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ORIGINAL RESEARCH

A note on the Diophantine equation $x^2 = 4p^n - 4p^m + \ell^2$

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Abstract Let ℓ be a fixed odd positive integer. In this paper, using some classical results on the generalized Ramanujan-Nagell equation, we completely derive all solutions (p, x, m, n) of the equation $x^2 = 4p^n - 4p^m + \ell^2$ with $\ell^2 < 4p^m$ for any $\ell > 1$, where p is a prime, x, m, n are positive integers satisfying $\gcd(x, \ell) = 1$ and $m < n$. Meanwhile we give a method to solve the equation with $l^2 > 4p^m$. As an example of using this method, we find all solutions (p, x, m, n) of the equation for $\ell \in \{5, 7\}$.

Keywords Polynomial-exponential Diophantine equation · Generalized Ramanujan-Nagell equation · Baker's method

Mathematics Subject Classification 11D61

1 Introduction

Let $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{P}$ be the sets of all integers, positive integers, rational numbers, real numbers, complex numbers and primes, respectively. Suppose that ℓ is a fixed odd positive integer.

The first work on the title equation was done by C. Skinner who was a high school student in 1987/88. He was only 15 years old. In 1989, C. Skinner [10] proved that if $p \neq 2$, then the equation

$$x^2 = 4p^n - 4p + 1, \quad p \in \mathbb{P}, \ x, n \in \mathbb{N}, \quad n \geq 1 \tag{1.1}$$

has only the solutions

$$(x, n) = \begin{cases} (1, 1), (5, 2), (31, 5), & \text{if } p = 3, \\ (1, 1), (9, 2), (559, 7), & \text{if } p = 5, \\ (1, 1), (2p - 1, 2), & \text{if } p > 5. \end{cases}$$

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To solve the equation (1.1) for all odd p primes, he used unique factorization of ideals along with linear recurrences and congruences. At the same time, P.-Z. Yuan was also interested in the title equation with $\ell = 1$ due to the group theoretical property of the solutions of generalized Ramanujan-Nagell equation. In the same year, P.-Z. Yuan [14] proved that if $\ell = 1$ and $p \neq 2$, then the equation

$$x^2 = 4p^n - 4p^m + \ell^2, \;\; p \in \mathbb{P}, \; x, m, n \in \mathbb{N}, \;\; \gcd(x, \ell) = 1, \;\; m < n \tag{1.2}$$

has only the solutions

$$(x, m, n) = \begin{cases} (5, 1, 2), (31, 1, 5), & \text{if } p = 3, \\ (9, 1, 2), (559, 1, 7), & \text{if } p = 5, \\ (2p^r - 1, r, 2r), \;\; \text{for any } r \in \mathbb{N}, & \text{otherwise.} \end{cases}$$

In 2002, F. Luca [9] considered the equation (1.2) where $\ell = 1$ and p is a prime power. Referring the solutions $m = n$, $x = 1$ and $n = 2m$, $x = 2p^m - 1$ for all $m \geq 0$ as *trivial*, he proved that the only non-trivial solutions of equation (1.2) with p a prime power and $n \geq m \geq 0$ but $(n, m) \neq (1, 0)$ are $(x, p, m, n) = (37, 7, 0, 3), (5, 2, 1, 3), (11, 2, 1, 5), (181, 2, 1, 13), (31, 3, 1, 5), (559, 5, 1, 7)$. The proof was an interesting combination of standard algebraic number theory with previous results, mainly found in the important paper of Y. F. Bilu, G. Hanrot and P. M. Voutier [2] concerning the primitive divisors of Lucas and Lehmer numbers. We also recall that all the solutions of the analogous Diophantine equation

$$x^2 = 4p^n + 4p^m + 1$$

where found, for $m = 1$ and $m = 2$, by N. Tzanakis and J. Wolfskill in [12], and for general m, by M.-H. Le in [7].

In 2017, M. A. Bennett and A. M. Scheerer [1] considered a more general equation with the form

$$x^2 = Np^n + Mp^m + \ell^2, \;\; p \in \mathbb{P}, \;\; x, \ell, m, n \in \mathbb{N}, \; m < n, \tag{1.3}$$

where M, N are nonzero integers with $N \geq 1$. They proved that if $p \neq 2$ and

$$\max\{|M|, N, \ell^2\} \leq p - 1, \tag{1.4}$$

then the solutions (p, x, m, n) of (1.3) satisfy either

$$n = 2m, \;\; \text{and} \;\; x = p^m \cdot x_0 \pm \ell, \;\; \ell, x_0 \in \mathbb{Z} \text{ with } \max\{x_0^2, 2\ell x_0\} < p$$

or

$$m \leq 3.$$

Their proofs were based upon Padé approximation to the binomial functions.

In this paper, we consider the equation (1.3) where $\ell > 1$ odd, $M = -4$ and $N = 4$. We first give the relation between (1.2) and generalized Ramanujan-Nagell equation as follows:

Theorem 1.1 *The Diophantine equation (1.2) has a solution (p, x, m, n) with $\ell^2 < 4p^m$ (or $\ell^2 > 4p^m$) if and only if the equation*

$$X^2 + (4p^m - \ell^2) = 4p^Z, \;\; X, Z \in \mathbb{N} \tag{1.5}$$

(or

$$X^2 - (\ell^2 - 4p^m) = 4p^Z, \;\; X, Z \in \mathbb{N} \tag{1.6}$$

has a solution (X, Z) with $Z > m$. Moreover, if the above condition holds, then $(x, n) = (X, Z)$.

Next, by Theorem 1.1, using some classical results on the generalized Ramanujan-Nagell equation, all solutions of (1.2) with $\ell^2 < 4p^m$ can be derived.

Theorem 1.2 *For $\ell > 1$, the Diophantine equation (1.2) has only the following solutions (p, x, m, n) with $\ell^2 < 4p^m$:*

$\ell = 3, \;\; p = 2, \;\; (x, m, n) = (5, 2, 3), (11, 2, 5), (181, 2, 13), (45, 3, 9).$

$\ell = 5,\ p = 2,\ (x, m, n) = (11, 3, 5), (181, 3, 13).$
$\ell = 5,\ p = 3,\ (x, m, n) = (31, 2, 5).$
$\ell = 9,\ p = 5,\ (x, m, n) = (559, 2, 7).$
$\ell = 11,\ p = 2,\ (x, m, n) = (181, 5, 13).$

For any fixed ℓ, let $p_1^{m_1}, \cdots, p_r^{m_r}$ denote all prime powers satisfying $\ell^2 > 4p_i^{m_i}$ and $p_i \nmid \ell$ $(i = 1, \cdots, r)$. By Theorem 1.1, all the solutions (p, x, m, n) of (1.2) with $\ell^2 > 4p^m$ and $(p, m) = (p_i, m_i)$ $(i = 1, \cdots, r)$ can be determined by solving the equations

$$X^2 - (\ell^2 - 4p_i^{m_i}) = p_i^Z,\ X, Z \in \mathbb{N},\ i = 1, \cdots, r. \tag{1.7}$$

Finally, as an example of using the above method, we find all solutions (p, x, m, n) of (1.2) for $\ell \in \{5, 7\}$.

Theorem 1.3 *For $\ell = 5$, the Diophantine equation (1.2) has only the following solutions:*

$p = 2,\ (x, m, n) = (11, 3, 5), (181, 3, 13), (7, 1, 3), (9, 1, 4), (23, 1, 7).$
$p = 3,\ (x, m, n) = (31, 2, 5), (7, 1, 2), (11, 1, 3).$

Theorem 1.4 *For $\ell = 7$, the Diophantine equation (1.2) has only the following solutions:*

$p = 2,\ (x, m, n) = (13, 1, 5), (17, 2, 6), (9, 3, 4), (23, 3, 7).$
$p = 3,\ (x, m, n) = (11, 2, 3), (19, 1, 4).$
$p = 11,\ (x, m, n) = (73, 1, 3).$

2 Preliminaries

Let D be an odd positive integer.

Lemma 2.1 *(Theorem 2 of [4]) The equation*

$$X^2 + D = 4 \cdot 2^Z,\ X, Z \in \mathbb{N}$$

has at most one solution (X, Z), except for the following cases:

$D = 7,\ (X, Z) = (1, 1), (3, 2), (5, 3), (11, 5), (191, 13).$
$D = 23,\ (X, Z) = (3, 3), (45, 9).$
$D = 2^{s+2} - 1,\ (X, Z) = (1, s), (2^{s+1} - 1, 2s)$ *where s is a positive integer with $s > 1$.*

Lemma 2.2 *(Theorem 2 of [4]) If p is an odd prime with $p \nmid D$, then the equation*

$$X^2 + D = 4p^Z,\ X, Z \in \mathbb{N}$$

has at most one solution (X, Z), except for the following cases:

$D = 11, p = 3,\ (X, Z) = (1, 1), (5, 2), (31, 5).$
$D = 19, p = 5,\ (X, Z) = (1, 1), (9, 2), (559, 7).$
$D = 4p^s - 1,\ (X, Z) = (1, s), (2p^s - 1, 2s)$ *where s is a positive integer with $s > 1$.*

Lemma 2.3 *([11]) For $D \in \{9, 17, 33, 41\}$, the equation*

$$X^2 - D = 4 \cdot 2^Z,\ X, Z \in \mathbb{N}$$

has only the following solutions:

$D = 9,\ (X, Z) = (5, 2).$
$D = 17,\ (X, Z) = (5, 1), (7, 3), (9, 4), (23, 7).$
$D = 33,\ (X, Z) = (7, 2), (17, 6).$
$D = 41,\ (X, Z) = (7, 1), (13, 5).$

Lemma 2.4 *([3]) The equation*

$$X^2 - 13 = 4 \cdot 3^Z,\ X, Z \in \mathbb{N}$$

has the only solutions $(X, Z) = (5, 1), (7, 2), (11, 3)$.

Lemma 2.5 *([12]) If* $q > 3$*, then the equation*

$$x^2 = 4q^n + 4q + 1, \ \ x, n \in \mathbb{N}, \ n > 1$$

has the only solution $(x, n) = (2q + 1, 2)$.

Lemma 2.6 *([7]) The equation*

$$x^2 = 4q^n + 4q^m + 1, \ \ x, m, n \in \mathbb{N}, \ \ n > m > 1, \ \gcd(m, n) = 1$$

has the only solution $(x, m, , n)$.

Lemma 2.7 *The equation*

$$X^2 - 37 = 4 \cdot 3^Z, \ \ X, Z \in \mathbb{N} \tag{2.1}$$

has the only solutions $(X, Z) = (7, 1), (19, 4)$.

Proof We now assume that (X, Z) is a solution of (2.1) with $(X, Z) \neq (7, 1)$ and $(19, 4)$. Then we have

$$X^2 = 4 \cdot 3^Z + 4 \cdot 3^2 + 1, \ Z > 2. \tag{2.2}$$

However, by [12], we see from (2.2) that $2 \nmid Z$, and by [7], it is impossible. Thus, the lemma is proved. □

To prove the subsequent Lemma 2.11, the following lemmas are introduced.

Lemma 2.8 *(Theorem 10.9.1 and 10.9.2 of [5])*

$$u^2 - Dv^2 = 1, \ \ u, v \in \mathbb{Z} \tag{2.3}$$

has positive integer solutions (u, v)*, and it has a unique positive integer solution* (u_1, v_1) *such that* $u_1 + v_1\sqrt{D} \leq u + v\sqrt{D}$*, where* (u, v) *through all positive integer solutions of* (2.3)*. Every solution* (u, v) *of* (2.3) *can be expressed as*

$$u + v\sqrt{D} = \pm(u_1 + v_1\sqrt{D})^r, \ \ r \in \mathbb{Z}.$$

Lemma 2.9 *([8,13]) If* D *is not a square,* p *is an odd prime with* $p \nmid D$ *and the equation*

$$A - DB^2 = p^C, \ \ A, B, C \in \mathbb{Z}, \ \gcd(A, B) = 1, \ C > 0 \tag{2.4}$$

has solutions (A, B, C)*, then it has a unique positive integer solution* (A_1, B_1, C_1) *such that* $C_1 \leq C$ *and* $1 < (A_1 + B_1\sqrt{D})/(A_1 - B_1\sqrt{D}) < u_1 + v_1\sqrt{D}$*, where* C *through all solutions of* (2.4)*,* (u_1, v_1) *is the least solution of* (2.3)*. The solution* (A_1, B_1, C_1) *is called the least solution of* (2.4)*. Every solution* (A, B, C) *of* (2.4) *can be expressed as*

$$C = C_1 t, \ t \in \mathbb{N},$$
$$A + B\sqrt{D} = (A_1 + \delta B_1\sqrt{D})^t(u + v\sqrt{D}), \ \delta \in \{1, -1\},$$

where (u, v) *is a solution of* (2.3).

For any algebraic number of α of degree k over $\mathbb{Q}$, let

$$h(\alpha) = \frac{1}{k}\left(\log|a_0| + \sum_{i=1}^{k} \log\max\left\{1, |\alpha^{(i)}|\right\}\right)$$

be the absolute logarithmic height of α, where a_0 is the leading coefficient of the minimal polynomial of α over $\mathbb{Z}$, and $\alpha^{(i)}$ $(i = 1, 2, \cdots, k)$ are the conjugates of α in $\mathbb{C}$. Let α_1, α_2 be two algebraic numbers with $\min\{|\alpha_1|, |\alpha_2|\} \geq 1$, and let $\log\alpha_1$, $\log\alpha_2$ be any determinations of their logarithms. Further, let b_1, b_2 be positive integers, and let $\Lambda = b_1\log\alpha_1 - b_2\log\alpha_2$.

Lemma 2.10 *If $\Lambda \neq 0$ and $\alpha_1, \alpha_2, \log \alpha_1, \log \alpha_2$ are real and positive, then*

$$\log |\Lambda| \geq -19.7 \times d^4 \times (\log A_1)(\log A_2) \left(\max \left\{ 1, \frac{20}{d}, 0.38 + \log E \right\} \right)^2,$$

where $d = [\mathbb{Q}(\alpha_1, \alpha_2) : \mathbb{Q}]/[\mathbb{R}(\alpha_1, \alpha_2) : \mathbb{R}]$,

$$\log A_j \geq \max \left\{ \frac{1}{d}, \frac{|\log \alpha_j|}{d}, h(\alpha_j) \right\}, \; j = 1, 2,$$

$$E = \frac{b_1}{d \log A_2} + \frac{b_2}{d \log A_1}.$$

Proof This lemma is the special case of Corollary 2 of [6] for $m = 20$. □

Lemma 2.11 *The equation*

$$X^2 - 5 = 4 \cdot 11^Z, \; X, Z \in \mathbb{N} \tag{2.5}$$

has the only solutions $(X, Z) = (7, 1), (73, 3)$.

Proof We assume that (X, Z) is a solution of (2.5) with $(X, Z) \neq (7, 1)$ and $(73, 3)$. So, it is clear that $2 \nmid X$. If $2 \mid Z$, then $5 = X^2 - 4 \cdot 11^Z = X^2 - (2 \cdot 11^{Z/2})^2 = (X - 2 \cdot 11^{Z/2})(X + 2 \cdot 11^{Z/2}) \geq X + 2 \cdot 11^{Z/2} > 5$, a contradiction. Hence we have $2 \nmid Z$, $Z \geq 5$ and $X > 73$. Let $\lambda = (-1)^{(X-1)/2}$. Since $X \equiv \lambda \pmod 4$, $(3X + 5\lambda)/4$ and $(X + 3\lambda)/4$ are coprime positive integers. By (2.5), we get

$$\left(\frac{3X + 5\lambda}{4} \right)^2 - 5 \left(\frac{X + 3\lambda}{4} \right)^2 = 11^Z. \tag{2.6}$$

We see from (2.6) that the equation

$$A - 5B^2 = 11^C, \; A, B, C \in \mathbb{Z}, \; \gcd(A, B) = 1, \; C > 0 \tag{2.7}$$

has a solution

$$(A, B, C) = \left(\frac{3X + 5\lambda}{4}, \frac{X + 3\lambda}{4}, Z \right). \tag{2.8}$$

Notice that $4^2 - 5 \cdot 1^2 = 11$ and $(u_1, v_1) = (9, 4)$ is the least solution of the equation

$$u^2 - 5v^2 = 1, \; u, v \in \mathbb{Z}. \tag{2.9}$$

By the definition given in Lemma 2.9, $(A_1, B_1, C_1) = (4, 1, 1)$ is the least solution of (2.7). Therefore, applying Lemma 2.9 to (2.8), we get either

$$\frac{3X + 5\lambda}{4} + \frac{X + 3\lambda}{4}\sqrt{5} = (4 + \sqrt{5})^Z (u + v\sqrt{5}) \tag{2.10}$$

or

$$\frac{3X + 5\lambda}{4} + \frac{X + 3\lambda}{4}\sqrt{5} = (4 - \sqrt{5})^Z (u + v\sqrt{5}) \tag{2.11}$$

where (u, v) is a solution of (2.9).

When (2.10) holds, we have

$$\frac{3X + 5\lambda}{4} - \frac{X + 3\lambda}{4} = (4 - \sqrt{5})^Z (u - v\sqrt{5}). \tag{2.12}$$

Since $(3X + 5\lambda)/4 + (X + 3\lambda)\sqrt{5}/4 > 0$ and $4 + \sqrt{5} > 0$, we see from (2.10) that $u + v\sqrt{5} > 0$. Hence, by Lemma 2.8, we get

$$u + v\sqrt{5} = (9 + 4\sqrt{5})^r, \; r \in \mathbb{Z}. \tag{2.13}$$

Further, since $X > 73$, by (2.10), (2.12) and (2.13), we have

$$1 < \frac{(3X+5\lambda)+(X+3\lambda)\sqrt{5}}{(3X+5\lambda)-(X+3\lambda)\sqrt{5}} = \left(\frac{X+\lambda\sqrt{5}}{X-\lambda\sqrt{5}}\right)\left(\frac{3+\sqrt{5}}{3-\sqrt{5}}\right) = \left(\frac{4+\sqrt{5}}{4-\sqrt{5}}\right)^{Z}(9+4\sqrt{5})^{2r} < 9+4\sqrt{5}. \tag{2.14}$$

Since $Z \geq 5$, we find from (2.14) that $r < 0$. Let

$$s = -r, \tag{2.15}$$

$$\alpha = 4+\sqrt{5},\ \bar{\alpha} = 4-\sqrt{5}\ \beta = 9+4\sqrt{5},\ \bar{\beta} = 9-4\sqrt{5} \tag{2.16}$$

Then, s is a positive integer, by (2.10), (2.12), (2.13), (2.15) and (2.16), we have

$$\frac{3X+5\lambda}{4} + \frac{X+3\lambda}{4}\sqrt{5} = \alpha^{Z}\bar{\beta}^{s},\ \frac{3X+5\lambda}{4} - \frac{X+3\lambda}{4}\sqrt{5} = \bar{\alpha}^{Z}\beta^{r}. \tag{2.17}$$

Further, eliminating X in (2.17), we get

$$\alpha^{Z}\bar{\beta}^{s}\left(\frac{3-\sqrt{5}}{2}\right) - \bar{\alpha}^{Z}\beta^{s}\left(\frac{3+\sqrt{5}}{2}\right) = \lambda\sqrt{5}. \tag{2.18}$$

Let

$$\rho = \frac{3+\sqrt{5}}{2},\ \bar{\rho} = \frac{3-\sqrt{5}}{2}. \tag{2.19}$$

Since $\beta = \rho^3$ and $\bar{\beta} = \bar{\rho}^3$, by (2.18) and (2.19), we obtain

$$\alpha^{Z}\bar{\rho}^{3s+1} - \bar{\alpha}^{Z}\rho^{3s+1} = \lambda\sqrt{5}. \tag{2.20}$$

Similarly, when (2.11) holds, we can deduce that

$$\alpha^{Z}\bar{\rho}^{3s-1} - \bar{\alpha}^{Z}\rho^{3s-1} = -\lambda\sqrt{5}, \tag{2.21}$$

where s is a positive integer. The combination of (2.20) and (2.21) yields

$$\left|\alpha^{Z}\bar{\rho}^{3s+\theta} - \bar{\alpha}^{Z}\rho^{3s+\theta}\right| = \sqrt{5},\ \theta \in \{1,-1\}. \tag{2.22}$$

Further, since $\log(1+z) < z$ for any $z > 0$, by (2.22), we have

$$0 < \left|Z\log\frac{\alpha}{\bar{\alpha}} - 2(3s+\theta)\log\rho\right| < \frac{\sqrt{5}}{\min\{\alpha^{Z}\bar{\rho}^{3s+\theta}, \bar{\alpha}^{Z}\rho^{3s+\theta}\}}. \tag{2.23}$$

Furthermore, since $\min\{\alpha^{Z}\bar{\rho}^{3s+\theta}, \bar{\alpha}^{Z}\rho^{3s+\theta}\} \geq \bar{\alpha}^{Z}\rho^{3s+\theta} - \sqrt{5} > \frac{1}{2}\bar{\alpha}^{Z}\rho^{3s+\theta}$ by (2.22), we get from (2.23) that

$$0 < \left|Z\log\frac{\alpha}{\bar{\alpha}} - 2(3s+\theta)\log\rho\right| < \frac{2\sqrt{5}}{\bar{\alpha}^{Z}\rho^{3s+\theta}}. \tag{2.24}$$

Let $\alpha_1 = \alpha/\bar{\alpha}$, $\alpha_2 = \rho$ and

$$\Lambda = Z\log\alpha_1 - 2(3s+\theta)\log\alpha_2. \tag{2.25}$$

By (2.16) and (2.19), we have $\mathbb{Q}(\alpha_1,\alpha_2) = \mathbb{Q}(\sqrt{5})$,

$$\frac{[\mathbb{Q}(\alpha_1,\alpha_2):\mathbb{Q}]}{[\mathbb{R}(\alpha_1,\alpha_2):\mathbb{R}]} = 2,\ h(\alpha_1) = \log(4+\sqrt{5}),\ h(\alpha_2) = \frac{1}{2}\log\left(\frac{3+\sqrt{5}}{2}\right). \tag{2.26}$$

Applying Lemma 2.10 to (2.26), we may choose that

$$d = 2,\ \ \log A_1 \geq \log(4 + \sqrt{5}),\ \ \log A_2 \geq \frac{1}{2}. \tag{2.27}$$

Therefore, using Lemma 2.10, by (2.24), (2.25) and (2.27), we have

$$\begin{aligned} &\log|\Lambda| \geq -19.7 \times 2^4 \times (\log(4+\sqrt{5})) \times \frac{1}{2}\left(\max\{10, 0.38 + \log E\}\right)^2 \\ &> -288.47\left(\max\{10, 0.38 + \log E\}\right)^2, \end{aligned} \tag{2.28}$$

where

$$E = Z + \frac{3s + \theta}{\log(4 + \sqrt{5})}. \tag{2.29}$$

By (2.24), (2.28) and (2.29), we get

$$\begin{aligned} &\log(2\sqrt{5}) + 288.47\left(\max\{10, 0.38 + \log E\}\right)^2 > \log\left(\bar{\alpha}^Z \rho^{3s+\theta}\right) \\ &= Z\log(4-\sqrt{5}) + (3s+\theta)\log\rho > \left(Z + \frac{3s+\theta}{\log(4+\sqrt{5})}\right)(\log(4-\sqrt{5})) \\ &> 0.56E. \end{aligned} \tag{2.30}$$

Therefore, by (2.30), we obtain

$$2.67 + 515.13\left(\max\{10, 0.38 + \log E\}\right)^2 > E. \tag{2.31}$$

If $10 < 0.38 + \log E$, then from (2.31) we get

$$2.67 + 515.13(0.38 + \log E)^2 > E,$$

whence we can deduce that

$$E < 68260. \tag{2.32}$$

If $10 \geq 0.38 + \log E$, then (2.32) is still true. Therefore, by (2.29) and (2.32), we get $Z < 68260$. But, using MAPLE 2016, (2.5) has no solution (X, Z) with $3 < Z < 68260$. Thus, the lemma is proved. □

3 Proofs of Theorems

Proof of Theorem 1.1 By comparing (1.1), (1.2), (1.5) and (1.6), the theorem follows easily. □

Proof of Theorem 1.2 Since $\ell > 1$, by Theorem 1.1, if (1.2) has a solution (p, x, m, n) with $\ell^2 < 4p^m$, then (1.5) or (1.6) has at least two solutions $(X, Z) = (X_1, Z_1)$ and (X_2, Z_2) such that $(X_1, Z_1) = (\ell, m)$ and $1 < X_1 < X_2$. Therefore, by Lemmas 2.1 and 2.2, we obtain the theorem immediately. □

Proof of Theorem 1.3 For $\ell = 5$, by Theorem 1.2, (1.2) has only the solutions

$$\begin{aligned} &p = 2,\ (x, m, n) = (11, 1, 5), (181, 3, 13) \text{ and} \\ &p = 3,\ (x, m, n) = (31, 2, 5) \end{aligned} \tag{3.1}$$

with $\ell^2 < 4p^m$.

On the other hand, since $p^m = 2, 3$ and 4 are all prime powers satisfying $\ell^2 > 4p^m$ and $p \nmid \ell$, by Theorem 1.1, all the solutions (p, x, m, n) of (1.2) with $\ell^2 > 4p^m$ can be determined by solving the equations

$$\begin{aligned} &X^2 - 17 = 4 \cdot 2^Z,\ X, Z \in \mathbb{N}, \\ &X^2 - 13 = 4 \cdot 3^Z\ X, Z \in \mathbb{N} \end{aligned}$$

and

$$X^2 - 9 = 4 \cdot 2^Z \; X, Z \in \mathbb{N}.$$

Therefore, by Lemmas 2.3 and 2.4, (1.2) has only the solutions

$$\begin{aligned} p &= 2, \; (x, m, n) = (7, 1, 3), (9, 1, 4), (23, 1, 7) \text{ and} \\ p &= 3, \; (x, m, n) = (7, 1, 2), (11, 1, 3) \end{aligned} \tag{3.2}$$

with $\ell^2 > 4p^m$. Thus, the combination of (3.1) and (3.2) yields the theorem. □

Proof of Theorem 1.4 Using the same method as in the proof of Theorem 1.3, by Lemmas 2.3, 2.4, 2.7 and 2.11, we can obtain the theorem immediately. □

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