\
&\quad - \left(\left[\frac{\rho^2}{(\rho-2)(\rho-3)} \frac{V^{(2)}(nyW(1))}{V(nyW(1))} - \frac{2\rho}{(\rho-2)} \frac{V^{(1)}(nyW(1))}{V(nyW(1))} + 1\right](y)^2\right. \\
&\quad + \left[\frac{\rho^2}{n(\rho-2)(\rho-3)} \left(\frac{2U^{(1)}(1)}{U(1)} + W^{(2)}(1) + 2\right) \frac{V^{(1)}(nyW(1))}{V(nyW(1))} - \right. \\
&\quad \left.\left.\frac{2\rho}{n(\rho-2)} \frac{U^{(1)}(1)}{U(1)}\right](y) + \left[\frac{\rho^2}{n^2(\rho-2)(\rho-3)} \left(2\frac{U^{(1)}(1)}{U(1)} + \frac{U^{(2)}(1)}{U(1)}\right)\right]\right) g^{(2)}(y) \Bigg| \\
&\leq \mathfrak{R}_n^{(\rho)}\left(|\mathfrak{E}(\mu, y)|; y\right)
\end{aligned} \tag{55}$$

Using the properties of $\mathfrak{W}(g; \mathfrak{d})$ from Lemma 5, we can write

$$\begin{aligned}
&\left| \frac{g^{(2)}(\mathfrak{p}) - g^{(2)}(y)}{2!} \right| \\
&\leq \frac{1}{2}\mathfrak{W}\left(g^{(2)}; |\mathfrak{p} - y|\right)\left(1 + (\mathfrak{p} - y)^2\right)\left(1 + y^2\right) \\
&\leq \frac{1}{2}\mathfrak{W}\left(g^{(2)}; |\mu - y|\right)\left(1 + (\mu - y)^2\right)\left(1 + y^2\right) \\
&\leq \left(1 + \frac{|\mu - y|}{\mathfrak{d}}\right)\left(1 + \mathfrak{d}^2\right)\mathfrak{W}\left(g^{(2)}; \mathfrak{d}\right)\left(1 + (\mu - y)^2\right)\left(1 + y^2\right)
\end{aligned}$$

Also [28]

$$\left| \frac{g^{(2)}(\mathfrak{p}) - g^{(2)}(y)}{2!} \right| \leq \begin{cases} 2\left(1 + \mathfrak{d}^2\right)^2\left(1 + y^2\right)\mathfrak{W}\left(g^{(2)}; \mathfrak{d}\right), & |\mu - y| \leq \mathfrak{d} \\ 2\left(1 + \mathfrak{d}^2\right)^2\left(1 + y^2\right)\frac{(\mu - y)^2}{\mathfrak{d}^4}\mathfrak{W}\left(g^{(2)}; \mathfrak{d}\right), & |\mu - y| \geq \mathfrak{d} \end{cases} \tag{56}$$

Therefore, for $0 \leq \mathfrak{d} \leq 1$, we get

$$\begin{aligned}
\left| \frac{g^{(2)}(\mathfrak{p}) - g^{(2)}(y)}{2!} \right| &\leq 2\left(1 + y^2\right)\left(1 + \frac{(\mu - y)^4}{\mathfrak{d}^4}\right)\left(1 + \mathfrak{d}^2\right)^2\mathfrak{W}\left(g^{(2)}; \mathfrak{d}\right) \\
&\leq 8\left(1 + y^2\right)\left(1 + \frac{(\mu - y)^4}{\mathfrak{d}^4}\right)\mathfrak{W}\left(g^{(2)}; \mathfrak{d}\right)
\end{aligned}$$

Therefore

$$|\mathfrak{E}(\mu, y)| = \frac{g^{(2)}(\mathfrak{p}) - g^{(2)}(y)}{2!}(\mu - y)^2 \leq$$

$$8\left(1+y^2\right)\left((\mu-y)^2+\frac{(\mu-y)^6}{\mathfrak{d}^4}\right)\mathfrak{W}\left(g^{(2)};\mathfrak{d}\right) \tag{57}$$

Using the linearity and positivity of $\mathfrak{R}_n^{(\rho)}$, and using the results from central moments from Lemma 4,

$$\begin{aligned}\mathfrak{R}_n^{(\rho)}\left(|\mathfrak{E}(\mu,y)|;y\right) &\le 8\left(1+y^2\right)\left\{\mathfrak{T}_{n,2}^{(\rho)}(y)+\frac{1}{\mathfrak{d}^4}\mathfrak{T}_{n,6}^{(\rho)}(y)\right\}\mathfrak{W}\left(g^{(2)};\mathfrak{d}\right)\\ &\le 8\left(1+y^2\right)\left\{O\left(\frac{1}{n}\right)+\frac{1}{\mathfrak{d}^4}O\left(\frac{1}{n^3}\right)\right\}\mathfrak{W}\left(g^{(2)};\mathfrak{d}\right)\end{aligned}$$

Choosing $\mathfrak{d}=n^{-1/2}$ gives

$$\mathfrak{R}_n^{(\rho)}\left(|\mathfrak{E}(\mu,y)|;y\right)\le 8\left(1+y^2\right)O\left(\frac{1}{n}\right)\mathfrak{W}\left(g^{(2)};n^{-1/2}\right) \tag{58}$$

Combining equations (55) and (58) leads to the required result. □

## 6 Grüss-Voronovskaja-type theorem

The following result brings out the non-multiplicativity of $\mathfrak{R}_n^{(\rho)}$. Similar recent studies include [3], [34], [5], [11], [29].

**Theorem 9** *Let $g(y), u(y)\in C_\zeta^*[0,\infty)$ such that $g^{(1)}(y)$, $g^{(2)}(y)$, $u^{(1)}(y)$, $u^{(2)}(y)$, $(gu)^{(1)}(y)$, $(gu)^{(2)}(y)\in C_\zeta^*[0,\infty)$, and $y,\mu\ge 0$. Also, let $\rho=\rho(n)$ be such that as $n\to\infty$, $\rho\to\infty$ and*

$$\lim_{n\to\infty}\frac{n}{\rho(n)}=q\in\mathbb{R}$$

*then:*

$$\begin{aligned}&\lim_{n\to\infty} n\Big[\mathfrak{R}_n^{(\rho)}\left((gu)(\mu);y\right)-\mathfrak{R}_n^{(\rho)}\left(g(\mu);y\right)\mathfrak{R}_n^{(\rho)}\left(u(\mu);y\right)\Big]\\ &=(\nu_1(y))\,g^{(1)}(y)u^{(1)}(y).\end{aligned}$$

*Proof* Consider the expression

$$n\Big[\mathfrak{R}_n^{(\rho)}\left((gu)(\mu);y\right)-\mathfrak{R}_n^{(\rho)}\left(g(\mu);y\right)\mathfrak{R}_n^{(\rho)}\left(u(\mu);y\right)\Big] \tag{59}$$

Using $(gu)^{(1)}(y)=g^{(1)}(y)u(y)+g(y)u^{(1)}(y)$ and $(gu)^{(2)}(y)=g^{(2)}(y)u(y)+2g^{(1)}(y)u^{(1)}(y)+g(y)u^{(2)}(y)$, it is easy to verify that the following expression is equivalent to the expression (59):

$$\begin{aligned}n\Big\{&\Phi_1-u(y)\{\Phi_2\}-\mathfrak{R}_n^{(\rho)}(g(\mu);y)\{\Phi_3\}+\mathfrak{R}_n^{(\rho)}(\mu-y;y)u^{(1)}(y)\{\Phi_4\}\\ &+\frac{1}{2!}\mathfrak{R}_n^{(\rho)}\left((\mu-y)^2;y\right)\left\{u^{(2)}(y)\{\Phi_4\}+\Phi_5\right\}\Big\}\end{aligned} \tag{60}$$

where

$$\begin{aligned}\Phi_1=&\mathfrak{R}_n^{(\rho)}((gu)(\mu);y)-(gu)(y)-(gu)^{(1)}(y)\mathfrak{R}_n^{(\rho)}(\mu-y;y)\\ &-\frac{(gu)^{(2)}(y)}{2!}\mathfrak{R}_n^{(\rho)}\left((\mu-y)^2;y\right)\end{aligned} \tag{61}$$

$$\Phi_2=\mathfrak{R}_n^{(\rho)}(g(\mu);y)-g(y)-g^{(1)}(y)\mathfrak{R}_n^{(\rho)}(\mu-y;y)-\frac{g^{(2)}(y)}{2!}\mathfrak{R}_n^{(\rho)}\left((\mu-y)^2;y\right) \tag{62}$$

$$\Phi_3=\mathfrak{R}_n^{(\rho)}(u(\mu);y)-u(y)-u^{(1)}(y)\mathfrak{R}_n^{(\rho)}(\mu-y;y)-\frac{u^{(2)}(y)}{2!}\mathfrak{R}_n^{(\rho)}\left((\mu-y)^2;y\right) \tag{63}$$

$$\Phi_4 = g(y) - \mathfrak{R}_n^{(\rho)}(g(\mu); y) \tag{64}$$

and

$$\Phi_5 = 2g^{(1)}(y)u^{(1)}(y). \tag{65}$$

Consider the terms $\Phi_1$, $\Phi_2$ and $\Phi_3$. In the limit as $n \to \infty$, by Theorem 8 and Lemma 5, for any $h \in C_\zeta^*[0, \infty)$, we have

$$\lim_{n\to\infty} n\Big[\mathfrak{R}_n^{(\rho)}(h(\mu); y) - h(y) - h^{(1)}(y)\mathfrak{R}_n^{(\rho)}(\mu - y; y) - \frac{h^{(2)}(y)}{2!}\mathfrak{R}_n^{(\rho)}\left((\mu - y)^2; y\right)\Big] = 0 \tag{66}$$

(for $\Phi_1$ this can be observed by considering $(gu)(\mu) = h(\mu)\ \forall\ \mu \geq 0$). Further, for $\Phi_4$, using Theorem 1, we have, $\forall\ y \geq 0$, $\mathfrak{R}_n^{(\rho)}(g(\mu); y) \to g(y)$. Therefore, we can conclude that

$$\lim_{n\to\infty} n\Phi_r = 0, \quad r = 1, 2, 3, 4. \tag{67}$$

Combining these results(67), and using Remark 5 in expressions (59) and (60), we get:

$$\begin{aligned}
&\lim_{n\to\infty} n\Big[\mathfrak{R}\left((gu)(\mu); y\right) - \mathfrak{R}\left(g(\mu); y\right)\mathfrak{R}\left(u(\mu); y\right)\Big] \\
&= \lim_{n\to\infty} n\frac{1}{2!}\mathfrak{R}_n^{(\rho)}\left((\mu - y)^2; y\right)\{\Phi_5\} \\
&= \lim_{n\to\infty} n\mathfrak{T}_{n,2}^{(\rho)}(y)\, g^{(1)}(y)u^{(1)}(y) \\
&= (\nu_1(y))\, g^{(1)}(y)u^{(1)}(y),
\end{aligned} \tag{68}$$

which is the required result. □

## 7 Conclusion

In this paper, we have presented a rich class of positive, linear approximation operators based on Miheşan's generalization of the Szász-Mirakjan operators, and incorporating the Boas-Buck polynomials. The proposed opeators form an important link in the field, as they reproduce several types of operators. Further, we have shown how some essential properties based on modulus of continuity, as well as the recently-acknowledged Quantitative Voronovskaja-type approximation theorems hold true for our operator.

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