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ORIGINAL RESEARCH

On the number of transitive relations on a set

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Abstract There is no known formula that counts the number of transitive relations on a set with n elements. In this paper, it is shown that no polynomial in n with integer coefficients can represent a formula for the number of transitive relations on a set with n elements. Several inequalities giving various useful recursions and lower bounds on the number of transitive relations on a set are also proved.

Keywords Combinatorics · Enumeration · Transitive relations

1 Introduction

Let S be a non-empty set. Any subset of $S \times S$ is a relation on S. A relation on S is transitive if and only if $\forall x, y, z \in S$, $(x, y) \in S \wedge (y, z) \in S \Rightarrow (x, z) \in S$.

OEIS [1] enlists the the number of transitive relations on sets with less than 19 elements. An explicit formula, if any, for the number of transitive relations on a set with n elements is still undiscovered.

Let $t(n)$ denote the number of transitive relations on a set with n elements. OEIS [1] enumerates $t(n), \forall n < 19$. The number of transitive relations on a set, $t(n)$, with $n < 19$ is tabulated as follows:

n	No. of transitive relations, $t(n)$
0	1
1	2
2	13
3	171
4	3994
5	154303
6	9415189
7	878222530
8	122207703623
9	24890747921947
10	7307450299510288
11	3053521546333103057
12	1797003559223770324237
13	1476062693867019126073312
14	1679239558149570229156802997

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n	No. of transitive relations, $t(n)$
15	2628225174143857306623695576671
16	5626175867513779058707006016592954
17	16388270713364863943791979866838296851
18	64662720846908542794678859718227127212465

2 Main Discussion

We prove that a formula, if any, for the number of transitive relations on a set cannot be a polynomial.

Theorem 1 $\nexists p(n) = \sum_{r=0}^{m} a_r n^r$, $a_i \in \mathbb{Z}$ *such that*, $p(n) = t(n), \forall n \in \mathbb{N}$.

Proof Let $p(n) = \sum_{r=0}^{m} a_r n^r$ be a polynomial in n.

If possible, let $p(n) = t(n), \forall n \in \mathbb{N}$.

Since $t(0) = 1, t(3) = 171$, we have

$t(0) = p(0) = \sum_{r=0}^{m} a_r 0^r = 1 \Rightarrow a_0 = 1$

$$a_0 = 1 \tag{1}$$

$t(3) = p(3) = \sum_{r=0}^{m} a_r 3^r = 171 \Rightarrow a_0 + 3a_1 + 3^2 a_2 + 3^3 a_3 + \ldots + 3^m a_m = 171 \Rightarrow 3a_1 + 3^2 a_2 + 3^3 a_3 + \ldots$ $+ 3^m a_m = 170$ (from (1))

$$a_1 + 3a_2 + 3^2 a_3 + \ldots + 3^{m-1} a_m = \frac{170}{3} \tag{2}$$

The proof is a direct consequence of (2). Since the sum of a finite number of integer terms cannot be a fraction, we conclude that at least one of $a_i, i \in \{1, 2, 3, \ldots, m\}$ is not an integer.

Thus, $\nexists p(n) = \sum_{r=0}^{m} a_r n^r$, $a_i \in \mathbb{Z}$ such that, $p(n) = t(n), \forall n \in \mathbb{N}$. □

We now state and prove a simple, but powerful inequality regarding $t(n)$, the number of transitive relations on a set.

Theorem 2 *Let* $n, n_1, n_2 \in \mathbb{N}$.

$$n = n_1 + n_2 \Rightarrow t(n) > t(n_1)t(n_2). \tag{3}$$

Proof Consider the set $A = \{a_1, a_2, a_3, \ldots, a_n\}$. We partition A into two sets A_1 and A_2, not necessarily non-empty, such that $A = A_1 \cup A_2$ and $A_1 \cap A_2 = \phi$. Let $|A_1| = n_1$ and $|A|_2 = n_2$. Since A_1 contains n_1 elements, there are $t(n_1)$ transitive relations on A_1. Similarly, there are $t(n_2)$ transitive relations on A_2. Now, if T_1 and T_2 are transitive relations on A_1 and A_2 respectively, then $T_1 \cup T_2$ is a transitive relation on A. Since there are $t(n_1)$ transitive relations on A_1 and $t(n_2)$ transitive relations on A_2, using the multiplication principle of counting, there are $t(n_1)t(n_2)$ possibilities for $T_1 \cup T_2$. Consequently, there are at least $t(n_1)t(n_2)$ transitive relations on A. □

2.1 Corollary 1

$$t(n) > 2 \times t(n-1), \forall n \mathbb{N}$$

Proof In (3) , put $n_1 = 1$ so that $n_2 = n - 1$. We get

$$t(n) > t(1)t(n-1)$$
$$t(n) > 2 \times t(n-1) \qquad (\because t(1) = 2)$$

□

2.2 Example:

$$t(4) = t(2+2) > t(2)t(2) = 13 \times 13 = 169$$
$$t(4) = t(1+3) > t(1)t(3) = 2 \times 171 = 342$$

The following theorem provides a larger lower bound for $t(n)$.

Theorem 3 *Let $n, n_1, n_2 \in \mathbb{N}$ such that $n = n_1 + n_2$. The following inequality holds*:

$$t(n) > t(n_1)t(n_2) + (2^{n_1} - 1)t(n_2) + (2^{n_2} - 1)t(n_1) \tag{4}$$

Proof Consider the set $A = \{a_1, a_2, a_3, ..., a_n\}$. As before, we partition A into two sets $A_1 = \{b_1, b_2, b_3, ..., b_{n_1}\}$ and $A_2 = \{c_1, c_2, c_3, ..., c_{n_2}\}$, not necessarily non-empty, such that $A = A_1 \cup A_2$ and $A_1 \cap A_2 = \phi$. From theorem 1, we get $t(n) > t(n_1)t(n_2)$.

Now, for each $b_\lambda \in A_1$, consider $B_\lambda = \{(b_\lambda, c_i), \forall i = 1, 2, 3, ..., n_2\}$. If T_2 is any transitive relation on A_2, then $B_\lambda \cup T_2$ is a transitive relation on A. Similarly, for each $c_\mu \in A_2$, consider $C_\mu = \{(b_i, c_\mu), \forall i = 1, 2, 3, ..., n_1\}$. If T_1 is any transitive relation on A_1, then $C_\mu \cup T_1$ is a transitive relation on A. Interestingly, if $b_{\lambda_1}, b_{\lambda_2} \in A_1$ and B_{λ_1}, B_{λ_2} are constructed the same way as that of B_λ, we observe that if T_2 is a transitive relation on A_2, then $B_{\lambda_1} \cup B_{\lambda_2} \cup T_2$ is a transitive relation on A. Similarly, if $c_{\mu_1}, c_{\mu_2} \in A_2$ and C_{μ_1}, C_{μ_2} are constructed the same way as that of C_μ, we observe that if T_1 is a transitive relation on A_1, then $C_{\mu_1} \cup C_{\mu_2} \cup T_1$ is a transitive relation on A. The same argument can be successfully continued to any number of elements $b_{\lambda_1}, b_{\lambda_2}, ..., b_{\lambda_r} \in A_1, r \leq n_1$ and $c_{\mu_1}, c_{\mu_2}, ..., c_{\mu_s} \in A_2, s \leq n_2$.

Consequently,

$$\begin{aligned} t(n) > t(n_1)t(n_2) &+ \binom{n_1}{1}t(n_2) + \binom{n_1}{2}t(n_2) + ... + \binom{n_1}{n_1}t(n_2) \\ &+ \binom{n_2}{1}t(n_1) + \binom{n_2}{2}t(n_1) + ... + \binom{n_2}{n_2}t(n_1) \\ \Rightarrow t(n) > t(n_1)t(n_2) &+ \left(\binom{n_1}{1} + \binom{n_1}{2} + ... + \binom{n_1}{n_1}\right)t(n_2) \\ &+ \left(\binom{n_2}{1} + \binom{n_2}{2} + ... + \binom{n_2}{n_2}\right)t(n_1) \end{aligned}$$

Using the fact that $\binom{n}{1} + \binom{n}{2} + ... + \binom{n}{n} = 2^n - 1$, we conclude that

$$t(n) > t(n_1)t(n_2) + \left(2^{n_1} - 1\right)t(n_2) + \left(2^{n_2} - 1\right)t(n_1)$$

□

The following corollary gives a useful recursive relation for $t(n)$.

2.3 Corollary 2

The following holds:

$$t(n) > 3 \times t(n-1) + 2^n - 2, \forall n\mathbb{N}$$

Proof In (4), choose $n_1 = 1$ so that $n_2 = n - 1$. We get

$$\begin{aligned} t(n) &> t(1)t(n-1) + (2^1 - 1)t(n-1) + (2^{n-1} - 1)t(n_1) \\ t(n) &> 2 \times t(n-1) + t(n-1) + (2^{n-1} - 1)2 \\ t(n) &> 3 \times t(n-1) + 2^n - 2 \end{aligned}$$

□

An even larger lower bound for $t(n)$ is obtained in (5) using the following result.

Theorem 4 *Let $n, n_1, n_2 \in \mathbb{N}$ such that $n = n_1 + n_2$. The following inequality holds*:

$$t(n) > t(n_1)t(n_2) + t(n_2)\left[\sum_{r=1}^{n_1}\binom{n_1}{r}t(n_1 - r)\right] + t(n_1)\left[\sum_{r=1}^{n_2}\binom{n_2}{r}t(n_2 - r)\right] \tag{5}$$

Proof Consider the set $A = \{a_1, a_2, a_3, \ldots, a_n\}$. As before, we partition A into two sets $A_1 = \{b_1, b_2, b_3, \ldots, b_{n_1}\}$ and $A_2 = \{c_1, c_2, c_3, \ldots, c_{n_2}\}$, not necessarily non-empty, such that $A = A_1 \cup A_2$ and $A_1 \cap A_2 = \phi$. From theorem 1, we get $t(n) > t(n_1)t(n_2)$. Now, for each $b_\lambda \in A_1$, consider $B_\lambda = \{(b_\lambda, c_i), \forall i = 1, 2, 3, \ldots, n_2\}$. If T_2 is any transitive relation on A_2 and T is any transitive relation on $A_1 - B_\lambda$, then $B_\lambda \cup T_2 \cup T$ is a transitive relation on A. Similarly, for each $c_\mu \in A_2$, consider $C_\mu = \{(b_i, c_\mu), \forall i = 1, 2, 3, \ldots, n_1\}$. If T_1 is any transitive relation on A_1 and S is a transitive relation on $A_2 - C_\lambda$, then $C_\mu \cup T_1 \cup S$ is a transitive relation on A. Interestingly, if $b_{\lambda_1}, b_{\lambda_2} \in A_1$ and B_{λ_1}, B_{λ_2} are constructed the same way as that of B_λ, we observe that if T_2 is a transitive relation on A_2 and T is a transitive relation on $A_1 - \{B_{\lambda_1}, B_{\lambda_2}\}$, then $B_{\lambda_1} \cup B_{\lambda_2} \cup T_2 \cup T$ is a transitive relation on A. Similarly, if $c_{\mu_1}, c_{\mu_2} \in A_2$ and C_{μ_1}, C_{μ_2} are constructed the same way as that of C_μ, we observe that if T_1 is a transitive relation on A_1 and S is a transitive relation on $A_2 - \{C_{\mu_1}, C_{\mu_2}\}$, then $C_{\mu_1} \cup C_{\mu_2} \cup T_1 \cup S$ is a transitive relation on A. The same argument can be successfully continued to any number of elements $b_{\lambda_1}, b_{\lambda_2}, \ldots, b_{\lambda_r} \in A_1, r \le n_1$ and $c_{\mu_1}, c_{\mu_2}, \ldots, c_{\mu_s} \in A_2, s \le n_2$.

Consequently,

$$\begin{aligned} t(n) &> t(n_1)t(n_2) \\ &\quad + \binom{n_1}{1}t(n_2)t(n_1 - 1) + \binom{n_1}{2}t(n_2)t(n_1 - 2) + \ldots + \binom{n_1}{n_1}t(n_2)t(0) \\ &\quad + \binom{n_2}{1}t(n_1)t(n_2 - 1) + \binom{n_2}{2}t(n_1)t(n_2 - 2) + \ldots + \binom{n_2}{n_2}t(n_1)t(0) \end{aligned}$$

This gives

$$\begin{aligned} t(n) &> t(n_1)t(n_2) \\ &\quad + \left[\binom{n_1}{1}t(n_1 - 1) + \binom{n_1}{2}t(n_1 - 2) + \ldots + \binom{n_1}{n_1}t(0)\right]t(n_2) \\ &\quad + \left[\binom{n_2}{1}t(n_2 - 1) + \binom{n_2}{2}t(n_2 - 2) + \ldots + \binom{n_2}{n_2}t(0)\right]t(n_1) \end{aligned}$$

This simplifies to

$$t(n) > t(n_1)t(n_2) + t(n_2)\left[\sum_{r=1}^{n_1}\binom{n_1}{r}t(n_1 - r)\right] + t(n_1)\left[\sum_{r=1}^{n_2}\binom{n_2}{r}t(n_2 - r)\right]$$

□

2.4 Corollary 3

The following holds:

$$t(n) > 3 \times t(n-1) + 2\left[\sum_{r=1}^{n-1}\binom{n-1}{r}t(n - 1 - r)\right]$$

Proof In (5), choose $n_1 = 1$ so that $n_2 = n - 1$. We get

$$t(n) > t(1)t(n-1) + t(n-1)\left[\sum_{r=1}^{1}\binom{1}{r}t(1-r)\right] + t(1)\left[\sum_{r=1}^{n-1}\binom{n-1}{r}t(n-1-r)\right]$$

$$t(n) > t(1)t(n-1) + t(n-1)t(0) + t(1)\left[\sum_{r=1}^{n-1}\binom{n-1}{r}t(n-1-r)\right]$$

$$t(n) > 2 \times t(n-1) + t(n-1) + 2\left[\sum_{r=1}^{n-1}\binom{n-1}{r}t(n-1-r)\right]$$

$$t(n) > 3 \times t(n-1) + 2\left[\sum_{r=1}^{n-1}\binom{n-1}{r}t(n-1-r)\right]$$

□

2.4.1 Example:

$$t(4) > 3 \times t(3) + 2\left[\sum_{r=1}^{3}\binom{3}{r}t(3-r)\right]$$

$$\Rightarrow t(4) > 3 \times t(3) + 2\left[\binom{3}{1}t(3-1) + \binom{3}{2}t(3-2) + \binom{3}{3}t(3-3)\right]$$

$$\Rightarrow t(4) > 3 \times t(3) + 2\left[\binom{3}{1}t(2) + \binom{3}{2}t(1) + \binom{3}{3}t(0)\right]$$

$$\Rightarrow t(4) > 3 \times 171 + 2\left[3 \times 13 + 3 \times 2 + 1 \times 1\right]$$

$$\Rightarrow t(4) > 605$$

Reference

1. OEIS, Sloane, Neil J. A. and The OEIS Foundation Inc., The on-line encyclopedia of integer sequences, 2020