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ORIGINAL RESEARCH

Diophantine approximation and continued fraction expansion for quartic power series over $\mathbb{F}_3$

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Abstract The main contribution of this paper is providing families of examples conjecturally generalizing the almost unique known so far example introduced first by Mills and Robbins (J Number Theory 23:388–404, 1986) of quartic power series over $\mathbb{F}_3(T)$ having an approximation exponent equal to 2 in relation with Roth's theorem as proved by Lasjaunias (J Number Theory 65:206–224 1997), and having a continued fraction expansion with an unbounded sequence of partial quotients.

Keywords Finite fields · Formal power series · Continued fraction

Mathematics Subject Classification 11J61 · 11J70

1 Introduction

Let p be a prime number and let $\mathbb{F}$ be a finite field of characteristic p. We let $\mathbb{F}[T]$, $\mathbb{F}(T)$ and $\mathbb{F}((T^{-1}))$ respectively denote, the ring of polynomials, the field of rational functions and the field of power series in $1/T$ over $\mathbb{F}$, where T is a formal indeterminate. These fields are valuated by the ultrametric absolute value introduced on $\mathbb{F}(T)$ by $|P/Q| = |T|^{\deg(P)-\deg(Q)}$, where $|T| > 1$ is a fixed real number. We recall that each irrational (rational) element α of $\mathbb{F}((T^{-1}))$ can be expanded as an infinite (finite) continued fraction. This will be denoted $\alpha = [a_0, a_1, \ldots, a_n, \ldots]$ where the $a_i \in \mathbb{F}[T]$, with $\deg(a_i) > 0$ for $i \geq 1$, are the partial quotients and the tail $\alpha_i = [a_i, a_{i+1}, \ldots] \in \mathbb{F}((T^{-1}))$ is the complete quotient. As in the classical theory, we define recursively the two sequences of polynomials $(P_n)_{n\geq 0}$ and $(Q_n)_{n\geq 0}$ by $P_n = a_n P_{n-1} + P_{n-2}$ and $Q_n = a_n Q_{n-1} + Q_{n-2}$, with the initial conditions $P_0 = a_0$, $P_1 = a_1 a_2 + 1$, $Q_0 = 1$ and $Q_1 = a_2$. We have $P_{n+1}Q_n - Q_{n+1}P_n = (-1)^n$, whence P_n and Q_n are coprime polynomials. The rational function P_n/Q_n is called a convergent to α and we have $P_n/Q_n = [a_0, a_1, \ldots, a_n]$ and $P_n/P_{n-1} = [a_n, a_{n-1}, \ldots, a_0]$. The following property of the continued fraction is easily checked: when B, C are nonzero polynomials in $\mathbb{F}[T]$, then

$$C\,[Ba_0, Ca_1, Ba_2, \ldots] = B\,[Ca_0, Ba_1, Ca_2, \cdots]. \tag{1.1}$$

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As for real numbers, the continued fraction expansion of formal power series is fundamental to measure the quality of their rational approximation. The irrationality measure (or the approximation exponent) of an irrational power series $\alpha \in \mathbb{F}((T^{-1}))$ is defined by:

$$\nu(\alpha) = -\limsup_{|Q|\longrightarrow\infty} \log(|\alpha - P/Q|)/\log(|Q|)$$

where $P, Q \in \mathbb{F}[T]$. It is directly related to the growth of the sequence of the degrees of the partial quotients in the continued fraction expansion of α. Indeed we have

$$\nu(\alpha) = 2 + \limsup_{n>1}(\deg(a_{n+1})/\sum_{1\leq i\leq n}\deg(a_i)). \tag{1.2}$$

Note that the irrationality measure is stable under a Möbius transformation.

For a general presentation of continued fractions and diophantine approximation in the function field case, the reader may consult [15] or ([16] Chap. 9).

We consider infinite continued fractions in $\mathbb{F}((T^{-1}))$ which are algebraic over $\mathbb{F}(T)$. The study of their rational approximation was initiated by Mahler [9]. The starting point in the study of rational approximation to algebraic real numbers is a famous theorem of Liouville established in 1850. This theorem was adapted by Mahler in the fields of power series with an arbitrary base field: if α is an element of $\mathbb{F}((T^{-1}))$, algebraic of degree $n > 1$ over $\mathbb{F}(T)$, then for all element P/Q of $\mathbb{F}(T)$, there exists a positive real number c such that

$$|\alpha - P/Q| \geq c/|Q|^n.$$

This result implies that $\nu(\alpha) \leq n$. In the case of real numbers, a well known improvement to Liouville's theorem was established in the form of Roth's theorem [14]. This improvement on the exponent is that for any irrational algebraic real number α, $\nu(\alpha) = 2$. It carries over to fields of power series if the base field has characteristic zero, as proved by Uchiyama in 1960 [17], the exponent of an irrational algebraic power series is still 2. In this case the exponent n in the right hand side of the above inequality can be replaced by $2+\epsilon$ for all $\epsilon > 0$. But a naive analog of Roth's theorem now fails in positive characteristic and consequently the study of rational approximation to algebraic elements becomes more complex. Mahler [9] gave an example showing that the approximation exponent $\nu(\alpha)$ could as large as n, the degree of α. He considered the irrational solution in $\mathbb{F}_p((T^{-1}))$ of the equation $x = 1/T + x^p$. For this element α, algebraic of degree p, we have rationals P/Q, with $|Q|$ arbitrarily large, and $|\alpha - P/Q| = |Q|^{-p}$.

Regarding diophantine approximation and continued fractions, a particular subset of elements in $\mathbb{F}((T^{-1}))$, algebraic over $\mathbb{F}(T)$ is worth considering. For $r = p^t$ with $t \geq 0$, we denote by $\mathcal{H}(r)$ the subset of irrational α belonging to $\mathbb{F}((T^{-1}))$ and satisfying an algebraic equation of the particular form $A\alpha^{r+1} + B\alpha^r + C\alpha + D = 0$, where A, B, C and D belong to $\mathbb{F}[T]$. Note that $\mathcal{H}(1)$ is simply the set of quadratic irrational elements in $\mathbb{F}((T^{-1}))$. The union of the subsets $\mathcal{H}(p^t)$, for $t \geq 0$, denoted by $\mathcal{H}$, is the set of hyperquadratic power series.

The rational approximation properties of the elements of $\mathcal{H}$, were studied independently by Voloch [18], and de Mathan [10]. They proved that:

If $\alpha \in \mathcal{H}$, and $P/Q \in \mathbb{F}(T)$, either we have

$$\liminf_{|Q|\longrightarrow\infty} |Q|^2|\alpha - P/Q| > 0 \tag{1.3}$$

or there exists a real number $\mu > 2$ such that

$$\liminf_{|Q|\longrightarrow\infty} |Q|^\mu|\alpha - P/Q| < \infty. \tag{1.4}$$

With respect to this, de Mathan and Lasjaunias [6], have shown that if an algebraic element does not belong to $\mathcal{H}$, then it cannot be too well approximated by rationals : if $\alpha \notin \mathcal{H}$ and it is algebraic of degree $n > 1$ over $\mathbb{F}(T)$, then, for all $\epsilon > 0$, we have $|\alpha - P/Q| > |Q|^{-([n/2]+1+\epsilon)}$, for all $P/Q \in \mathbb{F}(T)$ with $|Q|$ large enough. This last property highlights the peculiarity of the set $\mathcal{H}$. If rational approximation to certain hyperquadratic power series is well known, this is also due to the possibility of explicitly describing their continued fraction expansion. The first works in this area were undertaken by Baum and Sweet [3]. Later this has been done for many examples and for different subclasses of hyperquadratic elements (see in particular [15]). Nevertheless, the possibility of describing the continued fraction expansion for all hyperquadratic power series is yet an open problem. In [13] Mills and Robbins studied this problem by describing an algorithm to obtain, in certain cases, the continued fraction expansion for a hyperquadratic power series. They ultimately considered (p.403) the following algebraic equation:

$$x^4 + x^2 - Tx + 1 = 0. \tag{1.5}$$

They observed that this equation has a unique solution α in $\mathbb{F}_p((T^{-1}))$ for all primes p noting that for this solution, the continued fraction expansion has a remarkable pattern in both cases $p = 3$ and $p = 13$. The expansion in the case $p = 3$ was explicitly described by Buck and Robbins [4], and later by Lasjaunias [5] who used another somewhat easier method. Indeed, they recursively defined the following polynomial sequences:

$$\Omega_0 = \emptyset, \quad \Omega_1 = T, \quad \Omega_n = \Omega_{n-1}, -T, \Omega_{n-2}^{(3)}, -T, \Omega_{n-1} \quad for \ \ n \geq 2.$$

(here $\Omega_k^{(3)}$ denotes the sequence obtained by cubing each element of Ω_k and commas indicating juxtaposition of sequences); then they proved that $[0, \Omega_n]$ is the beginning for all $n > 0$ of the continued fraction expansion of this solution. This element satisfies, $\liminf_{|Q|\longrightarrow\infty} |Q|^2|\alpha - P/Q| = 0$ and $\liminf_{|Q|\longrightarrow\infty} |Q|^{\mu}|\alpha - P/Q| = \infty$ for all $\mu > 2$. So it satisfies neither (1.3) nor (1.4). Thus it does not belong to the set $\mathcal{H}$. This result was given by Lasjaunias in [5], by proving that there are two real positive constants λ_1 and λ_2 such that, for some rationals P/Q with $|Q|$ arbitrary large, we have $|\alpha - P/Q| \leq |Q|^{-(2+\lambda_1\sqrt{\log|Q|})}$, and for all rationals P/Q with $|Q| > 1$, we have $|\alpha - P/Q| \geq |Q|^{-(2+\lambda_2\sqrt{\log|Q|})}$. For instance, this element seems to be the first algebraic element for which the exponent approximation is equal to 2, although its partial quotients are unbounded.

Note that for each prime $p > 3$, the continued fraction expansion of the solution of (1.5) is remarkable and it has two different regular patterns and two different values of irrationality measure according to the remainder, 1 or 2, in the division of p by 3, see [2,8] for more details.

Our work is organized as follow. In the second Section we will extend the set of counter-examples initiated by Mahler [9]. We will compute the continued fraction and the approximation exponent of some quartic power series which are hyperquadratic over $\mathbb{F}_3$. For this, we will use an earlier theorem which allows us to determine the approximation exponent of an algebraic element when it is large enough, i.e, not close to 2. The basic idea of this theorem is due to Voloch [18]. It has been improved by de Mathan [11].

Theorem 1.1 ([7] p. 219) *Let $\alpha \in \mathbb{F}((T^{-1}))$. Assume that there is a sequence $(P_n, Q_n)_{n\geq 0}$, with $P_n, Q_n \in \mathbb{F}[T]$, satisfying the following conditions:*
(1) There are two real constants $\lambda > 0$ et $\mu > 1$, such that

$$|Q_n| = \lambda |Q_{n-1}|^{\mu} \ \ and \ \ |Q_n| > |Q_{n-1}| \ \ for \ \ all \ \ n \geq 1.$$

(2) There are two real constants $\rho > 0$ and $\gamma > 1 + \sqrt{\mu}$, such that

$$\left|\alpha - \frac{P_n}{Q_n}\right| = \rho |Q_n|^{-\gamma} \quad for \ \ all \ \ n \geq 0.$$

Then we have $\nu(\alpha) = \gamma$.

This Theorem allows us to find the approximation exponent of several examples of hyperquadratic elements(see [1,7]).

In Section 3 of this work, we will study the continued fraction expansion of the solution α of the quartic equation

$$C^2\alpha^4 + 2C\alpha^2 - A^2\alpha + 1 = 0 \tag{1.6}$$

where A and C are nonzero polynomials in $\mathbb{F}_3[T]$ such that A is not constant, C divides A and $\deg A \geq \deg C$. By computing the approximation exponent of the solution of this equation, we will prove that is not hyperquadratic. Our observation, based on computer calculation giving a finite number of partial quotients for many couples (A, C) of polynomials, implies that the solution of the equation (1.6) has very regular pattern in its continued fraction expansion. Note that this equation can be viewed as a generalization of the equation (1.5) introduced by Mills and Robbins. The properties of rational approximation of α were studied by Lasjaunias in [5] for the case $A = T$ and $C = -1$. For this case, the tools used to obtain a proof might well be applied in the general case, but we are aware that a different approach would be desirable. We will recall the steps of the proof and we will just give our result conjecturally. Thus we present a large family of algebraic power series having an approximation exponent value equal to 2, even though the degrees of their partial quotients are unbounded.

2 Diophantine approximation for some hyperquadratic power series of degree four over $\mathbb{F}_3(T)$

In this section we will study respectively the properties of rational approximations of the solutions of the equations $C\beta^4 - A\beta + 1 = 0$ and $-\beta^4 - A\beta + C = 0$, where A and C belong to $\mathbb{F}_3[T]$.

Theorem 2.1 *Let β be the irrational solution of the equation*

$$C\beta^4 - A\beta + 1 = 0 \tag{2.1}$$

such that $\deg A \geq \deg C$. *Assume that C divides A. Then the continued fraction expansion of β is*

$$[b_0, b_1, \ldots, b_n, \ldots]$$

such that $b_0 = 0$, $b_1 = A$ and for all $n \geq 2$:

$$b_n = \begin{cases} -Cb_{n-1}^3 & \text{if } n \text{ is odd;} \\ b_{n-1}^3/-C & \text{if } n \text{ is even.} \end{cases} \tag{2.2}$$

Furthermore, $\nu(\beta) = 4$.

Proof . Clear we have $|\beta| < 1$ then $b_0 = 0$. Let $\beta_1 = \beta^{-1}$ then β_1 satisfies the equation $\beta_1^4 - A\beta_1^3 + C = 0$. Clearly $[\beta_1] = b_1 = A$. In fact, as $|\beta_1| > 1$ then $|\beta_1^4| = |A\beta_1^3 + C| = |A\beta_1^3|$ so $|\beta_1| = |A|$, and since $|\beta_1 - A| = |C/\beta_1^3| < 1$ then we obtain that $[\beta_1] = A$. We can write the equation satisfied by β_1 as $\beta_1^3 = \dfrac{-C}{\beta_1 - A}$. So

$$\beta_1^3 = -C\beta_2. \tag{2.3}$$

Applying the Frobenius automorphism to both terms of the identity $\beta_1 = b_1 + 1/\beta_2$ and using $\beta_2 = b_2 + 1/\beta_3$ we obtain $b_2 + \dfrac{1}{\beta_3} = \dfrac{b_1^3}{-C} + \dfrac{1}{-C\beta_2^3}$. As C divides $A = b_1$ then C divides b_1^3, so we get that $b_2 = b_1^3/-C$ and

$$\beta_3 = -C\beta_2^3.$$

Again, this gives that $b_3 + \dfrac{1}{\beta_4} = -Cb_2^3 + \dfrac{-C}{\beta_3^3}$. So we obtain $b_3 = -Cb_2^3$ and

$$\beta_4 = \frac{\beta_3^3}{-C}. \tag{2.4}$$

This gives that C divides b_3 and (2.3) has the same shape as (2.4). We now claim that for all $k \geq 1$,

$$\begin{cases} b_{2k} = b_{2k-1}^3/-C,\, b_{2k+1} = -Cb_{2k}^3 \\ \beta_{2k+2} = \beta_{2k+1}^3/-C,\, \beta_{2k+1} = -C\beta_{2k}^3 \end{cases} \tag{2.5}$$

Clearly (2.5) is true for $k = 1$. So we assume (2.5) for $k = l \geq 1$. Then

$$\beta_{2l+2} = \left(\left(b_{2l+1}^3/-C\right) + \frac{1}{-C\beta_{2l+2}^3}\right).$$

From (2.5) we have C divides b_{2l+1}^3. This implies that $b_{2l+2} = b_{2l+1}^3/-C$ and $\beta_{2l+3} = -C\beta_{2l+2}^3$. Then

$$\beta_{2l+3} = -C\left(b_{2l+2}^3 + \frac{1}{\beta_{2l+3}^3}\right) = -Cb_{2l+2}^3 + \frac{-C}{\beta_{2l+3}^3},$$

which implies $b_{2l+3} = -Cb_{2l+2}^3$ and $\beta_{2l+4} = \beta_{2l+3}^3/-C$. Thus (2.5) is also true for $k = l + 1$. By induction, we see that (2.5) holds for all $k \geq 1$.

Furthermore, we can verify that the equality (2.2) gives that for all $n \geq 1$:

$$b_n = (-1)^{n-1} A^{3^{n-1}} C^{-\frac{3^{n-1}+(-1)^n}{4}}.$$

Thus the continued fraction expansion of β can be written as

$$\left[0, A, -A^3C^{-1}, A^{3^2}C^{-2}, -A^{3^3}C^{-7}, \ldots, (-1)^{n-1}A^{3^{n-1}}C^{-\frac{3^{n-1}+(-1)^n}{4}}, \ldots\right].$$

Now let $a = \deg A$ and $c = \deg C$. Knowing all the partial quotients of β, we can compute its approximation exponent by the formula (1.2):

$$\begin{aligned}\nu(\beta) &= 2 + \limsup \frac{3^n a - \frac{3^n+(-1)^{n+1}}{4}c}{\sum_{k=1}^{n}(3^{k-1}a - \frac{3^{k-1}+(-1)^k}{4}c)} \\ &= 2 + 2 = 4.\end{aligned}$$

□

In the next Theorem, we will give the value of $\nu(\beta)$ for β satisfying the equation (2.1) with the condition on the coefficients of this equation that is: C does not divides A.

Theorem 2.2 *Let β be the irrational solution of equation (2.1) such that $\deg A \geq \deg C$. Assume that C does not divide A. Then*

$$\nu(\beta) = 4 - \frac{\deg C}{\deg A}.$$

Proof Let β_1 and β_2 be the first and the second complete quotient of β. So β_1 satisfies the equation $\beta_1^4 - A\beta_1^3 + C = 0$. We have that $[\beta_1] = A$ and since $\beta_1 = A + 1/\beta_2$ then we can easily see that β_2 satisfies the equation $C\beta_2^4 + A^3\beta_2^3 + 1 = 0$. Hence $|\beta_1| = |A|$ and $|\beta_2| = |A^3/C|$. Let s be a positive rational number such that $|A| = |C|^s$. We consider the following sequence: $P_0 = 1$, $Q_0 = A$ and for $n \geq 1$

$$\begin{aligned}P_n &= Q_{n-1}^3 \\ Q_n &= AQ_{n-1}^3 - CP_{n-1}^3.\end{aligned}$$

Then for all $n \geq 0$:

$$\left|\beta - \frac{P_n}{Q_n}\right| = \left|\frac{1}{C\beta^3 - A} - \frac{Q_{n-1}^3}{AQ_{n-1}^3 - CP_{n-1}^3}\right| = \left|\frac{CP_{n-1}^3 - CQ_{n-1}^3\beta^3}{(C\beta^3 - A)(AQ_{n-1}^3 - CP_{n-1}^3)}\right|.$$

As $|C\beta^3 - A| = |A|$ and $|AQ_{n-1}^3 - CP_{n-1}^3| = |AQ_{n-1}^3|$ for all $n \geq 1$, then we get

$$\left|\beta - \frac{P_n}{Q_n}\right| = \frac{|C||P_{n-1} - Q_{n-1}\beta|^3}{|A|^2|Q_{n-1}|^3} = \frac{|C|}{|A|^2}\left|\beta - \frac{P_{n-1}}{Q_{n-1}}\right|^3.$$

We show by recursion that for all $n \geq 0$:

$$\left|\beta - \frac{P_n}{Q_n}\right| = \frac{|C|^{\frac{(3^n-1)}{2}}}{|A|^{3^n-1}}\left|\beta - \frac{P_0}{Q_0}\right|^{3^n}.$$

Since $\left|\beta - \frac{P_0}{Q_0}\right| = \left|\beta - \frac{1}{A}\right| = \frac{|A - \beta_1|}{|\beta_1||A|} = \frac{1}{|A|^2|\beta_2|} = \frac{|C|}{|A|^5}$ then $\left|\beta - \frac{P_0}{Q_0}\right|^{3^n} = |C|^{3^n}|A|^{-5.3^n}$. So

$$\left|\beta - \frac{P_n}{Q_n}\right| = |C|^{\frac{3^n-1}{2}}|C|^{3^n}|A|^{-(3^n-1)}|A|^{-5.3^n} = |C|^{\frac{3.3^n-1}{2}}|A|^{-(6.3^n-1)}.$$

Let $|A| = |C|^s$. Then $\left|\beta - \frac{P_n}{Q_n}\right| = |C|^{-\frac{(4s-1)3^{n+1}-2s+1}{2}}$. Secondly, we have for all $n \geq 1$ $Q_n = AQ_{n-1}^3 - CP_{n-1}^3$ then

$$|Q_n| = |A||Q_{n-1}|^3.$$

Again by recursion we show that

$$|Q_n| = |A|^{\frac{3^n-1}{2}}|Q_0|^{3^n} = |A|^{\frac{3^{n+1}-1}{2}} = |C|^{\frac{s3^{n+1}-s}{2}}.$$

So we obtain for all $n \geq 0$:

$$\left|\beta - \frac{P_n}{Q_n}\right| = \frac{1}{|C|^{2s}|Q_n|^{\frac{4s-1}{s}}}. \tag{2.6}$$

Since $\deg A \geq \deg C$ then $s \geq 1$. So $\frac{4s-1}{s} = 4 - \frac{1}{s} > 1 + \sqrt{3}$. Hence, if we put $\mu = 3, \lambda = |A|, \rho = 1/|C|^{2s}, \gamma = (4s-1)/s$ then $\gamma > 1 + \sqrt{\mu}$ and following Theorem 1.1 we conclude that $\nu(\beta) = 4 - \frac{1}{s}$. □

Theorem 2.3 *Let β be the irrational solution of the equation*

$$-\beta^4 - A\beta + C = 0 \tag{2.7}$$

such that $\deg A > \deg C$. *Assume that C divides A. Then the continued fraction expansion of β is*

$$[b_0, b_1, \ldots, b_n, \ldots]$$

such that $b_0 = 0$, $b_1 = A/C$ and for all $n \geq 2$:

$$b_n = \left(\frac{A}{C}\right)^{3^{n-1}}(C)^{\frac{3^{n-1}+(-1)^n}{4}}. \tag{2.8}$$

Furthermore, $\nu(\beta) = 4$.

Proof Clearly we have $|\beta| < 1$ then $b_0 = 0$. Let $\beta_1 = \beta^{-1}$ then β_1 satisfies the equation $C\beta_1^4 - A\beta_1^3 - 1 = 0$. Clearly $[\beta_1] = b_1 = A/C$. So the first partial quotient of β_1 is $b_1 = A/C$ and $\beta_1 = \frac{A}{C} + \frac{1}{\beta_2}$. We can easily see that β_1 satisfies

$$\beta_1^3 = \frac{1}{-A + C\beta_1} = \frac{\beta_2}{C},$$

then $C\beta_1^3 = \beta_2$. So $Cb_1^3 + \frac{C}{\beta_2^3} = \beta_2$. Hence $b_2 = Cb_1^3$ and $\beta_3 = \beta_2^3/C$. We apply again the same reasoning and we obtain that $\beta_3 = \frac{b_2^3}{C} + \frac{1}{C\beta_3^3}$, so $b_3 = b_2^3/C$ and $\beta_4 = C\beta_3^3$. By recurrence on k we prove easily that $\beta_{2k} = C\beta_{2k-1}^3$, $\beta_{2k+1} = \beta_{2k}^3/C$ and

$$b_k = \begin{cases} Cb_{k-1}^3 & \text{if } k \text{ is even;} \\ b_{k-1}^3/C & \text{if } k \text{ is odd.} \end{cases} \tag{2.9}$$

On the other hand, we have $b_3 = b_2^3/C = C^2 b_1^{3^2}$ and $b_4 = Cb_3^3 = C^{3^2-2}b_1^{3^3}$. We remark that $b_3 = C^{\frac{3^2-1}{4}}b_1^{3^2}$ and $b_4 = C^{\frac{3^3+1}{4}}b_1^{3^3}$. So by a simple recurrence on k we can prove that $b_k = C^{\frac{3^{k-1}+(-1)^k}{4}}b_1^{3^{k-1}}$. Then we deduce that the sequences of partial quotients of β is given by: $b_0 = 0$, $b_1 = A/C$ and for all $n \geq 2$:

$$b_n = (A/C)^{3^{n-1}}C^{\frac{3^{n-1}+(-1)^n}{4}}.$$

Let $a = \deg A$ and $c = \deg C$. We can compute the approximation exponent of β by the formula (1.2):

$$\nu(\beta) = 2 + \limsup \frac{3^n(a-c) - \frac{3^n+(-1)^{n+1}}{4}c}{\sum_{k=1}^{n}(3^{k-1}(a-c) - \frac{3^{k-1}+(-1)^k}{4}c)}$$
$$= 2 + 2 = 4.$$

□

In the following Theorem, we will give the value of $\nu(\beta)$ for β satisfying the equation (2.7) with the condition on the coefficients of this equation that is: C does not divides A.

Theorem 2.4 *Let β be the irrational solution of equation (2.7) such that $\deg A > \deg C$. Assume that C does not divide A. Suppose that $|A| = |C|^s$ with $s > \frac{3}{3-\sqrt{3}}$. Then*

$$\nu(\alpha) = 4 - \frac{3}{s}$$

Proof . Let β_1 be the first complete quotient of β. We can easily see that β_1 satisfies the equation $C\beta_1^4 - A\beta_1^3 - 1 = 0$ and $|\beta_1| = |A/C|$. So we have $|\beta_1| = |C|^{s-1}$.

We consider the following sequence: $P_0 = A$, $Q_0 = C$ and for $n \geq 1$

$$P_n = AP_{n-1}^3 + Q_{n-1}^3$$
$$Q_n = CP_{n-1}^3.$$

It is easily to see that $\beta_1 = \frac{1}{C\beta_1^3} + \frac{A}{C}$ and $\frac{P_n}{Q_n} = \frac{Q_{n-1}^3}{CP_{n-1}^3} + \frac{A}{C}$. Then for all $n \geq 0$:

$\left|\beta_1 - \frac{P_n}{Q_n}\right| = \left|\frac{1}{C\beta_1^3} - \frac{Q_{n-1}^3}{CP_{n-1}^3}\right| = \frac{1}{|C||\beta_1|^3}\left|\frac{\beta_1}{|\beta_1|} - \frac{P_{n-1}}{|\beta_1|Q_{n-1}}\right|^3 = \frac{1}{|C||\beta_1|^6}\left|\beta_1 - \frac{P_{n-1}}{Q_{n-1}}\right|^3$. We show by recursion that for all $n \geq 0$:

$$\left|\beta_1 - \frac{P_n}{Q_n}\right| = |C|^{-\frac{(3^n-1)}{2}}|\beta_1|^{-\frac{6(3^n-1)}{2}}\left|\beta_1 - \frac{P_0}{Q_0}\right|^{3^n}$$

since $\left|\beta_1 - \frac{P_0}{Q_0}\right| = \left|\beta_1 - \frac{A}{C}\right| = \frac{1}{|C||\beta_1|^3}$ then $\left|\beta_1 - \frac{P_0}{Q_0}\right|^{3^n} = |C|^{-3^n}|\beta_1|^{-3^{n+1}}$. So

$$\left|\beta_1 - \frac{P_n}{Q_n}\right| = |C|^{-\frac{3^{n+1}-1}{2}}|\beta_1|^{-\frac{3^{n+2}+3^{n+1}-6}{2}} = |C|^{-\frac{(s-1)3^{n+2}+s3^{n+1}-6(s-1)-1}{2}}.$$

On the other hand, we have for all $n \geq 1$ $Q_n = CP_{n-1}^3$ and since $|P_{n-1}| = |C|^{s-1}|Q_{n-1}|$ then

$$|Q_n| = |C|^{3s-2}|Q_{n-1}|^3.$$

Again by recursion we show that

$$|Q_n| = |C|^{\frac{(3s-2)(3^n-1)}{2}}|Q_0|^{3^n} = |C|^{\frac{s3^{n+1}-3s+2}{2}}.$$

So we obtain for all $n \geq 0$:

$$\left|\beta_1 - \frac{P_n}{Q_n}\right| = |C|^{-\frac{3(s-1)^2}{s}}|Q_n|^{-\frac{4s-3}{s}}.$$

We can verifies that if $s > \frac{3}{3-\sqrt{3}}$ then $\frac{4s-3}{s} > 1 + \sqrt{3}$. Hence by Theorem (1.1) we conclude that $\nu(\beta_1) = 4 - \frac{3}{s} = 4 - \frac{3 \deg C}{\deg A}$. □

3 Diophantine approximation of some not hyperquadratic power series of degree four over $\mathbb{F}_3(T)$

Now we will give a family of formal power series, defined by their continued fraction expansion, having a minimum value of approximation exponent. Before this, we recall some usual properties of continued fractions. If $\Omega_k = a_1, a_2, \ldots, a_k$ is a sequence of polynomials, we denote $\widetilde{\Omega_k}$ the sequence obtained by reversing the terms of Ω_k, i.e, $\widetilde{\Omega_k} = a_k, a_{k-1}, \ldots, a_1$. If B is nonzero element of $\mathbb{F}_3[T]$ such that B divides a_i for all odd i then $B^{-1}\Omega_k = B^{-1}a_1, Ba_2, \ldots,$

$B^{(-1)^k}a_k$. Also, if B is nonzero element of $\mathbb{F}_3[T]$ such that B divides a_i for all even i then $B\Omega_k = Ba_1, B^{-1}a_2, \ldots, B^{(-1)^{k-1}}a_k$. In particular, if ϵ is nonzero element of $\mathbb{F}_3$ then we write $\epsilon\Omega_k$ for $\epsilon a_1, \epsilon^{-1}a_2, \ldots, \epsilon^{(-1)^{k-1}}a_k$. Moreover, in $\mathbb{F}_3$ we have $\epsilon^{-1} = \epsilon$.

Theorem 3.1 *Let A and C be two nonzero polynomials in $\mathbb{F}_3[T]$ such that A is not constant,* $\deg C \le \deg A$ *and C divides A. Let us define the sequence $(\Omega_n)_{n\ge1}$ of finite sequences of elements of $\mathbb{F}_3[T]$ recursively by $\Omega_0 = \emptyset$, $\Omega_1 = A^2$ and for all $n \ge 0$*

$$\begin{cases} \Omega_{2n+1} = \Omega_{2n}, 2A^2, \dfrac{1}{C^2}\Omega^{(3)}_{2n-1}, 2A^2, \widetilde{\Omega}_{2n} \\ \Omega_{2n+2} = \Omega_{2n+1}, A^2/C, \dfrac{1}{2C}\Omega^{(3)}_{2n}, 2A^2, \dfrac{1}{2C}\Omega_{2n+1} \end{cases} \tag{3.1}$$

Let $\Omega_\infty = \lim_{n\longrightarrow\infty} \Omega_n$. Let $\theta \in \mathbb{F}_3((T^{-1}))$ such that $\theta = [0, a_1, \ldots, a_n, \ldots] = [0, \Omega_\infty]$. Then, there exist explicitly positive numbers λ_1 and λ_2 such that for some rationals P/Q with $|Q|$ arbitrarily large, we have

$$|\theta - P/Q| \le |Q|^{-(2+\lambda_1/\sqrt{\deg Q})} \tag{3.2}$$

and, for all rationals P/Q with $|Q|$ sufficiently large, we have

$$|\theta - P/Q| \ge |Q|^{-(2+\lambda_2/\sqrt{\deg Q})} \tag{3.3}$$

where $\lambda_1 = 2/\sqrt{3}$ and $\lambda_2 > 2/\sqrt{3}$.

Proof . We have $\Omega_2 = A^2, A^2, 2A^2, 2A^2/C$. Since C divides $a_1 = A^2$ and $a_3 = 2A^2$, then C divides the partial quotient of odd index in Ω_2. Suppose that C divides the partial quotients with odd index in Ω_n for an even n. From (3.1) we have $\Omega_{n+1} = \Omega_n, 2A^2, \frac{1}{C^2}\Omega^{(3)}_{n-1}, 2A^2, \widetilde{\Omega}_n$. As Ω_n has even number of partial quotients then $2A^2$ is a partial quotient with odd index and C divides it. Furthermore, $\frac{1}{C^2}\Omega^{(3)}_{n-1}$ has odd number of partial quotients and begins with a partial quotient with even index, then the partial quotient $2A^2$, coming after it, has an odd index and C divides it. Finally, as C divides all the partial quotients with odd index in Ω_n then it divides all partial quotients with even index in $\widetilde{\Omega}_n$. So we can compute all the partial quotients of $C^{-1}\Omega_{n+1}$ which is

$$C^{-1}\Omega_{n+1} = C^{-1}\Omega_n, 2A^2/C, \frac{1}{C}\Omega^{(3)}_{n-1}, 2A^2/C, C\widetilde{\Omega}_n.$$

By recursion, we prove that we can compute all partial quotients of $C^{-1}\Omega_n$ for all n.
We put $a = \deg A$ and $c = \deg C$. Let us define for each $n \ge 0$, the sequence Ω^*_n of the degrees of the elements of Ω_n. The sequence $c^{-1}\Omega^*_n$ is the sequence of degree of $C^{-1}\Omega_n$. We get, from the recursive definition (3.1), $\Omega^*_0 = \emptyset$ and

$\Omega^*_1 = 2a$
$\Omega^*_2 = 2a, 2a - c, 2a, 2a - c$
$\Omega^*_3 = 2a, 2a - c, 2a, 2a - c, 2a, (6a - 2c), 2a, 2a - c, 2a, 2a - c, 2a$
$\Omega^*_4 = \Omega^*_3, 2a - c, (6a - c, 6a - 2c, 6a - c, 6a - 2c), 2a, c^{-1}\Omega^*_3$
$\Omega^*_5 = \Omega^*_4, 2a, 6a - 2c, 6a - c, 6a - 2c, 6a - c, 6a - 2c, (18a - 4c), 6a - 2c, 6a - c, 6a - 2c,$
$6a - c, 6a - 2c, 2a, \widetilde{\Omega}^*_4$

From the definition of the approximation exponent, we see that we shall use, for all $k \ge 1$, Ω^*_{2k+1} to compute the value of the approximation exponent. Again, from (3.1) and by induction on k we see that Ω^*_{2k+1} has an odd number of terms, has $2(3^k a - \frac{3^k + (-1)^{k+1}}{4} c)$ as the central term, and is reversible.

For $k \geq 1$ we put $d_k = \deg a_k$ and $P/Q = [a_1, \ldots, a_k]$. We define $k_i = \inf\{k \geq 1; d_k = 2(3^i a - \frac{3^i + (-1)^{i+1}}{4} c)\}$. So we have

$$\sum_{a_k \in \Omega_{2i+1}} d_k = 2(3^i a - \frac{3^i + (-1)^{i+1}}{4} c) + 2 \sum_{k<k_i} d_k. \quad (3.4)$$

Now we put $D_n = \sum_{a_k \in \Omega_n} d_k$. Furthermore, we have $D_n = \deg \Omega_n = 2 \deg Q_n$.

$$D_{2i+1} = 2 \sum_{k=1}^{2i+1} (3^{k-1} a - \frac{3^{k-1} + (-1)^k}{4} c) = 2.(\frac{3^{2i+1} - 1}{2} a - \frac{3^{2i+1} - 3}{8} c).$$

Hence, if $(U_k/V_k)_{k \geq 0}$ is the sequence of convergents of θ, the relation (3.32) implies, for $i \geq 1$,

$$\begin{aligned} \deg V_{k_i - 1} &= \sum_{k<k_i} d_k \\ &= (D_{2i+1} - 2(3^i a - \frac{3^i + (-1)^{i+1}}{4} c))/2 \\ &= (\frac{3^{2i+1} - 23^i - 1}{2})(a - \frac{c}{4}) + (1 + (-1)^i)\frac{c}{4}. \end{aligned}$$

We can easily verify that $2(3^i a - \frac{3^i + (-1)^{i+1}}{4} c) \geq 2/\sqrt{3}\sqrt{\deg V_{k_i - 1}}$, which gives that

$$|T|^{-2(3^i a - \frac{3^i + (-1)^{i+1}}{4} c)} \leq |V_{k_i - 1}|^{2/\sqrt{3 \deg V_{k_i - 1}}}.$$

On the other hand, for $i \geq 1$, we have

$$|\theta - U_{k_i - 1}/V_{k_i - 1}| = |T|^{-2(3^i a - \frac{3^i + (-1)^{i+1}}{4} c)} |V_{k_i - 1}|^{-2}.$$

So, we obtain the desired inequality for $P/Q = U_{k_i - 1}/V_{k_i - 1}$ and for $i \geq 1$, with $\lambda_1 = 2/\sqrt{3}$.

Furthermore if U_k/V_k is a convergent to θ, then

$$\deg V_{k_i - 1} \leq \deg V_k < \deg V_{k_{i+1} - 1} \ \ implies \ \ |\theta - U_k/V_k| = |T|^{d_{k+1}} |U_k|^{-2}$$

As $\limsup \frac{2(3^i a - \frac{3^i + (-1)^{i+1}}{4} c)}{\sqrt{\deg V_{k_i - 1}}} = 2/\sqrt{3}$, then, if $\lambda_2 > 2/\sqrt{3}$, we can write

$$2(3^i a - \frac{3^i + (-1)^{i+1}}{4} c) < \lambda_2 \sqrt{\deg V_{k_i - 1}} \leq \lambda_2 \sqrt{\deg V_k}$$

for i large enough. It follows that (3.3) holds for U_k/V_k with k large enough. Since the convergents are the best rational approximation, this is also true for all P/Q with $|Q|$ large enough. □

Let $\beta \in \mathbb{F}_3((T^{-1}))$ be the solution of the equation (2.1) such that C divides A. We know from the Theorem 2.4 that the continued fraction expansion of β is:

$$[b_0, b_1, \ldots, b_n, \ldots]$$

such that $b_0 = 0$, $b_1 = A$ and for all $n \geq 2$:

$$b_n = (-1)^{n-1} A^{3^{n-1}} C^{-\frac{3^{n-1} + (-1)^n}{4}}.$$

In the next part, we will compute the continued fraction expansion and the approximation exponent of $\alpha = \beta^2 = [0, A, -A^3/C, \dots, (-1)^{n-1}A^{3^{n-1}}C^{-\frac{3^{n-1}+(-1)^n}{4}}, \dots]^2$. Note that, from the equation (2.1), β satisfies $\beta = (C\beta^4+1)/A$. So $\beta^2 = (C\beta^4+1)^2/A^2$, which gives that $C^2\beta^8 + 2C\beta^4 + 1 = A^2\beta^2$. Then we deduce that α satisfies the equation $C^2\alpha^4 + 2C\alpha^2 - A^2\alpha^2 + 1 = 0$ which is the equation (1.6).

We set $\alpha = [a_0, a_1, \dots, a_n, \dots]$. Observe that $a_0 = 0$ from the definition of α since $|\beta| < 1$. Then we introduce the usual two sequences of polynomials of $\mathbb{F}_3[T]$, defined inductively by

$$U_0 = 0, \quad U_1 = 1, \quad V_0 = 1, \quad V_1 = a_1,$$
$$U_n = a_nU_{n-1} + U_{n-2}, \quad V_n = a_nV_{n-1} + V_{n-2}$$

for $n \geq 2$. So $(U_n/V_n)_{n\geq 0}$ is the sequence of the convergents to α.

Now, in order to compute all the partial quotients of α, we need to introduce a series of Lemmas.

Lemma 3.1 *Let $(P_n/Q_n)_{n\geq 0}$ be the sequence of convergents of β. Then $P_0 = 0$, $Q_0 = 1$, $P_1 = 1$, $Q_1 = A$ and for all $n \geq 1$:*

$$\begin{cases} P_{2n+1} = Q_{2n}^3 \\ Q_{2n+1} = AQ_{2n}^3 - CP_{2n}^3 \end{cases} \quad \textit{and} \quad \begin{cases} P_{2n} = -Q_{2n-1}^3/C \\ Q_{2n} = -(A/C)Q_{2n-1}^3 + P_{2n-1}^3 \end{cases} \tag{3.5}$$

Proof . From the equality (2.8) defining the sequence of partial quotients of β we can easily check that $\frac{P_2}{Q_2} = [0, A, -A^3/C] = \frac{-A^3/C}{-A^4/C+1}$, and $\frac{P_3}{Q_3} = [0, A, -A^3/C, A^9/C^2] = \frac{-A^{12}/C^3+1}{-A^{13}/C^3+A^9/C^2+1}$. So $P_2 = -A^3/C = -Q_1^3/C$, $Q_2 = -A^4/C + 1 = -(A/C)Q_1^3 + P_1^3$, $P_3 = -A^{12}/C^3 + 1 = Q_2^3$ and $Q_3 = -A^{13}/C^3 + A^9/C^2 + 1 = AQ_2^3 - CP_2^3$. Hence (3.5) is satisfied for $n = 1$. Suppose that (3.5) is satisfied for $n = l > 1$. We know that $P_{2l+2} = b_{2l+2}P_{2l+1} + P_{2l}$ and $Q_{2l+2} = b_{2l+2}Q_{2l+1} + Q_{2l}$. Then

$$\begin{aligned} P_{2l+2} &= (b_{2l+1}^3/-C)Q_{2l}^3 - Q_{2l-1}^3/C = (b_{2l+1}^3Q_{2l}^3 + Q_{2l-1}^3)/-C \\ &= (b_{2l+1}Q_{2l} + Q_{2l-1})^3/-C = -Q_{2l+1}^3/C, \end{aligned}$$

and

$$\begin{aligned} Q_{2l+2} &= (b_{2l+1}^3/-C)(AQ_{2l}^3 - CP_{2l}^3) + (-(A/C)Q_{2l-1}^3 + P_{2l-1}^3) \\ &= -(A/C)(b_{2l+1}^3Q_{2l}^3 + Q_{2l-1}^3) + (b_{2l+1}^3P_{2l}^3 + P_{2l-1}^3) \\ &= -(A/C)Q_{2l+1}^3 + P_{2l+1}^3. \end{aligned}$$

So the right part of (3.5) is satisfied for $n = l + 1$. Samely, we can obtain the left part. By induction, we see that (3.5) holds for all $n \geq 1$. □

We note that the polynomials P_n and Q_n defined in the previous Lemma will be used throughout the rest of this section. Also, it is clear that C divides Q_n for all n odd integer. Moreover, for the proofs of the following Lemmas, we will follow [5] fairly closely.

Lemma 3.2 *Let P and Q be two polynomials of $\mathbb{F}_3[X]$, with $Q \neq 0$, and n a positive integer. Suppose that $PQ_n^2 - QP_n^2 \neq 0$. If*

$$|Q| \leq |Q_n|^2 \quad \textit{and} \quad |PQ_n^2 - QP_n^2| < \frac{|Q_n|^2}{|Q|} \tag{3.6}$$

then P/Q is a convergent to α. Moreover, if P and Q are coprime and the convergent P/Q is U_k/V_k, then we have

$$|a_{k+1}| = |PQ_n^2 - QP_n^2|^{-1}|Q|^{-1}|Q_n|^2. \tag{3.7}$$

Proof We have for $n \geq 0$

$$|\beta^2 - (P_n/Q_n)^2| = |\beta - (P_n/Q_n)||\beta + (P_n/Q_n)|.$$

Since $|\beta| = |P_n/Q_n| = |A|^{-1}$, we have two terms in the sum, each with the absolute value $|A|^{-1}$ and the same dominant coefficient. So this becomes

$$|\beta^2 - (P_n/Q_n)^2| = |\beta - (P_n/Q_n)||A|^{-1} = |Q_n Q_{n+1}|^{-1}|A|^{-1}.$$

So

$$|\beta^2 - (P_{2n}/Q_{2n})^2| = |Q_{2n} Q_{2n+1}|^{-1}|A|^{-1} = |Q_{2n}|^{-4}|A|^{-2}, \tag{3.8}$$
$$|\beta^2 - (P_{2n+1}/Q_{2n+1})^2| = |Q_{2n+1} Q_{2n+2}|^{-1}|A|^{-1} = |Q_{2n+1}|^{-4}|A^2/C|^{-1}. \tag{3.9}$$

From the equalities (3.8) and (3.9) we have:

$$|\alpha - (P_n/Q_n)^2| \leq \frac{1}{|Q_n|^4|A|} < \frac{1}{|Q_n|^4} \leq \frac{1}{|Q_n|^2|Q|} \leq \frac{|PQ_n^2 - QP_n^2|}{|Q_n|^2|Q|}.$$

Hence

$$|\alpha - (P_n/Q_n)^2| < |P/Q - (P_n/Q_n)^2|.$$

Therefore,

$$|\alpha - P/Q| = |\alpha - (P_n/Q_n)^2 + (P_n/Q_n)^2 - P/Q| = |P/Q - (P_n/Q_n)^2|$$

and by (3.6)

$$|\alpha - P/Q| < |Q|^{-2}.$$

This shows that P/Q is a convergent to α. Now if P and Q are coprime and $P/Q = U_k/V_k$, we have $|Q| = |V_k|$. Besides, we know that

$$|\alpha - U_k/V_k| = |V_k|^{-2}|a_{k+1}|^{-1}.$$

Since

$$|\alpha - U_k/V_k| = \left|P/Q - (P_n/Q_n)^2\right|,$$

then (3.7) holds and so we obtain the desired result. □

We denote by $a = \deg A$ and $c = \deg C$.

Lemma 3.3 *We consider the following sequences of rational functions:*

$$\frac{R_{1,n}}{S_{1,n}} = P_n^2/Q_n^2, \quad for \;\; all \;\; n \geq 1. \tag{3.10}$$

$$\frac{R_{2,n}}{S_{2,n}} = \begin{cases} C^{-1}P_n^2Q_n^2/(C^{-1}Q_n^4 + 1) & \text{if } n \text{ is odd;} \\ P_n^2Q_n^2/(Q_n^4 + 1) & \text{if } n \text{ is even.} \end{cases}$$
$$\frac{R_{3,n}}{S_{3,n}} = \begin{cases} P_n^2(-Q_n^4C^{-1} + 1)/-C^{-1}Q_n^6 & \text{if } n \text{ is odd;} \\ P_n^2(Q_n^4 + 2)/Q_n^6 & \text{if } n \text{ is even.} \end{cases}$$

Then for all $1 \leq i \leq 3$, $R_{i,n}/S_{i,n}$ is a convergent to α. Further $R_{i,n}$ and $S_{i,n}$ are coprime, and if we put $m(i,n)$ the integer such that $U_{m(i,n)}/V_{m(i,n)} = R_{i,n}/S_{i,n}$ then:

$$\begin{cases} \deg a_{m(1,n)+1} = 2a \;\; if \;\; n \;\; is \;\; even \;\; and \;\; \deg a_{m(1,n)+1} = 2a - c \;\; if \;\; n \;\; is \;\; odd, \\ \deg a_{m(2,n)+1} = 2a \;\; for \;\; all \;\; n, \\ \deg a_{m(3,n)+1} = 2a \;\; if \;\; n \;\; is \;\; even \;\; and \;\; \deg a_{m(3,n)+1} = 2a - c \;\; if \;\; n \;\; is \;\; odd. \end{cases}$$

Moreover, we have $\frac{R_{3,n}}{S_{3,n}}$ is the convergent which comes before $\frac{R_{1,n+1}}{S_{1,n+1}}$, i.e

$$U_{m(1,n+1)-1}/V_{m(1,n+1)-1} = R_{3,n}/S_{3,n} \tag{3.11}$$

Proof The equalities (3.8) and (3.9) gives that for all $n \geq 1$, $\frac{R_{1,n}}{S_{1,n}} = P_n^2/Q_n^2$ is a sequence convergent of α such that $\deg a_{m(1,n)+1} = 2a$ if n is even and $\deg a_{m(1,n)+1} = 2a - c$ if n is odd.

On the other hand we have $|S_{2,n}| = |C|^{-1}|Q_n|^4 \leq |Q_{n+1}|^2$ if n is odd and $|S_{2,n}| = |Q_n|^4 \leq |Q_{n+1}|^2$ if n is even. Moreover, for odd n we have $|S_{3,n}| = |Q_n|^6|C|^{-1} \leq |Q_{n+1}|^2$, and for even n we have $|S_{3,n}| = |Q_n|^6 \leq |Q_{n+1}|^2$ then we obtain the first part of the condition (3.6).

**)For odd n:*

$$\begin{aligned} R_{2,n}Q_{n+1}^2 - S_{2,n}P_{n+1}^2 &= P_n^2\frac{Q_n^2}{C}Q_{n+1}^2 - \left(\frac{Q_n^4}{C}+1\right)P_{n+1}^2 \\ &= (1 - P_{n+1}Q_n)^2\,\frac{Q_n^2}{C} - \left(\frac{Q_n^4}{C}+1\right)\frac{Q_n^6}{C^2} \\ &= \left(1 - \frac{Q_n^4}{C}\right)^2\frac{Q_n^2}{C} - \left(\frac{Q_n^4}{C}+1\right)\frac{Q_n^6}{C^2} \\ &= \frac{Q_n^2}{C}. \end{aligned}$$

Let H be a common divisor to $R_{2,n}$ and $S_{2,n}$ then H divides $\frac{Q_n^2}{C}$ and so H divides $\frac{Q_n^4}{C}$. Since H divides $S_{2,n} = \frac{Q_n^4}{C} + 1$ then H divides 1. Thus, $R_{2,n}$ and $S_{2,n}$ are coprime. On the other hand,

$$|R_{2,n}Q_{n+1}^2 - S_{2,n}P_{n+1}^2| = \frac{|Q_n|^2}{|C|} < \frac{|Q_{n+1}|^2}{|S_{2,n}|} = |A/C|^2|Q_n|^2$$

So $\frac{R_{2,n}}{S_{2,n}}$ is a convergent to α and

$$|a_{m(2,n)+1}| = |Q_n|^{-2}|C|^2|Q_n|^{-4}|Q_{n+1}|^2 = |Q_n|^{-2}|C|^2|Q_n|^{-4}|A/C|^2|Q_n|^6 = |A|^2.$$

**)For even n:*

$$\begin{aligned} |R_{2,n}Q_{n+1}^2 - S_{2,n}P_{n+1}^2| &= |P_n^2Q_n^2Q_{n+1}^2 - (Q_n^4+1)P_{n+1}^2| \\ &= |(1 + P_{n+1}Q_n)^2Q_n^2 - (Q_n^4+1)Q_n^6| \\ &= |(1 + Q_n^4)^2Q_n^2 - (Q_n^4+1)Q_n^6| \\ &= |Q_n|^2 < \frac{|Q_{n+1}|^2}{|S_{2,n}|} = |A||Q_n|^2. \end{aligned}$$

Moreover, it is clear that $R_{2,n}$ and $S_{2,n}$ are coprime. So $\frac{R_{2,n}}{S_{2,n}}$ is a convergent to α and

$$|a_{m(2,n)+1}| = |Q_n|^{-2}|Q_n|^{-4}|Q_{n+1}|^2 = |Q_n|^{-2}|Q_n|^{-4}|A|^2|Q_n|^6 = |A|^2.$$

**)For odd n:*

$$\begin{aligned} |R_{3,n}Q_{n+1}^2 - S_{3,n}P_{n+1}^2| &= \left|P_n^2(-\frac{Q_n^4}{C}+1)Q_{n+1}^2 + \frac{Q_n^6}{C}P_{n+1}^2\right| \\ &= \left|-(1 - P_{n+1}Q_n)^2(\frac{Q_n^4}{C} - 1) + \frac{Q_n^6}{C}\frac{Q_n^6}{C^2}\right| \end{aligned}$$

$$= \left| -(1-\frac{Q_n^4}{C})^2(\frac{Q_n^4}{C}-1)+\frac{Q_n^6}{C}\frac{Q_n^6}{C^2} \right|$$
$$= 1 \leq \frac{|Q_{n+1}|^2}{|S_{3,n}|} = |A^2C|.$$

This gives that $R_{3,n}$ and $S_{3,n}$ are coprime. So $\dfrac{R_{3,n}}{S_{3,n}}$ is a convergent to α and

$$|a_{m(3,n)+1}| = |Q_n|^{-6}|C||Q_{n+1}|^2 = |Q_n|^{-6}|C||A/C|^2|Q_n|^6 = |A^2/C|.$$

**)For even n:*

$$\begin{aligned}|R_{3,n}Q_{n+1}^2 - S_{3,n}P_{n+1}^2| &= |P_n^2(Q_n^4+2)Q_{n+1}^2 - Q_n^6P_{n+1}^2|\\ &= |(1-P_{n+1}Q_n)^2(Q_n^4+2) - Q_n^6Q_n^6|\\ &= |(1-Q_n^4)^2(Q_n^4+2) - Q_n^{12}|\\ &= 1 \leq \frac{|Q_{n+1}|^2}{|S_{3,n}|} = |A|^2.\end{aligned}$$

This gives that $R_{3,n}$ and $S_{3,n}$ are coprime. So $\dfrac{R_{3,n}}{S_{3,n}}$ is a convergent to α and

$$|a_{m(3,n)+1}| = |Q_n|^{-6}|Q_{n+1}|^2 = |Q_n|^{-6}|A|^2|Q_n|^6 = |A|^2.$$

Furthermore, we note that we have $|S_{1,n+1}| = |Q_{n+1}|^2 = |A|^2|Q_n^6| = |S_{3,n}||A|^2$ for even n and $|S_{1,n+1}| = |Q_{n+1}|^2 = |A/C|^2|Q_n|^6 = |S_{3,n}||A^2/C|$ for odd n, this leads to deduce that $R_{3,n}/S_{3,n}$ is the convergent coming before $R_{1,n+1}/S_{1,n+1}$. □

We introduce $\Omega_{1,n}$, $\Omega_{2,n}$ and $\Omega_{3,n}$ the sequences of partial quotients which represent respectively the convergents $\dfrac{R_{1,n}}{S_{1,n}}$, $\dfrac{R_{2,n}}{S_{2,n}}$ and $\dfrac{R_{3,n}}{S_{3,n}}$. Then:

$$R_{i,n}/S_{i,n} = [0, \Omega_{i,n}] \;\; and \;\; \Omega_{i,n} = a_1, \ldots, a_{m(i,n)} \;\; for \;\; n \geq 0 \;\; and \;\; 1 \leq i \leq 3.$$

We have $R_{1,1}/S_{1,1} = 1/A^2$ so $\Omega_{1,1} = a_1 = A^2$. Further,

$$R_{1,2}/S_{1,2} = P_2^2/Q_2^2 = [0, A^2, A^2/C, 2A^2, 2A^2/C]$$

so $\Omega_{1,2} = A^2, A^2/C, 2A^2, 2A^2/C$.

Note that for $n \geq 1$, we have $1 < |S_{1,n}| < |S_{2,n}| < |S_{3,n}|$ then $m(1,n) < m(2,n) < m(3,n)$. We put $\Lambda_{2,n} = a_{m(1,n)+2}, \ldots, a_{m(2,n)}$ and $\Lambda_{3,n} = a_{m(2,n)+2}, \ldots, a_{m(3,n)}$. Then, from the previous Lemma we can write for $n \geq 1$:

$$\begin{aligned}\Omega_{2,n} &= \Omega_{1,n}, a_{m(1,n)+1}, \Lambda_{2,n},\\ \Omega_{3,n} &= \Omega_{1,n}, a_{m(1,n)+1}, \Lambda_{2,n}, a_{m(2,n)+1}, \Lambda_{3,n}.\end{aligned}$$

So, from (3.11) we can write for $n \geq 1$:

$$\Omega_{1,n+1} = \Omega_{1,n}, a_{m(1,n)+1}, \Lambda_{2,n}, a_{m(2,n)+1}, \Lambda_{3,n}, a_{m(3,n)+1}. \tag{3.12}$$

On the other hand, observations by computers of the first few hundred of partial quotients of the solution α of (1.6) show that C divides all partial quotients with odd index of any sequence $\Omega_k = a_1, a_2, \ldots, a_k$ and we can compute the sequence of partial quotients of $C^{-1}\Omega_k$, as we have describe above. So, we admit this in the following Lemma, more precisely, equality (3.17) below. However, we are not able to provide a proof. For this reason, we will state our last result as a conjecture and we will expose this problem as an open question at the end of this section.

Lemma 3.4 *There exists, a sequences $(\epsilon_n)_{n\geq 1}$ of nonzero element of $\mathbb{F}_3$, such that* -)For even *n:*

$$a_{m(1,n)-k} = \epsilon_n \frac{1}{(2C)^{(-1)^k}} a_{k+1} \tag{3.13}$$

for each (k, n) with $0 \leq k \leq m(1, n) - 1$; $n \geq 1$. Further, we have for $n \geq 2$,

$$\begin{cases} \Lambda_{3,n}, a_{m(3,n)+1} = \epsilon_{n+1}\widetilde{\Omega}_{1,n} = \dfrac{1}{2C}\Omega_{1,n};\ \Lambda_{2,n} = \epsilon_{n+1}\widetilde{\Lambda}_{2,n} \\ a_{m(3,n)+1} = \epsilon_{n+1}A^2; \qquad a_{m(1,n)+1} = \epsilon_{n+1}a_{m(2,n)+1} \end{cases}$$

-)For odd *n:*

$$a_{m(1,n)-k} = \epsilon_n a_{k+1} \tag{3.14}$$

for each (k, n) with $0 \leq k \leq m(1, n) - 1$; $n \geq 1$. Further we have for $n \geq 2$,

$$\begin{cases} \Lambda_{3,n}, a_{m(3,n)+1} = \epsilon_{n+1}\widetilde{\Omega}_{1,n} = \Omega_{1,n};\ \Lambda_{2,n} = \epsilon_{n+1}\dfrac{1}{2C}\widetilde{\Lambda}_{2,n} \\ a_{m(3,n)+1} = \epsilon_{n+1}2A^2/C; \qquad a_{m(1,n)+1} = \epsilon_{n+1}\dfrac{1}{2C}a_{m(2,n)+1} \end{cases}$$

Proof *If n is even:* By (3.10) and (3.11), we can write

$$U_{m(1,n)} = \epsilon'_n P_n^2, \qquad V_{m(1,n)} = \epsilon'_n Q_n^2 \tag{3.15}$$

and

$$U_{m(1,n)-1} = \epsilon''_n P_{n-1}^2(-Q_{n-1}^4 C^{-1} + 1), \qquad V_{m(1,n)-1} = -\epsilon''_n Q_{n-1}^6/C = \epsilon''_n 2C P_n^2 \tag{3.16}$$

where ϵ'_n and ϵ''_n are nonzero elements of $\mathbb{F}_3$. We write $\epsilon_n = \epsilon'_n/\epsilon''_n$.

We can write $V_{m(1,n)}/V_{m(1,n)-1} = [a_{m(1,n)}, a_{m(1,n)-1}, \ldots, a_1]$. On the other hand, by (3.15) and (3.16), we have

$$\frac{V_{m(1,n)}}{V_{m(1,n)-1}} = \epsilon_n \frac{1}{2C}\frac{V_{m(1,n)}}{U_{m(1,n)}} = \epsilon_n \frac{1}{2C}\frac{1}{[0, a_1, \ldots, a_{m(1,n)}]};$$

therefore:

$$[a_{m(1,n)}, a_{m(1,n)-1}, \ldots, a_1] = \epsilon_n \frac{1}{2C}[a_1, \ldots, a_{m(1,n)}].$$

Admit that

$$\epsilon_n \frac{1}{2C}[a_1, \ldots, a_{m(1,n)}] = \epsilon_n[(2C)^{-1}a_1, \ldots, (2C)^{(-1)^i}a_i, \ldots, (2C)^{(-1)^{m(1,n)}}a_{m(1,n)}]. \tag{3.17}$$

Then we can write $\widetilde{\Omega}_{1,n} = \epsilon_n \frac{1}{2C}\Omega_{1,n}$ and we get equality (3.13).

If n is odd: Again by (3.10) and (3.11), we can write

$$U_{m(1,n)} = \epsilon'_n P_n^2, \qquad V_{m(1,n)} = \epsilon'_n Q_n^2 \tag{3.18}$$

and

$$U_{m(1,n)-1} = \epsilon''_n P_{n-1}^2(Q_n^4 - 1), \qquad V_{m(1,n)-1} = \epsilon''_n Q_{n-1}^6 = P_n^2 \tag{3.19}$$

Then we obtain

$$\frac{V_{m(1,n)}}{V_{m(1,n)-1}} = \epsilon_n \frac{V_{m(1,n)}}{U_{m(1,n)}} = \epsilon_n \frac{1}{[0, a_1, \ldots, a_{m(1,n)}]};$$

therefore:

$$[a_{m(1,n)}, a_{m(1,n)-1}, \ldots, a_1] = \epsilon_n[a_1, \ldots, a_{m(1,n)}]$$

Then we can write $\widetilde{\Omega}_{1,n} = \epsilon_n\Omega_{1,n}$ and we get equality (3.14). □

For $n \geq 1$, we put $\Omega_{1,n} = a_1, \Lambda_{1,n}$. Hence, for $n \geq 1$, (3.12) becomes

$$\Omega_{1,n+1} = A^2, \Lambda_{1,n}, a_{m(1,n)+1}, \Lambda_{2,n}, a_{m(2,n)+1}, \Lambda_{3,n}, a_{m(3,n)+1}. \tag{3.20}$$

For each finite sequence of nonzero polynomials, we define its degree as being the sum of the degrees of its terms. We have $\deg \Omega_{1,n} = \deg S_{1,n} = 2 \deg Q_n$.

*)If n is even then $\widetilde{\Omega}_{1,n+1} = \epsilon_{n+1}\Omega_{1,n+1}$. Further we have $\deg S_{2,n} = 4 \deg Q_n$ and $\deg S_{3,n} = 6 \deg Q_n$. As $\deg Q_n = 3 \deg Q_{n-1} + a - c$ then if we put $w_n = 6 \deg Q_{n-1} - 2c$ then $\deg \Omega_{1,n} = w_n + 2a$, $\deg S_{2,n} = 12 \deg Q_{n-1} + 4a - 4c = 2a + w_n + 2a + w_n$ and $\deg S_{3,n} = 18 \deg Q_{n-1} + 6a - 6c = 2a + w_n + 2a + w_n + 2a + w_n$. As $\deg a_{m(1,n)+1} = \deg a_{m(2,n)+1} = \deg a_{m(3,n)+1} = 2a$ then if we write the sequence of the degrees of the components in the right side of (3.20), we obtain the sequence, of 7 terms:$2a, w_n, 2a, w_n, 2a, w_n, 2a$. As this sequence is reversible and $\widetilde{\Omega}_{1,n+1} = \epsilon_{n+1}\Omega_{1,n+1}$, it is clear that $\widetilde{\Lambda}_{3,n} = \epsilon_{n+1}\Lambda_{1,n}$, $\Lambda_{2,n} = \epsilon_{n+1}\widetilde{\Lambda}_{2,n}$, $a_{m(3,n)+1} = \epsilon_{n+1}A^2$, $a_{m(1,n)+1} = \epsilon_{n+1}a_{m(2,n)+1}$.

*)If n is odd then $\widetilde{\Omega}_{1,n+1} = \epsilon_{n+1}\frac{1}{2C}\Omega_{1,n+1}$. Further we have $\deg S_{2,n} = 4 \deg Q_n - c$ and $\deg S_{3,n} = 6 \deg Q_n - c$. As $\deg Q_n = 3 \deg Q_{n-1} + a$ then if we put $w_n = 6 \deg Q_{n-1}$ then $\deg \Omega_{1,n} = w_n + 2a$, $\deg S_{2,n} = 12 \deg Q_{n-1} + 4a - c = 2a + w_n + 2a - c + w_n$ and $\deg S_{3,n} = 18 \deg Q_{n-1} + 6a - c = 2a + w_n + 2a - c + w_n + 2a + w_n$. As $\deg a_{m(1,n)+1} = 2a - c$, $\deg a_{m(2,n)+1} = 2a$ and $\deg a_{m(3,n)+1} = 2a - c$, then if we write the sequence of the degrees of the components in the right side of (3.20), we obtain the sequence, of 7 terms:$2a, w_n, 2a-c, w_n, 2a, w_n, 2a-c$. As $\widetilde{\Omega}_{1,n+1} = \epsilon_{n+1}\frac{1}{2C}\Omega_{1,n+1}$, it is clear that $\widetilde{\Lambda}_{3,n} = \epsilon_{n+1}2C\Lambda_{1,n}$, $\Lambda_{2,n} = \epsilon_{n+1}2C\widetilde{\Lambda}_{2,n}$, $a_{m(3,n)+1} = \epsilon_{n+1}A^2/2C$, $a_{m(1,n)+1} = \epsilon_{n+1}\frac{1}{2C}a_{m(2,n)+1}$.

Lemma 3.5 *There exists, a sequences $(\epsilon_n)_{n\geq 1}$ of nonzero element of $\mathbb{F}_3$, such that:*

-)For even n*: we have*

$$\Lambda_{2,n} = (\epsilon_n/C^2)\Omega_{n-1}^{(3)} \quad \textit{and} \quad a_{m(1,n)+1} = \epsilon_n 2A^2.$$

-)For odd n*: we have*

$$\Lambda_{2,n} = (\epsilon_n/2C)\Omega_{n-1}^{(3)} \quad \textit{and} \quad a_{m(1,n)+1} = \epsilon_n 2A^2/C.$$

Proof *)*If n is even:* We have $U_{m(1,n)}/V_{m(1,n)} = [0, \Omega_{1,n}]$, $U_{m(1,n)+1}/V_{m(1,n)+1} = [0, \Omega_{1,n}, a_{m(1,n)+1}]$ and $U_{m(2,n)}/V_{m(2,n)} = [0, \Omega_{1,n}, a_{m(1,n)+1}, \Lambda_{2,n}]$. If we put $x_{2,n}$, the element of $\mathbb{F}_3(T)$ defined by $[\Lambda_{2,n}]$, then we have

$$\frac{U_{m(2,n)}}{V_{m(2,n)}} = \frac{x_{2,n}U_{m(1,n)+1} + U_{m(1,n)}}{x_{2,n}V_{m(1,n)+1} + V_{m(1,n)}}. \tag{3.21}$$

We know that $U_{m(2,n)}/V_{m(2,n)} = R_{2,n}/S_{2,n} = P_n^2 Q_n^2/(Q_n^4 + 1)$. So if we put

$$P' = P_n^2 Q_n^2 \quad \textit{and} \quad Q' = Q_n^4 + 1 \tag{3.22}$$

the equality (3.21) gives that:

$$x_{2,n} = \epsilon_n' \frac{P_n^2 Q' - Q_n^2 P'}{V_{m(1,n)+1}P' - U_{m(1,n)+1}Q'}. \tag{3.23}$$

We should determine $U_{m(1,n)+1}/V_{m(1,n)+1}$. We use the fact that $R_{3,n-1}/S_{3,n-1}$ and $R_{(1,n)}/S_{(1,n)}$ are, from Lemma 3.3, the two reduced precedes it.

Hence we consider the polynomials P and Q of $\mathbb{F}_3[T]$, defined by:

$$P = 2A^2 P_n^2 + P_{n-1}^3 Q_n \quad \textit{and} \quad Q = 2A^2 Q_n^2 - C P_n^2. \tag{3.24}$$

We will apply Lemma 3.2, to prove that P/Q is a convergent to α. First we have $\deg Q = 2 \deg Q_n + 2a$ and then $Q \neq 0$. From (3.24) and (3.5), we have $PQ_n^2 - QP_n^2 = P_{n-1}^3 Q_n^3 + CP_n^4 = P_{n-1}^3 Q_n^3 - P_n^3 Q_{n-1}^3 = 1$, hence $\gcd(P, Q) = 1$. Since $2 \deg Q_n + 2a \leq 2 \deg Q_{n+1}$ for $n \geq 2$, the first part of condition; that is $|Q| < |Q_{n+1}|^2$, is satisfied. We should prove that $|PQ_{n+1}^2 - QP_{n+1}^2| < |Q_{n+1}|^2/|Q|$. We put

$$X_1 = Q_{n+1}^2 P_n^2 - Q_n^2 P_{n+1}^2 \quad \textit{and} \quad X_2 = P_{n-1}^3 Q_n Q_{n+1}^2 + C P_n^2 P_{n+1}^2.$$

From (3.24), we have $PQ_{n+1}^2 - QP_{n+1}^2 = 2A^2X_1 + X_2$. Since $P_{n+1}Q_n - Q_{n+1}P_n = 1$, and by (3.5), we have

$$X_1 = 2Q_nP_{n+1} + 1 = Q_n^4 + 1$$

then

$X_2 = Q_{n+1}^2P_{n-1}^3Q_n + CP_{n+1}^2P_n^2 = (Q_{n+1}/Q_n)^2(1 - CP_n^4) + CP_{n+1}^2P_n^2$

$X_2 = (Q_{n+1}/Q_n)^2 - C(P_n/Q_n)^2X_1$

$X_2 = (Q_{n+1}/Q_n)^2 - C(P_nQ_n)^2 - C(P_n/Q_n)^2$.

We put $X = PQ_{n+1}^2 - QP_{n+1}^2$. Since $X = 2A^2X_1 + X_2$, we have

$X = 2A^2 + 2A^2Q_n^4 + (Q_{n+1}/Q_n)^2 - C(P_nQ_n)^2 - C(P_n/Q_n)^2$

$X = 2A^2 + 2A^2Q_n^4 + (AQ_n^2 - CP_n^3/Q_n)^2 - C(P_n/Q_n)^2(Q_n^4 + 1)$

Since $Q_n^4 - AP_nQ_n^3 + CP_n^4 = P_{n+1}Q_n - Q_{n+1}P_n = 1$ then

$X = 2A^2 + (ACQ_nP_n^3 + C^2P_n^6/Q_n^2 - C(P_n/Q_n)^2(2Q_n^4 - AP_nQ_n^3 + CP_n^4))$

$X - 2A^2 = 2ACQ_nP_n^3 + CP_n^2Q_n^2 = CP_n^2Q_nP_{n-1}^3$. Since, for $n \geq 2$, $|P_{n-1}^3| < |Q_n|$ and $|C||P_n|^2 < |Q_n|^2$, this equality implies:

$$|X| < |Q_n^4| = \frac{|Q_{n+1}|^2}{|Q|}$$

Consequently, P/Q is a convergent to α, and since $\deg Q = \deg V_{m(1,n)} + 2a$, then it is next $U_{m(1,n)}/V_{m(1,n)}$. We can write

$$U_{m(1,n)+1} = \eta_nP \quad and \quad V_{m(1,n)+1} = \eta_nQ. \tag{3.25}$$

By (3.15), (3.16) and (3.24), and $\epsilon^{-1} = \epsilon$ for $\epsilon \in \mathbb{F}_3$, the first equality of (3.25) can be written

$$a_{m(1,n)+1}U_{m(1,n)} + U_{m(1,n)-1} = \eta_n\epsilon_n'2A^2U_{m(1,n)} + \eta_n\epsilon_n''U_{m(1,n)-1}.$$

Since we have $\deg U_{m(1,n)} > \deg U_{m(1,n)-1}$, it follows that $a_{m(1,n)+1} = \eta_n\epsilon_n'2A^2$ and $\eta_n\epsilon_n'' = 1$, i.e $\eta_n = \epsilon_n''$. Thus, since $\epsilon_n = \epsilon_n'\epsilon_n''$, we obtain:

$$a_{m(1,n)+1} = \epsilon_n2A^2.$$

So the equality (3.23) becomes:

$$x_{2,n} = \epsilon_n\frac{P_n^2Q' - Q_n^2P'}{QP' - PQ'}. \tag{3.26}$$

We are able to compute $x_{2,n}$.

$$P_n^2Q' - Q_n^2P' = P_n^2(Q_n^4 + 1) - Q_n^2(P_nQ_n)^2 = P_n^2 = Q_{n-1}^6/C^2.$$

From we have

$$QP' - PQ' = P_n^2Q_n^2(2A^2Q_n^2 - CP_n^2) - (Q_n^4 + 1)(2A^2P_n^2 + P_{n-1}^3Q_n)$$
$$QP' - PQ' = -CP_n^4Q_n^2 - Q_n^5P_{n-1}^3 - (2A^2P_n^2 + P_{n-1}^3Q_n)$$
$$QP' - PQ' = Q_n^2(P_nQ_{n-1} - Q_nP_{n-1})^3 - (2A^2P_n^2 + Q_n^2 - AP_nQ_n)$$
$$QP' - PQ' = A^2P_n^2 + Q_n^2 + AP_nQ_n$$
$$QP' - PQ' = (Q_n - AP_n)^2 = P_{n-1}^6.$$

So (3.26) gives that

$$x_{2,n} = \epsilon_n/C^2(Q_{n-1}/P_{n-1})^6.$$

Furthermore

$$[a_1, \ldots, a_{m(1,n-1)}] = (Q_{n-1}/P_{n-1})^2$$

then

$$(\epsilon_n/C^2)(Q_{n-1}/P_{n-1})^6 = \epsilon_n/C^2[a_1^3, \ldots, a_{m(1,n-1)}^3].$$

Thus, we conclude that (3.20)can be written as

$$\Omega_{1,n+1} = \Omega_{1,n}, \epsilon_n 2A^2, (\epsilon_n/C^2)\Omega_{1,n-1}^{(3)}, \epsilon_{n+1}\epsilon_n 2A^2, \epsilon_{n+1}\widetilde{\Omega}_{1,n}. \quad (3.27)$$

*)*If n is odd:*

We have $U_{m(1,n)}/V_{m(1,n)} = [0, \Omega_{1,n}]$, $U_{m(1,n)+1}/V_{m(1,n)+1} = [0, \Omega_{1,n}, a_{m(1,n)+1}]$ and $U_{m(2,n)}/V_{m(2,n)} = [0, \Omega_{1,n}, a_{m(1,n)+1}, \Lambda_{2,n}]$. If we put $x_{2,n}$, the element of $\mathbb{F}_3(T)$ defined by $[\Lambda_{2,n}]$, then we have

$$\frac{U_{m(2,n)}}{V_{m(2,n)}} = \frac{x_{2,n}U_{m(1,n)+1} + U_{m(1,n)}}{x_{2,n}V_{m(1,n)+1} + V_{m(1,n)}}. \quad (3.28)$$

We know that $U_{m(2,n)}/V_{m(2,n)} = R_{2,n}/S_{2,n} = C^{-1}P_n^2Q_n^2/(C^{-1}Q_n^4 + 1)$. So if we put

$$P' = C^{-1}P_n^2Q_n^2 \quad \textit{and} \quad Q' = C^{-1}Q_n^4 + 1 \quad (3.29)$$

the equality (3.28) gives that:

$$x_{2,n} = \epsilon_n' \frac{P_n^2Q' - Q_n^2P'}{V_{m(1,n)+1}P' - U_{m(1,n)+1}Q'}. \quad (3.30)$$

We should determine $U_{m(1,n)+1}/V_{m(1,n)+1}$. We use the fact that $R_{3,n-1}/S_{3,n-1}$ and $R_{(1,n)}/S_{(1,n)}$ are, from Lemma 3.3 , the two reduced precedes it.

Hence we consider the polynomials P and Q of $\mathbb{F}_3[T]$, defined by:

$$P = (A^2/C)P_n^2 + P_{n-1}^3Q_n \quad \textit{and} \quad Q = (A^2/C)Q_n^2 + P_n^2. \quad (3.31)$$

We will apply Lemma 3.2 to prove that P/Q is a convergent to α. First we have $\deg Q = 2\deg Q_n + 2a - c$ and then $Q \neq 0$. From (3.33) and (3.5), we have $PQ_n^2 - QP_n^2 = P_{n-1}^3Q_n^3 - P_n^4 = P_{n-1}^3Q_n^3 - P_n^3Q_{n-1}^3 = -1$, hence $\gcd(P, Q) = 1$. Since $2\deg Q_n + 2a - c \leq 2\deg Q_{n+1}$ for $n \geq 2$, the first part of condition; that is $|Q| < |Q_{n+1}|^2$, is satisfied. We should prove that $|PQ_{n+1}^2 - QP_{n+1}^2| < |Q_{n+1}|^2/|Q|$. We put

$$X_1 = Q_{n+1}^2P_n^2 - Q_n^2P_{n+1}^2 \quad \textit{and} \quad X_2 = P_{n-1}^3Q_nQ_{n+1}^2 - P_n^2P_{n+1}^2.$$

From (3.24), we have $PQ_{n+1}^2 - QP_{n+1}^2 = (A^2/C)X_1 + X_2$. Since $P_{n+1}Q_n - Q_{n+1}P_n = -1$, and by (3.5), we have

$$X_1 = -Q_nP_{n+1} + 1 = C^{-1}Q_n^4 + 1$$

then

$X_2 = Q_{n+1}^2P_{n-1}^3Q_n - P_{n+1}^2P_n^2 = (Q_{n+1}/Q_n)^2(-1 + P_n^4) - P_{n+1}^2P_n^2$

$X_2 = -(Q_{n+1}/Q_n)^2 + (P_n/Q_n)^2X_1$

$X_2 = -(Q_{n+1}/Q_n)^2 + C^{-1}(P_nQ_n)^2 + (P_n/Q_n)^2$.

We put $X = PQ_{n+1}^2 - QP_{n+1}^2$. Since $X = A^2/CX_1 + X_2$, we have

$X = A^2C^{-1} + A^2C^{-2}Q_n^4 - (Q_{n+1}/Q_n)^2 + C^{-1}(P_nQ_n)^2 + (P_n/Q_n)^2$

$X = A^2C^{-1} + A^2C^{-2}Q_n^4 - (-AC^{-1}Q_n^2 + P_n^3/Q_n)^2 + (P_n/Q_n)^2(Q_n^4/C + 1)$

Since $2Q_n^4C^{-1} + AC^{-1}P_nQ_n^3 - P_n^4 = P_{n+1}Q_n - Q_{n+1}P_n = -1$ then

$X = A^2C^{-1} + 2AC^{-1}Q_nP_n^3 - P_n^6/Q_n^2 + (P_n/Q_n)^2)(-Q_n^4C^{-1} - AC^{-1}P_nQ_n^3 + P_n^4)$

$X - A^2C^{-1} = AC^{-1}Q_nP_n^3 - P_n^2Q_n^2C^{-1} = P_n^2Q_n(AC^{-1}P_n - C^{-1}Q_n) = P_n^2Q_nP_{n-1}^3$. Since, for $n \geq 2$, $|P_{n-1}|^3 < |Q_n|$ and $|P_n|^2 < |Q_n|^2/C$, this equality implies:

$$|X| < |Q_n^4||C|^{-1} = \frac{|Q_{n+1}|^2}{|Q|}$$

Consequently, P/Q is a convergent to α, and since $\deg Q = \deg V_{m(1,n)} + 2a - c$, then it is next $U_{m(1,n)}/V_{m(1,n)}$. We can write

$$U_{m(1,n)+1} = \eta_n P \quad and \quad V_{m(1,n)+1} = \eta_n Q. \tag{3.32}$$

By (3.18), (3.19) and (3.32), and $\epsilon^{-1} = \epsilon$ for $\epsilon \in \mathbb{F}_3$, the first equality of (3.32) can be written

$$a_{m(1,n)+1}U_{m(1,n)} + U_{m(1,n)-1} = \eta_n\epsilon_n' A^2C^{-1}U_{m(1,n)} + \eta_n\epsilon_n'' U_{m(1,n)-1}.$$

Since we have $\deg U_{m(1,n)} > \deg U_{m(1,n)-1}$, it follows that $a_{m(1,n)+1} = \eta_n\epsilon_n' A^2C^{-1}$ and $\eta_n\epsilon_n'' = 1$, i.e $\eta_n = \epsilon_n''$. Thus, since $\epsilon_n = \epsilon_n'\epsilon_n''$, we obtain

$$a_{m(1,n)+1} = \epsilon_n A^2C^{-1}.$$

So the equality (3.30) becomes:

$$x_{2,n} = \epsilon_n \frac{P_n^2Q' - Q_n^2P'}{QP' - PQ'}. \tag{3.33}$$

We are able to compute $x_{2,n}$.

$$P_n^2Q' - Q_n^2P' = P_n^2(C^{-1}Q_n^4 + 1) - Q_n^2C^{-1}(P_nQ_n)^2 = P_n^2 = Q_{n-1}^6.$$

From we have

$$\begin{aligned}
QP' - PQ' &= C^{-1}P_n^2Q_n^2(A^2C^{-1}Q_n^2 + P_n^2) - (C^{-1}Q_n^4 + 1)(A^2C^{-1}P_n^2 + P_{n-1}^3Q_n)\\
QP' - PQ' &= C^{-1}P_n^4Q_n^2 - C^{-1}Q_n^5P_{n-1}^3 - (A^2C^{-1}P_n^2 + P_{n-1}^3Q_n)\\
QP' - PQ' &= C^{-1}Q_n^2(P_nQ_{n-1} - Q_nP_{n-1})^3 - (A^2C^{-1}P_n^2 - C^{-1}Q_n^2 + AP_nQ_n)\\
QP' - PQ' &= 2A^2C^{-1}P_n^2 + 2C^{-1}Q_n^2 + 2AP_nQ_n\\
QP' - PQ' &= 2C(C^{-1}Q_n - AC^{-1}P_n)^2 = 2CP_{n-1}^6.
\end{aligned}$$

So (3.33) gives that

$$x_{2,n} = \epsilon_n/2C(Q_{n-1}/P_{n-1})^6.$$

Furthermore

$$[a_1, \ldots, a_{m(1,n-1)}] = (Q_{n-1}/P_{n-1})^2$$

then

$$\epsilon_n/2C(Q_{n-1}/P_{n-1})^6 = (\epsilon_n/2C)[a_1^3, \ldots, a_{m(1,n-1)}^3].$$

Thus, we conclude that we can write (3.20) as:

$$\Omega_{1,n+1} = \Omega_{1,n}, \epsilon_nA^2C^{-1}, (\epsilon_n/2C)\Omega_{1,n-1}^{(3)}, \epsilon_{n+1}\epsilon_nA^2, (\epsilon_{n+1}/2C)\widetilde{\Omega}_{1,n}. \tag{3.34}$$

□

Finally, we have to determine ϵ_n for all $n \geq 1$. By Lemmas 3.3 and (3.14) we have simultaneously $\epsilon_n a_{m(1,n-1)}^3 = \epsilon_{n+1}\epsilon_n a_1^3$, which implies $a_{m(1,n-1)} = \epsilon_{n+1}a_1$ and $a_{m(1,n-1)} = \epsilon_{n-1}a_1$. Therefore, $\epsilon_{n+1} = \epsilon_{n-1}$ for even n. We can verify that we have also $\epsilon_{n+1} = \epsilon_{n-1}$ for odd n. Since $\Omega_{1,2} = 1/2C\widetilde{\Omega}_{1,2}$ and $\Omega_{1,3} = \widetilde{\Omega}_{1,3}$ then $\epsilon_2 = \epsilon_3 = 1$. So, we obtain $\epsilon_n = 1$ for all $n \geq 1$. Finally, by Lemma 3.4, the sequence $\Omega_{1,n}$ is reversible for all n odd, and so $\widetilde{\Omega}_{1,n} = \Omega_{1,n}$. The equality (3.34) becomes:

$$\Omega_{1,n+1} = \Omega_{1,n}, A^2/C, (1/2C)\Omega_{1,n-1}^{(3)}, 2A^2, (1/2C)\widetilde{\Omega}_{1,n}.$$

The equality (3.27) becomes

$$\Omega_{1,n+1} = \Omega_{1,n}, 2A^2, (1/C^2)\Omega_{1,n-1}^{(3)}, 2A^2, \widetilde{\Omega}_{1,n}.$$

So we can deduce the following result.

Conjecture 3.1 Let $\alpha \in \mathbb{F}_3((T^{-1}))$ be the formal power series, of strictly negative degree, satisfying (1.6). Let $(\Omega_n)_{n\geq 0}$ be a finite sequence of elements of $\mathbb{F}_3[T]$, defined by $\Omega_0 = \emptyset$, $\Omega_1 = A^2$ and for all $n \geq 0$:

$$\begin{cases} \Omega_{2n+1} = \Omega_{2n}, 2A^2, \dfrac{1}{C^2}\Omega_{2n-1}^{(3)}, 2A^2, \widetilde{\Omega}_{2n} \\ \Omega_{2n+2} = \Omega_{2n+1}, A^2/C, \dfrac{1}{2C}\Omega_{2n}^{(3)}, 2A^2, \dfrac{1}{2C}\Omega_{2n+1} \end{cases} \tag{3.35}$$

Let Ω_∞ be the infinite sequence beginning by Ω_n for all $n \geq 1$. Then, the continued fraction expansion of α is $\alpha = [0, \Omega_\infty]$.

We see that the equality (3.35) has the same shape as the equality (3.1). So this gives that the formal power series described in Theorem 3.1 is not other than the unique solution of the quartic equation (1.6).

Example 3.1 Let $(\Omega_n)_{n\geq 1}$ be a finite sequence of elements of $\mathbb{F}_3[T]$, defined by $\Omega_1 = T^2$, $\Omega_2 = T^2, T, 2T^2, 2T$ and for all $n \geq 0$

$$\begin{cases} \Omega_{2n+1} = \Omega_{2n}, 2T^2, \dfrac{1}{T^2}\Omega_{2n-1}^{(3)}, 2T^2, \widetilde{\Omega}_{2n} \\ \Omega_{2n+2} = \Omega_{2n+1}, T, \dfrac{1}{2T}\Omega_{2n}^{(3)}, 2T^2, \dfrac{1}{2T}\Omega_{2n+1} \end{cases}$$

Then, we have from conjecture (3.1): $\alpha = [0, \Omega_\infty] = [0, T, -T^2, T^7, \ldots, (-1)^{n-1}T^{\frac{3^n+(-1)^{n+1}}{4}}, \ldots]^2$ and according to theorem 3.1 we have $\nu(\alpha) = 2$.

In fact, α is the solution of the equation (1.6) with $A = C = T$. The partial quotients of Ω_5 are:

$[T^2, T, 2T^2, 2T, 2T^2, T^4, 2T^2, 2T, 2T^2, T, T^2, T, 2T^5, 2T^4, T^5, T^4, 2T^2, 2T, 2T^2, T, T^2, T, 2T^5, T,$
$T^2, T, 2T^2, 2T, 2T^2, T^4, T^5, 2T^4, 2T^5, 2T^4, T^{14}, 2T^4, 2T^5, 2T^4, T^5, T^4, 2T^2, 2T, 2T^2, T, T^2, T,$
$2T^5, T, T^2, T, 2T^2, 2T, 2T^2, T^4, T^5, 2T^4, 2T^5, T, T^2, T, 2T^2, 2T, 2T^2, T^4, 2T^2, 2T, 2T^2, T, T^2]$.

Note that in this case $A^2 = T^2$, $A^2/C = T$ and we have:

$$R_{1,1}/S_{1,1} = [0, T^2] = [0, \Omega_1], \qquad \Omega_1 = T^2.$$
$$R_{2,1}/S_{2,1} = [0, T^2, T];\ \ R_{3,1}/S_{3,1} = [0, T^2, T, 2T^2]$$
$$R_{1,2}/S_{1,2} = [0, T^2, T, 2T^2, 2T] = [0, \Omega_2] = [0, T^2, \Lambda_{1,2}]$$

So $\Omega_2 = T^2, T, 2T^2, 2T$ and we have $\widetilde{\Omega}_2 = (1/2T)\Omega_2$, $a_{m(1,1)+1} = a_2 = T$, $a_{m(2,1)+1} = a_3 = 2T^2$, $a_{m(3,1)+1} = a_4 = 2T$ and we see that for all $1 \leq k \leq 3$: $a_{4-k} = (2T)^{(-1)^{k+1}}a_{k+1}$.

$$\begin{aligned} R_{2,2}/S_{2,2} &= [0, T^2, T, 2T^2, 2T, 2T^2, T^4] = [0, \Omega_{2,2}] = [0, \Omega_2, 2T^2, \Lambda_{2,2}] \\ R_{3,2}/S_{3,2} &= [0, T^2, T, 2T^2, 2T, 2T^2, T^4, 2T^2, 2T, 2T^2, T] = [0, \Omega_{3,2}] \\ &= [0, \Omega_2, 2T^2, \Lambda_{2,2}, 2T^2, \Lambda_{3,2}] \\ R_{1,3}/S_{1,3} &= [0, T^2, T, 2T^2, 2T, 2T^2, T^4, 2T^2, 2T, 2T^2, T, T^2] = [0, \Omega_3] = [0, T^2, \Lambda_{1,3}] \end{aligned}$$

So $\Omega_3 = \Omega_{1,3} = \Omega_{1,2}, 2T^2, \Lambda_{2,2}, 2T^2, \Lambda_{3,2}, T^2$ and we have $a_{m(1,2)+1} = a_5 = 2T^2$, $a_{m(2,2)+1} = a_7 = 2T^2$, $a_{m(3,2)+1} = a_{11} = T^2$, $\Lambda_{2,2} = T^4 = T^{-2}\Omega_1^{(3)}$, $\Lambda_{3,2} = 2T, 2T^2, T = \widetilde{\Lambda}_{1,2}$ and we see that for all $1 \leq k \leq 10$: $a_{11-k} = a_{k+1}$.

$$\begin{aligned} R_{2,3}/S_{2,3} &= [0, T^2, T, 2T^2, 2T, 2T^2, T^4, 2T^2, 2T, 2T^2, T, T^2, T, 2T^5, 2T^4, T^5, T^4] \\ &= [0, \Omega_{2,3}] = [0, \Omega_3, T, \Lambda_{2,3}] \\ R_{3,3}/S_{3,3} &= [0, T^2, T, 2T^2, 2T, 2T^2, T^4, 2T^2, 2T, 2T^2, T, T^2, T, 2T^5, 2T^4, T^5, T^4, \\ &\quad 2T^2, 2T, 2T^2, T, T^2, T, 2T^5, T, T^2, T, 2T^2] \end{aligned}$$

$$= [0, \Omega_{3,3}] = [0, \Omega_3, 2T^3, \Lambda_{2,3}, 2T^2, \Lambda_{3,3}].$$

$$R_{1,4}/S_{1,4} = [0, T^2, T, 2T^2, 2T, 2T^2, T^4, 2T^2, 2T, 2T^2, T, T^2, T, 2T^5, 2T^4, T^5, T^4,$$

$$2T^2, 2T, 2T^2, T, T^2, T, 2T^5, T, T^2, T, 2T^2, 2T] = [0, \Omega_4]$$

So $\Omega_4 = \Omega_{1,4} = \Omega_{1,3}, T, \Lambda_{2,3}, 2T^2, \Lambda_{3,3}, 2T$ and we have $a_{m(1,3)+1} = a_{12} = T$, $a_{m(2,3)+1} = a_{17} = 2T^2$, $a_{m(3,3)+1} = a_{28} = 2T$, $\Lambda_{2,3} = 2T^5, 2T^4, T^5, T^4 = (1/2T)\Omega_2^{(3)} = 2T\widetilde{\Lambda}_{2,3}$, $\Lambda_{3,3} = 2T, 2T^2, T, T^2, T, 2T^5, T, T^2, T, 2T^2 = (1/2T)\widetilde{\Lambda}_{1,3}$ and we see that for all $1 \leq k \leq 27$: $a_{28-k} = (2T)^{(-1)^{k+1}} a_{k+1}$.

Remark 3.1 Note that the equation (2.1) can be written as $\alpha = (C\alpha^4 + 1)/A$, so $\nu(\alpha^4) = \nu(\alpha)$. Let $\beta = 1/\alpha^4$. We will determine the equation satisfied by β. We have $(A\alpha)^4 = (C\alpha^4 + 1)^4 = C^4\alpha^{16} + C^3\alpha^{12} + C\alpha^4 + 1$. Hence β satisfies the equation $\beta^4 + (-A^4 + C)\beta^3 + C^3\beta + C^4 = 0$. So it is clear that β is hyperquadratic. We can describe its continued fraction expansion as follow. We put $\gamma = 1/\alpha$. Then γ satisfies the equation

$$\gamma^4 = A\gamma^3 - C. \tag{3.36}$$

We know that the continued fraction expansion of γ is

$$\gamma = [A, -A^3C^{-1}, A^{3^2}C^{-2}, -A^{3^3}C^{-7}, \ldots, (-1)^{n-1}A^{3^{n-1}}C^{-\frac{3^{n-1}+(-1)^n}{4}}, \ldots].$$

From the property (1.1) of continued fractions and the equation (3.36) we get

$$\gamma^4 = \left[-C + A^4, -A^{3^2-1}C^{-3}, A^{3^3+1}C^{-6}, -A^{3^4-1}C^{-21}, \ldots, (-1)^{n-1}A^{3^n-(1)^n}C^{-\frac{3^n+3(-1)^n}{4}}, \ldots\right]$$
$$= \beta = 1/\alpha^4.$$

This led us to deduce the following curious relation between square of continued fractions:

$$[0, A, -A^3C^{-1}, A^{3^2}C^{-2}, -A^{3^3}C^{-7}, \ldots, (-1)^{n-1}A^{3^{n-1}}C^{-\frac{3^{n-1}+(-1)^n}{4}}, \ldots]^2 = [0, \Omega_\infty];$$
$$[0, \Omega_\infty]^2 = [0, -C + A^4, -A^{3^2-1}C^{-3}, A^{3^3+1}C^{-6}, \ldots, (-1)^{n-1}A^{3^n-(1)^n}C^{-\frac{3^n+3(-1)^n}{4}}, \ldots].$$

Remark 3.2 From Theorem 2.4, the continued fraction expansion of β solution of the equation (2.7) can be written as:

$$\beta = [0, A/C, (A/C)^3C, (A/C)^{3^2}C^2, \ldots, (A/C)^{3^{n-1}}C^{\frac{3^{n-1}+(-1)^n}{4}}, \ldots].$$

We wish to compute the continued fraction expansion and the approximation exponent of $\alpha = \beta^2 = [0, A/C, (A/C)^3C, (A/C)^{3^2}C^2, \ldots, (A/C)^{3^{n-1}}C^{\frac{3^{n-1}+(-1)^n}{4}}, \ldots]^2$. Note that, from the equation (2.7), β satisfies $\beta = (-\beta^4 + C)/A$. So $\beta^2 = (\beta^4 - C)^2/A^2$, which gives that $\beta^8 - 2C\beta^4 + C^2 = A^2\beta^2$. Then we deduce that α satisfies the equation

$$\alpha^4 + C\alpha^2 - A^2\alpha + C^2 = 0 \tag{3.37}$$

We have to state the following Conjecture.

Conjecture 3.2 Let $\alpha \in \mathbb{F}_3((T^{-1}))$ be the formal power series satisfying (3.37).
Let $(\Omega_n)_{n\geq 1}$ be a sequence of elements of $\mathbb{F}_3[T]$, defined by $\Omega_0 = \emptyset$, $\Omega_1 = A^2/C^2$ and for all $n \geq 0$

$$\begin{cases} \Omega_{2n+1} = \Omega_{2n}, 2A^2/C^2, C^2\Omega_{2n-1}^{(3)}, 2A^2/C^2, \widetilde{\Omega}_{2n} \\ \Omega_{2n+2} = \Omega_{2n+1}, 2A^2/C, C\Omega_{2n}^{(3)}, 2A^2/C^2, C\Omega_{2n+1} \end{cases} \tag{3.38}$$

Then $\alpha = [0, \Omega_\infty]$.

Note that obtaining this conjecture was achieved in the same way as conjecture 3.1. It is interesting to state that we can add the family of power series satisfying (3.37) to the set of elements admitting 2 as a value of their approximation exponents agreeing with Roth value.
At the end, we point out a question related to this work:

Open question: Let n be a positive integer and $(a_i)_{1\leq i\leq n}$ be a sequence of polynomials with coefficients in a finite field such that $\deg a_i > 0$. Let D be a nonzero polynomial with coefficients in a finite field and with strictly positive degree such that D divides a_1. Suppose that

$$[a_n, \ldots, a_1] = D^{-1}\,[a_1, \ldots, a_n]\,.$$

Then n is even and for all $0 \leq k \leq n-1$:

$$a_{n-k} = \frac{1}{D^{(-1)^k}} a_{k+1}.$$

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