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ORIGINAL RESEARCH

Relative Weak Injective and Weak Flat Modules with Respect to a Semidualizing Module

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Abstract Let R be a commutative ring, and let C be a semidualizing R-module. We introduce the notion of finitely presented C-injective modules, finitely presented C-flat modules, weak C-injective modules and weak C-flat modules. Some properties of these modules are investigated. It is proved that over weak C-injective ring R, if for a super finitely presented R-module M, $\mathrm{Hom}_R(M,R)$ is super finitely presented and $\mathrm{Hom}_R(M,R) \in \mathcal{A}_C(R)$, then the following statements hold:

(1) the R-module M is reflexive,
(2) the R-module M is Gorenstein projective, provided that $M \in \mathcal{A}_C(R)$.

Keywords Semidualizing · FP-injective · FP-flat · Weak injective · Weak flat

Mathematics Subject Classification 13D05 · 13D45 · 18G20

1 Introduction

Throughout this paper R is a commutative ring and all modules are unital. The notion of FP-injective modules (resp. FP-flat modules) is introduced in [20], as a generalization of injective modules (resp. flat modules). An R-module M is called *FP-injective* (resp. *FP-flat*) if $\mathrm{Ext}^1_R(F,M)=0$ (resp. $\mathrm{Tor}^R_1(M,F)=0$) for every finitely presented R-module F. The super finitely presented module originated from Grothendieck's notion of a pseudo-coherent module in [2]. In [4], the authors used the term "FP_∞-module" in the sense of a super finitely presented module. The super finitely presented modules might be an important tool in the Gorenstein homological algebra instead of the finitely presented modules. Recently, the notion of weak injective modules (resp. weak flat modules) is introduced in [10] as a generalization of FP-injective modules (resp. FP-flat modules). An R-module M is called *weak injective* (resp. *weak flat*) if $\mathrm{Ext}^1_R(F,M)=0$ (resp. $\mathrm{Tor}^R_1(M,F)=0$) for every super finitely presented R-module F.

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The notion of a "semidualizing module" is a central notion in relative homological algebra. This notion was first introduced by Foxby [9]. Then Vasconcelos [22] and Golod [11] rediscovered these modules using different terminology for different purposes. This notion has been investigated by many authors from different points of view; see for example [1, 5, 12, 16], and [21].

Among various research areas on semidualizing modules, one sometimes focuses on extending the "absolute" classical notion of homological algebra to the "relative" setting with respect to a semidualizing module. In this paper, we introduce the notion of relative *FP*-injective modules (resp. relative *FP*-flat modules) and relative weak injective modules (resp. weak flat modules) with respect to the semidualizing R-module C. We investigate some properties of these modules. It is proved that over weak C-injective ring R, the super finitely presented R-module M is reflexive, provided that $\mathrm{Hom}_R(M,R) \in \mathcal{A}_C(R)$ and $\mathrm{Hom}_R(M,R)$ is super finitely presented. Also, we investigate that for weak C-flat module $F \in \mathcal{B}_C(R)$ and for super finitely presented R-module M, the natural homomorphism

$$\eta_M : \mathrm{Hom}_R(M,R) \otimes_R F \to \mathrm{Hom}_R(M,F)$$

is an isomorphism, provided some special conditions. Finally, it is shown that over weak C-injective ring R, every super finitely presented module $M \in \mathcal{A}_C(R)$ is Gorenstein projective, provided that $\mathrm{Hom}_R(M,R) \in \mathcal{A}_C(R)$ and $\mathrm{Hom}_R(M,R)$ is super finitely presented.

2 Background Material

Throughout this paper $\mathcal{M}(R)$ denotes the category of R-modules. We use the term "subcategory" to mean a "full, additive subcategory $\mathcal{X} \subseteq \mathcal{M}(R)$ such that, for all R-modules M and N, if $M \cong N$ and $M \in \mathcal{X}$, then $N \in \mathcal{X}$". An *R-complex* is a sequence $Y = \cdots \xrightarrow{\partial^Y_{n+1}} Y_n \xrightarrow{\partial^Y_n} Y_{n-1} \xrightarrow{\partial^Y_{n-1}} \cdots$ of R-modules and R-homomorphisms such that $\partial^Y_{n-1}\partial^Y_n = 0$ for each integer n. Let $\mathcal{X}$ be a subcategory of $\mathcal{M}(R)$. The R-complex Y is $\mathrm{Hom}_R(\mathcal{X},-)$-*exact* if for each X in $\mathcal{X}$, the complex $\mathrm{Hom}_R(X,Y)$ is exact, and similarly for $\mathrm{Hom}_R(-,\mathcal{X})$- *exact*. Recall that an R-module M is called *super finitely presented*, if there exists a projective resolution P of M such that each P_i is a finitely generated projective. The super finitely presented module originated from Grothendieck [2], but was rediscovered by others using different terminology. Note that in the case that R is Noetherian, the class of super finitely presented R-modules is equal to the class of finitely presented R-modules.

The notion of semidualizing modules, defined next, was first introduced by Foxby [9]. Then Vasconcelos [22] and Golod [11] rediscovered these modules using different terminology for different purposes.

Definition 2.1 An R-module C is called *semidualizing* if

(*i*) C is super finitely presented,
(*ii*) the natural homothety homomorphism $\chi^R_C : R \to \mathrm{Hom}_R(C,C)$ is an isomorphism, and
(*iii*) $\mathrm{Ext}^{\geqslant 1}_R(C,C) = 0$.

A free R-module of rank one is semidualizing. If R is Noetherian and admits a dualizing module D, then D is a semidualizing. Note that this definition agrees with the established definition when R is Noetherian, in which case condition (i) is equivalent to C being finitely generated. An R-module is *C-projective* (resp. *C-flat* or *C-injective*) if it is isomorphic to a module of the form $P \otimes_R C$ for some projective R-module P (resp. $F \otimes_R C$ for some flat R-module F or $\mathrm{Hom}_R(C,I)$ for some injective R-module I). We let $\mathcal{P}_C(R)$, $\mathcal{F}_C(R)$ and $\mathcal{I}_C(R)$ denote the categories of C-projective, C-flat and C-injective R-modules, respectively. The *Auslander class* with respect to C is the class $\mathcal{A}_C(R)$ of R-modules M such that:

(*i*) $\mathrm{Tor}^R_i(C,M) = 0 = \mathrm{Ext}^i_R(C, C \otimes_R M)$ for all $i \geqslant 1$, and
(*ii*) the natural map $\gamma^M_C : M \to \mathrm{Hom}_R(C, C \otimes_R M)$ is an isomorphism.

The *Bass class* with respect to C is the class $\mathcal{B}_C(R)$ of R-modules M such that:

(*i*) $\mathrm{Ext}^i_R(C,M) = 0 = \mathrm{Tor}^R_i(C, \mathrm{Hom}_R(C,M))$ for all $i \geqslant 1$, and
(*ii*) the natural evaluation map $\xi^C_M : C \otimes_R \mathrm{Hom}_R(C,M) \underset{M}{\longrightarrow}$ is an isomorphism.

The notion of precovers and preenvelopes, defined next, are from [7].

Definition 2.2 Let $\mathcal{X}$ be a subcategory of $\mathcal{M}(R)$. An $\mathcal{X}$- *precover* of an R-module M is an R-module homomorphism $X \xrightarrow[M]{\varphi}$, where $X \in \mathcal{X}$, and such that the map $\mathrm{Hom}_R(X', \varphi)$ is surjective for every $X' \in \mathcal{X}$. If every R-module admits $\mathcal{X}$-precover, then the class $\mathcal{X}$ is *precovering*. The notions of *$\mathcal{X}$-preenvelope* and *preenveloping* are defined dually.

Let C be a semidualizing R-module. In [13], it is shown that the class $\mathcal{P}_C(R)$ is precovering. So, one can iteratively take precovers to construct an *augmented proper $\mathcal{P}_C$-projective resolution* for any R-module M, that is, a complex

$$X^+ = \cdots \longrightarrow C \otimes_R P_1 \longrightarrow C \otimes_R P_0 \longrightarrow M \longrightarrow 0$$

which is $\mathrm{Hom}_R(\mathcal{P}_C(R), -)$-exact. The truncated complex

$$X = \cdots \longrightarrow C \otimes_R P_1 \longrightarrow C \otimes_R P_0 \longrightarrow 0$$

is a *proper $\mathcal{P}_C$- projective resolution* of M.

Dually, in [13] it is proved that the class $\mathcal{I}_C(R)$ is enveloping. So, for an R-module N one can construct an *augmented proper $\mathcal{I}_C$-injective coresolution*, that is, a complex

$$Y^+ = 0 \longrightarrow N \longrightarrow \mathrm{Hom}_R(C, I^0) \longrightarrow \mathrm{Hom}_R(C, I^1) \longrightarrow \cdots$$

which is $\mathrm{Hom}_R(-, \mathcal{I}_C(R))$-exact. Also, in [13] it is shown that the class $\mathcal{F}_C(R)$ is covering. Similarly for an R-module M one can construct an *augmented proper $\mathcal{F}_C$-flat resolution*.

The following functors are studied in [19, 21], and [17]. We use them in the next section.

Definition 2.3 Let C be a semidualizing R-module, and let M and N be R-modules. Let J be a proper $\mathcal{I}_C$-coresolution of N. For each i, set

$$\mathrm{Ext}^i_{\mathcal{M}\mathcal{I}_C}(M, N) := \mathrm{H}_{-i}(\mathrm{Hom}_R(M, J)).$$

Let G be a proper $\mathcal{F}_C$-resolution of M. For each i, set:

$$\mathrm{Tor}_i^{\mathcal{F}_C\mathcal{M}}(M, N) := \mathrm{H}_i(G \otimes_R N).$$

Remark 2.4 In [17, Definition 3.5], the authors considered that R is Noetherian and C is a semidualizing R-module. Note that in this case, Definition 2.1 and [17, Definition 2.8] are equivalent, since the class of super finitely presented R-modules is equal to the class of finitely presented R-modules.

3 Main results

The notion of *FP*-injective modules (resp. *FP*-flat modules) is introduced in [20], as a generalization of injective modules (resp. flat modules). Recently, the weak injective modules (resp. weak flat modules) are introduced in [10] as a generalization of *FP*-injective modules (resp. *FP*-flat modules). In this section, we introduce the notion of relative *FP*-injective modules and weak injective modules with respect to the semidualizing R-module C. Also, we introduce the notion of relative *FP*-flat modules and weak flat modules and investigate some properties of these modules.

Definition 3.1 Let C be a semidualizing R-module.

(i) An R-module M is called *weak C-injective* (resp. *finitely presented C-injective*) if $\mathrm{Ext}^1_{\mathcal{M}\mathcal{I}_C}(F, M) = 0$ for any super finitely presented R-module F (resp. for any finitely presented R-module F).

(ii) An R-module N is called *weak C-flat* (resp. *finitely presented C-flat*) if $\mathrm{Tor}_1^{\mathcal{F}_C\mathcal{M}}(N, F) = 0$ for any super finitely presented R-module F (resp. for any finitely presented R-module F).

In the case $C = R$, we use the terminology "weak injective", "FP-injective", "weak flat" and "FP-flat" instead of "weak R-injective", "finitely presented R-injective", "weak R-flat" and "finitely presented R–flat".

Note that by [21, Theorem 3.1] and [17, Theorem 5.4], every C-injective module is finitely presented C-injective and every C-flat module is finitely presented C-flat, where C is a semidualizing R-module. It is clear that every super finitely presented R-module is finitely presented. Consequently:

(*i*) every finitely presented C-injective module is weak C-injective.
(*ii*) every finitely presented C-flat module is weak C-flat.

In [23, Proposition 2.2], it is proved that over coherent ring R, every finitely presented R-module is super finitely presented. Therefore, the class of weak C-injective modules and the class of weak C-flat modules coincide respectively with the class of finitely presented C-injective modules and finitely presented C-flat modules, where R is a coherent ring.

Remark 3.2 Let C be a semidualizing R-module, and let E be a faithfully injective R-module. Set $(-)^{\vee} = \mathrm{Hom}_R(-, E)$. By [17, Corollary 3.13], for any R-modules F and M, we have the isomorphism

$$\mathrm{Ext}^1_{\mathcal{MI}_C}(F, M^{\vee}) \cong \mathrm{Tor}_1^{\mathcal{F}_C\mathcal{M}}(M, F)^{\vee}.$$

Then an R-module M is weak C-flat (resp. finitely presented C-flat) if and only if $M^{\vee}$ is weak C-injective (resp. finitely presented C-injective). Consequently, an R-module M is weak C-flat (resp. finitely presented C-flat) if and only if $M^{+} = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is weak C-injective (resp. finitely presented C-injective).

Proposition 3.3 *Let C be a semidualizing R-module. Then the following statements hold.*

(*i*) Let $\{M_i\}_{i\in I}$ be a family of R-modules. Then $\prod_{i\in I} M_i$ is weak C-injective (resp. finitely presented C-injective) if and only if each M_i is weak C-injective (resp. finitely presented C-injective).
(*ii*) Let $\{N_i\}_{i\in I}$ be a family of R-modules. Then $\coprod_{i\in I} N_i$ is weak C-flat (resp. finitely presented C-flat) if and only if each N_i is weak C-flat (resp. finitely presented C-flat).
(*iii*) The class of weak C-injective (resp. finitely presented C-injective) R-modules is closed under direct limit.

Proof (*i*) Let F be a super finitely presented (resp. finitely presented) R-module. In the following sequence, the first and the last isomorphisms follow from [21, Thoerem 4.1] and the second isomorphism follows from [7, Theorem 3.2.22] and the third isomorphism follows from [7, Exercise 4, page 74].

$$\begin{aligned} \mathrm{Ext}^1_{\mathcal{MI}_C}(F, \prod_{i\in I} M_i) &\cong \mathrm{Ext}^1_R(C\otimes_R F, C\otimes_R \prod_{i\in I} M_i) \\ &\cong \mathrm{Ext}^1_R(C\otimes_R F, \prod_{i\in I}(C\otimes_R M_i)) \\ &\cong \prod_{i\in I}\mathrm{Ext}^1_R(C\otimes_R F, C\otimes_R M_i) \\ &\cong \prod_{i\in I}\mathrm{Ext}^1_{\mathcal{MI}_C}(F, M_i). \end{aligned}$$

So, we get the assertion.

(*ii*) Let F be a super finitely presented (resp. finitely presented) R-module. By [17, Corollary 3.11], we have

$$\mathrm{Tor}_1^{\mathcal{F}_C\mathcal{M}}(\coprod_{i\in I} N_i, F) \cong \coprod_{i\in I}\mathrm{Tor}_1^{\mathcal{F}_C\mathcal{M}}(N_i, F).$$

So, we get the assertion.

(*iii*) Let F be a super finitely presented (resp. finitely presented) R-module. In the following sequence, the first and the last isomorphisms follow from [21, Thoerem 4.1], the second isomorphism follows from [7, Theorem 1.5.7] and the third isomorphism follows from [3, Exercise 3, page 187].

$$\begin{aligned} \mathrm{Ext}^1_{\mathcal{MI}_C}(F, \varinjlim_{i\in I} M_i) &\cong \mathrm{Ext}^1_R(C\otimes_R F, C\otimes_R (\varinjlim_{i\in I} M_i)) \\ &\cong \mathrm{Ext}^1_R(C\otimes_R F, \varinjlim_{i\in I}(C\otimes_R M_i)) \\ &\cong \varinjlim_{i\in I}\mathrm{Ext}^1_R(C\otimes_R F, C\otimes_R M_i) \\ &\cong \varinjlim_{i\in I}\mathrm{Ext}^1_{\mathcal{MI}_C}(F, M_i). \end{aligned}$$

So, we get the assertion. □

Proposition 3.4 *Let C be a semidualizing R-module, and let $F \in \mathcal{A}_C(R)$ be a super finitely presented module. Then the following statements hold.*

(i) Let $M \in \mathcal{A}_C(R)$ be a weak C-injective module. Then $\mathrm{Ext}^n_{\mathcal{MI}_C}(F, M) = 0$ for all $n \geqslant 1$.

(ii) Let $N \in \mathcal{B}_C(R)$ be a weak C-flat module. Then $\mathrm{Tor}^{\mathcal{F}_C\mathcal{M}}_n(N, F) = 0$ for all $n \geqslant 1$.

Proof (i) Let $F \in \mathcal{A}_C(R)$ be a super finitely presented module. Then there exists an exact sequence

$$0 \to K \to P_{n-2} \to \cdots \to P_1 \to P_0 \to F \to 0$$

of R-modules such that for every $0 \leqslant i \leqslant n-2$, P_i is finitely generated projective. In the case that $n = 1$, take $K = F$. By [18, Corollary 3.1.8], $K \in \mathcal{A}_C(R)$, and by [14, Lemma 2.3] K is super finitely presented. Then $\mathrm{Ext}^1_{\mathcal{MI}_C}(K, M) = 0$ by assumption. By [21, Corollary 4.2], $\mathrm{Ext}^1_{\mathcal{MI}_C}(K, M) \cong \mathrm{Ext}^1_R(K, M)$, since $M \in \mathcal{A}_C(R)$ and $K \in \mathcal{A}_C(R)$. This implies that $\mathrm{Ext}^n_R(F, M) = 0$, and so $\mathrm{Ext}^n_{\mathcal{MI}_C}(F, M) = 0$ by [21, Corollary 4.2].

(ii) The argument of proof is similar to the proof (i). Note that $\mathrm{Tor}^{\mathcal{F}_C\mathcal{M}}_n(N, F) \cong \mathrm{Tor}^R_n(N, F)$, by [17, Proposition 4.3]. □

We set $M^* = \mathrm{Hom}_R(M, R)$, for any R-module M. In [6], it is stated that the ring R is a coherent ring if and only if M^* is finitely presented for any finitely presented R-module M. Naturally, the ring R is called a *generalized coherent* if M^* is super finitely presented for any super finitely presented R-module M. This notion are introduced in [23].

Theorem 3.5 *Let C be a semidualizing R-module. Then the following statements hold.*

(i) Let R be a weak C-injective R-module, and let M be a super finitely presented R-module such that M^* is super finitely presented R-module and $M^* \in \mathcal{A}_C(R)$. Then M is reflexive.

(ii) Let R be a generalized coherent ring, and suppose that every super finitely presented R-module belongs to $\mathcal{A}_C(R)$. Then R is weak C-injective if and only if every super finitely presented module is reflexive.

Proof (i) Let M be a super finitely presented R-module. Then there exists the exact sequence $P_1 \to P_0 \to M \to 0$ such that P_0 and P_1 are finitely generated projective R-modules. Assume that $N = \mathrm{Coker}((P_0)^* \to (P_1)^*)$. Then we have the exact sequence

$$0 \to M^* \to (P_0)^* \to (P_1)^* \to N \to 0.$$

Therefore, N is super finitely presented and $N \in \mathcal{A}_C(R)$ by [18, Corollary 3.1.8]. On the other hand, we have the exact sequence

$$0 \to \mathrm{Ext}^1_R(N, R) \to M \to M^{**} \to \mathrm{Ext}^2_R(N, R) \to 0,$$

by [15, Lemma 2.2]. By [21, Corollary 4.2], $\mathrm{Ext}^1_R(N, R) \cong \mathrm{Ext}^1_{\mathcal{MI}_C}(N, R) = 0$, since R is a weak C-injective R-module. Also, $\mathrm{Ext}^2_R(N, R) \cong \mathrm{Ext}^2_{\mathcal{MI}_C}(N, R) = 0$, by Proposition 3.4 and [21, Corollary 4.2]. So, we get the assertion.

(ii) "⇒" Let R be a weak C-injective R-module, and let M be a super finitely presented R-module. By assumption, M^* is a super finitely presented R-module, and $M^* \in \mathcal{A}_C(R)$. Then M is reflexive by (i), as desired.

"⇐" Let M be a super finitely presented R-module. It is sufficient to prove that $\mathrm{Ext}^1_{\mathcal{MI}_C}(M, R) = 0$. By [21, Theorem 4.2], $\mathrm{Ext}^1_{\mathcal{MI}_C}(M, R) \cong \mathrm{Ext}^1_R(M, R)$, since M and R belong to $\mathcal{A}_C(R)$. As the proof of item (i), we can get the exact sequence

$$0 \to M^* \to (P_0)^* \to (P_1)^* \to N \to 0,$$

where P_0 and P_1 are finitely generated projective R-modules. Since R is a generalized coherent ring, we get that N is super finitely presented and $N \in \mathcal{A}_C(R)$. Note that M, P_0, P_1, and N are reflexive, by assumption. Therefore, the exact sequence $P_1 \to P_0 \to M \to 0$ implies that $M \cong \mathrm{Coker}((P_1)^{**} \to (P_0)^{**}) = \mathrm{Coker}((P_1) \to (P_0))$. By [15, Lemma 2.2], we have the exact sequence $0 \to \mathrm{Ext}^1_R(M, R) \to N \to N^{**}$. Since N is reflexive, we get that $\mathrm{Ext}^1_R(M, R) = 0$, as desired. □

Corollary 3.6 *Let R be a self injective ring, and let C be a semidualizing R-module. Let $M \in \mathcal{B}_C(R)$ be a super finitely presented R-module such that M^* is super finitely presented R-module. Then M is reflexive, provided that R is a weak C-injective R-module.*

Proof Let $M \in \mathcal{B}_C(R)$ be a super finitely presented R-module. By [18, Proposition 3.3.16], we get that $M^* \in \mathcal{A}_C(R)$. So, we get the assertion by Theorem 3.5. □

Let M and F be R-modules. Consider the natural homomorphism

$$\eta_M : \mathrm{Hom}_R(M,R) \otimes_R F \to \mathrm{Hom}_R(M,F),$$

where $\eta_M(f \otimes z)(m) = f(m)z$ for $m \in M$, $z \in F$, and $f \in \mathrm{Hom}_R(M,R)$. By [18, Lemma A.1.2], η_M is an isomorphism whenever, M is finitely generated and projective or M is finitely presented and F is flat. In the following, we prove that η_M is an also isomorphism, whenever $F \in \mathcal{B}_C(R)$ is weak C-flat, provided some special conditions.

Proposition 3.7 *Let C be a semidualizing R-module, and let M be a super finitely presented R-module such that $M^* \in \mathcal{A}_C(R)$, and M^* is super finitely presented. Assume that $F \in \mathcal{B}_C(R)$ is weak C-flat. Then η_M is an isomorphism and $\mathrm{Ext}_R^{i>0}(M,F) = 0$, provided that $\mathrm{Ext}_R^{i>0}(M,R) = 0$.*

Proof Let M be a super finitely presented R-module. Then there exists an exact sequence $0 \to A \to P \to M \to 0$ where P is a finitely generated projective R-module and A is a super finitely presented R-module. Since $\mathrm{Ext}_R^{i>0}(M,R) = 0$, we get the exact sequence $0 \to M^* \to P^* \to A^* \to 0$. Hence A^* is a super finitely presented R-module, and $A^* \in \mathcal{A}_C(R)$ by [18, Proposition 3.1.7]. By assumption, $\mathrm{Tor}_1^{\mathcal{F}_C\mathcal{M}}(F,A^*) = 0$. By [17, Proposition 4.3], $\mathrm{Tor}_1^{\mathcal{F}_C\mathcal{M}}(F,A^*) \cong \mathrm{Tor}_1^R(F,A^*)$, and so we get the exact sequence $0 \to M^* \otimes_R F \to P^* \otimes_R F \to A^* \otimes_R F \to 0$. Consider the following diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \to & M^* \otimes_R F & \to & P^* \otimes_R F & \to & A^* \otimes_R F & \to & & & 0 \\
 & & \downarrow{\scriptstyle \eta_M} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \eta_A} & & & & \\
0 & \to & \mathrm{Hom}_R(M,F) & \to & \mathrm{Hom}_R(P,F) & \to & \mathrm{Hom}_R(A,F) & \to & \mathrm{Ext}_R^1(M,F) & \to & 0
\end{array}$$

Therefore, we get that η_M is a monomorphism. As the same argument, we get that η_A is a monomorphism, since A is a super finitely presented R-module such that $A^* \in \mathcal{A}_C(R)$ is super finitely presented. So, η_M is an isomorphism and $\mathrm{Ext}_R^1(M,F) = 0$. By shifting, we get that $\mathrm{Ext}_R^{i>0}(M,F) = 0$. □

Recall that an R-module M is called *Gorenstein projective* if there exists the complete projective resolution

$$\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

of R-modules such that $M \cong \mathrm{Ker}(P_0 \to P^0)$ and the functor $\mathrm{Hom}_R(-,Q)$ leaves the sequence exact whenever Q is a projective R-module. This notion is introduced in [8] as a generalization of projective modules. In the following, we show that if R is self weak C-injective, then every super finitely presented module $M \in \mathcal{A}_C(R)$ is Gorenstein projective, provided some special conditions.

Theorem 3.8 *Let C be a semidualizing R-module, and let R be a weak C-injective module. Then every super finitely presented module $M \in \mathcal{A}_C(R)$ is Gorenstein projective, provided that M^* is a super finitely presented module and $M^* \in \mathcal{A}_C(R)$.*

Proof Let $M \in \mathcal{A}_C(R)$ be a super finitely presented module such that $M^* \in \mathcal{A}_C(R)$ be a super finitely presented module. Then there exists the long exact sequence

$$\cdots \to F_n \to \cdots \to F_1 \to F_0 \to M^* \to 0$$

of R-modules such that F_i is finitely generated projective for each $i \geqslant 0$. For each $i \geqslant 0$, we set $K_i = \mathrm{Ker}(F_{i+1} \to F_i)$. Then K_i is a super finitely presented module and $K_i \in \mathcal{A}_C(R)$, by [18, Proposition 3.1.7]. By [21, Corollary 4.2], we have $\mathrm{Ext}_R^1(K_i,R) \cong \mathrm{Ext}^1_{\mathcal{MI}_C}(K_i,R) = 0$, since R is weak C-injective. Therefore, we have the long exact sequence

$$0 \to M^{**} \to F_0^* \to F_1^* \to \cdots \to F_n^* \to \cdots.$$

By Theorem 3.5, M is reflexive, so we have the long exact sequence

$$0 \to M \to F_0^* \to F_1^* \to \cdots \to F_n^* \to \cdots.$$

On the other hand, M is a super finitely presented R-module. Then there exists the long exact sequence

$$\cdots \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$$

of R-modules such that P_i is finitely generated projective for each $i \geqslant 0$. So we get the following complete projective resolution of M

$$\cdots \to P_n \to \cdots \to P_1 \to P_0 \to F_0^* \to F_1^* \to \cdots \to F_n^* \to \cdots,$$

where $M \cong \mathrm{Ker}(P_0 \to F_0^*)$. Assume that L is an arbitrary cosyzygy of this sequence. Then $L \in \mathcal{A}_C(R)$, and L is a super finitely presented R-module. Therefore, by [21, Corollary 4.2], we have $\mathrm{Ext}^1_R(L,R) \cong \mathrm{Ext}^1_{\mathcal{MI}_C}(L,R) = 0$, since R is weak C-injective. This means that for every projective R-module Q, $\mathrm{Hom}_R(-,Q)$ leaves this sequence exact. So, M is Gorenstein projective, as desired. □

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