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ORIGINAL RESEARCH

Approximating a common solution of extended split equality equilibrium and fixed point problems

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Abstract In this paper, we study an extension of the split equality equilibrium problem called the extended split equality equilibrium problem. We give an iterative algorithm for approximating a solution of extended split equality equilibrium and fixed point problems and obtained a strong convergence result in a real Hilbert space. We further applied our result to solve extended split equality monotone variational inclusion and equilibrium problems. The result of this paper complements and extends results on split equality equilibrium problems in the literature.

Keywords Strong convergence · Extended split equality equilibrium problem · Hilbert space · λ-demimetric mapping · Fixed point problem

Mathematics Subject Classification 49J53 · 65K10 · 49M37 · 90C25

1 Introduction

Let C be a nonempty, closed and convex subset of a Hilbert space H and let $T : C \to C$ be a mapping. A point $x \in C$ is said to be a fixed point of T if $Tx = x$. Denote the set of fixed points of the mapping T by $F(T)$.

Definition 1.1 ([36]) Let C be a nonempty, closed and convex subset of a Hilbert space H and let $\lambda \in (-\infty, 1)$. A mapping $T : C \to H$ with $F(T) \neq \emptyset$ is called λ-demimetric if for any $x \in C$ and $x^* \in F(T)$,

$$\langle x - x^*, x - Tx\rangle \geq \frac{1-\lambda}{2}||x - Tx||^2, \tag{1.1}$$

Clearly every λ-strictly pseudocontractive mapping T with $F(T) \neq \emptyset$ is a λ-demimetric mapping. Also, we

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recall that a mapping $T : C \to H$ is called (α, β)- generalised hybrid (see, [21]), if there exists $\alpha, \beta \in \mathbb{R}$ such that

$$\begin{aligned} \alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \\ \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2 \end{aligned} \tag{1.2}$$

for all $x, y \in C$. It has been shown that the class of (α, β)- generalised hybrid mappings generalises the nonexpansive mappings [37], nonspreading mappings [23, 24] and the hybrid mappings [35]. Moreover, if T is an (α, β)- generalised hybrid mapping and $F(T) \neq \emptyset$ [37], we have that for $x \in C$ and $x^* \in F(T)$,

$$\begin{aligned} \alpha\|x^* - Tx\|^2 + (1-\alpha)\|x^* - Tx\|^2 \\ \leq \beta\|x^* - x\|^2 + (1-\beta)\|x^* - x\|^2 \end{aligned} \tag{1.3}$$

and hence $\|Tx - x^*\| \leq \|x - x^*\|$. Therefore, we have that

$$2\langle x - x^*, x - Tx\rangle \geq \|x - Tx\|^2 \tag{1.4}$$

and thus

$$\langle x - x^*, x - Tx\rangle \geq \frac{1-0}{2}\|x - Tx\|^2, \tag{1.5}$$

which implies that every (α, β)- generalised hybrid mapping with $F(T) \neq \emptyset$ is 0-demimetric.

Let $f : C \times C \to \mathbb{R}$ be a bifunction, then the equilibrium problem associated with f and the set C is: find $x \in C$ such that

$$f(x, y) \geq 0, \forall y \in C. \tag{1.6}$$

A point $x \in C$ that satisfies (1.6) is called an equilibrium point. We shall in this work denote the set of solutions of equilibrium problem (1.6) by $EP(f, C)$. The equilibrium problem can be applied to solve problems from other fields such as physics and economics (see for example [34]). Also, many optimisation problems such as variational inequality, convex minimization, and Nash equilibrium problems can be transformed in the form of equilibrium problem (1.6).

The equilibrium problem because of its importance has attracted the interest of many mathematicians who have developed and studied numerous iterative algorithms for the approximation of solutions of equilibrium problems (see, [13]). Moreover, authors have also taken interest in the problems of finding a common element of fixed points of nonlinear mappings and the set of solutions of equilibriums, [1, 10, 15, 17]. The study of this kind of problem was inspired by certain problems arising from signal processing, network resource allocation, and image recovery results in mathematical models whose constraint can be expressed as fixed point problems and/or equilibrium problems [20, 29].

Let C_1 and C_2 be nonempty closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively and let $A : H_1 \to H_2$ be a bounded linear operator. The split feasibility problem (first introduced by Censor and Elfving [8], for modelling inverse problems in $\mathbb{R}^n$) is :

$$\text{find}\, x \in C_1 \; such\; that\; Ax \in C_2. \tag{1.7}$$

The split feasibility problem has been considered as a veritable area of study because of its applications in signal processing, image reconstruction and intensity modulated therapy [6, 7].

Let H_1 and H_2 be real Hilbert spaces and let $S : H_1 \to \mathbb{R} \cup \{+\infty\}$ and $g : H_2 \to \mathbb{R} \cup \{+\infty\}$ be closed convex proper functions. Attouch et al. [4] introduced the following convex optimization problem:

$$\min\{S(x) + g(y) + \frac{\mu}{2}Q(x, y), x \in H_1, y \in H_2\}, \tag{1.8}$$

where $Q : H_1 \times H_2 \to \mathbb{R}^+$ is a nonnnegative quadratic form which couples the two variables x and y, and μ is a positive parameter. An example of the nonnegative quadratic form $Q(x, y)$ is $Q(x, y) = \|Ax - Bx\|^2_{H_3}$, where $A \in L(H_1, H_3)$ and $B \in L(H_2, H_3)$ are Bounded linear operators acting from H_1 to H_3 and from H_2 to H_3 respectively (see, [3]). The optimization problem (1.8) can be applied to solve problems from various areas such as decision science and game theory, partial differential equations and mechanics, and optimal control and approximation theory [2, 4, 27].

Inspired by this kind of problem considered by Attouch et el. [3] and the interest to cover many situations such as decomposition methods for PDEs, Moudafi [32] introduced the following split equality problem. Let $A : H_1 \rightarrow H_3$, $B : H_2 \rightarrow H_3$ be two bounded linear operators, let C_1 and C_2 be nonempty closed and convex subsets of real Hilbert spaces H_1 and H_2 respectively. Find

$$x \in C_1, y \in C_2 \text{ such that } Ax = By. \tag{1.9}$$

Clearly the split equality problem (1.9) is a generalization of the split feasibility problem (1.7). Moreover, the spilt equality (1.9) allows asymmetric and partial relations between the variables x and y. Moudafi [31], further considered the following split equality fixed point theorem. Let $A : H_1 \rightarrow H_3$, $B : H_2 \rightarrow H_3$ be two bounded linear operators, let C_1 and C_2 be nonempty closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $S : C \rightarrow C$ and $T : Q \rightarrow Q$ be non linear operators. Find

$$x \in F(S), y \in F(T) \text{ such that } Ax = By. \tag{1.10}$$

Moudafi and Al-Shemas [33], proposed an iterative method for solving (1.10) for firmly quasi-nonexpansive operators as follows:

$$\begin{cases} x_{n+1} = S(x_n - \gamma_n A^*(Ax_n - By_n)), \\ y_{n+1} = T(y_n + \gamma_n A^*(Ax_n - By_n)), \forall n \geq 0, \end{cases} \tag{1.11}$$

where $\gamma_n \in (\epsilon, \frac{2}{\lambda_A + \lambda_B} - \epsilon)$, λ_A and λ_B stand for the spectral radius of A^*A and B^*B, respectively. Several iterative algorithms have been developed for solving the split equality problems and split convex feasibility problems [14, 16, 18, 19, 22, 25, 28, 30, 33, 38, 39].

In 2008, Atouch et al. [3] extended the convex optimization problem (1.8) to the case of n variables. Precisely, they considered the following general convex optimization problem:

$$\begin{aligned} \min\Bigg\{ &\sum_{i=1}^{n} f_i(x_i) \\ &+ \frac{1}{2} \sum_{1 \leq i \leq j \leq n} Q_{ij}(x_i, x_j), x_i \in H_i, i \in \{1, 2, \cdots, n\} \Bigg\}, \end{aligned} \tag{1.12}$$

where for $i = 1, 2, \cdots, n$, H_i is a real Hilbert spaces, $f_i : H_i \rightarrow \mathbb{R} \cup \{+\infty\}$ is convex, lower semicontinuous and proper functional, and $Q_{ij} : H_i \times H_j \rightarrow \mathbb{R}$ is a nonnegative continuous quadratic form. Similar to (1.8), one can choose $Q_{ij}(x_i, x_j) = ||A_i x_i - A_j x_j||_Z^2$, where $A_i \in L(H_i, Z)$ is a bounded linear operator mapping H_i to Z.

Recently, Che et al. [9], inspired by the work of Atouch et al. [3], introduced the following extended split equality problem (ESEP). Let H be a real Hilbert space. For $i = 1, 2, \cdots, n$, let C_i be a nonempty closed convex subset of real Hilbert spaces H_i respectively and $A_i : H_i \rightarrow H$ be bounded linear operators. Find

$$\begin{aligned} &x_1 \in C_1, x_2 \in C_2, \cdots, x_n \in C_n \\ &\text{such that } A_1 x_1 = A_2 x_2 = \cdots = A_n x_n. \end{aligned} \tag{1.13}$$

They proposed the following algorithm: Let $(x_{1,1}, x_{1,2}, \cdots, x_{1,n}) \in H_1 \times H_2 \times \cdots \times H_n$ be arbitrary. Calculate the (k+1)-th iterate via the following formula

$$\begin{cases} w_k = \dfrac{\sum_{i=1}^{n} A_i x_{k,i}}{n}, \\ x_{k+1,1} = P_{C_1}(x_{k,1} - \gamma_k A_1^*(A_1 x_{k,1} - w_k)), \\ x_{k+1,2} = P_{C_2}(x_{k,2} - \gamma_k A_2^*(A_2 x_{k,2} - w_k)), \\ \quad \vdots \\ x_{k+1,n} = P_{C_n}(x_{k,n} - \gamma_k A_n^*(A_n x_{k,n} - w_k)), \end{cases} \tag{1.14}$$

where the stepsize $\gamma_k \in (\epsilon, \min_{1 \leq i \leq n} \{\frac{1}{\lambda_{A_i}}\} - \epsilon)$, and λ_{A_i} stands for the spectral radius of $A_i^* A_i$. They obtained a weak convergence result.

In this paper, we study the following Extended Split Equality Fixed Point and Equilibrium Problems (ESEFPEP) which is to find

$$\begin{aligned} &x_1 \in F(T_1) \cap EP(f_1, C_1), x_2 \in F(T_2) \\ &\quad \cap EP(f_2, C_2), \cdots, x_n \in F(T_n) \cap EP(f_n, C_n) \\ &\quad \text{such that } A_1x_1 = A_2x_2 = \cdots = A_nx_n \end{aligned} \tag{1.15}$$

where $T_i : C_i \to C_i (i = 1, 2, \cdots, n)$ are demimetric mappings. We shall denote the solution set of (1.15) by Ω.

2 Preliminaries

Lemma 2.1 ([36, 37]) *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $\lambda \in (-\infty, 1)$ and let T be a λ-demimetric mapping of C into C. Then $F(T)$ is closed and convex.*

Lemma 2.2 ([13]) *Let C be a nonempty closed convex subset of a Hilbert space H and let $f : C \times C \to \mathbb{R}$ be a bi-function satisfying the following conditions*:

(A1) $f(x, x) = 0$ for all $x \in C$;
(A2) *f is monotone, that is, $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;*
(A3) *for each $x, y, z \in C$,*

$$\limsup_{t \downarrow 0} f(tz + (1-t)x, y) \leq f(x, y);$$

(A4) *for each $x \in C, y \mapsto f(x, y)$ is convex and lower semi-continuous.*

Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that

$$f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C.$$

Lemma 2.3 ([38]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\delta_n, n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ or $\sum_{n=1}^{\infty} |\alpha_n\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.4 ([13]) *Let C be a nonempty closed convex subset of a Hilbert space H and let f be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying $(A1) - (A4)$. For $r > 0$ and $x \in H$, define a mapping $T_r^f : H \to C$ as follows:*

$$T_r^f(x) = \{z \in C : f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}, \tag{2.1}$$

for all $x \in H$. Then the following hold:

(i) *T_r^f is single-valued;*
(ii) *T_r^f is firmly non-expansive, that is, for any $x, y \in H$,*

$$||T_r^f(x) - T_r^f(y)||^2 \leq \langle T_r^f(x) - T_r^f(y), x - y\rangle;$$

(iii) $F(T_r^f) = EP(C, f), \forall r > 0$;
(iv) *$EP(C, f)$ is closed and convex.*

Lemma 2.5 *Let H be a real Hilbert space. Then the following hold*:

(a) $||x+y||^2 \leq ||y||^2 + 2\langle x, x+y\rangle$ *for all* $x, y \in H$;
(b) $||x-y||^2 = ||x||^2 + ||y||^2 - 2\langle x, y\rangle$ *for all* $x, y \in H$.
(c) $||\alpha x + (1-\alpha)y||^2 = \alpha||x||^2 + (1-\alpha)||y||^2 - \alpha(1-\alpha)||x-y||^2$, *for all* $x, y \in H$ and $\alpha \in (0,1)$.

Definition 2.6 Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. A mapping $T : C \to H$ is called demiclosed if, for a sequence $\{x_n\}$ in C such that $x_n \rightharpoonup w$ and $x_n - T(x_n) \to 0$, then $w = Tw$ holds.

3 Main Results

Theorem 3.1 *Let H be a real Hilbert space. For $i = 1, 2, \cdots, n$, let C_i be a nonempty closed and convex subset of real Hilbert space H_i and let $A_i : H_i \to H$ be a bounded linear operator. Let $T_i : C_i \to C_i$ be λ_i-demimetric and demiclosed mappings and let $f_i : C_i \times C_i \to \mathbb{R}$ be bifunctions satisfying conditions $(A1)-(A4)$ such that $\Omega \neq \emptyset$. Let $(x_{1,1}, x_{1,2}, \cdots, x_{1,n}) \in H_1 \times H_2 \times \cdots \times H_n$ be arbitrary and let $u_i \in H_i (i = 1, 2 \cdots n)$ be arbitrary but fixed. Let the sequence $\{(x_{k,1}, x_{k,2}, \cdots x_{k,n})\}$ be generated as follows:*

$$\begin{cases} w_k = \dfrac{\sum_{i=1}^{n} A_i x_{k,i}}{n}, \\ y_{k,1} = T_{r_k}^{f_1}(x_{k,1} - \gamma_k A_1^*(A_1 x_{k,1} - w_k)), \\ x_{k+1,1} = \alpha_k u_1 + (1-\alpha_k)[(1-\beta_k)y_{k,1} + \beta_k T_1 y_{k,1}], \\ y_{k,2} = T_{r_k}^{f_2}(x_{k,2} - \gamma_k A_2^*(A_2 x_{k,2} - w_k)), \\ x_{k+1,2} = \alpha_k u_2 + (1-\alpha_k)[(1-\beta_k)y_{k,2} + \beta_k T_2 y_{k,2}], \\ \vdots \\ y_{k,n} = T_{r_k}^{f_n}(x_{k,n} - \gamma_k A_n^*(A_n x_{k,n} - w_k)), \\ x_{k+1,n} = \alpha_k u_n + (1-\alpha_k)[(1-\beta_k)y_{k,n} + \beta_k T_n y_{k,n}], k \geq 1. \end{cases} \tag{3.1}$$

where $\gamma_k \in (\epsilon, \min_{1\leq i\leq n}\{\frac{1}{\gamma_{A_i}}\} - \epsilon)$ and γ_{A_i} stands for the spectral radius of $A_i^* A_i$. Also, $\{\alpha_k\}$ and $\{\beta_k\}$ are sequences in (0, 1) *satisfying the following conditions*

(i) $\lim_{k\to\infty} \alpha_k = 0$, $\sum_{k=1}^{\infty} \alpha_k = \infty$;
(ii) $0 < \liminf_{k\to\infty} \beta_k < \limsup_{k\to\infty} \beta_k < 1 - \lambda$, $\lambda := \max_{1\leq i\leq n}\{\lambda_i\}$.
(iii) $r_k \geq r > 0$.

Then the sequence $\{(x_{k,1}, x_{k,2}, \cdots x_{k,n})\}$ converges strongly to $(z_1^*, z_2^*, \cdots, z_n^*) \in \Omega$.

Proof Let $(z_1, z_2, \cdots, z_n) \in \Omega$, and $\bar{z} = A_1 z_1 = A_2 z_2 = \cdots = A_n z_n$. Since T_i, $i = 1, 2, \cdots, n$ is λ_i-demimetric, we have,

$$\begin{aligned} &||y_{k,i} - z_i + \beta_k(T_i y_{k,i} - y_{k,i})||^2 \\ &\quad = ||y_{k,i} - z_i||^2 + \beta_k^2||T_i y_{k,i} - y_{k,i}||^2 \\ &\qquad + 2\beta_k\langle y_{k,i} - z_i, T_i y_{k,i} - y_{k,i}\rangle \\ &\quad \leq ||y_{k,i} - z_i||^2 + \beta_k^2||T_i y_{k,i} - y_{k,i}||^2 \\ &\qquad - \beta_k(1-\lambda_i)||y_{k,i} - T_i y_{k,i}||^2 \\ &\quad = ||y_{k,i} - z_i||^2 + \beta_k(\beta_k - (1-\lambda_i))||T_i y_{k,i} - y_{k,i}||^2 \\ &\quad \leq ||y_{k,i} - z_i||^2. \end{aligned} \tag{3.2}$$

Therefore, from (3.1) and (3.2), we obtain

$$\begin{aligned}||x_{k+1,i}-z_i||^2 &= ||\alpha_k u_i+(1-\alpha_k)[(1-\beta_k)y_{k,i} \\ &\quad +\beta_k T_i y_{k,i}]-z_i||^2 \\ &= \alpha_k||u_i-z_i||^2+(1-\alpha_k)||(1-\beta_k)y_{k,i} \\ &\quad +\beta_k T_i y_{k,i}-z_i||^2 \\ &\quad -\alpha_k(1-\alpha_k)||u_i-[(1-\beta_k)y_{k,i} \\ &\quad +\beta_k T_i y_{k,i}]||^2 \\ &\le \alpha_k||u_i-z_i||^2+(1-\alpha_k)||(1-\beta_k)y_{k,i} \\ &\quad +\beta_k T_i y_{k,i}-z_i||^2 \\ &\le \alpha_k||u_i-z_i||^2+(1-\alpha_k)||y_{k,i}-z_i||^2.\end{aligned} \tag{3.3}$$

Also from (3.1), we get

$$\begin{aligned}||y_{k,i}-z_i||^2 &= ||T_{r_k}^{f_i}(x_{k,i}-\gamma_k A_i^*(A_i x_{k,i}-w_k))-z_i||^2 \\ &\le ||x_{k,i}-z_i-\gamma_k A_i^*(A_i x_{k,i}-w_k)||^2 \\ &= ||x_{k,i}-z_i||^2+\gamma_k^2||A_i^*(A_i x_{k,i}-w_k)||^2 \\ &\quad -2\gamma_k\langle x_{k,i}-z_i, A_i^*(A_i x_{k,i}-w_k)\rangle \\ &= ||x_{k,i}-z_i||^2+\gamma_k^2||A_i^*(A_i x_{k,i}-w_k)||^2 \\ &\quad -2\gamma_k\langle A_i(x_{k,i}-z_i), A_i x_{k,i}-w_k\rangle \\ &= ||x_{k,i}-z_i||^2+\gamma_k^2||A_i^*(A_i x_{k,i}-w_k)||^2 \\ &\quad +\gamma_k[-||A_i x_{k,i}-A_i z_i||^2 \\ &\quad -||A_i x_{k,i}-w_k||^2+||A_i z_i-w_k||^2] \\ &\le ||x_{k,i}-z_i||^2-\gamma_k(1-\gamma_k||A_i||^2)||A_i x_{k,i}-w_k||^2 \\ &\quad -\gamma_k||A_i x_{k,i}-A_i z_i||^2+\gamma_k||A_i z_i-w_k||^2.\end{aligned} \tag{3.4}$$

Thus it follows from (3.3) and (3.4) that

$$\begin{aligned}||x_{k+1,i}-z_i||^2 &\le \alpha_k||u_i-z_i||^2 \\ &\quad +(1-\alpha_k)\Big[||x_{k,i}-z_i||^2-\gamma_k(1-\gamma_k||A_i||^2)||A_i x_{k,i}-w_k||^2 \\ &\quad -\gamma_k||A_i x_{k,i}-A_i z_i||^2+\gamma_k\frac{\sum_{i=1}^n||\bar{z}-A_i x_{k,i}||^2}{n}\Big],\end{aligned} \tag{3.5}$$

which implies

$$\begin{aligned}&\sum_{i=1}^n||x_{k+1,i}-z_i||^2 \\ &\le \alpha_k\sum_{i=1}^n||u_i-z_i||^2+(1-\alpha_k)[\sum_{i=1}^n||x_{k,i}-z_i||^2 \\ &\quad -\sum_{i=1}^n\gamma_k(1-\gamma_k||A_i||^2)||A_i x_{k,i}-w_k||^2].\end{aligned} \tag{3.6}$$

Hence

$$\begin{aligned}
&\sum_{i=1}^{n} ||x_{k+1,i} - z_i||^2 \\
&\quad \leq \alpha_k \sum_{i=1}^{n} ||u_i - z_i||^2 + (1-\alpha_k) \sum_{i=1}^{n} ||x_{k,i} - z_i||^2 \\
&\quad \leq \max\{\sum_{i=1}^{n} ||u_i - z_i||^2, \sum_{i=1}^{n} ||x_{k,i} - z_i||^2\} \\
&\quad \vdots \\
&\quad \leq \max\{\sum_{i=1}^{n} ||u_i - z_i||^2, \sum_{i=1}^{n} ||x_{1,i} - z_i||^2\}.
\end{aligned} \tag{3.7}$$

Therefore $\{\sum_{i=1}^{n} ||x_{k,i} - z_i||^2\}$ is bounded, which implies $\{x_{k,i}\}$ is bounded for each $i = 1, 2, \cdots, n$. We now consider two cases to obtain the strong convergence.

Case 1. Assume that $\{\sum_{i=1}^{n} ||x_{k,i} - z_i||^2\}$ is monotonically decreasing. Then it follows that $\{\sum_{i=1}^{n} ||x_{k,i} - z_i||^2\}$ is convergent and $\sum_{i=1}^{n} ||x_{k,i} - z_i||^2 - \sum_{i=1}^{n} ||x_{k+1,i} - z_i||^2 \to 0, k \to \infty$.

Now from (3.6), we have

$$\begin{aligned}
&(1-\alpha_k) \sum_{i=1}^{n} \gamma_k (1 - \gamma_k ||A_i||^2) ||A_i x_{k,i} - w_k||^2 \\
&\leq \alpha_k (\sum_{i=1}^{n} ||u_i - z_i||^2 - \sum_{i=1}^{n} ||x_{k,i} - z_i||^2) \\
&\quad + \sum_{i=1}^{n} ||x_{k,i} - z_i||^2 - \sum_{i=1}^{n} ||x_{k+1,i} - z_i||^2 \to 0, k \to \infty.
\end{aligned} \tag{3.8}$$

That is

$$\lim_{k\to\infty} (1-\alpha_k) \sum_{i=1}^{n} \gamma_k (1 - \gamma_k ||A_i||^2) ||A_i x_{k,i} - w_k||^2 = 0,$$

which implies

$$\lim_{k\to\infty} ||A_i x_{k,i} - w_k||^2 = 0, \text{ for each } i = 1, 2, \cdots, n. \tag{3.9}$$

But from Lemma 2.4 (ii), we have

$$\begin{aligned}
0 \leq & -\langle y_{k,i} - z_i, y_{k,i} - z_i \rangle \\
& + \langle y_{k,i} - z_i, x_{k,i} - \gamma_k (A_i^*(A_i x_{k,i} - w_k) - z_i \rangle \\
= & \langle y_{k,i} - z_i, x_{k,i} - \gamma_k A_i^*(A_i x_{k,i} - w_k) - y_{k,i} \rangle,
\end{aligned} \tag{3.10}$$

which implies

$$\begin{aligned}
&\langle y_{k,i} - x_{k,i}, y_{k,i} - z_i \rangle \\
&\quad \leq \gamma_k \langle A_i^*(A_i x_{k,i} - w_k), z_i - y_{k,i} \rangle \\
&\quad \leq \gamma_k ||A_i^*(A_i x_{k,i} - w_k)|| ||z_i - y_{k,i}||.
\end{aligned} \tag{3.11}$$

It then follows from Lemma 2.5(b), (3.3) and (3.11) that

$$\begin{aligned}||y_{k,i}-x_{k,i}||^2 =& ||x_{k,i}-z_i||^2 - ||y_{k,i}-z_i||^2 \\ &+ 2\langle y_{k,i}-x_{k,i}, y_{k,i}-z_i\rangle \\ \leq& ||x_{k,i}-z_i||^2 - (1-\alpha_k)||y_{k,i}-z_i||^2 \\ &+ 2\langle y_{k,i}-x_{k,i}, y_{k,i}-z_i\rangle \\ \leq& ||x_{k,i}-z_i||^2 - ||x_{k+1,i}-z_i||^2 \\ &+ \alpha_k||u_i-z_i||^2 \\ &+ 2\langle y_{k,i}-x_{k,i}, y_{k,i}-z_i\rangle \\ \leq& ||x_{k,i}-z_i||^2 - ||x_{k+1,i}-z_i||^2 \\ &+ \alpha_k||u_i-z_i||^2 \\ &+ 2\gamma_k||A_i^*(A_ix_{k,i}-w_k)||||z_i-y_{k,i}||.\end{aligned} \tag{3.12}$$

Thus

$$\begin{aligned}&\sum_{i=1}^n ||y_{k,i}-x_{k,i}||^2 \\ &\leq \sum_{i=1}^n ||x_{k,i}-z_i||^2 \\ &\quad - \sum_{i=1}^n ||x_{k+1,i}-z_i||^2 + \alpha_k\sum_{i=1}^n ||u_i-z_i||^2 \\ &\quad + 2\gamma_k\sum_{i=1}^n ||A_i^*(A_ix_{k,i}-w_k)||||z_i \\ &\quad - y_{k,i}|| \to 0, k\to\infty.\end{aligned} \tag{3.13}$$

Therefore,

$$\lim_{k\to\infty} ||y_{k,i}-x_{k,i}|| = 0, i = 1, 2, \cdots, n.$$

Again from Lemma 2.5 (c) and (3.2), we have

$$\begin{aligned}||x_{k+1,i}-z_i||^2 =& \alpha_k||u_i-z_i||^2 \\ &+ (1-\alpha_k)||y_{k,i}-z_i+\beta_k(T_iy_{k,i}-y_{k,i})||^2 \\ &- \alpha_k(1-\alpha_k)||y_{k,i} \\ &+ \beta_k(T_iy_{k,i}-y_{k,i}) - u_i||^2 \\ \leq& \alpha_k||u_i-z_i||^2 + (1-\alpha_k)||y_{k,i}-z_i \\ &+ \beta_k(T_iy_{k,i}-y_{k,i})||^2 \\ \leq& \alpha_k||u_i-z_i||^2 + (1-\alpha_k)||y_{k,i}-z_i||^2 \\ &+ (1-\alpha_k)\beta_k(\beta_k-(1-\lambda))||T_iy_{k,i}-y_{k,i}||^2.\end{aligned} \tag{3.14}$$

which implies

$$\begin{aligned}
&(1-\alpha_k)\beta_k((1-\lambda)-\beta_k)||T_iy_{k,i}-y_{k,i}||^2\\
&\quad\le \alpha_k||u_i-z_i||^2+(1-\alpha_k)||y_{k,i}-z_i||^2-||x_{k+1,i}-z_i||^2\\
&\quad\le \alpha_k[||u_i-z_i||^2-||x_{k,i}-z_i||^2]+(1-\alpha_k)||y_{k,i}-x_{k,i}||^2\\
&\qquad+||x_{k,i}-z_i||^2-||x_{k+1,i}-z_i||^2\\
&\qquad+2||y_{k,i}-x_{k,i}||||x_{k,i}-z_i||.
\end{aligned}\tag{3.15}$$

Therefore,

$$\begin{aligned}
&(1-\alpha_k)\beta_k((1-\lambda)-\beta_k)\sum_{i=1}^{n}||T_iy_{k,i}-y_{k,i}||^2\\
&\quad\le \alpha_k\sum_{i=1}^{n}[||u_i-z_i||^2-||x_{k,i}-z_i||^2]\\
&\qquad+(1-\alpha_k)\sum_{i=1}^{n}||y_{k,i}-x_{k,i}||^2\\
&\qquad+\sum_{i=1}^{n}||x_{k,i}-z_i||^2-\sum_{i=1}^{n}||x_{k+1,i}-z_i||^2\\
&\qquad+2\sum_{i=1}^{n}||y_{k,i}-x_{k,i}||||x_{k,i}-z_i||\to 0, k\to\infty,
\end{aligned}\tag{3.16}$$

which implies

$$||T_iy_{k,i}-y_{k,i}||\to 0, k\to\infty, \forall i=1,2,\cdots,n. \tag{3.17}$$

Again from (3.1), we have

$$\begin{aligned}
||x_{k+1,i}-y_{k,i}||\le{}&\alpha_k||u_i-y_{k,i}||\\
&+(1-\alpha_k)||(1-\beta_k)(y_{k,i}-y_{k,i})\\
&+\beta_k(T_iy_{k,i}-y_{k,i})||\\
={}&\alpha_k||u_i-y_{k,i}||\\
&+(1-\alpha_k)\beta_k||T_iy_{k,i}-y_{k,i}||\to 0, k\to\infty.
\end{aligned}\tag{3.18}$$

Now,

$$\begin{aligned}
||x_{k+1,i}-x_{k,i}||\le{}&||x_{k+1,i}-y_{k,i}||\\
&+||y_{k,i}-x_{k,i}||\to 0, k\to\infty.
\end{aligned}\tag{3.19}$$

Since $\{x_{k,i}\}$ is bounded for all $i=1,2,\cdots,n$ there exists a subsequence $\{x_{k_j,i}\}$ of $\{x_{k,i}\}$ for each $i=1,2,\cdots,n$ such that $\{x_{k_j,i}\}$ converges weakly to $z_i^*\in C_i$. From the assumption that $T_i(i=1,2,\cdots,n)$ is demiclosed, (3.13) and (3.17), we have that $z_i^*\in F(T_i)$ for each $i=1,2,\cdots,n$.

We now show that $z_i^*\in EP(f_i,C_i), i=1,2,\cdots,n$. From $y_{k_j,i}=T_{r_{k_j}}^{f_i}(x_{k_j,i}-\gamma_{k_j}A_i^*(A_ix_{k_j,i}-w_{k_j}))$, we have

$$\begin{aligned}
&f_i(y_{k_j,i},v_i)+\frac{1}{r_{k_j}}\langle v_i-y_{k_j,i},y_{k_j,i}-x_{k_j,i}\rangle\\
&\quad+\frac{1}{r_{k_j}}\langle v_i-y_{k_j,i},\gamma_{k_j}A_i^*(A_ix_{k_j,i}-w_{k_j}\rangle\\
&\quad\ge 0, \forall v_i\in C_i, i=1,2,\cdots,n.
\end{aligned}\tag{3.20}$$

It then follows from the monotonicity of $f_i(i=1,2,\cdots,n)$, that

$$\begin{aligned}&\frac{1}{r_{k_j}}\langle v_i - y_{k_j,i}, y_{k_j,i} - x_{k_j,i}\rangle\\&\quad+\frac{1}{r_{k_j}}\langle v_i - y_{k_j,i}, \gamma_{k_j}A_i^*(A_ix_{k_j,i} - w_{k_j})\rangle\\&\quad\geq f_i(v_i, y_{k_j,i}), \forall v_i \in C_i, i = 1,2,\cdots,n,\end{aligned} \tag{3.21}$$

and since $y_{k_j,i} \rightharpoonup z_i^*, i = 1,2,\cdots,n$ and $A4$ that $f_i(v_i, z_i^*) \leq 0, \forall v_i \in C_i, i = 1,2,\cdots,n.$

For $v_i \in C_i$, let $y_{i,t} := tv_i + (1-t)z_i^*$ for all $t \in (0,1)$. Then clearly $y_{i,t} \in C_i, i = 1,2,\cdots,n$. Therefore, from $A1$ and $A4$, we have

$$\begin{aligned}0 =&f_i(y_{i,t}, y_{i,t})\\\leq& tf_i(y_{i,t}, v_i) + (1-t)f_i(y_{i,t}, z_i^*)\\\leq& tf_i(y_{i,t}, v_i),\end{aligned} \tag{3.22}$$

which yields $f_i(y_{i,t}, v_i) \geq 0$.

Thus from $A3$, we obtain $f_i(z_i^*, v_i) \geq 0, i = 1,2,\cdots,n$.

Let $\bar{w} = \frac{\sum_{i=1}^n A_iz_i^*}{n}$, then it follows from (3.9) and the lower semicontinuity of the square norm that for $i = 1,2,\cdots,n$,

$$||A_iz_i^* - \bar{w}||^2 \leq \liminf_{k\to\infty} ||A_ix_{k,i} - w_k||^2 = 0.$$

Hence $A_iz_i^* - \bar{w} = 0$, which yields

$$\begin{aligned}&A_2z_2^* + A_3z_3^* + \cdots + A_nz_n^* = (n-1)A_1z_1^*,\\&A_1z_1^* + A_3z_3^* + \cdots + A_nz_n^* = (n-1)A_2z_2^*\\&\vdots\\&A_1z_1^* + A_2z_2^* + \cdots + A_{n-1}z_{n-1}^* = (n-1)A_nz_n^*,\end{aligned}$$

and solving, we obtain $A_1z_1^* = A_2z_2^* = \cdots = A_nz_n^*$. Therefore, $(z_1^*, z_2^*, \cdots, z_n^*) \in \Omega$. We now obtain the strong convergence

$$\begin{aligned}||x_{k+1,i} - z_i^*||^2 =&||\alpha_ku_i + (1-\alpha_k)[(1-\beta_k)y_{k,i}\\&+\beta_kT_iy_{k,i}] - z_i^*||^2\\\leq&(1-\alpha_k)^2||(1-\beta_k)y_{k,i} + \beta_kT_iy_{k,i} - z_i^*||^2\\&+2\alpha_k\langle u_i - z_i^*, x_{k+1,i} - z_i^*\rangle\\\leq&(1-\alpha_k)^2||y_{k,i} - z_i^*||^2 + 2\alpha_k\langle u_i\\&-z_i^*, x_{k+1,i} - z_i^*\rangle\\\leq&(1-\alpha_k)||x_{k,i} - z_i^*||^2\\&-(1-\alpha_k)[\gamma_k(1-\gamma_k||A_i||^2)||A_ix_{k,i} - w_k||^2\\&-\gamma_k||A_ix_{k,i} - z_i^*||^2\\&+\gamma_k\frac{\sum_{i=1}^n||\bar{z} - A_ix_{k,i}||^2}{n}]\\&+2\alpha_k\langle u_i - z_i^*, x_{k+1,i} - z_i^*\rangle.\end{aligned} \tag{3.23}$$

Therefore,

$$\sum_{i=1}^{n}||x_{k+1,i}-z_i^*||^2 \leq (1-\alpha_k)\sum_{i=1}^{n}||x_{k,i}-z_i^*||^2 + 2\alpha_k\sum_{i=1}^{n}\langle u_i - z_i^*, x_{k+1,i}-z_i^*\rangle. \tag{3.24}$$

Recall that in a real Hilbert space a sequence $\{x_n\}$ is said to converge weakly to a point $x \in H$ if $\langle x_n, y\rangle \to \langle x, y\rangle$ for all $y \in H$. Now since $||x_{k,i}-x_{k+1,i}|| \to 0$ and $x_{k,i} \rightharpoonup z_i^*$, we have that $x_{k+1,i} \rightharpoonup z_i^*$. Therefore, for each $i = 1, 2, \cdots, n$ we get

$$\begin{aligned}&\limsup_{k\to\infty}\langle u_i - z_i^*, x_{k+1,i} - z_i\rangle \\ &\quad = \lim_j \langle u_i - z_i^*, x_{k_j,i} - z_i^*\rangle \\ &\quad = \langle u_i - z_i^*, z_i^* - z_i^*\rangle = 0.\end{aligned}$$

Thus it is easy to see that $\limsup\limits_{k\to\infty} 2\sum_{i=1}^{n}\langle u_i - z_i^*, x_{k+1,i}-z_i^*\rangle = 0$, thus by Lemma 2.3, it follows that

$$\sum_{i=1}^{n}||x_{k+1,i}-z_i^*||^2 \to 0, k\to\infty,$$

which implies

$$||x_{k,i}-z_i^*|| \to 0, k \to \infty, \text{for each } i = 1, 2, \cdots, n.$$

That is

$$x_{k,i} \to z_i^*, i = 1, 2, \cdots, n.$$

Case 2: Asumme that $\{\sum_{i=1}^{n}||x_{k,i}-z_i||^2\}$ is not a monotonically decreasing sequence. Set $\Gamma_k = \sum_{i=1}^{n}||x_{k,i}-z_i||^2, \forall k \geq 1$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $k \geq k_0$ (for some k_0 large enough) by

$$\tau(k) := \max\{l \in \mathbb{N} : l \leq k, \Gamma_l \leq \Gamma_{l+1}\}.$$

Clearly, τ is a nondecreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and

$$0 \leq \Gamma_{\tau(k)} \leq \Gamma_{\tau(k)+1}, \textit{for all } k \geq k_0.$$

After a similar conclusion from (3.17) and (3.19) respectively, it is easy to see that

$$||T_i y_{\tau(k),i} - y_{\tau(k),i}|| \to 0, k \to \infty,$$

and

$$||x_{\tau(k)+1,i} - x_{\tau(k),i}|| \to 0, k \to \infty.$$

Because $\{x_{\tau(k),i}\}$ is bounded for all $i = 1, 2, \cdots, n$ there exists a subsequence of $\{x_{\tau(k),i}\}$, still denoted by $\{x_{\tau(k),i}\}$ which converges weakly to $z_i^* \in F(T_i)$, for $i = 1, 2, \cdots, n$. Also $\limsup\limits_{k\to\infty} 2\sum_{i=1}^{n}\langle u_i - z_i^*, x_{\tau(k)+1,i} - z_i^*\rangle = 0$.

Also from (3.24), we have

$$\sum_{i=1}^{n}||x_{\tau(k)+1,i}-z_i^*||^2 \leq (1-\alpha_{\tau(k)})\sum_{i=1}^{n}||x_{\tau(k),i}-z_i^*||^2 + 2\alpha_{\tau(k)}\sum_{i=1}^{n}\langle u_i - z_i^*, x_{\tau(k)+1,i}-z_i^*\rangle. \tag{3.25}$$

which implies

$$\alpha_{\tau(k)}\sum_{i=1}^{n}||x_{\tau(k),i}-z_i^*||^2 \leq \sum_{i=1}^{n}||x_{\tau(k),i}-z_i^*||^2 - \sum_{i=1}^{n}||x_{\tau(k)+1,i}-z_i^*||^2 + 2\alpha_{\tau(k)}\sum_{i=1}^{n}\langle u_i - z_i^*, x_{\tau(k)+1,i}-z_i^*\rangle. \tag{3.26}$$

That is

$$\sum_{i=1}^{n}||x_{\tau(k),i}-z_i^*||^2 \leq 2\sum_{i=1}^{n}\langle u_i - z_i^*, x_{\tau(k)+1,i}-z_i^*\rangle. \tag{3.27}$$

Therefore,

$$\sum_{i=1}^{n}||x_{\tau(k),i}-z_i^*||^2 \to 0, k\to\infty, \tag{3.28}$$

which implies

$$\lim_{k\to\infty}||x_{\tau(k)+1,i}-z_i^*|| = 0, \text{ for } i\ =\ 1,2,\cdots,n.$$

Furthermore, for $k\geq k_0$, it is easy to see that $\Gamma_{\tau(k)}\leq\Gamma_{\tau(k)+1}$ if $k\neq\tau(k)$ (that is $\tau(k)<k$), because $\Gamma_j\geq\Gamma_{j+1}$ for $\tau(k)+1\leq j\leq k$. As a consequence, we obtain for all $k\geq k_0$,

$$0\leq\Gamma_k\leq\max\{\Gamma_{\tau(k)},\Gamma_{\tau(k)+1}\}=\Gamma_{\tau(k)+1}.$$

Hence, $\lim_{k\to\infty}\Gamma_k=0$, thus

$$\lim_{k\to\infty}\sum_{i=1}^{n}||x_{k,i}-z_i^*||=0,$$

that is

$$||x_{k,i}-z_i^*||\to 0, k\to\infty, i=1,2,\cdots,n.$$

Thus we conclude that $\{(x_{k,1},x_{k,2},\cdots,x_{k,n})\}$ converges strongly to $(z_1^*,z_2^*,\cdots,z_n^*)\in\Omega$. this completes the proof. □

Corollary 3.2 *Let H be a real Hilbert space. For $i=1,2,\cdots,n$, let C_i be a nonempty closed and convex subset of real Hilbert space H_i and let $A_i:H_i\to H$ be bounded linear operators. For $\lambda_i\in(0,1)$, let $T_i:C_i\to C_i$ be λ_i -strictly pseudocontractive mappings $(i=1,2,\cdots,n)$ and let $f_i:C_i\times C_i\to\mathbb{R}$ be bifunctions satisfying conditions $(A1)-(A4)$ such that $\Omega\neq\emptyset$. Let $(x_{1,1},x_{1,2},\cdots,x_{1,n})\in H_1\times H_2\times\cdots\times H_n$ be arbitrary arbitrary and let $u_i\in H_i(i=1,2\cdots n)$ be arbitrary but fixed. Let the sequence $\{(x_{k,1},x_{k,2},\cdots x_{k,n})\}$ be generated as follows*:

$$\begin{cases} w_k=\dfrac{\sum_{i=1}^n A_i x_{k,i}}{n},\\ y_{k,1}=T_{r_k}^{f_1}(x_{k,1}-\gamma_k A_1^*(A_1x_{k,1}-w_k)),\\ x_{k+1,1}=\alpha_k u_1+(1-\alpha_k)[(1-\beta_k)y_{k,1}+\beta_k T_1 y_{k,1}],\\ y_{k,2}=T_{r_k}^{f_2}(x_{k,2}-\gamma_k A_2^*(A_2x_{k,2}-w_k)),\\ x_{k+1,2}=\alpha_k u_2+(1-\alpha_k)[(1-\beta_k)y_{k,2}+\beta_k T_2 y_{k,2}],\\ \vdots\\ y_{k,n}=T_{r_k}^{f_n}(x_{k,n}-\gamma_k A_n^*(A_nx_{k,n}-w_k)),\\ x_{k+1,n}=\alpha_k u_n+(1-\alpha_k)[(1-\beta_k)y_{k,n}+\beta_k T_n y_{k,n}],\ k\geq 1. \end{cases} \tag{3.29}$$

where $\gamma_k \in (\epsilon, \min_{1 \le i \le n} \{\frac{1}{\gamma_{A_i}}\} - \epsilon)$ *and* γ_{A_i} *stands for the spectral radius of* $A_i^* A_i$. *Also,* $\{\alpha_k\}$ *and* $\{\beta_k\}$ *are sequences in* $(0, 1)$ *satisfying the following conditions*

(i) $\lim_{k\to\infty} \alpha_k = 0, \sum_{k=1}^{\infty} \alpha_k = \infty;$

(ii) $0 < \liminf_{k\to\infty} \beta_k < \limsup_{k\to\infty} \beta_k < 1 - \lambda, \ \lambda := \max_{1 \le i \le n} \{\lambda_i\}$. *Then the sequence* $\{(x_{k,1}, x_{k,2}, \cdots x_{k,n})\}$ *converges strongly to* $(z_1^*, z_2^*, \cdots, z_n^*) \in \Omega$.

Corollary 3.3 *Let H be a real Hilbert space. For* $i = 1, 2, \cdots, n$, *let* C_i *be a nonempty closed and convex subset of real Hilbert space* H_i *and let* $A_i : H_i \to H$ *be bounded linear operators. For* $\delta_i, \eta_i \in \mathbb{R}$, *let* $T_i : C_i \to C_i$ *be* (δ_i, η_i) *- generalised hybrid mappings* $(i = 1, 2, \cdots, n)$ *and let* $f_i : C_i \times C_i \to \mathbb{R}$ *be bifunctions satisfying conditions* $(A1) - (A4)$ *such that* $\Omega \neq \emptyset$. *Let* $(x_{1,1}, x_{1,2}, \cdots, x_{1,n}) \in H_1 \times H_2 \times \cdots \times H_n$ *be arbitrary arbitrary and let* $u_i \in H_i (i = 1, 2 \cdots n)$ *be arbitrary but fixed. Let the sequence* $\{(x_{k,1}, x_{k,2}, \cdots x_{k,n})\}$ *be generated as follows*:

$$\begin{cases} w_k = \dfrac{\sum_{i=1}^n A_i x_{k,i}}{n}, \\ y_{k,1} = T_{r_k}^{f_1}(x_{k,1} - \gamma_k A_1^*(A_1 x_{k,1} - w_k)), \\ x_{k+1,1} = \alpha_k u_1 + (1 - \alpha_k)[(1 - \beta_k) y_{k,1} + \beta_k T_1 y_{k,1}], \\ y_{k,2} = T_{r_k}^{f_2}(x_{k,2} - \gamma_k A_2^*(A_2 x_{k,2} - w_k)), \\ x_{k+1,2} = \alpha_k u_2 + (1 - \alpha_k)[(1 - \beta_k) y_{k,2} + \beta_k T_2 y_{k,2}], \\ \vdots \\ y_{k,n} = T_{r_k}^{f_n}(x_{k,n} - \gamma_k A_n^*(A_n x_{k,n} - w_k)), \\ x_{k+1,n} = \alpha_k u_n + (1 - \alpha_k)[(1 - \beta_k) y_{k,n} + \beta_k T_n y_{k,n}], \ k \ge 1. \end{cases} \tag{3.30}$$

where $\gamma_k \in (\epsilon, \min_{1 \le i \le n} \{\frac{1}{\gamma_{A_i}}\} - \epsilon)$ *and* γ_{A_i} *stands for the spectral radius of* $A_i^* A_i$. *Also,* $\{\alpha_k\}$ *and* $\{\beta_k\}$ *are sequences in* $(0, 1)$ *satisfying the following conditions*

(i) $\lim_{k\to\infty} \alpha_k = 0, \sum_{k=1}^{\infty} \alpha_k = \infty;$

(ii) $0 < \liminf_{k\to\infty} \beta_k < \limsup_{k\to\infty} \beta_k < 1$. *Then the sequence* $\{(x_{k,1}, x_{k,2}, \cdots x_{k,n})\}$ *converges strongly to* $(z_1^*, z_2^*, \cdots, z_n^*) \in \Omega$.

4 Applications

Here, we apply the result of Theorem 3.1 to solve the following extended split equality monotone variational inclusion and equilibrium problems (ESEMVIEP) which is to find

$$\begin{aligned} & x_1 \in (B_1 + S_1)^{-1}(0) \cap EP(f_1, C_1), x_2 \in (B_2 + S_2)^{-1}(0) \\ & \cap EP(f_2, C_2), \cdots, x_n \in (B_n + f_n)^{-1}(0) \cap EP(f_n, C_n) \\ & \text{such that } A_1 x_1 = A_2 x_2 = \cdots = A_n x_n \end{aligned} \tag{4.1}$$

where $B_i : H_i \to 2^{H_i}$ $(i = 1, 2, \cdots, n)$ are maximal monotone mappings and $S_i : H_i \to H_i$ $(i = 1, 2, \cdots, n)$ are α_i-ism mappings. We shall denote the solution set of (4.1) by Ω_1.

A mapping $S : H \to H$ is said to be α-inverse strongly monotone (α-ism), if there exists a constant $\alpha > 0$ such that

$$\langle Sx - Sy, x - y \rangle \ge \alpha \|Sx - Sy\|^2, \textit{for all } x, y \in H.$$

A set valued mapping $B : H \to 2^H$ is called monotone if for all $x, y \in H$, with $u \in B(x)$ and $v \in B(y)$ then

$$\langle x - y, u - v \rangle \geq 0$$

and is maximal monotone if the graph of B denoted as $G(B)$ is not properly contained in the graph of any other monotone mapping. We recall that for multivalued mapping B,

$$G(B) = \{(x, y) : y \in B(x)\}.$$

The resolvent operator J_λ^B associated with B and $\rho > 0$ is the mapping $J_\rho^B : H \to H$ defined by

$$J_\rho^B(x) = (I + \rho B)^{-1}(x), \; x \in H. \tag{4.2}$$

The resolvent operator J_ρ^B is single valued, nonexpansive and 1-inverse strongly monotone (for example see [5]). Moreover $0 \in B(x) + S(x)$ if and only if $x = J_\rho^B(I - \rho S)(x)$, *for all* $\rho > 0$ (see [26]). If S is α-ism mapping with $0 < \rho < 2\alpha$, then $J_\rho^B(I - \rho f)$ is nonexpansive and $F(J_\rho^B(I - \rho S))$ is closed and convex.

Lemma 4.1 [11, 12] *Let H be a Hilbert space and $T : H \to H$ a nonexpansive mapping, then for all $x, y \in H$,*

$$\begin{aligned} &\langle (x - Tx) - (y - Ty), Ty - Tx \rangle \\ &\leq \frac{1}{2} ||(Tx - x) - (Ty - y)||^2 \end{aligned} \tag{4.3}$$

and consequently if $y \in F(T)$ then

$$\langle x - Tx, x - y \rangle \leq \frac{1}{2} ||Tx - x||^2. \tag{4.4}$$

Theorem 4.2 *Let H be a real Hilbert space. For $i = 1, 2, \cdots, n$, let C_i be a nonempty closed and convex subset of real Hilbert space H_i and let $A_i : H_i \to H$ be bounded linear operators. Let $B_i : H_i \to 2^{B_i}$ be maximal monotone mappings, $S_i : H_i \to H_i$ be μ_i -ism mappings with $0 < \rho < 2\mu_i$ and let $f_i : C_i \times C_i \to \mathbb{R}$ be bifunctions satisfying conditions $(A1) - (A4)$ such that $\Omega_1 \neq \emptyset$. Let $(x_{1,1}, x_{1,2}, \cdots, x_{1,n}) \in H_1 \times H_2 \times \cdots \times H_n$ be arbitrary and let $u_i \in H_i (i = 1, 2 \cdots n)$ be arbitrary but fixed. Let the sequence $\{(x_{k,1}, x_{k,2}, \cdots x_{k,n})\}$ be generated as follows:*

$$\begin{cases} w_k = \dfrac{\sum_{i=1}^n A_i x_{k,i}}{n}, \\ y_{k,i} = T_{r_k}^{f_1}(x_{k,1} - \gamma_k(A_1 x_{k,1} - w_k)), \\ x_{k+1,1} = \alpha_k u_1 + (1 - \alpha_k)[(1 - \beta_k) y_{k,1} + \beta_k J_\rho^{B_1}(I - \rho S_1) y_{k,1}], \; k \geq 1. \\ y_{k,2} = T_{r_k}^{f_2}(x_{k,2} - \gamma_k(A_2 x_{k,2} - w_k)), \\ x_{k+1,2} = \alpha_k u_2 + (1 - \alpha_k)[(1 - \beta_k) y_{k,2} + \beta_k J_\rho^{B_2}(I - \rho S_2) y_{k,2}], \\ \vdots \\ y_{k,n} = T_{r_k}^{f_n}(x_{k,n} - \gamma_k(A_n x_{k,n} - w_k)), \\ x_{k+1,n} = \alpha_k u_n + (1 - \alpha_k)[(1 - \beta_k) y_{k,n} + \beta_k J_\rho^{B_n}(I - \rho S_n) y_{k,n}], \; k \geq 1 \end{cases} \tag{4.5}$$

where $\gamma_k \in (\epsilon, \min_{1 \leq i \leq n} \{\frac{1}{\gamma_{A_i}}\} - \epsilon)$ and γ_{A_i} stands for the spectral radius of $A_i^* A_i$. Also, $\{\alpha_k\}$ and $\{\beta_k\}$ are sequences in $(0, 1)$ satisfying the following conditions

(i) $\lim_{k \to \infty} \alpha_k = 0, \sum_{k=1}^\infty \alpha_k = \infty$;

(ii) $0 < \liminf_{k \to \infty} \beta_k < \limsup_{k \to \infty} \beta_k < 1$. *Then the sequence $\{(x_{k,1}, x_{k,2}, \cdots x_{k,n})\}$ converges strongly to $(\bar{z}_1, \bar{z}_2, \cdots, \bar{z}_n) \in \Omega_1$.*

Proof From the assumption $0 < \rho < 2\mu_i$, we have that $J_\rho^{B_i}(I - \rho S_i)$ is nonexpansive for each $i = 1, 2, \cdots, n,$. Thus from Lemma 4.1 (4.4), we have that

$$\begin{aligned}&\langle x - J_\rho^{B_i}(I-\rho S_i)x, x-y\rangle \\ &\le \frac{1}{2}||J_\rho^{B_i}(I-\rho S_i)x - x||^2 \\ &= \frac{1-0}{2}||J_\rho^{B_i}(I-\rho S_i)x - x||^2,\end{aligned} \tag{4.6}$$

that is $J_\rho^{B_i}(I-\rho S_i)$ for each $i = 1, 2, \cdots, n$, is a λ-demimetric mapping with $\lambda = 0$. Thus the proof follows from Theorem 3.1. □

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