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ORIGINAL RESEARCH

On the Diophantine equation $x^2 + b^m = c^n$ with $a^2 + b^4 = c^2$

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Abstract Let a, b, c be pairwise relatively prime positive integers such that $a^2 + b^4 = c^2$ and b is odd. Then we show that the equation of the title has only one positive integer solution $(x, m, n) = (a, 4, 2)$ under some conditions.

Keywords Diophantine equation · Integer solution · Pythagorean numbers

Mathematics Subject Classification 11D61

1 Introduction

In 1956, Sierpiński[13] showed that the equation $3^x + 4^y = 5^z$ has only the positive integer solution $(x, y, z) = (2, 2, 2)$. Jeśmanowicz [8] conjectured that if a, b, c are Pythagorean numbers, i.e., positive integers satisfying $a^2 + b^2 = c^2$, then

$$a^x + b^y = c^z$$

has only the positive integer solution $(x, y, z) = (2, 2, 2)$. As an analogue of Jeśmanowicz' conjecture, the author [15] proposed the following:

Conjecture 1 *If a, b, c are positive integers satisfying $a^2 + b^2 = c^2$ with $\gcd(a, b, c) = 1$ and b odd, then the equation*

$$x^2 + b^m = c^n \tag{1.1}$$

has only one positive integer solution $(x, m, n) = (a, 2, 2)$.

The author[15] proved that if b and c are primes such that (i) $b^2 + 1 = 2c$ and (ii) $d = 1$ or even if $b \equiv 1 \pmod 4$, then Conjecture 1 is true, where d is the order of a prime divisor of (c) in the ideal class group of $\mathbb{Q}(\sqrt{-b})$. In [3], [9],[10] and [19], it was shown that if $b \not\equiv 1 \pmod 8$, and b or c is a prime power, then Conjecture 1 is true. It has been verified that Conjecture 1 holds for many other Pythagorean numbers. But Conjecture 1 is still unsolved. (See Cao[2], Terai[16], [17] and [18] for another analogue of Jeśmanowicz' conjecture.)

Related to Conjecture 1, in [5] and [6], Cenberci and Senay proposed the following:

Conjecture 2 *If a, b, c are positive integers satisfying $a^2 + b^2 = c^4$ with $\gcd(a, b, c) = 1$ and b odd, then equation (1.1) has only one positive integer solution $(x, m, n) = (a, 2, 4)$.*

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As another analogue of Conjecture 1, we also propose the following:

Conjecture 3 *If a, b, c are positive integers satisfying $a^2 + b^4 = c^2$ with $\gcd(a, b, c) = 1$ and b odd, then equation (1.1) has only one positive integer solution $(x, m, n) = (a, 4, 2)$.*

By Magma[1], we verified that Conjecture 3 is true in the range $c \leq 10^5$ and $\max\{m, n\} \leq 20$.

In this paper, when b has at most two distinct primes, we show that Conjecture 3 is true under some conditions. The proof is based on elementary methods and results concerning the equation $x^2 + 1 = 2y^n$ due to Ljunggren - Störmer and the equation $x^2 + y^4 = z^n$ due to Ellenberg. We also verify that when $b = 3p, 5p$ with p odd prime, Conjecture 3 is true under a certain condition.

2 Lemmas

As well-known, primitive Pythagorean numbers a, b, c with b odd can be parametrized as follows:

$$a = 2u_0v_0, \quad b = u_0^2 - v_0^2, \quad c = u_0^2 + v_0^2,$$

where u_0, v_0 are positive integers such that $\gcd(u_0, v_0) = 1$, $u_0 \not\equiv v_0 \pmod 2$ and $u_0 > v_0$.

In view of the above prametrization, we obtain all positive integers a, b, c satisfying $a^2 + b^4 = c^2$ with b odd. In fact, since $b^2 = u_0^2 - v_0^2$, we have

$$u^2 = u_0 + v_0, \quad v^2 = u_0 - v_0$$

with $b = uv$. Then

$$u_0 = \frac{u^2 + v^2}{2}, \quad v_0 = \frac{u^2 - v^2}{2}.$$

Thus we have shown the following:

Lemma 1 *All positive integer solutions of the equation $a^2 + b^4 = c^2$ with $\gcd(a, b, c) = 1$ and b odd are given by*

$$a = \frac{u^4 - v^4}{2}, \quad b = uv, \quad c = \frac{u^4 + v^4}{2}, \tag{2.1}$$

where u, v are positive integers such that $\gcd(u, v) = 1$, $u \equiv v \equiv 1 \pmod 2$ and $u > v$.

We also need the following lemmas to prove Theorems 1,2,3.

Lemma 2 *(Störmer[14]) The Diophantine equation*

$$x^2 + 1 = 2y^n$$

has no solutions in integers $x > 1$, $y \geq 1$ and n odd ≥ 3.

Lemma 3 *(Ljunggren[11]) The Diophantine equation*

$$x^2 + 1 = 2y^4$$

has only the positive integer solution $(x, y) = (1, 1), (239, 13)$.

Lemma 4 *(Ellenberg[7]) Let n be a positive integer with $n \geq 4$. Then the equation*

$$x^2 + y^4 = z^n$$

has no solutions in nonzero pairwise coprime integers x, y, z.

The following lemma is immediate from [4] and [12].

Lemma 5 (1) *The Diophantine equation*

$$x^2 + 9^m = 3281^n$$

has only the positive integer solution $(x, m, n) = (3280, 4, 2)$.

(2) *The Diophantine equation*

$$x^2 + 15^m = 25313^n$$

has only the positive integer solution $(x, m, n) = (25312, 4, 2)$.

(3) *The Diophantine equation*

$$x^2 + 25^m = 195313^n$$

has only the positive integer solution $(x, m, n) = (195312, 4, 2)$.

(4) *The Diophantine equation*

$$x^2 + 33^m = 7361^n$$

has only the positive integer solution $(x, m, n) = (7280, 4, 2)$.

3 The cases $b = p^\alpha,\ p^\alpha q^\beta$ with p, q odd primes.

In this section, when b has at most two distinct primes, we show that Conjecture 3 is true under some conditions.

Theorem 1 *Let a, b, c be as in (2.1). Suppose that* $b \equiv \pm 3 \pmod 8$ *and that b satisfies at least one of the following :*

(i) $b = u$ *with* $u = p^\alpha$ *and* $v = 1$.
(ii) $b = uv$ *with* $u = p^\alpha$ *and* $v = q^\beta$.
(iii) $b = u$ *with* $u = p^\alpha q^\beta$ *and* $v = 1$,

where p, q *are distinct odd primes and* α, β *are positive integers.*
Then Conjecture 3 is true.

Proof Let a, b, c be as in Theorem 1. Let (x, m, n) be a positive integer solution of equation (1.1).
Since $b \equiv \pm 3 \pmod 8$ and $c \equiv 1 \pmod 4$, we have

$$\left(\frac{b}{c}\right) = \left(\frac{c}{b}\right) = \left(\frac{2}{b}\right) = -1.$$

Hence we see that m and n are even.
Put $n = 2N$. From (1.1), we have

$$b^m = (c^N + x)(c^N - x).$$

(i) $b = u$ with $u = p^\alpha$ and $v = 1$. Then we have

$$b^m + 1 = 2c^N. \tag{3.1}$$

Note that $b^4 + 1 = 2c$, since $v = 1$. If N=1 or 2, then we easily see that equation (3.1) has only the solution $(m, N) = (4, 1)$. If $N \geq 3$, then it follows from Lemmas 2, 3 that equation (3.1) has no solutions.

(ii) $b = uv$ with $u = p^\alpha$ and $v = q^\beta$. Then we have

$$b^m + 1 = 2c^N, \tag{3.2}$$

or

$$u^m + v^m = 2c^N. \tag{3.3}$$

First consider equation (3.2). If $N \geq 3$, then it follows from Lemmas 2, 3 that equation (3.2) has no solutions. If N=1 or 2, then equation (3.2) can be written as

$$(uv)^m + 1 = u^4 + v^4 \ \text{ or } \ (u^4 + v^4)^2/2.$$

It is easy to show that the above equation has no solutions except for the equation

$$(uv)^6 + 1 = (u^4 + v^4)^2/2. \tag{3.4}$$

In view of a property of an elliptic curve, we see that equation (3.4) has no solutions. Indeed, by putting $X = 2u^2v^2$ and $Y = 2(u^4 + v^4)$, (3.4) can be reduced to the ellitptic curve

$$E : Y^2 = X^3 + 8$$

with rank $E(\mathbb{Q}) = 1$ and all integer points on E are $(X, Y) = (-2, 0), (1, \pm 3), (2, \pm 4), (46, \pm 312)$.

Next consider equation (3.3). Since m is even, the proof is divided into two cases (a) $m \equiv 0 \pmod 4$ and (b) $m \equiv 2 \pmod 4$.

Case (a) $m \equiv 0 \pmod 4$. If $n \geq 4$, then it follows from Lemma 4 that equation (1.1) has no solutions. If $n = 2$, i.e., $N = 1$, then the relation $c = (u^4 + v^4)/2$ yields $m = 4$ from (3.3), and hence $(x, m, n) = (a, 4, 2)$.

Case (b) $m \equiv 2 \pmod 4$, say $m = 2l$ with l odd. Then equation (3.3) becomes

$$\left(\frac{u^2 + v^2}{2}\right)\left(\frac{u^{2l} + v^{2l}}{u^2 + v^2}\right) = \left(\frac{u^4 + v^4}{2}\right)^N$$

Let r be an odd prime of $(u^2 + v^2)/2$. Then $u^4 + v^4 \equiv 0 \pmod r$. This implies that $u \equiv 0 \pmod r$ and $v \equiv 0 \pmod r$, which contradicts $\gcd(u, v) = 1$.

(iii) $b = u$ with $u = p^\alpha q^\beta$ and $v = 1$. Then we have

$$b^m + 1 = 2c^N, \tag{3.5}$$

or

$$u_1^m + v_1^m = 2\left(\frac{u_1^4 v_1^4 + 1}{2}\right)^N \tag{3.6}$$

with $u_1 = p^\alpha$ and $v_1 = q^\beta$.

First consider equation (3.5). Note that $b^4 + 1 = 2c$, since $v = 1$. As above, equation (3.5) has only the solution $(m, N) = (4, 1)$.

Next consider equation (3.6). In view of $b = u_1 v_1$ and $b \equiv \pm 3 \pmod 8$, we may suppose that $u_1 \equiv \pm 3 \pmod 8$ and $v_1 \equiv \pm 1 \pmod 8$. Since m is even, the proof is divided into two cases (a) $m \equiv 0 \pmod 4$ and (b) $m \equiv 2 \pmod 4$.

Case (a) $m \equiv 0 \pmod 4$. If $n \geq 4$, then it follows from Lemma 4 that equation (1.1) has no solutions. If $n = 2$, i.e., $N = 1$, then (3.6) has no solutions.

Case (b) $m \equiv 2 \pmod 4$, say $m = 2l$ with l odd. Then equation (3.6) becomes

$$\left(\frac{u_1^2 + v_1^2}{2}\right)\left(\frac{u_1^{2l} + v_1^{2l}}{u_1^2 + v_1^2}\right) = \left(\frac{u_1^4 v_1^4 + 1}{2}\right)^N \tag{3.7}$$

The right hand side of (3.7) is divisible by $(u_1^2 + v_1^2)/2 \equiv 5 \pmod 8$. This contradicts the fact that an odd prime factor r of $A^4 + B^4$ satisfies $r \equiv 1 \pmod 8$. This completes the proof of Theorem 1.

□

4 The cases $b = 3p, 5p$ with p odd prime.

In this section, when $b = 3p, 5p$ with p odd prime, we consider eqaution (1.1). We do not assume that $b \equiv \pm 3 \pmod 8$, i.e., $p \equiv \pm 1 \pmod 8$.

Theorem 2 (i) *Let p be an odd prime with $p > 3$. Put $u = p$ and $v = 3$ in (2.1). If $(3p + 1)/2$ has a prime factor other than 17 and 193, then Conjecture 3 is true. In particular, if p is an odd prime with $3 < p < 10^7$, then Conjecture 3 is true.*

(ii) *Let p be an odd prime. Put $u = 3p$ and $v = 1$ in (2.1). If $(p + 3)/2$ and $(p^2 + 3^2)/2$ have a prime factor other than 17 and 193, respectively, then Conjecture 3 is true. In particular, if p is an odd prime satisfying $3 \leq p < 10^7$ with $p \neq 31, 383, 167039$, then Conjecture 3 is true.*

Proof (i) $u = p$ and $v = 3$ in (2.1). Then $a = (p^4 - 3^4)/2,\ b = 3p,\ c = (p^4 + 3^4)/2$. Suppse that our assumptions are all satisfied. Let (x, m, n) be a positive integer solution of equation (1.1).

From (1.1), we have

$$\left(\frac{c}{3}\right) = \left(\frac{4c}{3}\right) = \left(\frac{2}{3}\right)\left(\frac{2c}{3}\right) = -1.$$

Hence we see that n are even, say $n = 2N$. (We do not know if m is even or not.) Thus

$$(3p)^m = (c^N + x)(c^N - x).$$

Then we have

$$(3p)^m + 1 = 2c^N \tag{4.1}$$

or

$$p^m + 3^m = 2c^N. \tag{4.2}$$

First consider equation (4.1). If m is odd, then equation (4.1) becomes

$$\left(\frac{3p+1}{2}\right)\left(\frac{(3p)^m+1}{3p+1}\right) = \left(\frac{p^4+3^4}{2}\right)^N.$$

Note that $(3p + 1)/2$ is odd, since $(p^4 + 3^4)/2$ is odd. Let r be an odd prime factor of $(3p + 1)/2$. Then $(p^4 + 3^4)/2 \equiv 0 \pmod r$. This implies that $3^8 + 1 \equiv 0 \pmod r$, so $r = 17, 193$, which contradicts our assumption. In the same way as in the proof of Theorem 1 (i), if m is even, we easily see that equation (4.1) has no solutions.

Next consider equation (4.2). If m is odd, then equation (4.2) becomes

$$\left(\frac{p+3}{2}\right)\left(\frac{p^m+3^m}{p+3}\right) = \left(\frac{p^4+3^4}{2}\right)^N.$$

Let r be an odd prime factor of $(p + 3)/2$. Then $p^4 + 3^4 \equiv 0 \pmod r$. This implies that $p \equiv 0 \pmod r$ and $3 \equiv 0 \pmod r$, which contradicts $\gcd(p, 3) = 1$. If $m \equiv 0 \pmod 4$, then it follows from Lemma 4 that equation (1.1) has only the solution $(x, m, n) = (a, 4, 2)$. We may suppose that $m \equiv 2 \pmod 4$, say $m = 2l$ with l odd. Then equation (4.2) becomes

$$\left(\frac{p^2+3^2}{2}\right)\left(\frac{p^{2l}+3^{2l}}{p^2+3^2}\right) = \left(\frac{p^4+3^4}{2}\right)^N.$$

Let r be an odd prime factor of $(p^2 + 3^2)/2$. Then $p^4 + 3^4 \equiv 0 \pmod r$. This implies that $p \equiv 0 \pmod r$ and $3 \equiv 0 \pmod r$, which contradicts $\gcd(p, 3) = 1$.

In particular, by Magma[1], we verified that $(3p + 1)/2$ has a prime other than 17 and 193 in the range $3 < p < 10^7$ with $p \neq 11$. When $p = 11$, Conjecture 3 is true from Lemma 5,(4).

(ii) $u = 3p$ and $v = 1$ in (2.1). Then $a = ((3p)^4 - 1)/2,\ b = 3p,\ c = ((3p)^4 + 1)/2$. Suppse that our assumptions are all satisfied. Let (x, m, n) be a positive integer solution of equation (1.1).

When $p = 3$, Conjecture 3 is true from Lemma 5,(1). We may suppose that $p > 3$. As in (i), equation (1.1) is reduced to solving the following:

$$(3p)^m + 1 = 2\left(\frac{(3p)^4+1}{2}\right)^N \tag{4.3}$$

or

$$p^m + 3^m = 2\left(\frac{(3p)^4+1}{2}\right)^N. \tag{4.4}$$

First consider equation (4.3). If m is odd, then an odd prime factor r of $(3p+1)/2$ divides $((3p)^4+1)/2$. This implies that $2 \equiv 0 \pmod r$, which is impossible. If m is even, then it follows from Lemmas 1,2 that equation (4.3) has only the solution $(m, N) = (4, 1)$.

Next consider equation (4.4). If m is odd, then equation (4.4) becomes

$$\left(\frac{p+3}{2}\right)\left(\frac{p^m+3^m}{p+3}\right)=\left(\frac{(3p)^4+1}{2}\right)^N.$$

Let r be an odd prime factor of $(p+3)/2$. Then $(3p)^4+1\equiv 0\pmod r$. This implies that $3^8+1\equiv 0\pmod r$, so $r=17, 193$, which contradicts our assumption. If $m\equiv 0\pmod 4$, then it follows from Lemma 4 that equation (1.1) has no solutions. We may suppose that $m\equiv 2\pmod 4$, say $m=2l$ with l odd. Then equation (4.4) becomes

$$\left(\frac{p^2+3^2}{2}\right)\left(\frac{p^{2l}+3^{2l}}{p^2+3^2}\right)=\left(\frac{(3p)^4+1}{2}\right)^N.$$

Let r be an odd prime factor of $(p^2+3^2)/2$. Then $(3p)^4+1\equiv 0\pmod r$. This implies that $3^8+1\equiv 0\pmod r$, so $r=17, 193$, which contradicts our assumption.

In particular, by Magma[1], we verified that $(p+3)/2$ has a prime other than 17 and 193 in the range $3<p<10^7$ with $p\neq 31, 383, 167039$, and that $(p^2+3^2)/2$ has a prime other than 17 and 193 in the range $3<p<10^7$ with $p\neq 5$. When $p=5$, Conjecture 3 is true from Lemma 5,(2). This completes the proof of Theorem 2. □

Similarly, when $b=5p$, we can show the following:

Theorem 3 *(i) Let p be an odd prime with $p>5$. Put $u=p$ and $v=5$ in (2.1). If $(5p+1)/2$ has a prime factor other than 17 and 11489, then Conjecture 3 is true. In particular, if p is a prime with $5<p<10^7$, then Conjecture 3 is true.*

(ii) Let p be an odd prime. Put $u=5p$ and $v=1$ in (2.1). If $(p+5)/2$ and $(p^2+5^2)/2$ have a prime factor other than 17 and 11489, respectively, then Conjecture 3 is true. In particular, if p is a prime satisfying $3\le p<10^7$ with $p\neq 29, 22973$, then Conjecture 3 is true.

Proof (i) $u=p$ and $v=5$ in (2.1). Then $a=(p^4-5^4)/2,\ b=5p,\ c=(p^4+5^4)/2$. Suppse that our assumptions are all satisfied. Let (x, m, n) be a positive integer solution of equation (1.1).

From (1.1), we have

$$\left(\frac{c}{5}\right)=\left(\frac{4c}{5}\right)=\left(\frac{2}{5}\right)\left(\frac{2c}{5}\right)=-1.$$

Hence we see that n are even, say $n=2N$. (We do not know if m is even or not.) Thus

$$(5p)^m=(c^N+x)(c^N-x).$$

Then we have

$$(5p)^m+1=2c^N \tag{4.5}$$

or

$$p^m+5^m=2c^N. \tag{4.6}$$

First consider equation (4.5). If m is odd, then equation (4.5) becomes

$$\left(\frac{5p+1}{2}\right)\left(\frac{(5p)^m+1}{5p+1}\right)=\left(\frac{p^4+5^4}{2}\right)^N.$$

Note that $(5p+1)/2$ is odd, since $(p^4+5^4)/2$ is odd. Let r be an odd prime factor of $(5p+1)/2$. Then $(p^4+5^4)/2\equiv 0\pmod r$. This implies that $5^8+1\equiv 0\pmod r$, so $r=17, 11489$, which contradicts our assumption. In the same way as in the proof of Theorem 1 (i), if m is even, we easily see that equation (4.5) has no solutions.

Next consider equation (4.6). If m is odd, then equation (4.6) becomes

$$\left(\frac{p+5}{2}\right)\left(\frac{p^m+5^m}{p+5}\right)=\left(\frac{p^4+5^4}{2}\right)^N.$$

Let r be an odd prime factor of $(p+5)/2$. Then $p^4+5^4\equiv 0 \pmod r$. This implies that $p\equiv 0 \pmod r$ and $5\equiv 0 \pmod r$, which contradicts $\gcd(p,5)=1$. If $m\equiv 0 \pmod 4$, then it follows from Lemma 4 that equation (1.1) has only the solution $(x,m,n)=(a,4,2)$. We may suppose that $m\equiv 2 \pmod 4$, say $m=2l$ with l odd. Then equation (4.6) becomes

$$\left(\frac{p^2+5^2}{2}\right)\left(\frac{p^{2l}+5^{2l}}{p^2+5^2}\right)=\left(\frac{p^4+5^4}{2}\right)^N.$$

Let r be an odd prime factor of $(p^2+5^2)/2$. Then $p^4+5^4\equiv 0 \pmod r$. This implies that $p\equiv 0 \pmod r$ and $5\equiv 0 \pmod r$, which contradicts $\gcd(p,5)=1$.

In particular, by Magma[1], we verified that $(5p+1)/2$ has a prime factor other than 17 and 11489 in the range $3<p<10^7$.

(ii) $u=5p$ and $v=1$ in (2.1). Then $a=((5p)^4-1)/2,\ b=5p,\ c=((5p)^4+1)/2$. Suppse that our assumptions are all satisfied. Let (x,m,n) be a positive integer solution of equation (1.1).

When $p=5$, Conjecture 3 is true from Lemma 5,(3). We may suppose that $p\neq 5$. As in (i), equation (1.1) is reduced to solving the following:

$$(5p)^m+1=2\left(\frac{(5p)^4+1}{2}\right)^N \tag{4.7}$$

or

$$p^m+5^m=2\left(\frac{(5p)^4+1}{2}\right)^N. \tag{4.8}$$

First consider equation (4.7). If m is odd, then an odd prime factor r of $(5p+1)/2$ divides $((5p)^4+1)/2$. This implies that $2\equiv 0 \pmod r$, which is impossible. If m is even, then it follows from Lemmas 1,2 that equation (4.7) has only the solution $(m,N)=(4,1)$.

Next consider equation (4.8). If m is odd, then equation (4.8) becomes

$$\left(\frac{p+5}{2}\right)\left(\frac{p^m+5^m}{p+5}\right)=\left(\frac{(5p)^4+1}{2}\right)^N.$$

Let r be an odd prime factor of $(p+5)/2$. Then $(5p)^4+1\equiv 0 \pmod r$. This implies that $5^8+1\equiv 0 \pmod r$, so $r=17, 11489$, which contradicts our assumption. If $m\equiv 0 \pmod 4$, then it follows from Lemma 4 that equation (1.1) has no solutions. We may suppose that $m\equiv 2 \pmod 4$, say $m=2l$ with l odd. Then equation (4.8) becomes

$$\left(\frac{p^2+5^2}{2}\right)\left(\frac{p^{2l}+5^{2l}}{p^2+5^2}\right)=\left(\frac{(5p)^4+1}{2}\right)^N.$$

Let r be an odd prime factor of $(p^2+5^2)/2$. Then $(5p)^4+1\equiv 0 \pmod r$. This implies that $5^8+1\equiv 0 \pmod r$, so $r=17, 11489$, which contradicts our assumption.

In particular, by Magma[1], we verified that $(p+5)/2$ has a prime factor other than 17 and 11489 in the range $3\le p<10^7$ with $p\neq 29, 22973$, and that $(p^2+5^2)/2$ has a prime factor other than 17 and 11489 in the range $3\le p<10^7$ with $p\neq 3$. When $p=3$, Conjecture 3 is true from Lemma 5,(2). This completes the proof of Theorem 3. ◻

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