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ORIGINAL RESEARCH

Mixed multiset topological space and Separation axioms

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Abstract Very recently the concept of separation axioms in multiset topological space and the concept of mixed multiset topological space were developed. In this article we redefine the definition of mixed multiset topological space.We introduce the separation axioms in it and investigate some of their properties.We also show that the properties of being T_0, T_1, T_2 spaces are hereditary.

Keywords Multiset · M-topology · Multipoint · Quasi-coincidence-Mixed M-topology

Mathematics Subject Classification Primary 03E70 · 03E15 · 03E20 · Secondary 54A05 · 54A10

1 Introduction

The theory of multiset (Bag or mset) as a generalization of set theory was developed by R.R. Yager [19] and later it was studied by W. D. Blizard [26, 27] and many others [1, 4–6, 11, 14, 22, 28]. After the introduction of fuzzy set by L.A. Jadeh [15] in 1965, the concepts of fuzzy multiset and fuzzy topology were developed and studied by many people [3, 7, 14, 21, 23, 24]. In the year 2012 Girish and John [10] developed the idea of multiset topology (M-topology). Many results of general topology were investigated in multiset topology [9, 12, 18]. Specially the separation axioms in multiset topological space were studied by Sheikh,Omar and Raafat [25]. Mixed topology is a technique of mixing two topologies on a set to get a third topology. The works on mixed topology were done by Buck [20], Cooper [8] and many more. In 1995 Das and Baishya [17] introduced the idea of mixed fuzzy topological space. This definition of Mixed fuzzy topological space was generalised by Tripathy and Ray [2]. Separation axioms in mixed fuzzy topological space were introduced and studied by M.H. Rashid and D.M. Ali [16] in 2008. In the year 2019, Shravan and Tripathy in [13] developed the concept of Mixed topological space on multisets (Mixed M-topological space) by using quasi-coincidence between a multipoint and a multiset. In this article we redefine definition of mixed M-topological space introduced by Shravan and Tripathy. We introduce the separation axioms in it and investigate some of their properties. We also show that the properties of being T_0, T_1, T_2 spaces are hereditary.

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2 Preliminaries

In classical set theory, a set is a well-defined collection of distinct objects. If repeated occurrences of any object is allowed in a set then that is known as multiset (mset or bag, for short).

Definition 2.1 A multiset M drawn from the set X is represented by a function Count M or C_M: $X \to N$, where N represents the set of non negative integers.

The mset M drawn from the set $X = \{x_1, x_2, ..., x_n\}$ is denoted by $M = \{k_1/x_1, k_2/x_2, ..., k_n/x_n\}$, where M is an mset with x_1 appearing k_1 times, x_2 appearing k_2 times and so on. In definition 2.1 , $C_M(x)$ is the number of occurrences of the element x in the mset M. However those elements which are not included in the mset M have zero count.

Clearly, a set is a special case of an mset.

Example 2.2. Let $X = \{a, b, c, d, e\}$ be any set. Then $M = \{a, a, b, b, d, d, b, d, d, e, d\}$ is a multiset drawn from X. The multiset M is also denoted by $M = \{2/a, 3/b, 5/d, 1/e\}$. Here $C_M(a) = 2, C_M(b) = 3, C_M(c) = 0, C_M(d) = 5, C_M(e) = 1$.

Definition 2.3. Let M be an mset drawn from a set X. The support set of M, denoted by M^*, is a subset of X and $M^* = \{x \in X : C_M(x) > 0\}$ i.e., M^* is an ordinary set and it is also called the root set.

An mset M is said to be an empty mset if for all $x \in X, C_M(x) = 0$.

Definition 2.4. A domain X is defined as a set of elements from which msets are constructed. The mset space $[X]^m$ is the set of all msets whose elements are in X such that no element in the mset occurs more than m times.

If $X = \{x_1, x_2, ..., x_k\}$ then $[X]^m = \{\{m_1/x_1, m_2/x_2, ..., m_k/x_k\} : \text{for } i = 1, 2, ..., k\,;\, m_i \in \{0, 1, 2, ..., m\}\}$.

Let $M, N \in [X]^m$. Then the following are defined :

1. $M = N$ if $C_M(x) = C_N(x)\ \forall x \in X$.
2. $M \subseteq N$ if $C_M(x) \leq C_N(x)\, \forall x \in X$.
3. $M \subset N$ if $C_M(x) \leq C_N(x)\, \forall x \in X$ and there exists an element $x \in X$ such that $C_M(x) < C_N(x)$
4. $P = M \cup N$ if $C_P(x) = \max\{C_M(x), C_N(x)\}\, \forall x \in X$.
5. $P = M \cap N$ if $C_P(x) = \min\{C_M(x), C_N(x)\}\, \forall x \in X$.
6. $P = M - N$ if $C_P(x) = \max\{C_M(x) - C_N(x),\, 0\}\, \forall x \in X$.

Let $[X]^m$ be am mset space and $\{M_1, M_2, M_3, ...\}$ be a collection of msets drawn from $[X]^m$. The following operations are possible under an arbitrary collection of msets.

1. The union $\cup_{i \in \triangle} M_i = \{C_{\cup M_i}(x)/x : C_{\cup M_i}(x) = Max\{C_{M_i}(x) : x \in X\}\}$.
2. The intersection $\cap_{i \in \triangle} M_i = \{C_{\cap M_i}(x)/x : C_{\cap M_i}(x) = Min\{C_{M_i}(x) : x \in X\}\}$.

Definition 2.5. Let X be a support set and $[X]^m$ be the mset space defined over X. Then for any mset $M \in [X]^m$, the complement M^c of M in $[X]^m$ is an element of $[X]^m$ such that $C_{M^c}(x) = m - C_M(x)$.

Definition 2.6. A submset N of M is a whole submset of M with each element in N having full multiplicity as in M, i.e., $C_N(x) = C_M(x)$ for every $x \in N$.

Definition 2.7. Let $M \in [X]^m$. The power whole mset of M, denoted by $PW(M)$, is defined as the set of all whole submsets of M.

Definition 2.8. Let $M \in [X]^m$. The power mset $P(M)$ of M is the set of all submsets of M .

Definition 2.9. Let $M \in [X]^m$. The power set of M is the support set of the power mset $P(M)$ and is denoted by $P^*(M)$.

Definition 2.10. Let $M \in [X]^m$ and $\tau \subseteq P^*(M)$. Then τ is called a multiset topology on M if τ satisfies the following properties.

1. ϕ and M are in τ.
2. The union of the elements of any subcollection of τ is in τ.
3. The intersection of any two elements of τ is in τ.

Mathematically, a multiset topological space is an ordered pair (M, τ) consisting of an mset $M \in [X]^m$ and a multiset topology $\tau \subseteq P^*(M)$ on M. The multiset topology is abbreviated as an M-topology.

Definition 2.11. Let $[X]^m$ be a space of multisets. A multipoint is a multiset M in $[X]^m$ such that

$$C_M(x) = \begin{cases} k, & \text{for } x \in X; \\ 0, & \text{for } y \neq x,\, y \in X \end{cases}$$

Remark 2.12. A multipoint $\{k/x\}$ is a subset of a multiset M or $\{k/x\} \in M$ if $k \leq C_M(x)$.

Definition 2.13. Let M be a multiset in the space $[X]^m$. Let $k/x \in M$. Then k/x is said to be quasi-coincident with $j/y \in M$ if $k + j > m$.

Definition 2.14. Let M be a multiset in the space $[X]^m$. Let $N \subseteq M$. Then k/x is said to be quasi-coincident with N if $k > C_{N^c}(x)$.

Definition 2.15. A multiset M is said to be quasi-coincident with N, i.e., MqN at x iff $C_M(x) > C_{N^c}(x)$.

Remark 2.16. If M and N are quasi-coincident at x then both $C_M(x)$ and $C_N(x)$ are non-zero and so M and N intersect at x.

Definition 2.17. A multiset N in an M-topological space (M, τ_1) is said to be Q-neighbourhood (Q-nbd)of k/x if and only if there exists an open mset P such that $k/xqP \subseteq N$.

Proposition 2.18. *Let $[X]^m$ be a space of multisets. Let $M, N \in [X]^m$. Then $M \subseteq N$ iff M and N^c are not quasi-coincident , i.e., $k/x \in M$ iff k/x is not quasi-coincident with M^c.*

In the year 2019, Shravan and Tripathy developed the concept of Mixed multiset topological space. By mixing two M-topologies they obtained a new M-topology in the following way.

Let $[X]^m$ be an mset space and $M \in [X]^m$. Suppose for each multipoint $k/x \in M$, there exists a collection $\mathcal{U}_{k/x}$ of submsets of M such that

1. k/xqU for all $U \in \mathcal{U}_{k/x}$.
2. $U, V \in \mathcal{U}_{k/x} \Rightarrow U \cap V \in \mathcal{U}_{k/x}$.
3. $U \in \mathcal{U}_{k/x}$ and $U \subseteq V \Rightarrow V \in \mathcal{U}_{k/x}$.

Then a new topology on M was defined in the following manner.

Definition 2.19. Let (M, τ_1) and (M, τ_2) be two M-topological spaces in $[X]^m$. Define $\tau_1(\tau_2) = \{A \subseteq [X]^m :$ For every multipoint k/x with k/xqA, there exists a τ_2-Q-nbd B of k/x such that k/xqB and τ_1-closure $\overline{B} \subseteq A\}$. This collection $\tau_1(\tau_2)$ is an M-topology on M known as mixed multiset topology (mixed M-topology) on M and the pair $(M, \tau_1(\tau_2))$ as mixed M-topological space.

Definition 2.20. Let (M, τ_1) be an M-topological space and $A \subseteq M$. The closure of A is defined as the mset intersection of all the closed msets containing A and is denoted as $Cl(A)$ or $\overline{A}$, i.e., $Cl(A) = \min\{C_K(x) : A \subseteq K, K \text{is a closed mset}\}$.

3 Main Results

Definition 3.1. Let $M, N \in [X]^m$. Then M is said to be quasi-coincident with N, i.e., MqN iff there exists an element x in X such that $C_M(x) > C_{N^c}(x)$.

Clearly MqN iff $C_M(x) + C_N(x) > m$.

Shravan and Tripathy in [13] developed the concept of Mixed topological space on multisets (Mixed M-topological space) by using quasi-coincidence between a multipoint and a multiset. Here we redefine the mixed multiset topological space by using quasi-coincidence between two multisets. So our definition will generalise the existing definition.

Theorem 3.2. *Let (M, τ_1) and (M, τ_2) be two M-topological spaces on M. Let $\tau_1(\tau_2) = \{A \subseteq M :$ For any submset B of M with BqA, there exists a τ_2-open mset C such that BqC and τ_1-closure $\overline{C} \subseteq A\}$. This collection $\tau_1(\tau_2)$ will form an M-topology on M. We shall call it mixed multiset topology (Mixed M-topology) on M and the pair $(M, \tau_1(\tau_2)$ as mixed multiset topological space (Mixed M-topological space).*

Proof.

(1) Clearly $\phi, M \in \tau_1(\tau_2)$.

(2) Let $A_1, A_2 \in \tau_1(\tau_2)$. Let A be a submset of M such that $Aq(A_1 \cap A_2)$. Clearly $Aq(A_1 \cap A_2)$ implies that AqA_1 and AqA_2. Since $A_1 \in \tau_1(\tau_2)$, so, there exists a τ_2-open mset B_1 such that AqB_1 and τ_1-closure $\overline{B_1} \subseteq A_1$. Again since $A_2 \in \tau_1(\tau_2)$ so, there exists a τ_2-open mset B_2 such that AqB_2 and τ_1-closure $\overline{B_2} \subseteq A_2$. Obviously $B_1 \cap B_2 \in \tau_2$ and $\overline{B_1 \cap B_2} \subseteq \overline{B_1} \cap \overline{B_2} \subseteq A_1 \cap A_2$. Also AqB_1 , AqB_2 $\Rightarrow Aq(B_1 \cap B_2)$. Thus for any submset A of M with $Aq(A_1 \cap A_2)$, there exists a τ_2-open mset $B_1 \cap B_2$ such that $Aq(B_1 \cap B_2)$ and τ_1-closure $\overline{B_1 \cap B_2} \subseteq A_1 \cap A_2$. Therefore $A_1 \cap A_2 \in \tau_1(\tau_2)$.

(3) Let $\{A_i : i \in \triangle\}$ be a collection of elements of $\tau_1(\tau_2)$. Let A be a submset of M such that $Aq(\bigcup_{i\in\triangle} A_i)$. Now $Aq(\bigcup_{i\in\triangle} A_i) \Rightarrow C_A(x) + C_{\bigcup_{i\in\triangle} A_i}(x) > m$ for some $x \Rightarrow C_A(x) + \max_{i\in\triangle}\{C_{A_i}(x)\} > m \Rightarrow$ there exists a $A_{i_0} \in \{A_i : i \in \triangle\}, i_0 \in \triangle$ such that $C_A(x) + C_{A_{i_0}}(x) > m \Rightarrow AqA_{i_0}$. Since $A_{i_0} \in \tau_1(\tau_2)$, so there exists a τ_2-open mset B_{i_0} such that AqB_{i_0} and τ_1-closure $\overline{B_{i_0}} \subseteq A_{i_0}$. But $A_{i_0} \subseteq \bigcup_{i\in\triangle} A_i$. Therefore $\overline{B_{i_0}} \subseteq \bigcup_{i\in\triangle} A_i$. Thus for any submset A of M with $Aq(\bigcup_{i\in\triangle} A_i)$, there exists a τ_2-open mset B_{i_0} such that AqB_{i_0} and τ_1-closure $\overline{B_{i_0}} \subseteq \bigcup_{i\in\triangle} A_i$. Therefore $\bigcup_{i\in\triangle} A_i \in \tau_1(\tau_2)$.

Hence $\tau_1(\tau_2)$ is an M-topology on M, i.e. $(M, \tau_1(\tau_2))$ is an M-topological space. □

***Example** 3.3.* Let $X = \{a, b, c\}$ and $M = \{2/a, 3/b, 1/c\} \in [X]^3$. Let $\tau_1 = \{\phi, M, \{2/a\}, \{3/b\}, \{2/a, 3/b\}\}$ and $\tau_2 = \{\phi, M, \{2/a\}, \{1/c\}, \{2/a, 1/c\}\}$. Clearly τ_1 and τ_2 are two M-topologies on M. From theorem 3.2, we obtain $\tau_1(\tau_2) = \{\phi, M, \{1/c\}, \{2/a, 1/c\}\}$, which is clearly a mixed M-topology on M, i.e.,$(M, \tau_1(\tau_2))$ is a mixed M-topological space.

Definition 3.4. Let $(M, \tau_1(\tau_2))$ be a mixed M-topological space. The complement of a submset N of M is the set $M - N$. The complement of N is denoted by N^c.

Definition 3.5. Let $(M, \tau_1(\tau_2))$ be a mixed M-topological space. A submset N of M is called a $\tau_1(\tau_2)$-open mset if $N \in \tau_1(\tau_2)$. A submset N of M is called a $\tau_1(\tau_2)$-closed mset if $M - N$ is open, i.e., if $N^c \in \tau_1(\tau_2)$.

Definition 3.6. Let $(M, \tau_1(\tau_2))$ be a mixed M-topological space and let $N \subseteq M$. Then the collection $\tau_1(\tau_2)_N = \{N \cap G : G \in \tau_1(\tau_2)\}$ is a mixed M-topology on N. We shall call $\tau_1(\tau_2)_N$ a subspace mixed M-topology and $(N, \tau_1(\tau_2)_N)$ a subspace of $(M, \tau_1(\tau_2))$.

***Example** 3.7.* Let $X = \{a, b, c\}$ and $M = \{2/a, 3/b, 1/c\} \in [X]^3$. Let $\tau_1 = \{\phi, M, \{2/a\}, \{3/b\}, \{2/a, 3/b\}\}$ and $\tau_2 = \{\phi, M, \{2/a\}, \{1/c\}, \{2/a, 1/c\}\}$. Then $\tau_1(\tau_2) = \{\phi, M, \{1/c\}, \{2/a, 1/c\}\}$. Let $N = \{1/a, 1/c\}$. Then $\tau_1(\tau_2)_N = \{\phi, N, \{1/a\}, \{1/c\}\}$ is a subspace mixed M-topology.

Remark 3.8. Throughout this article a multipoint $\{k/x\}$ of M in a mixed M-topological space $(M, \tau_1(\tau_2))$ will simply be denoted by *k/x*.

Theorem 3.9. *Let $M \in [X]^m$. Let (M, τ_1) and (M, τ_2) be two M-topological spaces on M. If every τ_1-Q-neighbourhood of k/x is a τ_2-Q-neighbourhood of k/x for all multipoints k/x then τ_1 is coarser than τ_2.*

Proof. Let $A \in \tau_1$. We show that $A \in \tau_2$.
Now $A \in \tau_1$
$\Rightarrow A$ is a τ_1-Q-neighbourhood of every multipoint *k/x* quasi-coincident with A.
$\Rightarrow A$ is a τ_2-Q-neighbourhood of every multipoint *k/x* quasi-coincident with A.
$\Rightarrow$ there exists a $B_{k/x} \in \tau_2$ such that $k/xqB_{k/x}$ and $B_{k/x} \subseteq A$.
We now show that $A = \bigcup_{k/x} B_{k/x}$, union being taken over all such $B_{k/x} \subseteq A$.
We have $C_A(x) + m - C_A(x) + 1 > m$
$\Rightarrow m - C_A(x) + 1 > m - C_A(x)$
$\Rightarrow m - C_A(x) + 1 > C_{A^c}(x)$
$\Rightarrow [m - C_A(x) + 1]/xqA$
$\Rightarrow [m + 1 - C_A(x)]/xqD_{k/x}$ for some $D_{k/x} \in \tau_2$ with $D_{k/x} \subseteq A$
$\Rightarrow m + 1 - C_A(x) + C_{D_{k/x}}(x) > m$
$\Rightarrow C_{D_{k/x}}(x) > C_A(x) - 1$
$\therefore C_A(x) = max_{k/x}\{C_{B_{k/x}}(x)\}$, i.e., $A = \bigcup_{k/x} B_{k/x}$.
Since every $B_{k/x}$ is a τ_2-open mset, so A is a τ_2-open mset. Therefore $\tau_1 \subseteq \tau_2$. Hence proved. □

Theorem 3.10. *Let $M \in [X]^m$. Let (M, τ_1) and (M, τ_2) be two M-topological spaces on M. Then the mixed M-topology $\tau_1(\tau_2)$ is coarser than τ_2.*

Proof. Let A be a $\tau_1(\tau_2)$-Q-neighbourhood of *k*/*x*. Then there exists $B \in \tau_1(\tau_2)$ such that *k*/*xqB* and $B \subseteq A$. So, there exists a τ_2-open mset B_2 such that k/xqB_2 and τ_1-closure $\overline{B_2} \subseteq B$. Therefore there exists a τ_2-open mset B_2 such that k/xqB_2 and $B_2 \subseteq A$.[Because $B_2 \subseteq \overline{B_2} \subseteq B \subseteq A$] and so A is a τ_2-Q-neighbourhood of *k*/*x*. Therefore, by theorem 3.9 , we have $\tau_1(\tau_2) \subseteq \tau_2$. Hence proved. □

Definition 3.11. Let $(M, \tau_1(\tau_2))$ be a mixed *M*-topological space. Then *M* is said to be a T_0-space iff for any pair of multipoints k_1/x_1, k_2/x_2 in *M* such that $x_1 \neq x_2$, there exists a $\tau_1(\tau_2)$-open mset *G* such that $k_1/x_1 \in G$, $k_2/x_2 \notin G$ or there exists a $\tau_1(\tau_2)$-open mset *H* such that $k_1/x_1 \notin H$, $k_2/x_2 \in H$.

Theorem 3.12. *Let $(M, \tau_1(\tau_2))$ be a T_0-space. Then every subspace of it is a T_0-space and hence the property is hereditary.*

Proof. Let $N \subseteq M$. Then $(N, \tau_1(\tau_2)_N)$ is a subspace mixed *M*-topology, where $\tau_1(\tau_2)_N = \{N \cap G : G \in \tau_1(\tau_2)\}$. We want to show $(N, \tau_1(\tau_2)_N)$ is T_0-space. Let k_1/x and k_2/y be any two multipoints in *N* such that $x \neq y$. Then $k_1/x, k_2/y \in M$, $x \neq y$. Since $(M, \tau_1(\tau_2))$ is T_0-space, so there exists a $\tau_1(\tau_2)$-open mset *H* such that $(k_1/x \in H$, $k_2/y \notin H)$ or there exists a $\tau_1(\tau_2)$-open mset *K* such that $(k_1/x \notin K$, $k_2/y \in K)$. Therefore $(k_1/x \in N \cap H$ and $k_2/y \notin N \cap H)$ or $(k_1/x \notin N \cap K$ and $k_2/y \in N \cap K)$. Also $N \cap H, N \cap K \in \tau_1(\tau_2)_N$. Thus for any two multipoints k_1/x and k_2/y in *N* ,there exists a $\tau_1(\tau_2)_N$-open mset $N \cap H$ such that $(k_1/x \in N \cap H$, $k_2/y \notin N \cap H)$ or there exists a $\tau_1(\tau_2)_N$-open mset $N \cap K$ such that $(k_1/x \notin N \cap K$, $k_2/y \in N \cap K)$. Hence $(N, \tau_1(\tau_2)_N)$ is T_0-space and the property is hereditary. □

Theorem 3.13. *Let (M, τ_1) and (M, τ_2) be two M-topological spaces. If $(M, \tau_1(\tau_2))$ is T_0-space then (M, τ_2) is also T_0-space.*

Proof. Let *r*/*x* and *s*/*y*, $x \neq y$ be two multipoints of *M*. Since $(M, \tau_1(\tau_2))$ is T_0, so there exists a $\tau_1(\tau_2)$-open mset *G* such that $(r/x \in G$ and $s/y \notin G)$ or there exists a $\tau_1(\tau_2)$-open mset *H* such that $(r/x \notin H$ and $s/y \in H)$. Since $\tau_1(\tau_2) \subseteq \tau_2$, so $G, H \in \tau_2$. Thus there exists a τ_2-open mset *G* such that $(r/x \in G$ and $s/y \notin G)$ or there exists a τ_2-open mset *H* such that $(r/x \notin H$ and $s/y \in H)$. Therefore (M, τ_2) is T_0-space. □

Remark 3.14. The converse of the theorem 3.13 is not true. The following counter example establishes that.

Example 3.15. Let $X = \{a, b, c\}$ and $M = \{2/a, 3/b\} \in [X]^3$. Let $\tau_1 = \{\phi, M, \{2/a\}\}$ and $\tau_2 = \{\phi, M, \{2/a\}, \{3/b\}\}$. It is Clear that τ_1 and τ_2 are two M-topologies on *M*. Also from the definition of mixed M-topology we obtain that $\tau_1(\tau_2) = \{\phi, M, \{1/a\}\}$. We can observe that (M, τ_2) is a T_0-space but $(M, \tau_1(\tau_2))$ is not a T_0-space.

Theorem 3.16. *Let $M \in [X]^m$. Let (M, τ_1) and (M, τ_2) be two M-topological spaces. If (M, τ_1) is T_0-space and $\tau_1 \subseteq \tau_2$ then $(M, \tau_1(\tau_2))$ is a T_0-space.*

Proof. Let k_1/x_1 and k_2/x_2 be any two multipoints of *M* such that $x_1 \neq x_2$. Since (M, τ_1) is T_0, so there exists a τ_1-open mset *G* such that $k_1/x_1 \in G$ and $k_2/x_2 \notin G$ or there exists a τ_1-open mset *H* such that $k_1/x_1 \notin H$ and $k_2/x_2 \in H$. Suppose that $G \in \tau_1$ exists such that $k_1/x_1 \in G$ and $k_2/x_2 \notin G$. Let *U* be a τ_1-Q-nbd of k_1/x_1 such that $k_2/x_2 \notin U$. Then for $G \in \tau_1$, we have $k_1/x_1qG \subseteq U$. So, $k_1 > C_{G^c}(x_1) \Rightarrow k_1 + C_G(x_1) > m \Rightarrow k_1 + C_U(x_1) > m \Rightarrow k_1/x_1qU$. Since $\tau_1 \subseteq \tau_2$ and $G \in \tau_1$, so *G* a is τ_2-open mset such that k_1/x_1qG and τ_1-closure $\overline{G} \subseteq U$. Therefore $U \in \tau_1(\tau_2)$. Also $k_1/x_1 \in G \Rightarrow k_1/x_1 \in U$. So $U \in \tau_1(\tau_2)$ such that $k_1/x_1 \in U$ and $k_2/x_2 \notin U$. Again if $H \in \tau_1$ exists such that $k_1/x_1 \notin H$ and $k_2/x_2 \in H$ and if *V* is a τ_1-Q-nbd of k_2/x_2 such that $k_1/x_1 \notin V$ then exactly in the same way we can show that $V \in \tau_1(\tau_2)$ such that $k_1/x_1 \notin V$ and $k_2/x_2 \in V$. Thus there exists a $\tau_1(\tau_2)$-open mset *U* such that $k_1/x_1 \in U$ and $k_2/x_2 \notin U$ or there exists a $\tau_1(\tau_2)$-open mset *V* such that $k_1/x_1 \notin V$ and $k_2/x_2 \in V$. Hence $(M, \tau_1(\tau_2))$ is a T_0-space. □

Definition 3.17. Let $(M, \tau_1(\tau_2))$ be a mixed *M*-topological space. Then *M* is said to be a T_1-space iff for any pair of multipoints k_1/x_1, k_2/x_2 in *M* such that $x_1 \neq x_2$, there exist $\tau_1(\tau_2)$-open msets *G* , *H* such that $(k_1/x_1 \in G$, $k_2/x_2 \notin G)$ and $(k_1/x_1 \notin H$, $k_2/x_2 \in H)$.

Theorem 3.18. *Let $(M, \tau_1(\tau_2))$ be a T_1-space. Then every subspace of it is a T_1-space and hence the property is hereditary.*

Proof. Let $N \subseteq M$. Then $(N, \tau_1(\tau_2)_N)$ is subspace mixed M-topology, where $\tau_1(\tau_2)_N = \{N \cap G : G \in \tau_1(\tau_2)\}$. We want to show $(N, \tau_1(\tau_2)_N)$ is T_1-space. Let k_1/x and k_2/y be any two multipoints in N such that $x \neq y$. Then $k_1/x, k_2/y \in M$ such that $x \neq y$. Since $(M, \tau_1(\tau_2))$ is T_1-space, so there exist two $\tau_1(\tau_2)$-open msets H and K such that $(k_1/x \in H$ and $k_2/y \notin H)$ and $(k_1/x \notin K$ and $k_2/y \in K)$. Therefore $(k_1/x \in N \cap H$ and $k_2/y \notin N \cap H)$ and $(k_1/x \notin N \cap K$ and $k_2/y \in N \cap K)$. Also $N \cap H, N \cap K \in \tau_1(\tau_2)_N$. Therefore $(N, \tau_1(\tau_2)_N)$ is T_1-space and hence the property is hereditary. □

Theorem 3.19. *Let (M, τ_1) and (M, τ_2) be two M-topological spaces. If $(M, \tau_1(\tau_2))$ is T_1-space then (M, τ_2) is also T_1-space.*

Proof. Let r/x and s/y, $x \neq y$ be two multipoints of M. Since $(M, \tau_1(\tau_2))$ is T_1, so there exist two $\tau_1(\tau_2)$-open msets G and H such that $(r/x \in G$, $s/y \notin G)$ and $(r/x \notin H$, $s/y \in H)$. Since $\tau_1(\tau_2) \subseteq \tau_2$, so $G, H \in \tau_2$. Thus there exist two τ_2-open msets G and H such that $(r/x \in G$, $s/y \notin G)$ and $(r/x \notin H$ and $s/y \in H)$. Therefore (M, τ_2) is T_1-space. □

Remark 3.20. The converse of the theorem 3.19 is not true. We show this in the following example.

Example 3.21. Let $X = \{a, b, c\}$ and $M = \{2/a, 3/b, 1/c\} \in [X]^3$. Let $\tau_1 = \{\phi, M, \{2/a\}, \{3/b\}, \{2/a, 3/b\}\}$ and $\tau_2 = \{\phi, M, \{2/a\}, \{3/b\}, \{1/c\}, \{2/a, 3/b\}, \{1/c, 3/b\}, \{2/a, 1/c\}\}$. Clearly τ_1 and τ_2 are two M-topologies on M. Also from the definition of mixed M-topology we obtain that $\tau_1(\tau_2) = \{\phi, M, \{1/c\}, \{1/c, 3/b\}, \{2/a, 1/c\}\}$. We can observe that (M, τ_2) is a T_1-space but $(M, \tau_1(\tau_2))$ is not a T_1-space.

Theorem 3.22. *Let $M \in [X]^m$. Let (M, τ_1) and (M, τ_2) be two M-topological spaces. If (M, τ_1) is T_1-space and $\tau_1 \subseteq \tau_2$ then $(M, \tau_1(\tau_2))$ is a T_1-space.*

Proof. Let k_1/x_1 and k_2/x_2 be any two multipoints of M such that $x_1 \neq x_2$. Since (M, τ_1) is T_1, so there exist τ_1-open msets G , H such that $(k_1/x_1 \in G$, $k_2/x_2 \notin G)$ and $(k_1/x_1 \notin H$, $k_2/x_2 \in H)$ Let U be a τ_1-Q-nbd of k_1/x_1 such that $k_2/x_2 \notin U$. Then for $G \in \tau_1$, we have $k_1/x_1 qG \subseteq U$. So, $k_1 > C_{G^c}(x_1) \Rightarrow k_1 + C_G(x_1) > m \Rightarrow k_1 + C_U(x_1) > m \Rightarrow k_1/x_1 qU$. Since $\tau_1 \subseteq \tau_2$ and $G \in \tau_1$, so G a is τ_2-open mset such that $k_1/x_1 qG$ and τ_1-closure $\overline{G} \subseteq U$. Therefore $U \in \tau_1(\tau_2)$. Also $k_1/x_1 \in G \Rightarrow k_1/x_1 \in U$. So $U \in \tau_1(\tau_2)$ such that $k_1/x_1 \in U$ and $k_2/x_2 \notin U$. Again if V is a τ_1-Q-nbd of k_2/x_2 such that $k_1/x_1 \notin V$ then exactly in the same way we can show that $V \in \tau_1(\tau_2)$ such that $k_1/x_1 \notin V$ and $k_2/x_2 \in V$. Thus there exist $\tau_1(\tau_2)$-open msets U , V such that $(k_1/x_1 \in U$ and $k_2/x_2 \notin U)$ and $(k_1/x_1 \notin V$ and $k_2/x_2 \in V)$. Hence $(M, \tau_1(\tau_2))$ is a T_1-space. □

Theorem 3.23. *If $(M, \tau_1(\tau_2))$ is a T_1-space then it is a T_0-space.*

Proof. Let m_1/x_1 and m_2/x_2 be any two multipoints of M such that $x_1 \neq x_2$. Since $(M, \tau_1(\tau_2))$ is a T_1-space, so there exist $\tau_1(\tau_2)$-open msets H and K such that $(m_1/x_1 \in H$, $m_2/x_2 \notin H)$ and $(m_1/x_1 \notin K$, $m_2/x_2 \in K)$. Hence $(M, \tau_1(\tau_2))$ is a T_0-space. □

Remark 3.24. The converse of the theorem 3.23 is not true. The following counter example reveals that.

Example 3.25. Let $X = \{a, b, c\}$ and $M = \{2/a, 3/b, 1/c\} \in [X]^3$. Let $\tau_1 = \{\phi, M, \{2/a\}, \{3/b\}, \{2/a, 3/b\}\}$ and $\tau_2 = \{\phi, M, \{2/a\}, \{1/c\}, \{2/a, 1/c\}\}$. Clearly τ_1 and τ_2 are two M-topologies on M. Also we obtain $\tau_1(\tau_2) = \{\phi, M, \{1/c\}, \{2/a, 1/c\}\}$. We see that $(M, \tau_1(\tau_2))$ is a T_0-space but not T_1-space.

Theorem 3.26. *Let $(M, \tau_1(\tau_2))$ be a mixed M-topological space. If k/x is $\tau_1(\tau_2)$-closed mset for all $x \in M^*$, $k = C_M(x)$ then $(M, \tau_1(\tau_2))$ is T_1-space.*

Proof. Let m_1/x_1 and m_2/x_2 be any two multipoints of M such that $x_1 \neq x_2$. Let $C_M(x_1) = k_1$ and $C_M(x_2) = k_2$. Then by hypothesis, k_1/x_1 and k_2/x_2 are $\tau_1(\tau_2)$-closed msets of M. Let $k_1/x_1 = r$ and $k_2/x_2 = s$. Then $r^c, s^c \in \tau_1(\tau_2)$ and so $m_1/x_1 \in s^c, m_2/x_2 \notin s^c$ and $m_1/x_1 \notin r^c, m_2/x_2 \in r^c$. Hence $(M, \tau_1(\tau_2))$ is T_1-space. □

Definition 3.27. Let $(M, \tau_1(\tau_2))$ be a mixed M-topological space. Then M is said to be a T_2-space iff for any pair of multipoints k_1/x_1, k_2/x_2 in M such that $x_1 \neq x_2$, there exist $\tau_1(\tau_2)$-open msets G , H such that $k_1/x_1 \in G, k_2/x_2 \in H$ and $G \cap H = \phi$.

Theorem 3.28. *Let $(M, \tau_1(\tau_2))$ be a T_2-space. Then every subspace of it is a T_2-space and hence the property is hereditary.*

Proof. Let $N \subseteq M$. Then $(N, \tau_1(\tau_2)_N)$ is subspace mixed M-topology, where $\tau_1(\tau_2)_N = \{N \cap G : G \in \tau_1(\tau_2)\}$. We want to show $(N, \tau_1(\tau_2)_N)$ is T_2-space. Let k_1/x and k_2/y be any two multipoints in N such that $x \neq y$. Then $k_1/x, k_2/y \in M$ such that $x \neq y$. Since $(M, \tau_1(\tau_2))$ is T_2-space, so there exist two $\tau_1(\tau_2)$-open msets H and K such that $k_1/x \in H$, $k_2/y \in K$ and $H \cap K = \phi$. Therefore $k_1/x \in N \cap H$, $k_2/y \in N \cap K$ and $(N \cap H) \cap (N \cap K) = \phi$. Also $N \cap H, N \cap K \in \tau_1(\tau_2)_N$. Therefore $(N, \tau_1(\tau_2)_N)$ is T_2-space and hence the property is hereditary. □

Theorem 3.29. *Let (M, τ_1) and (M, τ_2) be two M-topological spaces. If $(M, \tau_1(\tau_2))$ is T_2-space then (M, τ_2) is also a T_2-space.*

Proof. Let m_1/x_1 and m_2/x_2 be any two multipoints of M such that $x_1 \neq x_2$. Since $(M, \tau_1(\tau_2))$ is a T_2-space, so there exist $\tau_1(\tau_2)$-open msets H and K such that $m_1/x_1 \in H$, $m_2/x_2 \in K$ and $H \cap K = \phi$. Since $\tau_1(\tau_2) \subseteq \tau_2$, so $H, K \in \tau_2$. Therefore (M, τ_2) is T_2-space. □

Remark 3.30. The converse of the theorem 3.29 is not true. We establish that in the following example.

Example 3.31. Let $X = \{a, b, c\}$ and $M = \{2/a, 3/b, 1/c\} \in [X]^3$. Let $\tau_1 = \{\phi, M, \{2/a\}, \{3/b\}, \{2/a, 3/b\}\}$ and $\tau_2 = \{\phi, M, \{2/a\}, \{3/b\}, \{1/c\}, \{2/a, 3/b\}, \{1/c, 3/b\}, \{2/a, 1/c\}\}$. Clearly τ_1 and τ_2 are two M-topologies on M. Also from the definition of mixed M-topology we obtain that $\tau_1(\tau_2)) = \{\phi, M, \{1/c\}, \{1/c, 3/b\}, \{2/a, 1/c\}\}$. We see that (M, τ_2) is a T_2-space but $(M, \tau_1(\tau_2))$ is not a T_2-space.

Theorem 3.32. *Let $M \in [X]^m$. Let (M, τ_1) and (M, τ_2) be two M-topological spaces. If (M, τ_1) is T_2-space and $\tau_1 \subseteq \tau_2$ then $(M, \tau_1(\tau_2))$ is a T_2-space.*

Proof. Let k_1/x_1 and k_2/x_2 be any two multipoints of M such that $x_1 \neq x_2$. Since (M, τ_1) is T_2, so there exist τ_1-open msets G , H such that $k_1/x_1 \in G$, $k_2/x_2 \in H$ and $G \cap H = \phi$. Let U be a τ_1-Q-nbd of k_1/x_1. Then for $G \in \tau_1$, we have $k_1/x_1 qG \subseteq U$. So, $k_1 > C_{G^c}(x_1) \Rightarrow k_1 + C_G(x_1) > m \Rightarrow k_1 + C_U(x_1) > m \Rightarrow k_1/x_1 qU$. Since $\tau_1 \subseteq \tau_2$ and $G \in \tau_1$, so G a is τ_2-open mset such that $k_1/x_1 qG$ and τ_1-closure $\overline{G} \subseteq U$. Therefore $U \in \tau_1(\tau_2)$. Also $k_1/x_1 \in G \Rightarrow k_1/x_1 \in U$. Again if V is a τ_1-Q-nbd of k_2/x_2 then exactly in the same way we can show that $V \in \tau_1(\tau_2)$ such that $k_2/x_2 \in V$. Also $G \cap H = \phi \Rightarrow U \cap V = \phi$. Thus there exist $\tau_1(\tau_2)$-open msets U , V such that $k_1/x_1 \in U$, $k_2/x_2 \in V$ and $U \cap V = \phi$.. Hence $(M, \tau_1(\tau_2))$ is a T_2-space. □

Theorem 3.33. *If $(M, \tau_1(\tau_2))$ is a T_2-space then it is a T_1-space.*

Proof. Let m_1/x_1 and m_2/x_2 be any two multipoints of M such that $x_1 \neq x_2$. Since $(M, \tau_1(\tau_2))$ is a T_2-space, so there exist $\tau_1(\tau_2)$-open msets H and K such that $m_1/x_1 \in H$, $m_2/x_2 \in K$ and $H \cap K = \phi$. Since $H \cap K = \phi$, so $m_1/x_1 \notin K$ and $m_2/x_2 \notin H)$ Thus there are $\tau_1(\tau_2)$-open msets H and K such that $(m_1/x_1 \in H$, $m_2/x_2 \notin H)$ and $(m_1/x_1 \notin K$, $m_2/x_2 \in K)$. Hence $(M, \tau_1(\tau_2))$ is a T_1-space. □

Remark 3.34. The converse of the theorem 3.33 is not true. The following example confirms that.

Example 3.35. Let $X = \{a, b, c\}$ and $M = \{2/a, 3/b, 1/c\} \in [X]^3$. Let $\tau_1 = \{\phi, M, \{2/a\}, \{3/b\}, \{1/c\}\}$ and $\tau_2 = \{\phi, M, \{2/a, 3/b\}, \{1/c, 3/b\}, \{2/a, 1/c\}\}$. Then $\tau_1(\tau_2) = \{\phi, M, \{2/a, 3/b\}, \{1/c, 3/b\}, \{2/a, 1/c\}\}$. We see that $(M, \tau_1(\tau_2))$ is a T_1-space but not a T_2-space.

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