



A parametric solution method for a generalized fractional programming problem

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Received: 8 October 2019 / Accepted: 3 September 2020 / Published online: 14 June 2021
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Abstract This paper proposes a parametric method for solving a generalized fractional programming problem which is called sum-of-ratios problem. The sum-of-ratios problems occur in many fields including computer vision, finance, engineering and management. Compared with other methods based on branch-and-bound procedure, our algorithm is based on Newton-like method for solving a system of nonlinear equations with parameters and it needs to solve convex programming problem in each iteration. We showed the global linear and local superlinear/quadratic rate of convergence of the algorithm. We demonstrated the practical efficiency of the algorithm by numerical experiments for various kinds of sum-of-ratios problem. In the numerical experiments, our method exhibited better solution quality and better convergence rate than other methods.

Keywords Fractional programming · Sum-of-ratios problem · Parametric convex programming approach

1 Introduction

The sum-of-ratios problem, which is to minimize (maximize) a sum of several fractional functions in convex set, is a non-convex optimization problem that is difficult to solve by the traditional optimization methods. The sum-of-ratios problem has attracted great interest over the 40 years for its practical and theoretical significance. The problem occurs in many important applications such as finite element method, computer graphics, finance, engineering and management [5, 6, 12, 15, 17]. In [12], many problems in projective geometry such as multi-view triangulation, camera resectioning and homography estimation were formulated as the sum-of-ratios problem and the branch-and-bound method, which is based on the recent developments in fractional programming and the theory of convex underestimators [2–5, 9, 14, 16, 17, 21, 24–26], was employed to find its global solution. In the method of [12], the number of variables increases as twice as the number of fractional functions and it requires to solve a second-order cone programming problem to obtain a lower bound of the optimal value in each iteration. In recent years, some algorithms have been developed for the global solution to these optimization problems

Communicated by T S S R K Rao, Phd.

This work was supported by the National Natural Science Foundation under the fourth Five Years Program for Development of Science and Technology of DPR Korea.

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[2, 3, 9, 11, 18, 19, 25]. Given any tolerance ε , if the optimization problem is feasible, the algorithms return a solution which is at most ε far from the global optimum. But the methods based on branch-and-bound procedure require a lot of computations and have low convergence rate. Furthermore, it is not easy to find a reasonable branching strategy.

In this paper, we propose an efficient algorithm which transforms the sum-of-ratios problem into parametric convex programming problem and solves the convex programming problem in each iteration. Unlike the methods based on the branch-and-bound, our method does not need to introduce new variables and constraints, and it also does not need to use additional complicated computations for obtaining the upper bounds and lower bounds for numerators and denominators of every ratio, so it is easier to implement than the previous methods. This paper is organized as follows. In Section 2, we show that the sum-of-ratios problem is equivalent to some kind of parametric convex programming problem. In Section 3, it is about a Newton-like algorithm for finding optimal parameters associated with the optimal solution of the sum-of-ratios problem and its convergence characteristics were amply considered. In Section 4, several test problems in the previous studies are used to show the advantage of our algorithm over the other methods, and the experiment results show that our algorithm has better performance than other methods. Finally, Section 5 provides some discussions.

2 Equivalent parametric convex programming

The sum-of-ratios problem is to minimize the sum of fractional functions subject to convex constraints, which is formulated as follows.

$$\min \sum_{i=1}^p F_i(x), \quad \text{s.t. } x \in X \quad (1)$$

where $F_i(x) = \frac{f_i(x)}{h_i(x)}$, $i = 1, \dots, p$, $X = \{x \in \mathbb{R}^n | g_i(x) \leq 0, i = 1, \dots, m\}$, and $f_i(x)$, $g_i(x)$ and $-h_i(x)$ are twice continuously differentiable convex functions.

It is assumed that the feasible set X is bounded, $f_i(x) \geq 0$ and $h_i(x) > 0$ for every $x \in X$, and that $\text{int}X = \{x \in \mathbb{R}^n | g_i(x) < 0, i = 1, \dots, m\} \neq \emptyset$ (Slater condition). Even with these restrictions the above problem is NP-complete [8].

It is easy to see that the problem (1) is equivalent to the following problem.

$$\begin{aligned} \min \sum_{i=1}^p \beta_i, \quad \text{s.t.} \\ F_i(x) \leq \beta_i, i = 1, \dots, p, \quad x \in X, \quad \beta \in \mathbb{R}^p \end{aligned} \quad (2)$$

Let $\beta = (\beta_1, \dots, \beta_p)$ and $\gamma = (\gamma_1, \dots, \gamma_p)$.

Theorem 1. *If $(\bar{x}, \bar{\beta})$ is the solution of the problem (2), then there exists $\bar{\gamma}$ such that \bar{x} is a solution of the following convex programming problem for fixed $\gamma = \bar{\gamma}$ and $\beta = \bar{\beta}$.*

$$\min \sum_{i=1}^p \gamma_i (f_i(x) - \beta_i h_i(x)), \quad \text{s.t. } x \in X \quad (3)$$

And \bar{x} also satisfies the following system of equations for $\gamma = \bar{\gamma}$ and $\beta = \bar{\beta}$:

$$\gamma_i = \frac{1}{h_i(x)}, i = 1, \dots, p \quad (4)$$

$$f_i(x) - \beta_i h_i(x) = 0, i = 1, \dots, p \quad (5)$$

Proof. The constraint $F_i(x) \leq \beta_i$ is equivalent to $f_i(x) - \beta_i h_i(x) \leq 0$. Let's define the following function for the problem (2).



$$L(x, \beta, w, \gamma, v) = w \sum_{i=1}^p \beta_i + \sum_{i=1}^p \gamma_i (f_i(x) - \beta_i h_i(x)) + \sum_{i=1}^m v_i g_i(x).$$

By Fritz-John optimality condition (Theorem 4.2.8 of [1]), there exist $\bar{w}, \bar{\gamma} = (\bar{\gamma}_1, \dots, \bar{\gamma}_p)$ and $\bar{v} = (\bar{v}_1, \dots, \bar{v}_m)$ such that

$$\frac{\partial L}{\partial x} = \sum_{i=1}^p \bar{\gamma}_i (\nabla f_i(\bar{x}) - \bar{\beta}_i \nabla h_i(\bar{x})) + \sum_{i=1}^m \bar{v}_i \nabla g_i(\bar{x}) = 0 \tag{6}$$

$$\frac{\partial L}{\partial \beta_i} = \bar{w} - \bar{\gamma}_i h_i(\bar{x}) = 0, i = 1, \dots, p \tag{7}$$

$$\bar{\gamma}_i \frac{\partial L}{\partial \gamma_i} = \bar{\gamma}_i (f_i(\bar{x}) - \bar{\beta}_i h_i(\bar{x})) = 0, i = 1, \dots, p \tag{8}$$

$$\bar{v}_i \frac{\partial L}{\partial v_i} = \bar{v}_i g_i(\bar{x}) = 0, i = 1, \dots, m \tag{9}$$

$$g_i(\bar{x}) \leq 0, \bar{v}_i \geq 0, i = 1, \dots, m \tag{10}$$

$$f_i(\bar{x}) - \bar{\beta}_i h_i(\bar{x}) \leq 0, \bar{\gamma}_i \geq 0, i = 1, \dots, p \tag{11}$$

$$\bar{w} \geq 0, (\bar{w}, \bar{\gamma}, \bar{v}) \neq (0, 0, 0) \tag{12}$$

Suppose that $\bar{w} = 0$. Then, by (7), we have $\bar{\gamma} = 0$ because $h_i(x) > 0, i = 1, \dots, p$ for all $x \in X$. Hence, it follows from (6), (9), (10) and (12) that

$$\sum_{i \in I(\bar{x})} \bar{v}_i \nabla g_i(\bar{x}) = 0, \tag{13}$$

$$\sum_{i \in I(\bar{x})} \bar{v}_i > 0, \bar{v}_i \geq 0, i \in I(\bar{x}), \tag{14}$$

where $I(\bar{x}) = \{i | g_i(\bar{x}) = 0, 1 \leq i \leq m\}$. By Slater condition, there exists a point x' such that

$$g_i(x') < 0, i = 1, \dots, m. \tag{15}$$

Since $g_i(x), i = 1, \dots, m$ are convex, it follows from (15) that

$$\nabla g_i(\bar{x})^T (x' - \bar{x}) \leq g_i(x') - g_i(\bar{x}) < 0, i \in I(\bar{x}) \tag{16}$$

Letting $d = x' - \bar{x}$, from (16) and (14), we have $\left(\sum_{i \in I(\bar{x})} \bar{v}_i \nabla g_i(\bar{x}) \right)^T d < 0$, which contradicts (13). Thus, we have $\bar{w} > 0$.

Denoting $\frac{\bar{w}}{\bar{w}}$ and $\frac{\bar{v}_i}{\bar{w}}$ by $\bar{\gamma}$ and \bar{v} , respectively, we see that (7) is equivalent to (4) and so (8) is equivalent to (5) because $\bar{\gamma}_i > 0, i = 1, \dots, p$ by (4).

Given $\gamma = \bar{\gamma}$ and $\beta = \bar{\beta}$, (6), (9) and (10) is just the KKT condition for the problem (3). Since the problem (3) is convex programming for parameters $\gamma > 0$ and $\beta \geq 0$, the KKT condition is also sufficient optimality condition and then \bar{x} is the solution of (3) for $\gamma = \bar{\gamma}$ and $\beta = \bar{\beta}$. \square

Let $\alpha = (\beta, \gamma)$ denote parameter vector and let

$$\Omega = \{ \alpha = (\beta, \gamma) \in R^{2p} | 0 \leq \beta \leq \beta^u, \gamma^l \leq \gamma \leq \gamma^u \},$$

where



$$\beta^u = (\beta_1^u, \dots, \beta_p^u), \gamma^l = (\gamma_1^l, \dots, \gamma_p^l), \gamma^u = (\gamma_1^u, \dots, \gamma_p^u),$$

$$\beta_i^u = \max_{x \in X} \frac{f_i(x)}{h_i(x)}, \gamma_i^u = \max_{x \in X} \frac{1}{h_i(x)},$$

$$\gamma_i^l = \min_{x \in X} \frac{1}{h_i(x)}, i = 1, \dots, p.$$

and it is obvious that $\gamma_i^l > 0, i = 1, \dots, p$.

Remark 1.

- (i) Since the feasible set X is bounded by the assumption, the β^u, γ^l and γ^u are well defined, and the set Ω is bounded.
- (ii) The β^u, γ^l and γ^u involved in the definition of Ω are only necessary for theoretical consideration but in solving the problem (1) at all.

Let $x(\alpha)$ be the solution of the problem (3) for fixed $\alpha \in \Omega$ and let

$$\psi_i^1(\alpha) \equiv \phi_i(x(\alpha), \alpha) = -f_i(x(\alpha)) + \beta_i h_i(x(\alpha)), i = 1, \dots, p, \tag{17}$$

$$\psi_i^2(\alpha) \equiv \phi_{p+i}(x(\alpha), \alpha) = -1 + \gamma_i h_i(x(\alpha)), i = 1, \dots, p \tag{18}$$

Let

$$\hat{\Omega} = \{\alpha \in \Omega \mid \psi_i^1(\alpha) = 0, \psi_i^2(\alpha) = 0, i = 1, \dots, p\}.$$

and consider the following system of nonlinear equations:

$$\psi_i^1(\alpha) = 0, i = 1, \dots, p, \tag{19}$$

$$\psi_i^2(\alpha) = 0, i = 1, \dots, p \tag{20}$$

Corollary 1. If \bar{x} is the solution for the problem (1), then there exists $\bar{\alpha} \in \Omega$ such that $\bar{x} = x(\bar{\alpha})$ and $\bar{\alpha} \in \hat{\Omega}$.

Proof. The proof follows from Theorem 1 because (5) and (4) implies (19) and (20), respectively. \square

Corollary 2. If the problem (1) has a solution \bar{x} , then $\hat{\Omega} \neq \emptyset$. And if $\hat{\Omega}$ consists of only one point α^* , i.e. $\hat{\Omega} = \{\alpha^*\}$, then $x(\alpha^*)$ is the solution of the problem (1).

Proof. It follows from Corollary 1 that $\hat{\Omega} \neq \emptyset$. If $\hat{\Omega} = \{\alpha^*\}$, the $x(\alpha^*)$ is the solution for the problem (1) because $\bar{x} = x(\alpha^*)$ by Corollary 1. \square

It follows from Corollary 2 that the problem (1) is reduced to find the solution $x(\alpha)$ of the problem (3) satisfying (19) and (20) for parameter vector α . If $\hat{\Omega} = \{\alpha^*\}$, then solving the problem (1) is equivalent to finding the $\alpha^* \in \hat{\Omega}$.

In the next section, we establish the uniqueness of $\hat{\Omega}$ under some assumptions and propose an algorithm to find the unique element of $\hat{\Omega}$, i.e. the global optimal solution for the problem (1).

3 Algorithm and its convergence

In this section, first, we consider the system of equations to be satisfied by the parameters corresponding to the optimal solution of the problem (1) in Lemma 2 and Lemma 3. We show the existence and uniqueness of the solution to the system (Theorem 2, Corollary 3) and then construct a Newton-like algorithm ((46) or (59)) to solve the system. Second, we prove the global linear and local superlinear/quadratic convergence of the algorithm (Corollary 4, Theorem 3). Finally, we show that the solution of the convex programming problem (3) corresponding to the optimal parameter vector, which is any accumulation point of the sequence generated by the algorithm, is the solution of the original problem (1) under some assumptions (Theorem 5). Let

$$\psi_i(\alpha) = \psi_i^1(\alpha), \quad \psi_{p+i}(\alpha) = \psi_i^2(\alpha), i = 1, \dots, p. \tag{21}$$

Then (19) and (20) can be rewritten as



$$\psi(\alpha) = 0 \tag{22}$$

Now, let's derive Jacobian matrix $\psi'(\alpha)$ to construct a Newton-like algorithm. If $x(\alpha)$ is differentiable, we have the followings by (17) and (18):

$$\frac{\partial \psi_i(\alpha)}{\partial \alpha_j} = \nabla_x \phi_i^T \frac{\partial x(\alpha)}{\partial \alpha_j} + \frac{\partial \phi_i(\alpha)}{\partial \alpha_j}, i = 1, \dots, 2p, j = 1, \dots, 2p \tag{23}$$

$$\frac{\partial \phi_i(\alpha)}{\partial \alpha_j} = \begin{cases} h_i(x(\alpha)) & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \tag{24}$$

$$\frac{\partial \phi_{p+i}(\alpha)}{\partial \alpha_j} = \begin{cases} h_i(x(\alpha)) & \text{if } p + i = j, \\ 0 & \text{if } p + i \neq j \end{cases} \tag{25}$$

for $i = 1, \dots, p, j = 1, \dots, 2p$. Therefore, defining A and B as

$$A = \left(\nabla_x \phi_i^T \frac{\partial x(\alpha)}{\partial \alpha_j} \right)_{i,j=1,\dots,2p}, \tag{26}$$

$$B = \begin{pmatrix} \text{diag}(h_i(x(\alpha)))_{i=1,\dots,p} & 0 \\ 0 & \text{diag}(h_i(x(\alpha)))_{i=1,\dots,p} \end{pmatrix},$$

we have

$$\psi'(\alpha) = A + B. \tag{27}$$

To compute A, we need to find derivatives of $x(\alpha)$. Since $x(\alpha)$ is the optimal solution of the problem (3) for given $\alpha = (\beta, \gamma)$, there exists a Lagrange multiplier vector $v(\alpha) \geq 0$ such that $x(\alpha)$ and $v(\alpha)$ satisfy the following equations:

$$\sum_{i=1}^p \gamma_i (\nabla f_i(x(\alpha)) - \beta_i \nabla h_i(x(\alpha))) + \sum_{i=1}^m v_i(\alpha) \nabla g_i(x(\alpha)) = 0,$$

$$v_i(\alpha) g_i(x(\alpha)) = 0, i = 1, \dots, m.$$

letting $F(x, \alpha) = \sum_{i=1}^p \gamma_i (f_i(x) - \beta_i h_i(x)), x \in X, \alpha = (\beta, \gamma), \nabla G(x(\alpha)) = (\nabla g_1(x(\alpha)), \dots, \nabla g_m(x(\alpha)))$, we have

$$\nabla_x F(x(\alpha), \alpha) + \nabla G(x(\alpha)) v(\alpha) = 0, \tag{28}$$

$$v_i(\alpha) g_i(x(\alpha)) = 0, i = 1, \dots, m \tag{29}$$

Let

$$I(x) = \{i | g_i(x) = 0, 1 \leq i \leq m\} = \{i_1, \dots, i_l\},$$

$$G^1(x) = (g_{i_1}(x), \dots, g_{i_l}(x))^T \text{ and } v^1 = (v_{i_1}, \dots, v_{i_l})^T.$$

Then (28) and (29) can be written by

$$\nabla_x F(x(\alpha), \alpha) + \nabla G^1(x(\alpha)) v^1(\alpha) = 0, \tag{30}$$

$$G^1(x(\alpha)) = 0. \tag{31}$$

By differentiating (30) and (31) with variables $\alpha_j, j = 1, \dots, 2p$, we have

$$[\nabla_x^2 F + (\nabla^2 G^1)^T v^1] \frac{\partial x}{\partial \alpha_j} + \nabla G^1 \frac{\partial v^1}{\partial \alpha_j}$$

$$= - \frac{\partial^2 F}{\partial x \partial \alpha_j} - \left(\frac{\partial^2 G^1}{\partial x \partial \alpha_j} \right)^T v^1, \tag{32}$$



$$(\nabla G^1)^T \frac{\partial x}{\partial \alpha_j} = -\frac{\partial G^1}{\partial \alpha_j}, \tag{33}$$

where

$$(\nabla^2 G^1)^T v^1 = \sum_{i \in I(x(\alpha))} v_i^1 \nabla_x^2 g_i(x(\alpha)). \tag{34}$$

Since $g_i(x), i = 1, \dots, m$ don't involve any parameters, we have

$$\frac{\partial G^1}{\partial \alpha_j} = 0, \frac{\partial^2 G^1}{\partial x \partial \alpha_j} = 0, j = 1, \dots, 2p. \tag{35}$$

Therefore, by denoting

$$\begin{aligned} y &= \begin{pmatrix} x \\ v^1 \end{pmatrix}, \\ \Delta y^j &= \begin{pmatrix} \frac{\partial x}{\partial \alpha_j} \\ \frac{\partial v^1}{\partial \alpha_j} \end{pmatrix}, \\ J(\alpha) &= \begin{pmatrix} \nabla_x^2 F + (\nabla_x^2 G^1)^T v^1 & \nabla G^1 \\ (\nabla G^1)^T & 0 \end{pmatrix}, \\ J_j(\alpha) &= -\begin{pmatrix} \frac{\partial^2 F}{\partial x \partial \alpha_j} \\ 0 \end{pmatrix}, \end{aligned}$$

we can write (32) as

$$J(\alpha) \Delta y^j(\alpha) = J_j(\alpha), j = 1, \dots, 2p.$$

Hence, if $J(\alpha)$ is nonsingular, we have

$$\Delta y^j(\alpha) = \begin{pmatrix} \frac{\partial x(\alpha)}{\partial \alpha_j} \\ \frac{\partial v^1(\alpha)}{\partial \alpha_j} \end{pmatrix} = J^{-1}(\alpha) J_j(\alpha), j = 1, \dots, 2p. \tag{36}$$

Thus, we have the following result.

Lemma 1. *If $\nabla_x^2 F + (\nabla_x^2 G^1)^T v^1$ is nonsingular, we have for every $j = 1, \dots, 2p$*

$$\begin{aligned} \frac{\partial x(\alpha)}{\partial \alpha_j} &= [\nabla_x^2 F \\ &\quad + (\nabla_x^2 G^1)^T v^1]^{-1} \left\{ \nabla G^1 M^{-1} (\nabla G^1)^T [\nabla_x^2 F + (\nabla_x^2 G^1)^T v^1]^{-1} - I \right\} \\ &\quad \frac{\partial^2 F}{\partial x \partial \alpha_j}, \end{aligned} \tag{37}$$

where $M = (\nabla G^1)^T [\nabla_x^2 F + (\nabla_x^2 G^1)^T v^1]^{-1} \nabla G^1$.

Proof. If $\nabla_x^2 F + (\nabla_x^2 G^1)^T v^1$ is nonsingular, we have

$$\frac{\partial x(\alpha)}{\partial \alpha_j} = -[\nabla_x^2 F + (\nabla_x^2 G^1)^T v^1]^{-1} \left[\nabla G^1 \frac{\partial v^1}{\partial \alpha_j} + \frac{\partial^2 F}{\partial x \partial \alpha_j} \right] \tag{38}$$

by (32) and (35). By substituting (38) in (33) and taking (35) into account, we have

$$(\nabla G^1)^T [\nabla_x^2 F + (\nabla_x^2 G^1)^T v^1]^{-1} \left[\nabla G^1 \frac{\partial v^1}{\partial \alpha_j} + \frac{\partial^2 F}{\partial x \partial \alpha_j} \right] = 0.$$

Therefore, we have



$$\frac{\partial v^1}{\partial \alpha_j} = -M^{-1}(\nabla G^1)^T[\nabla_x^2 F + (\nabla_x^2 G^1)^T v^1]^{-1} \frac{\partial^2 F}{\partial x \partial \alpha_j}. \tag{39}$$

By substituting (39) in (38), we have (37). □

Remark 2. Since $\gamma > 0$ and $\beta \geq 0$ for every $\alpha = (\beta, \gamma) \in \Omega$, if there exists a $k \in \{1, \dots, p\}$ such that $f_k(x)$ is strictly convex, then $\nabla_x^2 F$ is positive definite and $\nabla^2 g_i(x), i \in I(x)$ are positive semidefinite because of the convexity of $f_i(x), -h_i(x), i = 1, \dots, p$ and $g_i(x), i = 1, \dots, m$. It follows from (34) that $\nabla_x^2 F + (\nabla_x^2 G^1)^T v^1$ is also positive definite, which shows that (38) is well-defined.

In the above, we exploited the differentiability of $x(\alpha)$ and $v(\alpha)$. Now, for the differentiability of $x(\alpha)$ and $v(\alpha)$, it needs the following assumption.

(H1) In the problem (3) for each $\alpha \in \Omega$, LICQ(linear independence constraint qualification), second-order sufficiency condition and strict complementarity condition are satisfied at $x(\alpha)$

Remark 3. The assumption (H1) is often required when applying the classical implicit function theorem to obtain differentiability of the optimal solution in parametric optimization problem. Under the assumption (H1), we have the existence and the uniqueness of differentiable $x(\alpha)$ and $v(\alpha)$ satisfying (28) and (29) and have the non-singularity of the matrix $J(\alpha)$ (Theorem 2.1 of [7]). Therefore, it follows that the matrix M defined in Lemma 1 is nonsingular.

For the sake of convenient analysis for a Newton-like algorithm to be presented below, the following assumption is needed.

(H2) ∇G^1 is nonsingular for each $\alpha \in \Omega$, i.e. $|I(x(\alpha))| = n$ for each $\alpha \in \Omega$.

Remark 4. The assumption (H2) implies that $\frac{\partial x}{\partial \alpha_j} = 0, j = 1, \dots, 2p$ from (33) and (35). Therefore, it follows from (26) and (27) that $\psi'(\alpha) = B$. As we will show below, the assumption (H1) and (H2) is a sufficient condition under which the system $\psi(\alpha) = 0$ has a unique solution and Newton method for the system always finds the global minimizer of the initial problem. It may be possible to apply the Newton method for the problems which don't satisfy the assumption (H1) and (H2).

Lemma 2. Under the assumption (H1) and (H2), $\psi(\alpha)$ is strongly monotone with constant $\delta > 0$ in Ω , where

$$\delta = \min_i \delta_i, \delta_i = \min_{x \in X} h_i(x), i = 1, \dots, p$$

Proof. By (26) and Remark 4, the Jacobian matrix of $\psi(\alpha)$ is as follows.

$$\psi'(\alpha) = \begin{pmatrix} h_1(x(\alpha)) & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & h_2(x(\alpha)) & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & h_N(x(\alpha)) & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & h_1(x(\alpha)) & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & h_2(x(\alpha)) & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & h_N(x(\alpha)) \end{pmatrix} \tag{40}$$

Then $\psi'(\alpha)$ is positive definite because $x(\alpha) \in X$ and $h_i(x(\alpha)) > 0, i = 1, \dots, p$. Therefore, for any $z \in R^{2p}$, we have



$$\begin{aligned} z^T \psi'(\alpha) z &= \sum_{i=1}^p z_i^2 h_i(x(\alpha)) + \sum_{i=1}^p z_{i+p}^2 h_i(x(\alpha)) \\ &= \sum_{i=1}^p (z_i^2 + z_{i+p}^2) h_i(x(\alpha)) \geq \sum_{i=1}^p (z_i^2 + z_{i+p}^2) \delta_i \\ &\geq \delta \sum_{i=1}^{2p} z_i^2 = \delta \|z\|^2, \end{aligned}$$

which completes the proof. □

Let

$$\begin{aligned} A_i(\alpha) &= \frac{f_i(x(\alpha))}{h_i(x(\alpha))}, \quad A_{p+i}(\alpha) = \frac{1}{h_i(x(\alpha))}, \\ i &= 1, \dots, p, A(\alpha) = (A_1(\alpha)), \\ &\dots, A_p(\alpha), A_{p+1}(\alpha), \dots, A_{2p}(\alpha) \end{aligned}$$

Lemma 3. *The equation $\psi(\alpha) = 0$ is equivalent to the equation $\alpha = A(\alpha)$. If the problem (1) has an optimal solution, the equation (22) has at least one solution in Ω .*

Proof. The first proposition is obvious from the definition of $A(\alpha)$ and $\psi(\alpha)$. It follows from Corollary 1 that there is $\alpha \in \Omega$ such that $\psi(\alpha) = 0$. □

Let's introduce a mapping $B_\lambda(\alpha) = \pi_\Omega(\alpha - \lambda\psi(\alpha))$ to establish the existence and uniqueness of the solution of (22), where λ is a positive constant and $\pi_\Omega(a)$ denotes the projection of a onto Ω , i.e. $\pi_\Omega(a)$ is the solution of the following problem:

$$\min \|x - a\|^2, \quad \text{s.t. } x \in \Omega$$

Lemma 4. *For all $a \in R^p$ and for all $x \in \Omega$,*

$$(\pi_\Omega(a) - a)^T (x - \pi_\Omega(a)) \geq 0, \tag{41}$$

$$\text{and for all } a, b \in R^p \|\pi_\Omega(a) - \pi_\Omega(b)\| \leq \|a - b\| \tag{42}$$

Proof. See Kinderlehrer and Stampcchia [13]. □

Lemma 5. *Assume that $\psi(\alpha)$ is differentiable, and satisfies (40) and Lipschitz condition with the constant M in Ω . Then $B_\lambda : \Omega \rightarrow \Omega$ is a contractive mapping for all $\lambda \in (0, 2\delta/M^2)$.*

Proof. Since $\psi(\alpha)$ is strongly monotone with the constant δ by Lemma 2,

$$(\psi(\alpha') - \psi(\alpha))^T (\alpha' - \alpha) \geq \delta \|\alpha' - \alpha\|^2. \tag{43}$$

By (42), (43) and the Lipschitz condition, we have



$$\begin{aligned}
 \|B_\lambda(\alpha') - B_\lambda(\alpha)\|^2 &= \|\pi_\Omega(\alpha' - \lambda\psi(\alpha')) \\
 &\quad - \pi_\Omega(\alpha - \lambda\psi(\alpha))\|^2 \\
 &\leq \|\alpha' - \lambda\psi(\alpha') - (\alpha - \lambda\psi(\alpha))\|^2 = \\
 &\|\alpha' - \alpha - \lambda(\psi(\alpha') - \psi(\alpha))\|^2 \\
 &= \|\alpha' - \alpha\|^2 - 2\lambda(\alpha' - \alpha)^T(\psi(\alpha') - \psi(\alpha)) \\
 &\quad + \lambda^2\|\psi(\alpha') - \psi(\alpha)\|^2 \\
 &\leq \|\alpha' - \alpha\|^2 - 2\lambda\delta\|\alpha' - \alpha\|^2 \\
 &\quad + \lambda^2\|\psi(\alpha') - \psi(\alpha)\|^2 \\
 &\leq (1 - 2\lambda\delta + \lambda^2M^2)\|\alpha' - \alpha\|^2,
 \end{aligned}$$

which implies that

$$\|B_\lambda(\alpha') - B_\lambda(\alpha)\| \leq q\|\alpha' - \alpha\|$$

for all $\lambda \in (0, 2\delta/M^2)$, where $q = \sqrt{1 - 2\lambda\delta + \lambda^2M^2} < 1$. □

Theorem 2. Assume that the problem (1) has an optimal solution. Suppose that $\psi(\alpha)$ is differentiable, and satisfies (40) and the Lipschitz condition in Ω . The equation $\alpha = B_\lambda(\alpha)$ for all $\lambda \in (0, 2\delta/M^2)$ is equivalent to the equation $\psi(\alpha) = 0$ and the equation (22) has a unique solution.

Proof. By Lemma 5 and the contractive mapping principle, $B_\lambda(\alpha)$ has only one fixed point α^* in Ω for all $\lambda \in (0, 2\delta/M^2)$, i.e. $B_\lambda(\alpha^*) = \alpha^*$. From (41), we have

$$\begin{aligned}
 &[\pi_\Omega(\alpha^* - \lambda\psi(\alpha^*)) - (\alpha^* - \lambda\psi(\alpha^*))]^T[\alpha - \pi_\Omega(\alpha^* - \lambda\psi(\alpha^*))] = \\
 &= [\alpha^* - (\alpha^* - \lambda\psi(\alpha^*))]^T(\alpha - \alpha^*) \geq 0
 \end{aligned}$$

for $\alpha \in \Omega$, i.e. for every $\alpha \in$

$$\Omega, \text{ we have } \psi(\alpha^*)^T(\alpha - \alpha^*) \geq 0. \tag{44}$$

Since the problem (1) has an optimal solution, there is a $\bar{\alpha} \in \Omega$ such that $\psi(\bar{\alpha}) = 0$ by Lemma 3. Then, it follows from (44) that

$$[\psi(\alpha^*) - \psi(\bar{\alpha})]^T(\bar{\alpha} - \alpha^*) \geq 0. \tag{45}$$

By Lemma 2, $\psi(\alpha)$ is strongly monotone with the constant δ and so we have

$$[\psi(\alpha^*) - \psi(\bar{\alpha})]^T(\alpha^* - \bar{\alpha}) \geq \delta\|\alpha^* - \bar{\alpha}\|^2.$$

This inequality and (45) together implies $0 \leq [\psi(\alpha^*) - \psi(\bar{\alpha})]^T(\bar{\alpha} - \alpha^*) \leq -\delta\|\alpha^* - \bar{\alpha}\|^2$. Therefore, $\|\alpha^* - \bar{\alpha}\| = 0$, i.e. $\alpha^* = \bar{\alpha}$. Thus, $\psi(\alpha^*) = 0$, which means that α^* is a solution of (22).

By the definition of $B_\lambda(\alpha)$, it is obvious that $\alpha \in \Omega$ with $\psi(\alpha) = 0$ is also the solution of $\alpha = B_\lambda(\alpha)$.

Suppose that $\alpha' \in \Omega$ is a solution of (22) that is different from α^* . Then, $\alpha' = B_\lambda(\alpha')$, which is contradiction because α^* is an unique fixed point of $B_\lambda(\alpha)$ in Ω . Thus, α^* is a unique solution of the equation (22). □

Corollary 3. Suppose that the assumptions of Theorem 2 are satisfied. Then, the equation $\alpha = A(\alpha)$ has an unique solution which is the unique solution of the equation (22).

Proof. By Lemma 3, $\alpha = A(\alpha)$ is equivalent to $\psi(\alpha) = 0$ which is in turn equivalent to $\alpha = B_\lambda(\alpha)$ by Theorem 2. Therefore, it follows from Theorem 2 that equation $\alpha = A(\alpha)$ has a unique solution which is the unique solution of the equation (22). □

As shown in Lemma 3, α is a fixed point of $A(\alpha)$ if and only if α is a root of $\psi(\alpha)$. We can construct the following fixed-point iteration to find a fixed point of $A(\alpha)$.



$$\begin{aligned}\beta^{k+1} &= \left(\frac{f_1(x^k)}{h_1(x^k)}, \dots, \frac{f_p(x^k)}{h_p(x^k)} \right)^T, \\ \gamma^{k+1} &= \left(\frac{1}{h_1(x^k)}, \dots, \frac{1}{h_p(x^k)} \right)^T,\end{aligned}\quad (46)$$

where $x^k = x(\alpha^k)$.

Corollary 4. *Suppose that the assumptions of Theorem 2 are satisfied. Then the algorithm (46) is just the Newton method to solve the equation (22) and its local rate of convergence is superlinear or quadratic.*

Proof. The Newton method for the equation (22) is as following.

$$\alpha^{k+1} = \alpha^k - [\psi'(\alpha^k)]^{-1} \psi(\alpha^k) \quad (47)$$

By (40), (17) and (18), the right-hand side of (47) is equal to $A(\alpha^k)$, i.e. the right-hand side of (46). Therefore, (47) means $\alpha^{k+1} = A(\alpha^k)$, that is, the fixed-point iteration method to find a fixed point of $A(\alpha)$ is just the Newton method for solving the equation (22). Hence, the algorithm (46) has local superlinear or quadratic convergence rate. \square

Theorem 3. *Suppose that the assumptions of Theorem 2 are satisfied and that there exists $L > 0$ such that for every $\alpha, \alpha' \in \Omega$*

$$\|\psi'(\alpha) - \psi'(\alpha')\| \leq L \|\alpha - \alpha'\|, \quad (48)$$

and that there exists $\tilde{M} > 0$ such that for every $\alpha \in \Omega$

$$\|[\psi'(\alpha)]^{-1}\| \leq \tilde{M}. \quad (49)$$

Let

$$\alpha^{k+1} = \alpha^k + \lambda_k d^k, \quad d^k = -[\psi'(\alpha^k)]^{-1} \psi(\alpha^k), \quad (50)$$

where λ_k is the greatest ξ^i satisfying

$$\begin{aligned}\|\psi(\alpha^k) \\ + \xi^i d^k\| \leq (1 - \varepsilon \xi^i) \|\psi(\alpha^k)\|\end{aligned}\quad (51)$$

and $i \in \{0, 1, 2, \dots\}$, $\xi \in (0, 1)$, $\varepsilon \in (0, 1)$. Then, the modified Newton method defined by (50) and (51) converges to the unique solution α^* of $\psi(\alpha) = 0$ with linear rate for any starting point $\alpha^0 \in \Omega$ and the rate in the neighborhood of the solution is quadratic.

Proof. We have already shown the existence and uniqueness of the solution $\alpha \in \Omega$ for the equation $\psi(\alpha) = 0$. If there is k such that $\psi(\alpha^k) = 0$, then α^k is a solution. So, it is assumed that $\psi(\alpha^k) \neq 0$ for every k . For $\lambda \in [0, 1]$, we have the following by the Newton-Leibniz formula and (48).

$$\begin{aligned}\|\psi(\alpha^k + \lambda d^k)\| &= \left\| \psi(\alpha^k) + \lambda \int_0^1 \psi'(\alpha^k + \theta \lambda d^k) d^k d\theta \right\| \\ &= \left\| \psi(\alpha^k) + \lambda \int_0^1 [\psi'(\alpha^k + \theta \lambda d^k) - \psi'(\alpha^k)] d^k d\theta - \lambda \psi(\alpha^k) \right\| \\ &\leq (1 - \lambda) \|\psi(\alpha^k)\| + \lambda^2 L \|d^k\|^2,\end{aligned}\quad (52)$$

where

$$\|\psi'(\alpha^k + \theta \lambda d^k) - \psi'(\alpha^k)\| \leq L \theta \lambda \|d^k\|$$

by the Lipschitz condition, and $\psi'(\alpha^k) d^k = -\psi(\alpha^k)$ by (50). In view of (49), it follows from (50) and (52) that

$$\|\psi(\alpha^k + \lambda d^k)\| \leq \left[1 - \lambda(1 - \lambda L \tilde{M}^2) \|\psi(\alpha^k)\| \right] \|\psi(\alpha^k)\|.$$

Letting $\bar{\lambda}_k = \frac{1-\varepsilon}{L \tilde{M}^2 \|\psi(\alpha^k)\|}$, we have

$$\|\psi(\alpha^k + \lambda d^k)\| \leq (1 - \varepsilon \lambda) \|\psi(\alpha^k)\| \tag{53}$$

for every $\lambda \in (0, \min\{1, \bar{\lambda}_k\})$. Then, (51), the definition of λ_k , implies that

$$\lambda_k \geq \min\{1, \xi \bar{\lambda}_k\}. \tag{54}$$

Since it follows from (53) that

$$\|\psi(\alpha^{k+1})\| \leq (1 - \varepsilon \lambda_k) \|\psi(\alpha^k)\|, \tag{55}$$

$\{\|\psi(\alpha^k)\|\}$ is a monotonically decreasing sequence and so $\{\bar{\lambda}_k\}$ increases monotonically. From (54), it follows that

$$1 - \varepsilon \lambda_k \leq 1 - \varepsilon \min\{1, \xi \bar{\lambda}_k\} \leq 1 - \varepsilon \min\{1, \xi \bar{\lambda}_0\}.$$

Letting $q = 1 - \varepsilon \min\{1, \xi \bar{\lambda}_0\}$, then $q < 1$ and by (55) we have

$$\|\psi(\alpha^k)\| \leq q^k \|\psi(\alpha^0)\|. \tag{56}$$

Therefore, $\{\|\psi(\alpha^k)\|\}$ converges to zero with linear rate, which means that $\{\alpha^k\}$ converges to a solution of $\psi(\alpha) = 0$ for any starting point $\alpha^0 \in \Omega$. Since $\|\psi(\alpha^k)\| \rightarrow 0$ as $k \rightarrow \infty$, we have $\bar{\lambda}_k \rightarrow \infty$ as $k \rightarrow \infty$ and so, by (54), there exists k_0 such that $\lambda_k = 1$ for every $k \geq k_0$. Thus, in this case, (50) becomes the Newton method and the rate of convergence in the neighborhood of the solution is quadratic by the Lipschitz property (48). \square

Let us consider sufficient condition under which the assumptions of Theorem 3 are satisfied.

Theorem 4. *Suppose that $\psi(\alpha)$ is differentiable and satisfies (40) in Ω . If $h_i(x), i = 1, \dots, p$ and $x(\alpha)$ are Lipschitz continuous in the feasible set X and Ω , respectively, then $\psi'(\alpha)$ is Lipschitz continuous and $[\psi'(\alpha)]^{-1}$ is bounded in Ω .*

Proof. By the Lipschitz continuity of $h_i(x)$ and $x(\alpha)$, there is a $L_i > 0$ and $C > 0$ such that, for all $\alpha, \alpha' \in \Omega$,

$$|h_i(x(\alpha)) - h_i(x(\alpha'))| \leq L_i |x(\alpha) - x(\alpha')|, \tag{57}$$

$$|x(\alpha) - x(\alpha')| \leq C |\alpha - \alpha'| \tag{58}$$

Since, by (40), we have

$$\psi'(\alpha) - \psi'(\alpha') = \begin{bmatrix} \text{diag}(h_i(x(\alpha)) - h_i(x(\alpha'))) & 0 \\ 0 & \text{diag}(h_i(x(\alpha)) - h_i(x(\alpha'))) \end{bmatrix},$$

there exists a constant $L > 0$ by (57) and (58) such that $|\psi'(\alpha) - \psi'(\alpha')| \leq L |\alpha - \alpha'|$, which means the Lipschitz continuity of $\psi'(\alpha)$.

It is easy to see

$$[\psi'(\alpha)]^{-1} = \begin{bmatrix} \text{diag}\left(\frac{1}{h_i(x(\alpha))}\right) & 0 \\ 0 & \text{diag}\left(\frac{1}{h_i(x(\alpha))}\right) \end{bmatrix}.$$

Thus, from $\frac{1}{h_i(x(\alpha))} \leq \gamma_i^u, i = 1, \dots, p$ and Remark 1, it follows that there exists a $\tilde{M} > 0$ such that for every $\alpha \in \Omega, \quad \|[\psi'(\alpha)]^{-1}\| \leq \tilde{M}$, that is, (49) is satisfied. \square

Under the assumption of Theorem 4, the modified Newton method defined by

$$\alpha^{k+1} = \alpha^k - \lambda_k [\psi'(\alpha^k)]^{-1} \psi(\alpha^k)$$

can be rewritten component-wise as following.



$$\begin{cases} \beta_i^{k+1} = \beta_i^k - \frac{\lambda_k}{h_i(x^k)} [\beta_i^k h_i(x^k) - f_i(x^k)] = (1 - \lambda_k)\beta_i^k + \lambda_k \frac{f_i(x^k)}{h_i(x^k)}, & i = 1, \dots, p \\ \gamma_i^{k+1} = \gamma_i^k - \frac{\lambda_k}{h_i(x^k)} [\gamma_i^k h_i(x^k) - 1] = (1 - \lambda_k)\gamma_i^k + \lambda_k \frac{1}{h_i(x^k)}, & i = 1, \dots, p \end{cases} \tag{59}$$

On the basis of the above consideration, we construct an algorithm to find global solution of the problem (1) under the assumptions of Theorem 4 as following.

[Algorithm MN] Step 0. Choose $\zeta \in (0, 1)$, $\varepsilon \in (0, 1)$ and $y^0 \in X$. Let

$$\begin{aligned} \beta_i^0 &= \frac{f_i(y^0)}{h_i(y^0)}, & \gamma_i^0 &= \frac{1}{h_i(y^0)}, & i &= 1, \dots, p, & k &= 0, \\ \beta^k &= (\beta_1^k, \dots, \beta_p^k), & \gamma^k &= (\gamma_1^k, \dots, \gamma_p^k), & \alpha^k &= (\beta^k, \gamma^k). \end{aligned}$$

Step 1. Find a solution $x^k = x(\alpha^k)$ of the problem (3) for $\alpha = \alpha^k$.

Step 2. If $\psi(\alpha^k) = 0$, then x^k is a global solution and so stop the algorithm. Otherwise, let i_k denote the smallest integer among $i \in \{0, 1, 2, \dots\}$ satisfying

$$|\psi(\alpha^k + \zeta^i d^k)| \leq (1 - \varepsilon \zeta^i) |\psi(\alpha^k)|$$

and let $\lambda_k = \zeta^{i_k}$, where

$$\begin{aligned} d^k &= (d_1^k, \dots, d_p^k, d_{p+1}^k, \dots, d_{2p}^k), & d_i^k &= -\frac{\psi_i(\alpha^k)}{h_i(x^k)}, \\ d_{p+i}^k &= -\frac{\psi_{p+i}(\alpha^k)}{h_i(x^k)}, & i &= 1, \dots, p. \end{aligned}$$

Compute $\alpha^{k+1} = \alpha^k + \lambda_k d^k$ which is equal to (59).

Step 3. Let $k = k + 1$ and go to step 1. □

Remark 5. Under the assumptions of Theorem 4, the step 2 of Algorithm MN is just the modified Newton method (50), (51). If the stepsize $\lambda_k = \zeta^{i_k}$ in the step 2 is replaced by $\lambda_k = 1$, the step 2 is just the Newton method (47) or (46), and so we denote Algorithm MN as Algorithm N in this case. As shown in Theorem 3, Algorithm MN has global linear and local quadratic rate of convergence.

Remark 6. If the assumption (H1) is satisfied, we can construct a Newton-like algorithm similar to Algorithm MN without the assumption (H2). In this case, the Newton direction in the step 2 is determined by $d^k = -(\psi'(\alpha^k))^{-1} \psi(\alpha^k)$ using the formula (26), (27) and (37).

Even without differentiability of $\psi(\alpha)$, we obtain the following result.

Theorem 5. Suppose that $\lambda_k \equiv 1$ (or $\lambda_k \geq \bar{\lambda} > 0$) for every k . Then any accumulation point α^* of $\{\alpha^k\}$ generated by The algorithm N, i.e. (46) (or The algorithm MN, i.e. (59)) is the solution of the equation (22). Moreover, if the equation (22) has a unique solution, the $x(\alpha^*)$ is the solution of the problem (1).

Proof. As shown in Remark 1, the parameter set Ω is bounded. Therefore, the sequence $\{\alpha^k\}$ is bounded and it has a convergent subsequence by Bolzano-Weierstrass theorem. Then any accumulation point α^* of the subsequence is the solution of the equation (22) by (46) or (59). The Corollary 2 completes the proof. □

Table 1 Computation result for Example 1

method	iter	optimal point	optimal value	time
N	6	(1.0000000, 1.7435214)	4.0608191	0.0156
MN	6	(1.0000000, 1.7435180)	4.0608191	0.0158
[5]	27	(1.0200000, 1.7100000)	4.0319159	
[18]	18	(1.0000000, 1.7500000)	4.0608042	



Table 2 Computation result for Example 2

method	iter	ε	time
N	3	1e-12	0.0084
MN	3	1e-12	0.0083
[11]	1205	1e-03	
[19]	1615	1e-03	
[27]	1016	1e-02	

Table 3 Computation result for Example 3

method	iter	optimal point	optimal value	time
N	2	(0.00000000, 3.33333333, 0.00000000)	3.002923976608	0.0244
MN	3	(0.00000000, 3.33333333, 0.00000000)	3.002923976608	0.0347
[14]	21	(0.000, 0.000, 1.25)	2.9312	
[10]	119	(0, 3.33333, 0)	3.002923972	
[21]	25	(0, 3.33333, 0)	3.00292	

4 Numerical experiments

Our algorithms N and MN were coded in MATLAB and numerical experiments had been carried out on a Pentium (R) 4 CPU 2 GHz RAM 0.97 GB personal computer using the optimization toolbox of MATLAB R2009a. We set $\|\psi(\alpha^k)\| < 1e - 6$ as the stopping criterion in numerical experiments. In the tables below, “iter” is the total number of the convex programming problems solved during the algorithm’s work, and the row “N” and “MN” show the average performance of 100 runs with random initial points in our algorithm N and MN.

Example 1 ([5, 18]).

$$\begin{aligned} \max \quad & \frac{-x_1^2 + 3x_1 - x_2^2 + 3x_2 + 3.5}{x_1 + 1} \\ & + \frac{x_2}{x_1^2 - 2x_1 + x_2^2 - 8x_2 + 20}, \\ \text{s.t.} \quad & 2x_1 + x_2 \leq 6, \quad 3x_1 + x_2 \leq 8, \\ & x_1 - x_2 \leq 1, \quad x_1 \geq 1, \quad x_2 \geq 1 \end{aligned}$$

Table 1 shows the computation result, which demonstrates the advantage of our algorithm over the previous algorithms based on the branch-and-bound method.

Example 2 ([11, 19, 27]).

$$\begin{aligned} \max \quad & \frac{-x_1^2 + 3x_1 - x_2^2 + 3x_2 + 3.5}{x_1 + 1} \\ & + \frac{x_2}{x_1^2 - 2x_1 + x_2^2 - 8x_2 + 20}, \\ \text{s.t.} \quad & 2x_1 + x_2 \leq 6, \quad 3x_1 + x_2 \leq 8, \\ & x_1 - x_2 \leq 1, \quad x_1 \geq 1, \quad x_2 \geq 2. \end{aligned}$$

The optimal solution and optimal value is (1, 2) and 4.0357143 , respectively. The results of our methods and previous methods [11, 19, 27] are shown in Table 2. This shows that our methods are much better than the previous methods.

Example 3 ([10, 14, 21]).



Table 4 Computation results for Example 4

method	iter	optimal point	optimal value	time
N	2	(1.11111111, 0.00000000, 0.00000000)	4.0907029478	0.0963
MN	3	(1.11111111, 0.00000000, 0.00000000)	4.0907029478	0.1059
[14]	31	(0.000, 1.111, 0.000)	3.7984	
[21]	3	(1.11111, 0, -3.33067e-016)	4.0907	
[22]	38	(1, 0, 0)	4.081481	

$$\begin{aligned}
 &\max \quad \frac{3x_1 + 5x_2 + 3x_3 + 50}{3x_1 + 4x_2 + 5x_3 + 50} \\
 &+ \frac{3x_1 + 4x_2 + 50}{4x_1 + 3x_2 + 2x_3 + 50} \\
 &+ \frac{4x_1 + 2x_2 + 4x_3 + 50}{5x_1 + 4x_2 + 3x_3 + 50}, \\
 &\text{s.t. } 6x_1 + 3x_2 + 3x_3 \leq 10, \\
 &10x_1 + 3x_2 + 8x_3 \leq 10, \quad x_1, x_2, x_3 \geq 0.
 \end{aligned}$$

The test results are shown in the following table.

Example 4 ([14, 21, 22]).

$$\begin{aligned}
 &\max \quad \frac{4x_1 + 3x_2 + 3x_3 + 50}{3x_2 + 3x_3 + 50} \\
 &+ \frac{3x_1 + 4x_3 + 50}{4x_1 + 4x_2 + 5x_3 + 50} \\
 &+ \frac{x_1 + 2x_2 + 5x_3 + 50}{x_1 + 5x_2 + 5x_3 + 50} \\
 &+ \frac{x_1 + 2x_2 + 4x_3 + 50}{5x_2 + 4x_3 + 50}, \\
 &\text{s.t. } 2x_1 + x_2 + 5x_3 \leq 10, \\
 &x_1 + 6x_2 + 3x_3 \leq 10, \\
 &5x_1 + 9x_2 + 2x_3 \leq 10, \\
 &9x_1 + 7x_2 + 3x_3 \leq 10, \quad x_1, x_2, x_3 \geq 0.
 \end{aligned}$$

The results of numerical experiments are given in Table 4, which shows the advantage of our methods over the previous methods.

Example 5 ([9, 16, 25]).

Table 5 Computation result for Example 5

method	iter	optimal point	optimal value	time
MN	4	(1.00000000, 1.23129885, 1.00000000)	0.81856565	0.0412
[25]	10	(1.0000, 1.0000, 1.0000)	0.8333	
[9]	108	(1.62988, 1.64453, 1.0025)	0.8929	
[16]	84	(1.32547, 1.42900, 1.20109)	0.9020	



Table 6 Computation result for Example 6

method	iter	optimal point	optimal value	time(s)
N	3	(2.2500000, 1.2500000)	2.3284023	0.0376
MN	4	(2.2500000, 1.2500000)	2.3284023	0.0483
[25]	19	(0.1000, 0.1000)	3.7142402	

$$\begin{aligned} \min & \frac{x_1^2 + x_2^2 + 2x_1x_3}{x_3^2 + 5x_1x_2} \\ & + \frac{x_1 + 1}{x_1^2 - 2x_1 + x_2^2 - 8x_2 + 20}, \\ \text{s.t. } & x_1^2 + x_2^2 + x_3 \leq 5, \quad (x_1 - 2)^2 + x_2^2 + x_3^2 \leq 5, \\ & 1 \leq x_1 \leq 3, \\ & 1 \leq x_2 \leq 3, \quad 1 \leq x_3 \leq 2. \end{aligned}$$

The numerical experiment results are shown in Table 5. It shows that the method MN is much better than previous methods. The algorithm N failed to find any global solution.

Example 6 ([25]).

$$\begin{aligned} \min & \frac{-x_1^2 + 3x_1 - x_2^2 + 3x_2 + 3.5}{x_1 + 1} \\ & + \frac{x_2}{x_1^2 + 2x_1 + x_2^2 - 8x_2 + 20}, \\ \text{s.t. } & 2x_1 + x_2 \leq 6, \quad 3x_1 + x_2 \leq 8, \\ & x_1 - x_2 \leq 1, \quad 0.1 \leq x_1, x_2 \leq 3 \end{aligned}$$

The numerical experiment results are shown in Table 6, which shows the advantage of our algorithms over the previous methods.

Example 7 ([4, 26]).

$$\begin{aligned} \min & \frac{x_1^2 - 4x_1 + 2x_2^2 - 8x_2 + 3x_3^2 - 12x_3 - 56}{x_1^2 - 2x_1 + x_2^2 - 2x_2 + x_3 + 20} \\ & + \frac{2x_1^2 - 16x_1 + x_2^2 - 8x_2 - 2}{2x_1 + 4x_2 + 6x_3}, \\ \text{s.t. } & x_1 + x_2 + x_3 \leq 10, \quad -x_1 - x_2 + x_3 \leq 4, \\ & 1 \leq x_1, x_2, x_3 \end{aligned}$$

The numerical experiment result is shown in Table 7.

Example 8 ([9, 16]).

Table 7 Computation result for Example 7

method	iter	optimal point	optimal value	time(s)
N	4	(1.82160941, 1, 1)	- 6.11983418	0.0189
MN	5	(1.82160941, 1, 1)	- 6.11983418	0.0295
[4]	24	(1.81, 1.00, 1.00)	- 6.1197786	
[26]		(1.8216, 1, 1)	- 6.1198	



Table 8 Computation result for Example 8

method	iter	optimal point	optimal value	time(s)
N	1	(1.6666666, 3.0000000)	1.8833333	0.0213
MN	4	(1.6666666, 3.0000000)	1.8833333	0.0431
[16]	142	(1.66598, 2.99899)	1.8867	
[9]	114	(1.66649, 2.99998)	1.8835281	

Table 9 Computation result for Example 9

method	iter	optimal point	optimal value	time(s)
N	2	(1.61803398708, 1.00000000000)	0.88386868337	0.0266
MN	4	(1.61803398708, 1.00000000000)	0.88386868337	0.0432
[23]	420	(1.6180339, 1.0000000)	0.883868686	
[18]	5802	(1.6180339885, 1.0000000046)	0.8838686849	
[11]	129	(1.617547466, 1.000000000)	0.8840633217	

$$\begin{aligned}
 \min \quad & \frac{x_2}{x_1^2 - 2x_1 + x_2^2 - 8x_2 + 20} \\
 & - \frac{x_1^2 + 2x_2^2 - 3x_1 - 3x_2 - 10}{x_1 + 1}, \\
 \text{s.t.} \quad & 2x_1 + x_2^2 - 6x_2 \leq 0, \\
 & 3x_1 + x_2 \leq 8, \quad x_1^2 - x_2 - x_1 \leq 0, \\
 & 1 \leq x_1 \leq 3, \quad 1 \leq x_2 \leq 3.
 \end{aligned}$$

The numerical experiment results are shown in Table 8. Our results are better than those of previous methods.

Example 9 ([11, 18, 23]).

$$\begin{aligned}
 \min \quad & a_1 \frac{-x_1^2 + 3x_1 + 2x_2^2 + 3x_2 + 3.5}{x_1 + 1} \\
 & + a_2 \frac{x_2}{x_1^2 - 2x_1 + x_2^2 - 8x_2 + 20}, \\
 \text{s.t.} \quad & 3x_1 + x_2 \leq 8, \quad x_1 - x_1^{-1}x_2 \leq 1, \\
 & 2x_1x_2^{-1} + x_2 \leq 6, \quad 1 \leq x_1, x_2 \leq 3,
 \end{aligned}$$

where $a_1 = 0.25$ and $a_2 = -1.75$. The numerical experiment results are shown in Table 9.

Example 10 ([11, 20]).

Table 10 Computation result for Example 10

method	iter	optimal point	optimal value	time(s)
N	1	(1.000000000, 1.41421967047)	0.48560364402	0.0833
MN	5	(1.000000000, 1.41421967047)	0.48560364402	0.1693
[20]	90	(1.0000, 1.4142)	0.4856	
[11]	140	(1.000000000, 1.414213525)	0.48560317968	



$$\begin{aligned} \min \quad & \frac{2x_1 + x_2}{x_1 + 10} + \frac{2}{x_2 + 10}, \quad \text{s.t.} \\ & -x_1^2 - x_2^2 + 3 \leq 0, \quad -x_1^2 - x_2^2 + 8x_2 - 14 \leq 0, \\ & 3x_1 + x_2 \leq 8, \quad 2x_1 + x_2 \leq 6, \quad x_1 - x_2 \leq 1, \\ & 1 \leq x_1 \leq 3, \quad 1 \leq x_2 \leq 4, \end{aligned}$$

The numerical experiment results are shown in Table 10.

Remark 7. Example 5,6,8 and 9 do not satisfy the assumption of the problem (1), which is, that numerators are convex and denominators are concave in all ratios. In addition, Example 5,8 and 10 have nonconvex constraints. The solutions in [11] and [20] are nonfeasible. For those problems, the algorithm MN obtained much more improved optimal solutions than the existing methods based on branch-and-bound algorithm. Our algorithm MN always found global solution for any starting point in all the examples 1-10 and the algorithm N also gave the same good results for all the examples but for the example 5 which does not satisfy the assumption about objective and constraint functions. This shows that our algorithms can find more accurate global solutions for more general nonlinear sum of ratios problems.

Example 11. We carried out numerical experiments for randomly generated minimization problems as following. The numerator and denominator of each term in objective function is

$$\begin{aligned} f_i(x) &= \frac{1}{2}x^T A_{0i}x + q_{0i}^T x \quad \text{and} \\ h_i(x) &= c_i^T x, \quad i = 1, \dots, p, \end{aligned}$$

respectively, where

$$\begin{aligned} A_{0i} &= U_i D_{0i} U_i^T, \quad D_{0i} = \text{diag}(d), \\ U_i &= Q_1 Q_2 Q_3, \quad d = i - i \cdot \text{rand}(n, 1), \\ Q_j &= I - 2 \frac{w_j w_j^T}{\|w_j\|^2}, \\ j &= 1, 2, 3, \quad w_1 = -i + \text{rand}(n, 1), \\ w_2 &= -2i + 2 \text{rand}(n, 1), \\ w_3 &= -3i + 3 \text{rand}(n, 1), \quad c_i = i - i \cdot \text{rand}(n, 1), \\ q_{0i} &= i + i \cdot \text{rand}(n, 1), \quad i = 1, \dots, p. \end{aligned}$$

The feasible set is given by

$$X = \{x \in R^n | Ax \leq b, \quad 1 \leq x_i \leq 5, \quad i = 1, \dots, n\},$$

where

$$A = -1 + 2 \cdot \text{rand}(5, n), \quad b = 2 + 3 \cdot \text{rand}(5, 1).$$

Starting with randomly generated point in $[1, 5]^n$, we carried out 50 runs of Algorithm N and Algorithm MN for fixed n and p . Both of algorithms N and MN were successful for all the problems. It is shown in Table 11 and Table 12.

As shown in the above tables, the number of iteration of Algorithm N and Algorithm MN was independent of the number of variables and fractional functions, and the run-time was proportional to the number of variables.

Remark 8. It is worth mentioning that both of $\|\psi(x)\|$ and the objective function decreased monotonically at each iteration of our algorithms N and MN in all numerical experiments for Examples 1-11.



Table 11 Computation result of Algorithm N, i.e.(47) or (46) for Example 11

n	p = 5		p = 10		p = 50		p = 100		p = 200	
	iter	time	iter	time	iter	time	iter	time	iter	time
10	7	0.0324	7	0.0386	6	0.0388	5	0.0314	4	0.0398
50	5	0.3347	5	0.3712	5	0.3738	5	0.3759	5	0.4146
100	5	1.4143	5	1.4437	5	1.4525	5	1.5356	5	1.5572
200	5	7.6532	5	7.6625	5	7.7274	5	7.7532	5	8.1661

Table 12 Computation result of Algorithm MN, i.e.(50),(51) or (59) for Example 11

n	p = 5		p = 10		p = 50		p = 100		p = 200	
	iter	time	iter	time	iter	time	iter	time	iter	time
10	7	0.0327	7	0.0398	6	0.0392	5	0.0317	4	0.0401
50	5	0.3354	6	0.3756	5	0.3772	5	0.3794	5	0.4167
100	6	1.5481	5	1.4488	5	1.4563	5	1.5362	5	1.5684
200	6	8.3422	5	7.6808	5	7.7436	5	7.7623	5	9.2416

5 Discussions

In this paper, we presented a parametric optimization algorithm for the sum-of-ratios problem, which is a non-convex fractional programming problem that occurs in many fields, including computer graphics and management. Our approach is based on transforming the sum-of-ratios problem into parametric convex programming problem and it applied Newton-like algorithm to update parameters. The algorithm has global linear and local superlinear/quadratic rate of convergence. Through numerical experiments, we demonstrated that our method always finds the global optimum with a fast convergence rate. In fact, the experiments showed that the optimal point is usually obtained within the first few iterations and it also showed that the algorithm can find global solution for larger class of sum-of-ratios problem than those considered in this paper. Although there is no theoretical guarantee that our method can always find a global optimal solution, we expect that our approach will solve successfully several linear and nonlinear sum-of-ratios problems in practice.

Acknowledgements The authors are grateful to the anonymous referees for their helpful comments and suggestions, which have greatly improved the earlier version of this paper.

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