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ORIGINAL RESEARCH

Edge-connectivity in hypergraphs

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Abstract The edge-connectivity of a connected hypergraph H is the minimum number of edges (named as edge-cut) whose removal makes H disconnected. It is known that the edge-connectivity of a hypergraph is bounded above by its minimum degree. H is super edge-connected, if every edge-cut consists of edges incident with a vertex of minimum degree. A hypergraph H is linear if any two edges of H share at most one vertex. We call H uniform if all edges of H have the same cardinality. Sufficient conditions for equality of edge-connectivity and minimum degree of graphs and super edge-connected graphs are known. In this paper, we present a generalization of some of these sufficient conditions to linear and/or uniform hypergraphs.

Keywords Edge-connectivity · Hypergraph · Maximally edge-connected · Super edge-connected

1 Introduction

As one of the classical parameters that indicate how reliable a graph G is, the edge-connectivity $\lambda(G)$, defined as the minimum number of edges whose removal renders G disconnected, has attracted much attention in recent years. In 1932, Whitney [18] established one of the basic foundations of edge-connectivity for graphs: the edge-connectivity $\lambda(G)$ of a connected graph G is bounded above by the minimum degree $\delta(G)$. Thus, in order to study reliability and fault tolerance of graphs, sufficient conditions for graphs satisfying $\lambda(G) = \delta(G)$ (so called maximally edge-connected) are of great interest. For other results the reader is referred to, for example, [5] and the survey [12].

Hypergraphs are a natural generalization of graphs in which "edges" may consist of more than 2 vertices. More precisely, a *hypergraph* $H = (V, E)$ consists of a set V and a collection E of non-empty subsets of V. The elements of V are called *vertices* and the elements of E are called *hyperedges*, or simply *edges*. We define the *order* and *size* of H by $n = |V(H)|$ and $m = |E(H)|$, respectively. Unless specified otherwise, we consider only simple hypergraphs, i.e., hypergraphs whose edges are distinct. An *r-uniform* hypergraph H is a hypergraph such that all edges of H have cardinality r. We use K_n^r to denote the *complete r-uniform hypergraph of order n*, i.e., the hypergraph on n vertices whose edge set consists of all possible r-

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subsets of the vertex set. A hypergraph is called *linear* if any two edges of the hypergraph share at most one vertex. Obviously, every (simple) graph is a (linear) 2-uniform hypergraph.

For $v, w \in V$, v and w are said to be *adjacent*, if there exists an edge $e \in E$ such that $\{v, w\} \subseteq e$. A vertex v and an edge e are said to be *incident* if $v \in e$. The *degree* of a vertex v, denoted by $d_H(v)$, is the number of edges which are incident to v. The *minimum degree* and *maximum degree* among the vertices of H are denoted by $\delta(H)$ and $\Delta(H)$, respectively. The *neighborhood* of a vertex v, denoted by $N_H(v)$, is the set of all vertices different from v that are adjacent to v. If H is clear from the contest, we denote $d_H(v)$ and $N_H(v)$ by $d(v)$ and $N(v)$, respectively. For $X \subseteq V$, use $H[X]$ to denote the subgraph of H induced by X.

A *walk* in a hypergraph H is a finite alternating sequence $W = (v_0, e_1, v_1, e_2, ..., e_k, v_k)$, where $v_i \in V$ for $i \in \{0, 1, ..., k\}$ and $e_j \in E$ such that $\{v_{j-1}, v_j\} \in e_j$ for $j \in \{1, 2, ..., k\}$. A walk W is a *path* if all the vertices v_i for $i \in \{0, 1, ..., k\}$ and all the edges e_j for $j \in \{1, 2, ..., k\}$ in W are distinct. The *length* of a path, is the number of edges that it contains. We define the *distance* between two vertices u and v, denoted by $d_H(u, v)$, as the length of a shortest path between u and v. A hypergraph is *connected*, if there is a walk between any pair of its vertices, otherwise it is *disconnected*. The *diameter* $D(H)$ of a connected hypergraph H is defined by $D(H) = max_{u,v \in V(H)} d_H(u, v)$.

We can extend the concept of edge-connectivity from graph theory to hypergraphs in a natural way in which the concept can be generalized. For a subset $S \subseteq E(H)$, we define $H - S$ to be the hypergraph obtained from H by deleting the edges in S without affecting the rest of the hypergraph. When $H - S$ is disconnected, we say that S is an *edge-cut*. The minimum cardinality of an edge-cut in a connected hypergraph H is called its *edge-connectivity*, denoted by $\lambda(H)$.

There has been several papers investigating the connectivity of the hypergraphs. In [19], Zykov presented a Menger-type theorem for hypergraphs. Edge augmentation of hypergraphs are studied in the literature (see e.g. [1, 2, 4]). Gu and Lai [10] gave necessary and sufficient conditions for an r-uniform hypergraphic sequence to have a k-edge-connected relazation. Jami et al. [13] provided a generalization of a result on edge-connectivity of permutation graphs for hypergraphs. In [6], Dankelmann and Meierling observed that $\lambda(H) \le \delta(H)$ for general hypergraphs, and generalized some well-known sufficient conditions for graphs G satisfying $\lambda(G) = \delta(G)$ to hypergraphs. In [7], the authors investigated vertex-connectivity of hypergraphs. For a subset $X \subset V(H)$, $H - X$ denotes the hypergraph obtained by removing the vertices X from H and removing all the edges that intersect X. The vertex-connectivity $\kappa(H)$, is defined as the minimum cardinality of such X whose removal makes G disconnected. In [7], they also defined another vertex-connectivity for hypergraphs, and considered the complexity of the two kinds of vertex-connectivity for hypergraphs. The following result in [7] provided a generalization of a result of Whitney [18] on connectivity of graphs to hypergraphs.

Theorem 1.1 ([7]). *Let H be a hypergraph with at least two vertices. Then $\kappa(H) \le \lambda(H) \le \delta(H)$.*

Thus, we call a hypergraph H satisfying $\lambda(H) = \delta(H)$ (resp. $\kappa(H) = \delta(H)$) *maximally edge-connected* (resp. *maximally vertex-connected*). If, furthermore, every minimum edge-cut consists of edges incident with one vertex, then H is said to be *super edge-connected*, or simply, *super-* λ. Our main work is to investigate how some sufficient conditions for graphs to be maximally edge-connected or super-λ can be generalized to uniform and/or linear hypergraphs. In Section 2, we present results that will be useful in our arguments. In Section 3, two kinds of degree conditions for equality of edge-connectivity and minimum degree for graphs are generalized to uniform linear hypergraphs. In Section 4, we generalize a sufficient condition for maximally edge-connected graphs depending on the order, the maximum degree and the minimum degree as well as on the diameter, to uniform linear hypergraphs. In Section 5, we generalize a sufficient condition for maximally edge-connected graphs and super-λ graphs depending on the size, the order, the minimum degree and a parameter (as defined in Section 5) to uniform hypergraphs.

2 Preliminary lemmas

In this section, we will list or prove some lemmas which will be used in our later proofs.

In a connected hypergraph $H = (V, E)$, let $S \subseteq E$ be a minimum edge-cut of H and H_1 be a component of $H - S$. A vertex v of H_1 is *internal* if v is not incident with any edge of S; otherwise, v will be *external*. In 1981, Goldsmith confirmed a very useful lemma in [9] when he studied the n-th order edge-connectivity of graphs. Now we present the special case of his lemma as follows.

Lemma 2.1 ([9]). *Let S be a minimum edge-cut of a graph G. If $\lambda(G) < \delta(G)$, then each component of $G - S$ contains at least two internal vertices.*

And we will give a similar result with Lemma 2.1 for uniform linear hypergraphs.

Lemma 2.2 *Let H be an r-uniform linear hypergraph and S be a minimum edge-cut of H. If $\lambda(H) < \delta(H)$, then each component of $H - S$ contains at least one internal vertex.*

Proof Let H_1 be a component of $H - S$ and x be an external vertex of H_1. Set $E_1 = \{e \in E \mid x \in e$ and $e \in E(H_1)\}$, and $E_2 = \{e \in E \mid x \in e$ and $e \in S\}$. Obviously, $E_2 \neq \emptyset$. If $E_1 = \emptyset$, then $\delta(H) > \lambda(H) = |S| \geq |E_2| = d(x) \geq \delta(H)$, a contradiction. Thus, $E_1 \neq \emptyset$. It follows that $|E_1| + |E_2| = d(x) \geq \delta(H) > \lambda(H) = |S| = |E_2| + |S - E_2|$, which implies that $|E_1| > |S - E_2|$. Since H is linear and each edge of $S - E_2$ is incident with at most $r - 1$ vertices in $V(H_1)$, we have

$$|(\bigcup_{e \in S - E_2} e) \cap V(H_1)| \leq (r-1)|S - E_2| < (r-1)|E_1| = |\bigcup_{e \in E_1}(e - \{x\})|,$$

and $(\bigcup_{e \in E_1}(e - \{x\})) \cap (\bigcup_{e \in E_2}(e - \{x\})) = \emptyset$, which implies that there exists at least one vertex $w \in N(x) \cap (\bigcup_{e \in E_1} e)$ that is not covered by any edge of S, i.e., w is an internal vertex of H_1. □

Our lemma implies the following two results of [6] for linear r-uniform hypergraphs to be maximally edge-connected.

Theorem 2.3 ([6]). *Let H be an r-uniform linear hypergraph with $D(H) \leq 2$. Then $\lambda(H) = \delta(H)$.*

Proof Let S be a minimum edge-cut of H. The distance condition implies that there exists at least one component of $H - S$ that contains no internal vertex. Then by Lemma 2.2, we have $\lambda(H) \geq \delta(H)$ and the result holds. □

If $\lambda(H) < \delta(H)$, then by Lemma 2.2, each component of $H - S$ contains at least one internal vertex w. It follows that each component of $H - S$ contains at least $1 + (r-1)\delta(H)$ vertices ($N(w) \cup \{w\} \subseteq V(H_1)$ and thus $|V(H_1)| \geq 1 + (r-1)d(w) \geq 1 + (r-1)\delta(H)$). Hence, $|V(H)| \geq 2 + 2(r-1)\delta(H)$, and we obtain the following condition for linear uniform hypergraphs to be maximally edge-connected.

Theorem 2.4 ([6]). *Let H be an r-uniform linear hypergraph of order n. If $n \leq 1 + 2(r-1)\delta(H)$, then $\lambda(H) = \delta(H)$.*

The special case $r = 2$ is the classical result as the following.

Corollary 2.5 *Let G be a connected graph of order n. Then $\lambda(G) = \delta(G)$, if*

(1) $n \leq 2\delta(G) + 1$; *Chartrand* [3]
(2) $D(G) \leq 2$; *Plesník* [15].

3 Degree conditions

We now work towards a generalization of some degree conditions for equality of edge-connectivity and minimum degree for graphs to linear uniform hypergraphs. We point out here that we present the same generalizations as that of [16], but use a different method from [16]. For the sake of completeness, we also give the complete proof in our paper. In 1974, Lesniak [14] proved the following strengthening result of Corollary 2.5 (1) for graphs.

Theorem 3.1 ([14]). *If G is a graph of order n with $d(u) + d(v) \geq n - 1$ for all distinct non-adjacent vertices u and v, then $\lambda(G) = \delta(G)$.*

Below we present a generalization of the above result for r-uniform linear hypergraphs.

Theorem 3.2 *Let H be an r-uniform linear hypergraph of order n. If $d(u) + d(v) \geq \lceil \frac{n-1}{r-1} \rceil$ for all distinct non-adjacent vertices u and v, then $\lambda(H) = \delta(H)$.*

Proof Let $H=(V,E)$ and let $S\subseteq E$ be a minimum edge-cut of H. Then $H-S$ consists of two parts H_1 and H_2 such that there are no edges between H_1 and H_2. Denote the vertex set of H_i by V_i for $i=1,2$. We claim that there exists no internal vertex in H_1 or H_2. Suppose, on the contrary, that there exist internal vertices $x_i\in V_i$ for $i=1,2$. Then x_1 and x_2 are non-adjacent and $d(x_1)+d(x_2)\le\lfloor\frac{|V_1|-1}{r-1}\rfloor+\lfloor\frac{|V_2|-1}{r-1}\rfloor\le\frac{n-2}{r-1}<\frac{n-1}{r-1}$, contradict to the hypothesis. Thus, by Lemma 2.2, we have that $\lambda(H)=\delta(H)$. □

Note that Theorem 3.1 is a special case of Theorem 3.2 when $r=2$. Fig. 1 shows that Theorem 3.2 is in a sense a best possible result, since $d(u)+d(v)\ge\lceil\frac{n-1}{r-1}\rceil-1$ for all pairs of non-adjacent vertices of H_1 and H_1 is not maximally edge-connected.

Theorem 3.3 *Let H be an r-uniform linear hypergraph of order n. If for each edge e there exist at least $r-1$ vertices incident with e such that each degree is at least $\lceil\frac{\lfloor\frac{n}{2}\rfloor}{r-1}\rceil$, then $\lambda(H)=\delta(H)$.*

Proof Let S be a minimum edge-cut of H and let H_1 be a component of $H-S$ with the minimum cardinality. Then $|V(H_1)|\le\lfloor\frac{n}{2}\rfloor$. If $|V(H_1)|=1$, then the result follows. In the following, we assume that $|V(H_1)|\ge 2$. Let $v\in V(H_1)$ such that $d(v)=min\{d(x)\mid x\in V(H_1)\}$. Set $E_1=\{e\in E\mid v\in e$ and $e\in E(H_1)\}$, and $E_2=\{e\in E\mid v\in e$ and $e\in S\}$. We consider the following two cases.

Case 1: $d(v)\ge\lceil\frac{\lfloor\frac{n}{2}\rfloor}{r-1}\rceil$ It follows that $d(x)\ge\lceil\frac{\lfloor\frac{n}{2}\rfloor}{r-1}\rceil$ for all $x\in V(H_1)$. Since $d_{H_1}(x)\le\lfloor\frac{|V(H_1)|-1}{r-1}\rfloor\le\lfloor\frac{\lfloor\frac{n}{2}\rfloor-1}{r-1}\rfloor$, we see that H_1 contains no internal vertices. And by Lemma 2.2, we have $\lambda(H)=\delta(H)$.

Case 2: $d(v)<\lceil\frac{\lfloor\frac{n}{2}\rfloor}{r-1}\rceil$

If $E_1=\emptyset$, then $\delta(H)\le d(v)=|E_2|\le|S|=\lambda(H)$, and the result follows. Now, we assume that $E_1\ne\emptyset$. In this condition, by our hypothesis, we have $d(x)\ge\lceil\frac{\lfloor\frac{n}{2}\rfloor}{r-1}\rceil$ for any $x\in X$, where $X=\bigcup_{e\in E_1}(e-\{v\})$, which implies that each vertex x in X is an external vertex. Set $Y=\{e_u\in E\mid u\in e_u\in S$ and $u\in X\}$, then $Y\cap E_2=\emptyset$ since H is linear. It follows that $(r-1)|E_1|=|X|=|(\bigcup_{e\in Y}e)\cap X|\le(r-1)|Y|$ and we have $|E_1|\le|Y|$. Thus, $\delta(H)\le d(v)=|E_1|+|E_2|\le|Y|+|E_2|\le|S|=\lambda(H)\le\delta(H)$, and the proof is complete. □

It is easy to check that H_1 in Fig. 1 is a regular hypergraph and Fig. 1 can also show that Theorem 3.3 is a best possible result in a sense. Now, we give an irregular hypergraph H_2 (see Fig. 2), which is not maximally edge-connected, and there exist at least $r-1$ vertices incident with each edge such that each degree is at least $\lceil\frac{\lfloor\frac{n}{2}\rfloor}{r-1}\rceil-1$.

When $r=2$, as a special case of Theorem 3.3, we can get the following degree condition for maximally edge-connected graphs.

Theorem 3.4 ([11]). *Let G be a connected graph. If for each edge e there exists at least one vertex v incident with e such that $d(v)\ge\lfloor\frac{n}{2}\rfloor$, then $\lambda(G)=\delta(G)$.*

4 A sufficient condition about order

In this section, we present a generalization of the following result by Esfahanian.

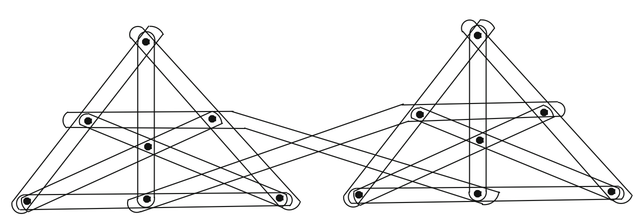

Fig. 1 A 3-uniform linear hypergraph H_1

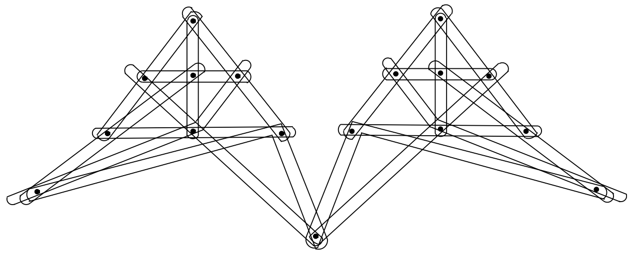

Fig. 2 A 3-uniform linear hypergraph H_2

Theorem 4.1 ([8]). *Let G be a graph with maximum degree* $\Delta \geq 3$, *minimum degree* δ, *diameter D and order n. Then* $\lambda(G) = \delta(G)$, *when*

$$n \geq (\delta - 1)[\frac{(\Delta - 1)^{D-1} + \Delta(\Delta - 2) - 1}{\Delta - 2}] + 1.$$

Now, consider an r-uniform linear hypergraph $H = (V, E)$ with maximum degree $\Delta(H) = \Delta$ and let $X_0 \subset V$ with $X_0 = \{x_1, x_2, ..., x_p\}$, where $|X_0| = p$. Denote by $\overline{X_0} = V \backslash X_0$. For each $x_i \in X_0$, let $X_i = N(x_i) \cap \overline{X_0}$, where $i \in \{1, 2, ..., p\}$. For a vertex $x \in \overline{X_0}$, $d(x, X_0)$ denotes $\min\{d(x, u) \mid u \in X_0\}$. Define $k = max\{d(x, X_0) \mid x \in \overline{X_0}\}$. We claim that n, the order of H, is bounded by:

$$\begin{aligned} n \leq |X_0| &+ |X_1|[1 + (\Delta - 1)(r - 1) + (\Delta - 1)^2(r - 1)^2 + \cdots + (\Delta - 1)^{k-1}(r - 1)^{k-1}] \\ &+ |X_2|[1 + (\Delta - 1)(r - 1) + (\Delta - 1)^2(r - 1)^2 + \cdots + (\Delta - 1)^{k-1}(r - 1)^{k-1}] \\ &+ \cdots + |X_p|[1 + (\Delta - 1)(r - 1) + (\Delta - 1)^2(r - 1)^2 + \cdots + (\Delta - 1)^{k-1}(r - 1)^{k-1}] \end{aligned}$$

which is equivalent to

$$n \leq |X_0| + [\sum_{i=1}^{p} |X_i|][\sum_{i=0}^{k-1} (\Delta - 1)^i (r - 1)^i]. \tag{1}$$

To see the validity of the above claim, observe that for each vertex $u \in \overline{X_0}$, there exists a vertex $x_j \in X_0$ such that $d(u, x_j) \leq k$. And, in the right-hand side of the inequality (1), for each $x_i \in X_0$, the maximum number of vertices in $\overline{X_0}$, which are at distance less than or equal to k from x_i, is computed.

Using the discussion above we now compute the upper-bound on n, as a function of other hypergraph parameters.

Theorem 4.2 *Let* n, λ, δ, Δ *and D respectively be the order, the edge-connectivity, the minimum degree, the maximun degree and the diameter of an r-uniform linear hypergraph* $H = (V, E)$. *If* $\lambda < \delta$ *and* $\Delta > 2$, *then*

$$n \leq (\delta - 1)[\frac{(\Delta - 1)^{D-1}(r - 1)^D + (\Delta - 1)^2(r - 1)^2 - r}{(\Delta - 1)(r - 1) - 1}].$$

Proof Let $S \subseteq E$ be a minimum edge-cut of H. We can partition V into two disjoint non-empty sets Y and $\overline{Y}$ such that $H - S$ contains no edges between Y and $\overline{Y}$. Let Y_0 and $\overline{Y_0}$ be the sets of external vertices respectively in Y and $\overline{Y}$. Let $D_Y = max\{d(y, Y_0) \mid y \in Y\}$, and $D_{\overline{Y}} = max\{d(y', \overline{Y_0}) \mid y' \in \overline{Y}\}$. Since $\lambda < \delta$, then $D_Y \geq 1$ and $D_{\overline{Y}} \geq 1$ by Lemma 2.2. And it is easy to see that $D_Y + D_{\overline{Y}} + 1 \leq D$.

Set $Y_0 = \{x_1, x_2, ..., x_p\}$, and $\overline{Y_0} = \{x'_1, x'_2, ..., x'_q\}$ where $p = |Y_0|$ and $q = |\overline{Y_0}|$. Let $X_i = N(x_i) \cap (Y - Y_0)$ and $X'_i = N(x'_i) \cap (\overline{Y} - \overline{Y_0})$, where $x_i \in Y_0$ and $x'_i \in \overline{Y_0}$. Combining with the above claim of n, we have

$$n = |Y| + |\overline{Y}| \leq |Y_0| + [\sum_{i=1}^{|Y_0|} |X_i|][\sum_{i=0}^{D_Y-1} (\Delta-1)^i (r-1)^i] + |\overline{Y_0}| + [\sum_{i=1}^{|\overline{Y_0}|} |X_i'|][\sum_{i=0}^{D_{\overline{Y}}-1} (\Delta-1)^i (r-1)^i].$$

Without loss of generality, we assume that $D_{\overline{Y}} \leq D_Y$. Thus, we have:

$$\begin{aligned} n \leq & |Y_0| + |\overline{Y_0}| + [\sum_{i=1}^{|Y_0|} |X_i|][\sum_{i=0}^{D_{\overline{Y}}-1} (\Delta-1)^i (r-1)^i + \sum_{i=D_{\overline{Y}}}^{D_Y-1} (\Delta-1)^i (r-1)^i] \\ & + [\sum_{i=1}^{|\overline{Y_0}|} |X_i'|][\sum_{i=0}^{D_{\overline{Y}}-1} (\Delta-1)^i (r-1)^i] \\ = & |Y_0| + |\overline{Y_0}| + (\sum_{i=1}^{|Y_0|} |X_i| + \sum_{i=1}^{|\overline{Y_0}|} |X_i'|)[\sum_{i=0}^{D_{\overline{Y}}-1} (\Delta-1)^i (r-1)^i] \\ & + [\sum_{i=1}^{|Y_0|} |X_i|][\sum_{i=D_{\overline{Y}}}^{D_Y-1} (\Delta-1)^i (r-1)^i] \end{aligned}$$

Since each edge of S is incident with at most $(r-1)$ vertices of Y_0, we have $|Y_0| \leq (r-1)\lambda$. And it is easy to see that $|Y_0| + |\overline{Y_0}| \leq \lambda r$, $|X_i| \leq (d(x_i)-1)(r-1) \leq (\Delta-1)(r-1)$ and $|X_i'| \leq (\Delta-1)(r-1)$. It follows that

$$\begin{aligned} n \leq & |Y_0| + |\overline{Y_0}| + (|Y_0| + |\overline{Y_0}|)(\Delta-1)(r-1)[\sum_{i=0}^{D_{\overline{Y}}-1} (\Delta-1)^i (r-1)^i] \\ & + |Y_0|(\Delta-1)(r-1)[\sum_{i=D_{\overline{Y}}}^{D_Y-1} (\Delta-1)^i (r-1)^i] \\ \leq & \lambda r + \lambda r(\Delta-1)(r-1)[\sum_{i=0}^{D_{\overline{Y}}-1} (\Delta-1)^i (r-1)^i] + (r-1)\lambda(\Delta-1)(r-1)[\sum_{i=D_{\overline{Y}}}^{D_Y-1} (\Delta-1)^i (r-1)^i] \\ = & \lambda r + (r-1)\lambda(\Delta-1)(r-1)[\sum_{i=0}^{D_Y-1} (\Delta-1)^i (r-1)^i] + \lambda(\Delta-1)(r-1)[\sum_{i=0}^{D_{\overline{Y}}-1} (\Delta-1)^i (r-1)^i] \\ = & \lambda r + \lambda(\Delta-1)(r-1)^2 \frac{1-(\Delta-1)^{D_Y}(r-1)^{D_Y}}{1-(\Delta-1)(r-1)} + \lambda(\Delta-1)(r-1)\frac{1-(\Delta-1)^{D_{\overline{Y}}}(r-1)^{D_{\overline{Y}}}}{1-(\Delta-1)(r-1)} \end{aligned}$$

Let $a = (\Delta-1)(r-1)$, one has

$$\begin{aligned} n \leq & \lambda r + \lambda(r-1)a\frac{1-a^{D_Y}}{1-a} + \lambda a \frac{1-a^{D_{\overline{Y}}}}{1-a} \\ = & \lambda\{r + \frac{a}{a-1}[(r-1)a^{D_Y} - r + a^{D_{\overline{Y}}}]\} \end{aligned}$$

Using the fact that $D_Y \geq 1$, $D_{\overline{Y}} \geq 1$ and $D_Y + D_{\overline{Y}} + 1 \leq D$, one can show that

$$\begin{aligned} n \leq & \lambda\{r + \frac{a}{a-1}[(r-1)a^{D-2} + a - r]\} \\ = & \lambda \frac{(r-1)a^{D-1} + a^2 - r}{a-1} \end{aligned}$$

We remind that the above relation has been computed with the assumption that $\lambda < \delta$. Thus, we have

$$n \le (\delta-1)\frac{(\Delta-1)^{D-1}(r-1)^D + (\Delta-1)^2(r-1)^2 - r}{(\Delta-1)(r-1)-1}.$$

This completes the proof. □

See Fig. 3, we give an r-uniform linear hypergraph H_3 which is not maximally edge-connected and reaches the upper bound presented in Theorem 4.2. H_3 is constructed by r copies of H_0 and adding a new edge consisting of all the vertices of degree r, where H_0 is also an r-uniform linear hypergraph. Theorem 4.2 yields the following result whose special case $r = 2$ is Theorem 4.1.

Theorem 4.3 *Let H be an r-uniform linear hypergraph with maximum degree* $\Delta \ge 3$, *minimum degree* δ *and diameter D. Then* $\lambda(H) = \delta(H)$ *when*

$$|V(H)| \ge (\delta-1)\left[\frac{(\Delta-1)^{D-1}(r-1)^D + (\Delta-1)^2(r-1)^2 - r}{(\Delta-1)(r-1)-1}\right] + 1.$$

5 A sufficient condition about size

In this section, we will work towards a sufficient condition for maximally edge-connected and super-λ r-uniform hypergraphs. And, we need the following definition. For two integers n and k, we define $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ when $k \le n$ and $\binom{n}{k} = 0$ when $k > n$.

Definition 5.1 For two integers δ and r with $r \ge 2$, define $t = t(\delta, r)$ to be the largest integer such that $\binom{t-1}{r-1} \le \delta$. That is, t is the integer satisfying $\binom{t-1}{r-1} \le \delta < \binom{t}{r-1}$.

Lemma 5.2 *Let H be a connected r-uniform hypergraph of order n, size m, minimum degree* δ *and edge-connectivity* λ. *If* $\lambda < \delta$, *then*

$$m \le \binom{n-t}{r} + \binom{t}{r} + \delta - 1,$$

where $t = t(\delta, r)$.

Proof Let S be an arbitrary minimum edge-cut and let X denote the vertex set of the component of $H - S$ with minimum number of vertices, and $Y = V(H) - X$. We first show that $|Y| \ge |X| \ge t$. Suppose $|X| \le t - 1$. Then we obtain

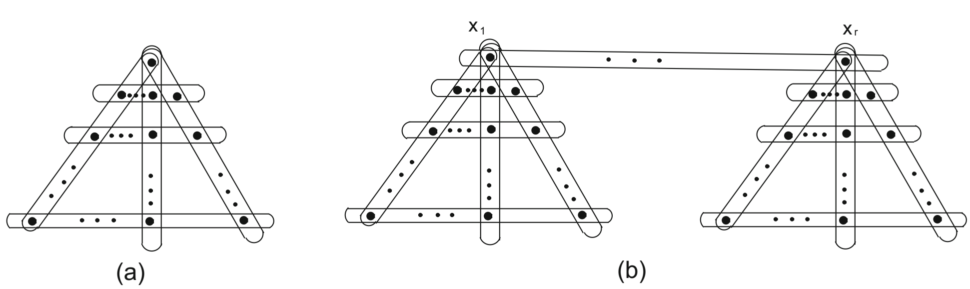

Fig. 3 (a) H_0; (b) H_3

$$|X|\delta \leq \sum_{x\in X} d(x) \leq |X|\binom{|X|-1}{r-1} + (r-1)\lambda$$
$$\leq |X|\binom{|X|-1}{r-1} + (r-1)(\delta-1),$$

and thus,

$$\binom{t-1}{r-1} \leq \delta \leq \binom{|X|}{r-1} - \frac{r-1}{|X|-r+1} \leq \binom{t-1}{r-1} - \frac{r-1}{|X|-r+1}. \tag{2}$$

If $|X| = 1$, then $\lambda = \delta$, contradicting $\lambda < \delta$. So, $|X| \geq r$, by the fact that $H[X]$ is connected, and the inequality (2) is impossible. Therefore, we have $t \leq |X| \leq |Y| \leq n-t$, and

$$m - |S| \leq \binom{n}{r} - \sum_{i=1}^{r-1}\binom{|X|}{i}\binom{|Y|}{r-i}$$
$$= \binom{n}{r} - [\binom{n}{r} - \binom{|X|}{r} - \binom{|Y|}{r}]$$
$$= \binom{|X|}{r} + \binom{n-|X|}{r}.$$

This bound leads to

$$m \leq \binom{|X|}{r} + \binom{n-|X|}{r} + (\delta-1).$$

Since $f(x) = \binom{x}{r} + \binom{n-x}{r}$ is a decreasing function when $1 \leq x \leq \frac{n}{2}$ and a increasing function when $\frac{n}{2} \leq x \leq n$, we obtain

$$m \leq \binom{t}{r} + \binom{n-t}{r} + (\delta-1).$$

□

Theorem 5.3 *Let H be a connected r-uniform hypergraph of order n, size m, minimum degree δ and edge-connectivity λ. If*

$$m \geq \binom{n-t}{r} + \binom{t}{r} + \delta - 1,$$

then $\lambda = \delta$, unless that H is a hypergraph obtained from $K_t^r \cup K_{n-t}^r$ by adding $\delta - 1$ edges, where $t = t(\delta, r)$.

Proof If $m > \binom{n-t}{r} + \binom{t}{r} + \delta - 1$, then by Lemma 5.2, we have $\lambda = \delta$. If $m = \binom{n-t}{r} + \binom{t}{r} + \delta - 1$, then by the proof of Lemma 5.2, we have that the inequalities in the proof of Lemma 5.2 must by equalities, which implies that $\lambda = \delta - 1$, $|X| = t$, $|Y| = n - t$, $H[X] \cong K_t^r$ and $H[Y] \cong K_{n-t}^r$. This completes the proof. □

Corollary 5.4 ([17]). *Let G be a connected graph of order n, size m, minimum degree δ and edge-connectivity λ. If*

$$m \geq \binom{n-\delta-1}{2} + \binom{\delta+1}{2} + \delta - 1,$$

then $\lambda = \delta$, *unless that G is a graph obtained from* $K_{\delta+1} \cup K_{n-\delta-1}$ *by adding* $\delta - 1$ *edges.*

Proof When $r = 2$, then $t = \delta + 1$ by the Definition 5.1. Thus, the result follows from Theorem 5.3. □

Lemma 5.5 *Let H be a connected r-uniform hypergraph of order n, size m, minimum degree δ and edge-connectivity λ. If H is not super- λ, then*

$$m \leq \binom{n-t+1}{r} + \binom{t-1}{r} + \delta,$$

where $t = t(\delta, r)$.

Proof Let S be an arbitrary minimum edge-cut such that every component of $H - S$ has at least two vertices. Let X denote the vertex set of the component of $H - S$ with minimum number of vertices, and $Y = V(H) - X$. Clearly, $|Y| \geq |X| \geq 2$. We first show that $|Y| \geq |X| \geq t - 1$. Suppose $|X| \leq t - 2$. Then we obtain

$$|X|\delta \leq \sum_{x \in X} d(x) \leq |X|\binom{|X|-1}{r-1} + (r-1)\lambda$$
$$\leq |X|\binom{|X|-1}{r-1} + (r-1)\delta,$$

and thus,

$$\binom{t-1}{r-1} \leq \delta \leq \binom{|X|}{r-1} \leq \binom{t-2}{r-1}, \tag{3}$$

a contradiction. Therefore, we have $t - 1 \leq |X| \leq |Y| \leq n - t + 1$, and

$$m - |S| \leq \binom{|X|}{r} + \binom{n-|X|}{r},$$

and so

$$m \leq \binom{|X|}{r} + \binom{n-|X|}{r} + \delta$$
$$\leq \binom{n-t+1}{r} + \binom{t-1}{r} + \delta.$$

□

By the same argument as that of Theorem 5.3 and Corollary 5.4, the following results follows.

Theorem 5.6 *Let H be a connected r-uniform hypergraph of order n, size m, minimum degree δ and edge-connectivity λ. If*

$$m \geq \binom{n-t+1}{r} + \binom{t-1}{r} + \delta,$$

then H is super- λ, unless $t \geq 3$ *and H is a hypergraph obtained from* $K^r_{t-1} \cup K^r_{n-t+1}$ *by adding δ edges between* K^r_{t-1} *and* K^r_{n-t+1} *such that* $\delta(H) = \delta$, *where* $t = t(\delta, r)$.

Corollary 5.7 ([17]). *Let G be a connected graph of order n, size m, minimum degree δ and edge-connectivity λ. If*

$$m \geq \binom{n-\delta}{2} + \binom{\delta}{2} + \delta,$$

then G is super- λ, unless $\delta \geq 2$ *and G is a graph obtained from* $K_\delta \cup K_{n-\delta}$ *by adding δ edges between* K_δ *and* $K_{n-\delta}$ *such that* $\delta(G) = \delta$.

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