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ORIGINAL RESEARCH

Regularity in weighted oriented graphs

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Abstract Let D be a weighted oriented graph with the underlying graph G and $I(D)$, $I(G)$ be the edge ideals corresponding to D and G respectively. We show that the regularity of edge ideal of a certain class of weighted oriented graph remains same even after adding certain kind of new edges to it. We also establish the relationship between the regularity of edge ideal of weighted oriented path and cycle with the regularity of edge ideal of their underlying graph when vertices of V^+ are sinks.

Keywords Weighted oriented graph · labeled hypergraph · edge ideal · Castelnuovo-Mumford regularity

Mathematics Subject Classification 13D02 · 13F20 · 13C10 · 05C22 · 05E40 · 05C20

1 Introuduction

A weighted oriented graph is a triplet $D = (V(D), E(D), w)$, where $V(D)$ is the vertex set, $E(D)$ is the edge set and w is a weight function $w : V(D) \longrightarrow \mathbb{N}^+$, where $\mathbb{N}^+ = \{1, 2, \ldots\}$. Specifically $E(D)$ consists of ordered pairs of the form (x_i, x_j) which represents a directed edge from the vertex x_i to the vertex x_j. The weight of a vertex $x_i \in V(D)$ is $w(x_i)$, denoted by w_i or w_{x_i}. We set $V^+(D) := \{x \in V(D) | w(x) \geq 2\}$ and it is denoted by V^+. The underlying graph of D is the simple graph G whose vertex set is same as the vertex set of D and whose edge set is $\{\{x, y\} | (x, y) \in E(D)\}$. If $V(D) = \{x_1, \ldots, x_n\}$ we can regard each vertex x_i as a variable and consider the polynomial ring $R = k[x_1, \ldots, x_n]$ over a field k. The edge ideal of D is defined as

$$I(D) = (x_i x_j^{w_j} | (x_i, x_j) \in E(D)).$$

If a vertex x_i of D is a source, we shall always assume $w_i = 1$ because in this case the definition of $I(D)$ does not depend on the weight of x_i. If $w(x) = 1$ for all $x \in V$, then $I(D)$ recovers the usual edge ideal of the underlying graph G, which has been extensively studied in the literature in [1–3, 5, 14, 15]. The interest in edge ideals of weighted digraphs comes from coding theory, in the study of Reed-Muller types codes. The edge ideal of weighted digraph appears as initial ideals of vanishing ideals of projective spaces over finite fields [17].

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Algebraic invariants like Cohen-Macaulayness and unmixedness of edge ideals of weighted oriented graphs have been studied in [9, 11, 18]. In [18], Pitones et al. have characterised the minimal strong property of D when vertices of V^+ are sinks. Recently, the invariants like Castelnuvo-Mumford regularity and projective dimension of weighted oriented graphs have drawn the attention of many researchers. In [20], Zhu et al. have expressed the projective dimension and regularity of edge ideals of some class of weighted oriented forests or cycles and in [4], Beyarslan et al. gave the formula for projective dimension and regularity of edge ideals of weighted oriented graphs having the property P defined as follows:

A weighted oriented graph D is said to have property P if there is at most one edge oriented into each vertex and suppose that for all non-leaf, non-source vertices, x_j, either $w_j \geq 2$ or the unique edge (x_i, x_j) into the vertex x_j has the property that x_i is a leaf.

In general it is a difficult problem to give a general formula for the regularity of edge ideal of an arbitrary weighted oriented graph even if the regularity of edge ideal of its underlying graph is known as the edge ideal changes according to the orientation of its edges and its weight function. In this paper we study the regularity of weighted oriented graphs arising by adding new edges to the weighted oriented graphs having property P. By studying the regularity of edge ideal of weighted oriented graphs we partially answer one question asked by H.T. Hà in [12]. Also we establish the relation between the regularity of edge ideal of weighted oriented graph D and its underlying graph G when D is a weighted oriented path or cycle with vertices of V^+ are sinks.

This paper is structured as follows. In section 2, we recall all the definitions and results that will be required for the rest of the paper. In section 3, we prove that the regularity of edge ideal of one or more weighted oriented graphs with property P remains unchanged even after adding new edges among the connected and disconnected components (Theorem 3.3). As some applications of Theorem 3.3, we compute the regularity of edge ideals of some weighted oriented graphs whose underlying graphs are dumbbell graph, complete graph, join of two cycles and complete $m-$partite graph. In Theorem 3.10, we prove that the regularity of edge ideal of a weighted oriented graph with property P remains same even after adding certain type of oriented edges from new vertices towards a single vertex of it. By using Proposition 3.11, we able to give the combinatorial conditions for one question asked by H.T. Hà in [12]. In section 4, we compute the regularity of edge ideal of a weighted oriented path or cycle in terms of regularity of edge ideal of their underlying graph when vertices of V^+ are sinks.

2 Preliminaries

In this section we present some of the definitions and results that will be needed throughout the paper. Let $D = (V(D), E(D), w)$ be a weighted oriented graph with underlying graph $G = (V(G), E(G))$. For a vertex u in a graph G, let $N_G(u) = \{v \in V(G) | \{u, v\} \in E(G)\}$ be the set of neighbours of u and set $N_G[u] := N_G(u) \cup \{u\}$. For a subset $W \subseteq V(G)$ of the vertices in G, define $G \setminus W$ to be the subgraph of G with the vertices in W (and their incident edges) deleted. Let x be a vertex of the weighted oriented graph D, then the sets $N_D^+(x) = \{y : (x, y) \in E(D)\}$ and $N_D^-(x) = \{y : (y, x) \in E(D)\}$ are called the out-neighbourhood and the in-neighbourhood of x respectively. Further, $N_D(x) = N_D^+(x) \cup N_D^-(x)$ is the set of neighbourhoods of x and set $N_D[u] := N_D(u) \cup \{u\}$. For $T \subset V$, we define the *induced subgraph* $\mathcal{D} = (V(\mathcal{D}), E(\mathcal{D}), w)$ of D to be the weighted oriented graph such that $V(\mathcal{D}) = T$ and for any $u, v \in V(\mathcal{D})$, $(u, v) \in E(\mathcal{D})$ if and only if $(u, v) \in E(D)$. Here $\mathcal{D} = (V(\mathcal{D}), E(\mathcal{D}), w)$ is a weighted oriented graph with the same orientation as in D and for any $u \in V(\mathcal{D})$, if u is not a source in $\mathcal{D}$, then its weight equals to the weight of u in D, otherwise, its weight in $\mathcal{D}$ is 1. For a subset $W \subseteq V(D)$ of the vertices in D, define $D \setminus W$ to be the induced subgraph of D with the vertices in W (and their incident edges) deleted. For $Y \subset E(D)$, we define $D \setminus Y$ to be a subgraph of D with all edges in Y deleted (but its vertices remained). If $Y = \{e\}$ for some $e \in E(D)$, we write $D \setminus e$ in place of $D \setminus \{e\}$. Define $\deg_D(x) = |N_D(x)|$ for $x \in V(D)$. A vertex $x \in V(D)$ is called a leaf vertex if $\deg_D(x) = 1$. A vertex $x \in V(D)$ is called a source vertex if $N_D(x) = N_D^+(x)$. A vertex $x \in V(D)$ is called a sink vertex if $N_D(x) = N_D^-(x)$.

Now we give some algebraic definitions and results. Let k be a field and $R = k[x_1, \ldots, x_n]$ be the polynomial ring in n variables over k. Suppose that M is a non zero graded R-module with minimal free resolution

$$0 \longrightarrow \cdots \longrightarrow \bigoplus_j R(-j)^{\beta_{1,j}(M)} \longrightarrow \bigoplus_j R(-j)^{\beta_{0,j}(M)} \longrightarrow M \longrightarrow 0$$

where $\beta_{i,j}(M)$ denote the (i, j)-th graded Betti number of M, is an invariant of M that equals the number of minimal generators of degree j in the $i-$th syzygy module of M. The invariant which measures the complexity of the module is Castelnuovo-Mumford regularity denoted by $\operatorname{reg}(M)$ and defined as

$$\operatorname{reg}(M) := \max\{j - i \mid \beta_{i,j}(I) \neq 0\}.$$

Let $I \subset R$ be a monomial ideal. Then $\mathcal{G}(I)$ denotes the set of minimal generators of I. In general, it is difficult to find the regularity even for monomial ideals. With the help of Betti splitting we can compute this type of invariant for certain class of ideals. The Betti splitting is defined as follows:

Definition 2.1 Let I be a monomial ideal and suppose that there exist monomial ideals J and K such that $\mathcal{G}(I)$ is the disjoint union of $\mathcal{G}(J)$ and $\mathcal{G}(K)$. Then $I = J + K$ is a Betti splitting if

$$\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i,j}(K) + \beta_{i-1,j}(J \cap K)$$

for all $i, j \geq 0$, where $\beta_{i-1,j}(J \cap K) = 0$ if $i = 0$.

This formula was first obtained for the total Betti numbers by Eliahou and Kervaire [6] and extended to the graded case by Fatabbi [7]. In [8], the authors describe the following sufficient conditions for an ideal I to have a Betti splitting.

Theorem 2.2 [8, Corollary 2.7] *Suppose that $I = J + K$ where $\mathcal{G}(J)$ contains all the generators of I divisible by some variable x_i and $\mathcal{G}(K)$ is a nonempty set containing the remaining generators of I. If J has a linear resolution, then $I = J + K$ is a Betti splitting.*

When I is having a Betti splitting, Definition 2.1 implies the following result:

Corollary 2.3 *If $I = J + K$ is a Betti splitting, then*

$$\operatorname{reg}(I) = \max\{\operatorname{reg}(J), \operatorname{reg}(K), \operatorname{reg}(J \cap K) - 1\}.$$

Let $u \in R$ be a monomial, we set $\operatorname{Supp}(u) = \{x_i : x_i | u\}$. Let I be a monomial ideal, $\mathcal{G}(I) = \{u_1, \ldots, u_m\}$ denote the unique minimal set of monomial generators of I and we set $\operatorname{Supp}(I) := \bigcup_{i=1}^{m} \operatorname{Supp}(u_i)$. The following lemmas are well known.

Lemma 2.4 [19, Lemma 3.4] *Let $R_1 = k[x_1, \ldots, x_m]$ and $R_2 = k[x_{m+1}, \ldots, x_n]$ be two polynomial rings, $I \subseteq R_1$ and $J \subseteq R_2$ be two nonzero homogeneous ideals. Then*

$$\operatorname{reg}(I + J) = \operatorname{reg}(I) + \operatorname{reg}(J) - 1.$$

Lemma 2.5 [10, Lemma 2.3] *Let I, J be two monomial ideals such that $\operatorname{Supp}(I) \cap \operatorname{Supp}(J) = \phi$. Then* $\operatorname{reg}(IJ) = \operatorname{reg}(I) + \operatorname{reg}(J)$.

Lemma 2.6 [10, Lemma 1.2] *Let $0 \to A \to B \to C \to 0$ be short exact sequence of finitely generated graded R-modules. Then*
$\operatorname{reg}(B) \leq \max\{\operatorname{reg}(A), \operatorname{reg}(C)\}$ *and the equality holds if* $\operatorname{reg}(A) - 1 \neq \operatorname{reg}(C)$.

Lemma 2.7 [12, Lemma 3.1] *Let $G = (V, E)$ be a simple graph. If G' is an induced subgraph of G, then* $\operatorname{reg}(I(G')) \leq \operatorname{reg}(I(G))$.

The following two corollaries are based on the regularity of edge ideal in path and cycle.

Corollary 2.8 [3, Theorem 4.7] *Let G be a path of length n denoted as P_n. Then*

(a) $\operatorname{reg}(I(P_n)) = \lfloor \frac{n+2}{3} \rfloor + 1$,

(b) $\operatorname{reg}(I(P_n)) = \operatorname{reg}(I(P_{n-3})) + 1$ *for* $n \geq 4$.

Corollary 2.9 [3, Theorem 4.7, Theorem 5.2] *Let G be a cycle of length n denoted as C_n. Then*

(a) *if* $n \equiv 0, 1 (\bmod 3)$, *then* $\operatorname{reg}(I(G)) = \operatorname{reg}(I(G \setminus \{x\})) = \operatorname{reg}(I(G \setminus N[x])) + 1$ *except* $n = 3, 4$ *and* $\operatorname{reg}(I(G)) = \operatorname{reg}(I(G \setminus \{x\})) = 2$ *for* $n = 3, 4$.

(b) *if* $n \equiv 2 (\bmod 3)$, *then* $\operatorname{reg}(I(G)) = \operatorname{reg}(I(G \setminus \{x\})) + 1 = \operatorname{reg}(I(G \setminus N[x])) + 1$.

In order to deal with non square-free monomial ideals, polarization is proved to be a powerful process to obtain a square-free monomial ideal from a given monomial ideal.

Definition 2.10 Suppose that $u = x_1{}^{a_1} \cdots x_n{}^{a_n}$ is a monomial in R. Then we define the polarization of u to be the square-free monomial

$$\mathcal{P}(u) = x_{11}x_{12}\cdots x_{1a_1}x_{21}x_{22}\cdots x_{2a_2}\cdots x_{n1}x_{n2}\cdots x_{na_n}$$

in the polynomial ring $R^{\mathcal{P}} = k[x_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq a_i]$. If $I \subseteq R$ is a monomial ideal with $\mathcal{G}(I) = \{u_1, \ldots, u_m\}$, the polarization of I, denoted by $I^{\mathcal{P}}$ is defined as:

$$I^{\mathcal{P}} = (\mathcal{P}(u_1), \ldots, \mathcal{P}(u_m))$$

which is a square-free monomial ideal in the polynomial ring $R^{\mathcal{P}}$.

The following lemma shows that the regularity is preserved under polarization.

Lemma 2.11 [13, Corollary 1.6.3] *Let $I \subset R$ be a monomial ideal and $I^{\mathcal{P}} \subset R^{\mathcal{P}}$ its polarization. Then*

(a) $\beta_{ij}(I) = \beta_{ij}(I^{\mathcal{P}})$ *for all i and j,*
(b) $\operatorname{reg}(I) = \operatorname{reg}(I^{\mathcal{P}})$.

Next we see the connection of square-free monomial ideals with hypergraphs and labeled hypergraphs.

2.1 Hypergraph

A hypergraph $\mathcal{H}$ over $X = \{x_1, \ldots, x_n\}$ is a pair $\mathcal{H} = (X, \mathscr{E})$ where X is the set of elements called vertices and $\mathscr{E}$ is a set of non-empty subsets of X called hyperedges or edges. A hypergraph $\mathcal{H}$ is simple if there is no nontrivial containment between any pair of its edges.

The following construction gives a one-to-one correspondence between square-free monomial ideals in $R = k[x_1, \ldots, x_n]$ and simple hypergraphs over X.

Definition 2.12 Let $\mathcal{H}$ be a simple hypergraph on X. For a subset $E \subset X$, let x^E denote the monomial $\prod_{x_i \in E} x_i$. Then the edge ideal of $\mathcal{H}$ is defined as

$$I(\mathcal{H}) = (x^E | E \subseteq X \text{ is an edge in } \mathcal{H}) \subset R.$$

2.2 Labeled Hypergraph

The labeled hypergraph associated to a given square-free monomial ideal I introduced in [16]. In the definition of labeled hypergraph, generators of the ideal correspond to vertices of the hypergraph and the edges of the hypergraph correspond to variables which are obtained by the divisibility relations between the minimal generators of the ideal.

Definition 2.13 [16] Let $I \subset R = k[x_1, \ldots, x_n]$ be a square-free monomial ideal with minimal monomial generating set $\{f_1, \ldots, f_\mu\}$. The labeled hypergraph of I is the tuple $H(I) = (V, X, E, \mathcal{E})$. The set $V = [\mu]$ is called the vertex set of H. The set $\mathcal{E}$ is called the edge set of $H(I)$ and is the image of the function $E : \{x_1, \ldots, x_n\} \longrightarrow \mathscr{P}(V)$ defined by $E(x_i) = \{j : x_i \text{ divides } f_j\}$ where $\mathscr{P}(V)$ represents the power set of V. Here the set $X = \{x_i : E(x_i) \neq \emptyset\}$.

The label of an edge $F \subseteq \mathcal{E}$ is defined as the collection of variables $x_i \in \{x_1, \ldots, x_n\}$ such that $E(x_i) = F$. The number $|X|$ counts the number of labels appearing in $H(I)$ while $|\mathcal{E}|$ counts the number of distinct edges. A vertex $v \in V$ is closed if $\{v\} \in \mathcal{E}$, otherwise, v is open. An edge $F \in \mathcal{E}$ of $H(I)$ is called simple if $|F| \geq 2$ and F has no proper subedges other than $\emptyset$. If every open vertex is contained in exactly one simple edge, then we say that $H(I)$ has isolated simple edges.

Example 2.14 Let $I = (x_1x_3x_5, x_1x_2x_3, x_3x_4x_5, x_4x_5x_6) \subset k[x_1, \ldots, x_6]$. Let $f_1 = x_1x_3x_5, f_2 = x_1x_2x_3, f_3 = x_3x_4x_5$ and $f_4 = x_4x_5x_6$. Then $V = \{1, 2, 3, 4\}$, $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, and $\mathcal{E} = \{\{1, 2\}, \{2\}, \{1, 2, 3\}, \{3, 4\}, \{1, 3, 4\}, \{4\}\}$. See Figure 1.

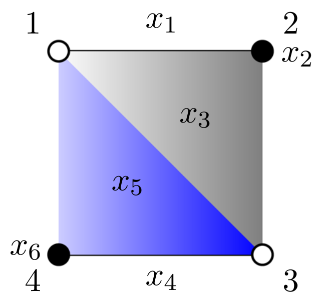

Figure 1 The labeled hypergraph of $I = (x_1x_3x_5, x_1x_2x_3, x_3x_4x_5, x_4x_5x_6)$

3 Some results of regularity in weighted Oriented graphs

In this section, we compute the regularity of $R/I(D)$ for certain class of weighted oriented graph D by connecting their polarized edge ideal with the labeled hypergraph and using the technique of Betti splitting. In this section we have considered a particular type of weighted oriented graph having property P as defined in the introudcion.

The regularity of edge ideal of weighted oriented graph having property P was first studied by Beyarslan et al. in the following result.

Proposition 3.1 [4, Corollary 3.1] *Let D be a weighted oriented graph having property P with weight function w on the vertices $x_1, \ldots, x_n$. Then*

$$\operatorname{reg}(R/I(D)) = \sum_{i=1}^{n} w_i - |E(D)|.$$

Beyarslan et al. have proved the above result using the concept of labeled hypergraph described in [16]. We noticed that the following result of Lin and McCullough using the concept of isolated simple edges of labeled hypergraphs will be useful for calculating the regularity of some new class of weighted oriented graphs.

Proposition 3.2 [16, Theorem 4.12] *Let $I \subset R$ be a square-free monomial ideal and suppose that $H(I) = (V, X, E, \mathcal{E})$ has isolated simple edges. Then*

$$\operatorname{reg}(R/I) = |X| - |V| + \sum_{\substack{F \in \mathcal{E} \\ F \text{simple}}} (|F| - 1).$$

The following theorem shows that the regularity of edge ideal of one or more weighted oriented graphs with property P remains unchanged even after adding new edges among the connected and disconnected components.

Theorem 3.3 *Let $D_1, D_2, \ldots, D_s$ for $s \geq 1$ are the weighted oriented graphs having property P with weight function w on vertex sets $\{x_{1_1}, \ldots, x_{n_{11}}\}$,$\{x_{1_2}, \ldots, x_{n_{22}}\}, \ldots, \{x_{1_s}, \ldots, x_{n_{ss}}\}$ respectively. Let D be a weighted oriented graph obtained by adding k new oriented edges among $D_1, D_2, \ldots, D_s$ where every edge is of the form (x_{a_i}, x_{b_j}) for some $x_{a_i} \in V(D_i)$, $x_{b_j} \in V(D_j)$ (i may equal with j) with $w_{a_i}, w_{b_j} \geq 2$ and no vertex of $N^-_{D_j}(x_{b_j})$ is a leaf vertex in D_j. Then*

$$\operatorname{reg}(R/I(D)) = \operatorname{reg}(R/I(D_1)) + \cdots + \operatorname{reg}(R/I(D_s)).$$

Proof Here $V(D) = V(D_1) \cup \cdots \cup V(D_s) = \{x_{1_1}, \ldots, x_{n_{11}}, \ldots, x_{1_s}, \ldots, x_{n_{ss}}\}$. Let $|E(D_1)| = e_1, \ldots, |E(D_s)| = e_s$, then $|E(D)| = e_1 + \cdots + e_s + k$. Let $I(D_1), \ldots, I(D_s), I(D)$ be the edge ideals of the weighted oriented graphs $D_1, \ldots, D_s, D$ respectively. Let $m_1, \ldots, m_{e_1+\cdots+e_s+k}$ be the minimal generators of the polarized ideal $I(D)^{\mathcal{P}}$. Suppose $k_1, \ldots, k_s$ number of new edges are oriented towards $D_1, \ldots, D_s$ where

$k_1+\cdots+k_s=k$. Let the k_1 new edges are oriented towards r_1 vertices of D_1 where for any vertex x_{j_1} among those r_1 verices $w_{j_1}\geq 2$ and no vertex of $N^-_{D_1}(x_{j_1})$ is a leaf vertex in D_1. Now we consider the labeled hypergraph of $I(D)^{\mathcal{P}}$ i.e. $H(I(D)^{\mathcal{P}})=(V,X,E,\mathcal{E})$ where $V=[e_1+\cdots+e_s+k]$. Without loss of generality let x_{q_1} is one of those r_1 vertices and l_{1_1} number of new edges are oriented towards x_{q_1}. Since D_1 has property P, $|N^-_{D_1}(x_{q_1})|=1$ and so $|N^-_D(x_{q_1})|=l_{1_1}+1$. Let the generators corresponding to those $l_{1_1}+1$ edges numbered as $d_o,d_1,d_2,\ldots,d_{l_{1_1}}$ where each $d_i\in[e_1+\cdots+e_s+k]$. Here $E(x_{q_1 i})=\{d_o,d_1,d_2,\ldots,d_{l_{1_1}}\}$ for $2\leq i\leq w_{q_1}$. Let $F_{1_1}=E_{x_{q_1 2}}$. Then $F_{1_1}\subseteq\mathcal{E}$ with label $\{x_{q_1 i},2\leq i\leq w_{q_1}\}$. Since no vertex of $N^-_{D_1}(x_{q_1})$ is a leaf vertex in D_1, then no vertex of $N^-_D(x_{q_1})$ is a leaf vertex in D. Thus there does not exist any element of X which lies in the generators of $I(D)^{\mathcal{P}}$ corresponding to some proper subset of F_{1_1} which implies F_{1_1} is a simple edge. Let us assume $l_{2_1},\ldots,l_{r_{1_1}}$ number of new edges are oriented towards remaining r_1-1 vertices of D_1, then similarly we get $F_{2_1},\ldots,F_{r_{1_1}}$ are the simple edges with cardinality $l_{2_1}+1,\ldots,l_{r_{1_1}}+1$ respectively. Thus $|F_{j_1}|=l_{j_1}+1$ for $1\leq j\leq r_1$ where $l_{1_1}+\cdots+l_{r_{1_1}}=k_1$. Let k_i new edges are oriented towards r_i vertices of D_i for $2\leq i\leq s$ by the definition of new edges. If we assume $l_{1_i},\ldots,l_{r_{i_i}}$ number of new edges are oriented towards r_i vertices of D_i, then similarly we get $F_{1_i},\ldots,F_{r_{i_i}}$ are the simple edges with cardinality $l_{1_i}+1,\ldots,l_{r_{i_i}}+1$ respectively for $2\leq i\leq s$ i.e. $|F_{j_i}|=l_{j_i}+1$ for $1\leq j\leq r_i$, $2\leq i\leq s$ where $l_{1_i}+\cdots+l_{r_{i_i}}=k_i$ for each i. Let $F=F_{1_1}\cup\cdots\cup F_{r_{1_1}}\cup\cdots\cup F_{1_s}\cup\cdots\cup F_{r_{s_s}}$ and $C=V\setminus F$. Let $V_i\subset V$ be the set of vertices corresponding to the minimal generators of $I(D_i)^{\mathcal{P}}$ for $1\leq i\leq s$ in $H(I(D)^{\mathcal{P}})$.

For $c\in C\cap V_1$, let $m_c=x_{i_1 1}\prod_{t=1}^{w_{j_1}}x_{j_1 t}$ i.e. a minimal generator of $I(D)^{\mathcal{P}}$ corresponding to some edge (x_{i_1},x_{j_1}) of D_1. If x_{j_1} is a leaf in both D_1 and D, then m_c is the only minimal generator of $I(D)^{\mathcal{P}}$ which is divisible by $x_{j_1 1}$ and therefore $\{c\}\in\mathcal{E}$ with label $\{x_{j_1 t},1\leq t\leq w_{j_1}\}$. In case of x_{j_1} is a leaf in D_1 but not in D, atleast one new edge is oriented away from x_{j_1}, then by definition of new edges $w_{j_1}\geq 2$ and m_c is the only minimal generator of $I(D)^{\mathcal{P}}$ which is divisible by $x_{j_1 2}$. Therefore $\{c\}\in\mathcal{E}$ with label $\{x_{j_1 t},2\leq t\leq w_{j_1}\}$. If x_{j_1} is not a leaf in D_1, then by assumption since x_{j_1} is not a source, either $w_{j_1}\geq 2$ or x_{i_1} is a leaf in D_1. If x_{i_1} is a leaf in D_1, then $w_{i_1}=1$. Thus none of the new edges are connected with x_{i_1} and x_{i_1} is a leaf in D, then m_c is the only minimal generator of $I(D)^{\mathcal{P}}$ which is divisible by $x_{i_1 1}$ and $\{c\}\in\mathcal{E}$ with label $\{x_{i_1 1}\}$. If x_{i_1} is not a leaf in D_1 then $w_{j_1}\geq 2$ in D_1 and so is in D. By the property P of D_1, at most one edge is oriented into the vertex x_{j_1} in D_1 and so is in D because no new edge is oriented towards x_{j_1}. Then m_c is divisible by $x_{j_1 2}$ and none of any other generator of $I(D)^{\mathcal{P}}$ is divisible by $x_{j_1 2}$. Thus $\{c\}\in\mathcal{E}$ with label $\{x_{j_1 t},2\leq t\leq w_{j_1}\}$. Therefore for every $c\in C\cap V_1$, $\{c\}\in\mathcal{E}$. By the similar arguement for every $c\in C\cap V_i$, $\{c\}\in\mathcal{E}$ where $2\leq i\leq s$. So every $c\in C$ is closed. Here each of the remaining edges of $\mathcal{E}$ is some image $E(x_{p_i 1})$ where x_{p_i} is one of the non-leaf vertex of D_i for some $i\in[s]$, $p\in[n_i]$ and it contains either one F_{j_i} for some $j\in[r_i]$ or atleast one {c} for some $c\in C\cap V_i$ as a proper subset. Thus they are not simple. Therefore F_{j_i}'s are the only simple edges in the labeled hypergraph $H(I(D)^{\mathcal{P}})$ and by the definition of F_{j_i}'s no two F_{j_i}'s have a common element which implies every open vertex is contained in exactly one simple edge i.e. $H(I(D)^{\mathcal{P}})$ has isolated simple edges. Hence by Lemma 2.11, Proposition 3.2 and Proposition 3.1, we have

$$\begin{aligned}
\operatorname{reg}(R/I(D)) &= |X|-|V|+\sum_{i=1}^{r_1}(|F_{i_1}|-1)+\cdots+\sum_{i=1}^{r_s}(|F_{i_s}|-1)\\
&= \sum_{v\in V(D_1)}w(v)+\cdots+\sum_{v\in V(D_s)}w(v)-(e_1+\cdots+e_s+k)\\
&\quad+(l_{1_1}+\cdots+l_{r_{1_1}})+\cdots+(l_{1_s}+\cdots+l_{r_{s_s}})\\
&= \sum_{v\in V(D_1)}w(v)+\cdots+\sum_{v\in V(D_s)}w(v)-(e_1+\cdots+e_s+k)+k_1+\cdots+k_s\\
&= \sum_{v\in V(D_1)}w(v)-e_1+\cdots+\sum_{v\in V(D_s)}w(v)-e_s\\
&= \operatorname{reg}(R/I(D_1))+\cdots+\operatorname{reg}(R/I(D_s)).
\end{aligned}$$

□

Corollary 3.4 *Let D be a weighted oriented graph having property P with weight function w on the vertices $x_1,\ldots,x_n$. Let D' be a weighted oriented graph obtained by adding k new oriented edges where each edge is of the form (x_i,x_j) for some $x_i,x_j\in V(D)$ with $w_i,w_j\geq 2$ and no vertex of $N_D^-(x_j)$ is a leaf vertex in D. Then*

$$\mathrm{reg}(R/I(D'))=\mathrm{reg}(R/I(D)).$$

Proof The proof directly follows from Theorem 3.3 for $s=1$. □

In the next two corollaries we give application of Corollary 3.4 into some particular kind of weighted oriented graphs.

A graph G is called a dumbbell graph if G contains two cycles C_n and C_m of length n and m respectively joined by a path P_r of length r and we denote it by $C_n\cdot P_r\cdot C_m$.

A path or cycle is said to be naturally oriented if all of its edges oriented in same direction. In a naturally oriented unicyclic graph, the cycle is naturally oriented and each edge of the tree connected with the cycle oriented away from the cycle. A naturally oriented dumbbell graph is the union of two naturally oriented cycles and a naturally oriented path joining them.

Corollary 3.5 *Let $D'=(V(D'),E(D'),w)$ be a weighted naturally oriented dumbbell graph whose underlying graph is $G=C_n\cdot P_1\cdot C_m$ where $C_n=x_1\ldots x_nx_1, P_1=x_1y_1$ and $C_m=y_1\ldots y_my_1$ with $w(x)\geq 2$ for any vertex x. Then*

$$\mathrm{reg}(R/I(D'))=\sum_{x\in V(D')}w(x)-|E(D')|+1.$$

Proof Here $V(D')=\{x_1,\ldots,x_n,y_1,\ldots,y_m\}$. Without loss of generality we give orientation to D' as shown in Figure 2. Let $D=D'\setminus e$ where $e=(y_m,y_1)$. Since D is a weighted naturally oriented unicyclic graph, it has property P. Thus by Proposition 3.1, we have $\mathrm{reg}(R/I(D))=\sum_{x\in V(D)}w(x)-|E(D)|=\sum_{x\in V(D')}w(x)-(|E(D')|-1)$. By adding the oriented edge e to D we get D'. Hence by Corollary 3.4, $\mathrm{reg}(R/I(D'))=\mathrm{reg}(R/I(D))=\sum_{x\in V(D')}w(x)-|E(D')|+1$. □

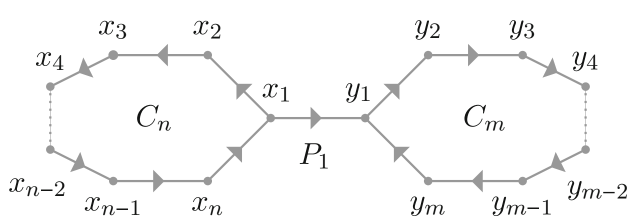

Figure 2 Weighted naturally oriented Dumbbell graph($G=C_n\cdot P_1\cdot C_m$)

Remark 3.6 Similarly we can find the regularity of edge ideal of weighted naturally oriented dumbbell graph when the two naturally oriented cycles are joined by a naturally oriented path of length r for $r \geq 2$.

Corollary 3.7 *Let D be a weighted naturally oriented cycle whose underlying graph is $C_n = x_1 \ldots x_n x_1$ with $w(x) \geq 2$ for any vertex x. Let D_k be a weighted oriented graph we get after addition of k diagonals in any direction to D for $1 \leq k \leq \binom{n}{2} - n$ and here $D_{\binom{n}{2}-n}$ is a weighted oriented complete graph. Then for each k,*

$$\operatorname{reg}(R/I(D_k)) = \operatorname{reg}(R/I(D)) = \sum_{i=1}^{n} w_i - n.$$

Proof Here $V(D_k) = V(D) = \{x_1, \ldots, x_n\}$ for each k. Since D is a weighted naturally oriented cycle, it has property P. Thus by Proposition 3.1, $\operatorname{reg}(R/I(D)) = \sum_{i=1}^{n} w_i - n$. Here D_k is obtained by adding k diagonals with any direction to D, for $1 \leq k \leq \binom{n}{2} - n$. Hence by Corollary 3.4, we have $\operatorname{reg}(R/I(D_k)) = \operatorname{reg}(R/I(D)) = \sum_{i=1}^{n} w_i - n$ for each k. □

As some application of Theorem 3.3, we derive the formulas for regularity of edge ideals of some weighted oriented graphs whose underlying graphs are the join of two cycles and complete m-partite graph.

The join of two simple graphs G_1 and G_2, denoted by $G_1 * G_2$ is a graph on the vertex set $V(G_1) \sqcup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$ together with all the edges joining $V(G_1)$ and $V(G_2)$.

A graph G is $m-$partite graph if $V(G) = V_1 \sqcup \cdots \sqcup V_m$ where V_i's are independent set and this $m-$partite graph is complete $m-$partite graph if $\{x, y\} \in E(G)$ if and only if $x \in V_i$, $y \in V_{i+1}$ for $1 \leq i \leq m$ where $V_{m+1} = V_1$.

Corollary 3.8 *Let D_1 and D_2 be two weighted naturally oriented cycles whose underlying graphs are $C_n = x_1 \ldots x_n x_1$ and $C_m = y_1 \ldots y_m y_1$ respectively with $w(v) \geq 2$ for any vertex v. Let D'_k be a weighted oriented graph we get after addition of k oriented edges joining $V(G_1)$ and $V(G_2)$ in any direction between D_1 and D_2 for $1 \leq k \leq mn$ and here D'_{mn} is a weighted oriented graph whose underlying graph is $C_n * C_m$. Then for each k,*

$$\operatorname{reg}(R/I(D'_k)) = \operatorname{reg}(R/I(D_1)) + \operatorname{reg}(R/I(D_2)) = \sum_{i=1}^{n} w_{x_i} + \sum_{i=1}^{m} w_{y_i} - (n+m).$$

Proof Here $V(D'_k) = V(D_1) \cup V(D_2) = \{x_1, \ldots, x_n, y_1, \ldots, y_m\}$ for $1 \leq k \leq mn$. Since D_1 and D_2 are weighted naturally oriented cycle, they have property P. Thus by Proposition 3.1, $\operatorname{reg}(R/I(D_1)) = \sum_{i=1}^{n} w_{x_i} - n$ and $\operatorname{reg}(R/I(D_2)) = \sum_{i=1}^{m} w_{y_i} - m$. Here D'_k is obtained by adding k new oriented edges joining $V(C_n)$ to $V(C_m)$ in any direction between D_1 and D_2 for $1 \leq k \leq mn$. Hence by Theorem 3.3 for $s = 2$, we have $\operatorname{reg}(R/I(D'_k)) = \operatorname{reg}(R/I(D_1)) + \operatorname{reg}(R/I(D_2)) = \sum_{i=1}^{n} w_{x_i} + \sum_{i=1}^{m} w_{y_i} - (n+m)$ for each k. □

In the following corollary, we give a short proof of [21, Theorem 5.1] using Theorem 3.3.

Corollary 3.9 *Let $D = (V(D), E(D), w)$ is a weighted oriented complete m-partite graph for $m \geq 3$ with vertex set $V(D) = \bigsqcup_{i=1}^{m} V_i$ and edge set $E(D) = \bigsqcup_{i=1}^{m} E(D_i)$ where D_i is a weighted oriented complete bipartite graph on $V_i \sqcup V_{i+1}$ and every edge of $E(D_i)$ is of the form (u, v) with $u \in V_i$, $v \in V_{i+1}$ for $1 \leq i \leq m$ by setting $V_{m+1} = V_1$. If $w(x) \geq 2$ for all $x \in V(D)$, then*

$$\operatorname{reg}(R/I(D)) = \sum_{x\in V(D)} w(x) - |V(D)|.$$

Proof Let $V_i = \{x_{1i}, x_{2i}, \ldots, x_{n_i i}\}$ for $1 \leq i \leq m$. For $1 \leq i \leq m$, let D'_i be the oriented graph over vertex set $V(D'_i) = V_i \sqcup V_{i+1}$ and the edge set $E(D'_i) = \{(x_{1i}, x_{1i+1}), (x_{2i}, x_{2i+1}), \ldots, (x_{n_i i}, x_{n_i i+1})\} \cup \{(x_{1i}, x_{n_i+1\, i+1}), (x_{1i}, x_{n_i+2\, i+1}), \ldots, (x_{1i}, x_{n_{i+1} i+1})\}$ if $n_i < n_{i+1}$ or the edge set $E(D'_i) = \{(x_{1i}, x_{1i+1}), (x_{2i}, x_{2i+1}), \ldots, (x_{n_{i+1} i}, x_{n_{i+1} i+1})\}$ if $n_i \geq n_{i+1}$.

Let $D' = (V(D'), E(D'), w)$ be the weighted oriented $m-$partite graph over the vertex set $V(D') = \bigsqcup_{i=1}^{m} V_i$ and the edge set $E(D') = \bigsqcup_{i=1}^{m} E(D'_i)$ with the same weight function as in D. Observe that in each D'_i, there is exactly one edge oriented into each vertex of V_{i+1} which implies in D', exactly one edge oriented into each vertex of $V(D')$. Thus each component of D' is with property P. Hence by Proposition 3.1, $\operatorname{reg}(R/I(D')) = \sum_{x\in V(D')} w(x) - |E(D')| = \sum_{x\in V(D')} w(x) - |V(D')| = \sum_{x\in V(D)} w(x) - |V(D)|$. Here D is obtained by adding all the edges of the set $E(D) \setminus E(D')$ to D'. If there is s components in D' for some $s \geq 1$, then by Theorem 3.3, we have

$$\operatorname{reg}(R/I(D)) = \operatorname{reg}(R/I(D')) = \sum_{x\in V(D)} w(x) - |V(D)|.$$

□

In the following theorem, we show that the regularity of edge ideal of a weighted oriented graph D with property P remains same even after adding certain type of edges from new vertices oriented towards a single vertex of it.

Theorem 3.10 *Let D be a weighted oriented graph having property P with weight function w on the vertices $x_1, \ldots, x_n$. Let D'_k be a weighted oriented graph after adding k new oriented edges to D at x_p with $w_p \geq 2$ for a fixed $p \in [n]$ where each edge is of the form (x_{n+i}, x_p) for some $i \in [k]$ and each x_{n+i} is a new vertex. Then*

$$\operatorname{reg}(I(D'_k)) = \operatorname{reg}(I(D)) = \sum_{i=1}^{n} w_i - |E(D)| + 1.$$

Proof Here $V(D) = \{x_1, \ldots, x_n\}$. Without loss of generality let $x_p = x_n$. We prove this theorem by applying induction on the number of new oriented edges added to D at x_n.

Base case: If $k = 0$, then the proof follows trivially.

For $k \geq 1$, let D'_k be a weighted oriented graph after adding the k new oriented edges $(x_{n+1}, x_n), (x_{n+2}, x_n), \ldots, (x_{n+k}, x_n)$ from new vertices to x_n in D where $w_n \geq 2$. Here $I(D'_k) = I(D'_{k-1}) + x_{n+k} x_n^{w_n}$ where $D'_{k-1} = D'_k \setminus \{x_{n+k}\}$. Then $I(D'_k)^{\mathcal{P}} = I(D'_{k-1})^{\mathcal{P}} + x_{n+k,1} \prod_{j=1}^{w_n} x_{nj}$. Note that in D'_{k-1}, there are $k-1$ new oriented edges added to D at x_n. Let $J = x_{n+k,1} \prod_{j=1}^{w_n} x_{nj}$ and $K = I(D'_{k-1})^{\mathcal{P}}$. Since J has linear resolution, $I(D'_k)^{\mathcal{P}} = J + K$ is a Betti splitting. Here $\operatorname{reg}(J) = w_n + 1$. By Lemma 2.11, Proposition 3.1 and induction hypothesis, we have $\operatorname{reg}(K) = \operatorname{reg}(I(D'_{k-1})) = \operatorname{reg}(I(D)) = \sum_{x\in V(D)} w(x) \ -|E(D)| + 1$. Now we want to compute $\operatorname{reg}(J \cap K) - 1$ (Fig. 3).

Let $N^-_{D'_k}(x_n) = \{x_{n-1}, x_{n+1}, x_{n+2}, \ldots, x_{n+k}\}$ where $x_{n-1} \in V(D)$ and $x_{n+1}, x_{n+2}, \ldots, x_{n+k}$ are the new vertices in D'_k. Let $N^+_D(x_n) = \{x_{n_1}, x_{n_2}, \ldots, x_{n_r}, x_{n_{r+1}}, \ldots, x_{n_s}\}$ among which $w_{n_i} = 1$ for $1 \leq i \leq r$ and $w_{n_i} \geq 2$ for $r+1 \leq i \leq s$ in D. Let $N^-_D(x_{n-1}) = \{x_{n-2}\}$ and $N^+_D(x_{n-1}) = \{x_n, x_{n-1_1}, x_{n-1_2}, \ldots, x_{n-1_p}, x_{n-1_{p+1}}, \ldots, x_{n-1_t}\}$ such that $x_{n-1_1}, x_{n-1_2}, \ldots, x_{n-1_p}$ are leaf vertices and $x_{n-1_{p+1}}, x_{n-1_{p+2}}, \ldots, x_{n-1_t}$ are non-leaf vertices in D. Here the r vertices $x_{n_1}, x_{n_2}, \ldots, x_{n_r}$ are leaf vertices and the $t-p$ vertices $x_{n-1_{p+1}}, x_{n-1_{p+2}}, \ldots, x_{n-1_t}$ are of weight ≥ 2 in D by the property P.

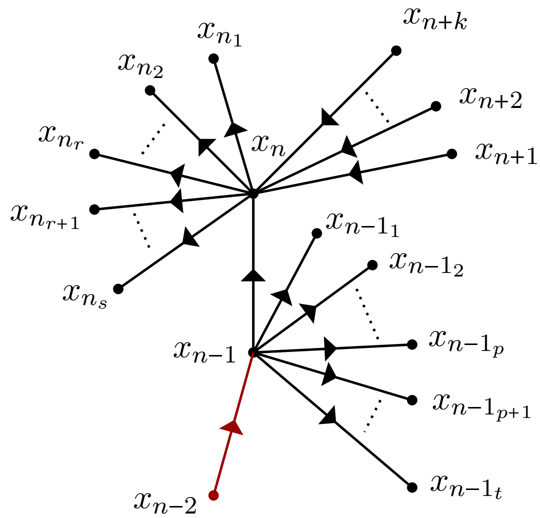

Figure 3 Neighbourhood of x_{n-1} and x_n in weighted oriented graph D'_k

Let $J \cap K = JL = (x_{n+k,1}\prod_{j=1}^{w_n} x_{nj})((x_{n+2,1}, x_{n+3,1}, \ldots, x_{n+k,1}, x_{n-1,1}, x_{n_1,1}, x_{n_2,1}, \ldots, x_{n_r,1}, \prod_{j=1}^{w_{n_{r+1}}} x_{n_{r+1},j}, \prod_{j=1}^{w_{n_{r+2}}} x_{n_{r+2},j}, \ldots, \prod_{j=1}^{w_{n_s}} x_{n_s,j}) + (I(D \setminus \{x_n, x_{n-1}\})^{\mathcal{P}})$. Let $L_1 = (\prod_{j=1}^{w_{n_{r+1}}} x_{n_{r+1},j}\ ,\ \prod_{j=1}^{w_{n_{r+2}}} x_{n_{r+2},j}, \ldots, \prod_{j=1}^{w_{n_s}} x_{n_s,j})$ and $L_2 = (I(D \setminus \{x_n, x_{n-1}\}))^{\mathcal{P}}$. Note that $\operatorname{reg}(L) = \operatorname{reg}(L_1 + L_2)$. By expressing L_1 as $(x_{n_{r+1},1}\prod_{j=2}^{w_{n_{r+1}}} x_{n_{r+1},j}, x_{n_{r+2},1}\prod_{j=2}^{w_{n_{r+2}}} x_{n_{r+2},j}\ ,\ldots, x_{n_s,1}\prod_{j=2}^{w_{n_s}} x_{n_s,j})$, we can think of $L_1 + L_2$ as the polarized edge ideal of the weighted oriented graph with $|E(D)| - (t + r + 2)$ edges obtained from $D \setminus \{x_n, x_{n-1}\}$ by adding one leaf of weight $w_{n_i} - 1$ to each x_{n_i} for $i = r + 1, \ldots, s$. Observe that in this graph the $s - r$ vertices $x_{n_{r+1}}, \ldots, x_{n_s}$, the $t - p$ vertices $x_{n-1_{p+1}}, \ldots, x_{n-1_t}$ become source vertices and each of its component is with property P. So we can apply Proposition 3.1 to compute the $\operatorname{reg}(L_1 + L_2)$.

Case-I: Let $x_{n-2} \in N_D^+(x_n)$. Then by the property P, $x_{n-2} \in \{x_{n_{r+1}}, x_{n_{r+2}}, \ldots, x_{n_s}\}$ and $w_{n-1} \geq 2$. Thus by Lemma 2.5 and Proposition 3.1, we have

$$\begin{aligned}
\operatorname{reg}(J \cap K) - 1 &= \operatorname{reg}(J) + \operatorname{reg}(L) - 1\\
&= \operatorname{reg}(J) + \operatorname{reg}(L_1 + L_2) - 1\\
&= (w_n + 1) + \sum_{x \in V(D)\setminus V_1} w(x) + (1 + w_{n_{r+1}} - 1) + (1 + w_{n_{r+2}} - 1)\\
&\quad + \cdots + (1 + w_{n_s} - 1) + (t - p) - [|E(D)| - (t + r + 2)] + 1 - 1\\
&= \sum_{x \in V(D)\setminus V_2} w(x) - |E(D)| + (t - p) + (t + r + 2) + 1
\end{aligned}$$

where $V_1 = \{x_n, x_{n-1}, x_{n_1}, x_{n_2}, \ldots, x_{n_r}, x_{n_{r+1}}, \ldots, x_{n_s}, x_{n-1_1}, \ldots, x_{n-1_p}, x_{n-1_{p+1}}, \ldots, x_{n-1_t}\}$ and $V_2 = V_1 \setminus \{x_n, x_{n_{r+1}}, \ldots, x_{n_s}\}$.

Since the sum of the weights of vertices of $V_2 = w_{n-1} + (w_{n_1} + w_{n_2} + \cdots + w_{n_r}) + (w_{n-1_1} + w_{n-1_2} + \cdots + w_{n-1_p}) + (w_{n-1_{p+1}} + w_{n-1_{p+2}} + \cdots + w_{n-1_t}) \geq 2 + r + p + 2(t - p) = (t - p) + (t + r + 2)$, $\operatorname{reg}(J \cap K) - 1 \leq \operatorname{reg}(K)$. Thus by Lemma 2.11 and Corollary 2.3, we have

$$\operatorname{reg}(I(D'_k)) = \operatorname{reg}(I(D'_k)^{\mathcal{P}}) = \max\{\operatorname{reg}(J), \operatorname{reg}(K), \operatorname{reg}(J \cap K) - 1\} = \operatorname{reg}(K) = \operatorname{reg}(I(D)).$$

Case-II: Let $x_{n-2} \notin N_D^+(x_n)$. If x_{n-2} is a leaf, then x_{n-2} become a source vertex which implies $w_{n-2} = 1$ and by the property P, $w_{n-1} \geq 1$. Then we follow the same process of Case-I and see that the value of $\operatorname{reg}(J \cap K)$ remains same as in Case-I where only V_2 is replaced by $V_2 \cup \{x_{n-2}\}$. If x_{n-2} is not a leaf, $w_{n-2} \geq 1$ and by the property P, $w_{n-1} \geq 2$. Again we follow the same process of Case-I and see that the value of $\operatorname{reg}(J \cap K)$

remains same as in Case-I where V_2 also remains same. Wheather x_{n-2} is a leaf or non-leaf vertex, sum of the weights of vertices of $V_2 \geq (t-p)+(t+r+2)$. Therefore by the similar arguement as in Case-I, $\operatorname{reg}(I(D'_k)) = \operatorname{reg}(I(D))$. □

Proposition 3.11 *Let $D_1, D_2, \ldots, D_s$ for $s \geq 2$ are the weighted oriented graphs having property P with weight function w on vertex sets $\{x_{1_1}, \ldots, x_{n_{11}}\}$, $\{x_{1_2}, \ldots, x_{n_{22}}\}, \ldots, \{x_{1_s}, \ldots, x_{n_{ss}}\}$ respectively. Let D be a weighted oriented graph obtained by adding k new oriented edges among $D_1, D_2, \ldots, D_s$ where every edge is of the form (x_{a_i}, x_{b_j}) for some $x_{a_i} \in V(D_i)$, $x_{b_j} \in V(D_j)$, $i \neq j$ with $w_{a_i}, w_{b_j} \geq 2$ such that no vertex of $N^-_{D_j}(x_{b_j})$ is a leaf vertex in D_j and the set of new edges oriented towards D_t go to a single vertex of D_t for $t = 1, \ldots, s$. Let $k_1, \ldots, k_s$ number of new edges are oriented towards $D_1, \ldots, D_s$ where $k_1 + \cdots + k_s = k$ and D'_t be the new weighted oriented graph after addition of the k_t new oriented edges to D_t which are oriented towards a single vertex of it for $t = 1, \ldots, s$. Then*

$$\operatorname{reg}(R/I(D)) = \operatorname{reg}(R/I(D'_1)) + \cdots + \operatorname{reg}(R/I(D'_s)).$$

Proof By Theorem 3.3, we have $\operatorname{reg}(R/I(D)) = \operatorname{reg}(R/I(D_1)) + \cdots + \operatorname{reg}(R/I(D_s))$ and by Theorem 3.10, $\operatorname{reg}(R/I(D_t)) = \operatorname{reg}(R/I(D'_t))$ for $t = 1, \ldots, s$. Hence $\operatorname{reg}(R/I(D)) = \operatorname{reg}(R/I(D'_1)) + \cdots + \operatorname{reg}(R/I(D'_s))$. □

The above proposition partially answer the following question asked by H.T. Hà in [12].

Question 1 [12, Problem 6.8] *Let $\mathcal{H}, \mathcal{H}_1, \ldots, \mathcal{H}_s$ be simple hypergraphs over the same vertex set X and assume that $\mathscr{E}(\mathcal{H}) = \bigcup_{i=1}^{s} \mathscr{E}(\mathcal{H}_i)$. Find combinatorial conditions for the following equality to hold:*

$$\operatorname{reg}(S/I(\mathcal{H})) = \sum_{i=1}^{s} \operatorname{reg}(S/I(\mathcal{H}_i)).$$

Observation: Let $D, D'_1, D'_2, \ldots, D'_s$ and R are same as defined in Proposition 3.11. Let

$$X = \{x_{1_1 1}, \ldots, x_{1_1 w_{1_1}}, \ldots, x_{n_{11}1}, \ldots, x_{n_{11} w_{n_{11}}}, \ldots, x_{1_s 1}, \ldots, x_{1_s w_{1_s}}, \ldots, x_{n_{ss}1}, \ldots, x_{n_{ss} w_{n_{ss}}}\}$$

and $S = R^{\mathcal{P}}$. If we assume that $\mathcal{H}, \mathcal{H}_1, \ldots, \mathcal{H}_s$ be the simple hypergraphs over X such that $I(D)^{\mathcal{P}}, I(D'_1)^{\mathcal{P}}, \ldots, I(D'_s)^{\mathcal{P}}$ are the square-free monomial edge ideals $I(\mathcal{H}), I(\mathcal{H}_1), \ldots, I(\mathcal{H}_s)$ respectively then $\mathscr{E}(\mathcal{H}) = \bigcup_{i=1}^{s} \mathscr{E}(\mathcal{H}_i)$ and by Proposition 3.11 and Lemma 2.11,

$$\operatorname{reg}(S/I(\mathcal{H})) = \sum_{i=1}^{s} \operatorname{reg}(S/I(\mathcal{H}_i)).$$

4 Regularity in weighted Oriented Paths and Cycles

In this section, we relate the regularity of edge ideals of weighted oriented paths or cycles when vertices of V^+ are sinks with the regularity of edge ideals of their underlying graphs. First we compute the regularity of edge ideals of weighted oriented paths when vertices of V^+ are sinks.

We divide the set T of all weighted oriented paths when vertices of V^+ are sinks into two sets:

T_1: Set of all weighted oriented paths where the two end vertices are in V^+ and the distance between any two consecutive vertices of V^+ is 3.

Note that the length of any weighted oriented path in T_1 is multiple of 3. (See Figure 4.)

T_2: Set of remaining weighted oriented paths when the vertices of V^+ are sinks i.e.

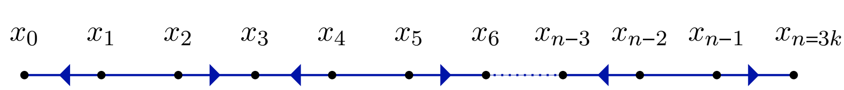

Figure 4 A weighted oriented path in T_1

$$T_2 = T \setminus T_1.$$

Remark 4.1 Let D be a weighted oriented path of length n for $n \geq 4$ in T_2 with underlying graph $G = P_n = x_0x_1\ldots x_n$. Let $D_1 = D \setminus \{x_n\}$, $D_2 = D \setminus \{x_{n-2}, x_{n-1}, x_n\}, D_1' = D \setminus \{x_0\}$ and $D_2' = D \setminus \{x_0, x_1, x_2\}$.

Case-I: Assume $n \equiv 1(\mod 3)$. Here $n \geq 4$ and $n = 3k+1$ for some $k \in \mathbb{N}$. Then the length of D_2 or D_2' is $3k-2$ which is not a multiple of 3. Thus both D_2 and D_2' are in T_2. If D_1 is in T_1, then $x_{n-1} \in V^+$ which implies $x_n \notin V^+$. So one end vertex of D_1' i.e. $x_n \notin V^+$. Hence D_1' is in T_2.

Case-II: Assume $n \equiv 2(\mod 3)$. Here $n \geq 5$ and $n = 3k+2$ for some $k \in \mathbb{N}$. Then the length of D_1 and D_2 are $3k+1$ and $3k-1$ respectively. Note that none of them is a multiple of 3. Hence both D_1 and D_2 are in T_2.

Case-III: Assume $n \equiv 0(\mod 3)$. Here $n \geq 6$ and $n = 3k$ for some $k \in \mathbb{N}$. Then the length of D_1 or D_1' is $3k-1$ which is not a multiple of 3. Thus both D_1 and D_1' are in T_2. If D_2 is in T_1, then $x_{n-3} \in V^+$. Since D is in T_2, $x_n \notin V^+$ which implies one end vertex of D_2' i.e. $x_n \notin V^+$. Hence D_2' is in T_2.

Observe that if D is in T_2 then either D_1 and D_2 or D_1' and D_2' are in T_2 in either cases. Thus without loss of generality we can rename the vertices and always assume that D_1 and D_2 are in T_2.

Theorem 4.2 *Let D be a weighted oriented path of length n in T_2 with underlying graph $G = P_n = x_0x_1\ldots x_n$. Then* $\text{reg}(I(D)) = \text{reg}(I(G)) + \sum_{x_i \in V^+} (w_i - 1)$ *where $w_i = w(x_i)$ for $x_i \in V^+$.*

Proof Here $V(D) = \{x_0, x_1, \ldots, x_n\}$. We use the method of induction on the number of edges of D and prove this theorem in different cases depending upon the position of the vertices of V^+.

Base Case: $|E(D)| \leq 3$.

Assume that $|E(D)| = 3$ and $V(D) = \{x_0, x_1, x_2, x_3\}$. If $x_0, x_1 \notin V^+$ and $x_3 \in V^+$, then $I(D) = (x_0x_1, x_1x_2, x_2x_3^{w_3})$. Let $J = (x_2x_3^{w_3})$ and $K = (x_0x_1, x_1x_2)$. Since J has linear resolution, $I(D) = J + K$ is a Betti splitting. Here $\text{reg}(J) = w_3 + 1$ and $\text{reg}(K) = 2$. Let $J \cap K = JL$ where $L = (x_0x_1, x_1)$ which implies $\text{reg}(J \cap K) = w_3 + 2$. Thus by Corollary 2.3, we have $\text{reg}(I(D)) = \max\{\text{reg}(J), \text{reg}(K), \text{reg}(J \cap K) - 1\}$ $= w_3 + 1 = \text{reg}(I(G)) + w_3 - 1$. Similarly depending upon the position of the vertices of V^+ using the Betti splitting technique for any weighted oriented path D in T_2 with $|E(D)| \leq 3$, we can show that $\text{reg}(I(D)) = \text{reg}(I(G)) + \sum_{x_i \in V^+} (w_i - 1)$.

Now we consider D to be a weighted oriented path of length $n \geq 4$ and $V(D) = \{x_0, \ldots, x_n\}$. Let $D_1 = D \setminus \{x_n\}$, $D_2 = D \setminus \{x_{n-2}, x_{n-1}, x_n\}$, $H_1 = G \setminus \{x_n\}$ and $H_2 = G \setminus \{x_{n-2}, x_{n-1}, x_n\}$ i.e. H_1 and H_2 are the corresponding underlying graphs of D_1 and D_2 respectively. Without loss of generality by Remark 4.1, we can fix x_n in one end of D such that D_1 and D_2 are in T_2.

Case-I: Assume that $x_{n-2} \notin V^+$ and $x_n \in V^+$. Let $J = (x_{n-1}x_n^{w_n})$ and $K = I(D_1)$. As J has linear resolution, $I(D) = J + K$ is a Betti splitting and $\text{reg}(J) = w_n + 1$. Since D_1 is the weighted oriented path of length $n-1$ in T_2, by induction hypothesis we get $\text{reg}(K) = \text{reg}(I(D_1)) = \text{reg}(I(H_1)) + \sum_{x_i \in V^+ \setminus \{x_n\}} (w_i - 1)$.

Let $J \cap K = JL$ where $L = (I(D_2), x_{n-2})$. Since D_2 is the weighted oriented path of length $n-3$ in T_2, by induction hypothesis we have $\text{reg}(L) = \text{reg}(I(D_2)) = \text{reg}(I(H_2)) + \sum_{x_i \in V^+ \setminus \{x_n\}} (w_i - 1)$. By Corollary 2.8, $\text{reg}(I(G)) = \text{reg}(I(H_2)) + 1$. Thus by Lemma 2.5, we have

$$\begin{aligned}
\text{reg}(J \cap K) &= \text{reg}(J) + \text{reg}(L) \\
&= \text{reg}(J) + \text{reg}(I(D_2)) \\
&= (w_n + 1) + \text{reg}(I(H_2)) + \sum_{x_i \in V^+ \setminus \{x_n\}} (w_i - 1) \\
&= \text{reg}(I(H_2)) + 1 + \sum_{x_i \in V^+ \setminus \{x_n\}} (w_i - 1) + w_n \\
&= \text{reg}(I(G)) + \sum_{x_i \in V^+ \setminus \{x_n\}} (w_i - 1) + w_n
\end{aligned}$$

By Lemma 2.7, $\text{reg}(I(H_1)) \leq \text{reg}(I(G))$. Thus by Corollary 2.3, we get

$$\begin{aligned}\operatorname{reg}(I(D)) &= \max\{\operatorname{reg}(J), \operatorname{reg}(K), \operatorname{reg}(J\cap K)-1\}\\ &= \max\left\{w_n+1, \operatorname{reg}(I(H_1))+\sum_{x_i\in V^+\setminus\{x_n\}}(w_i-1), \operatorname{reg}(I(G))+\sum_{x_i\in V^+}(w_i-1)\right\}\\ &= \operatorname{reg}(I(G))+\sum_{x_i\in V^+}(w_i-1).\end{aligned}$$

Case-II: Assume that x_{n-2} and $x_n \in V^+$.

Let $J=(x_{n-1}x_n^{w_n})$ and $K=I(D_1)$. Since J has linear resolution, $I(D)=J+K$ is a Betti splitting. Here $\operatorname{reg}(J)=w_n+1$ and by the same arguement as in Case-I, $\operatorname{reg}(K)=\operatorname{reg}(I(H_1))+\sum_{x_i\in V^+\setminus\{x_n\}}(w_i-1)$ and $\operatorname{reg}(J\cap K)=\operatorname{reg}(I(G))+\sum_{x_i\in V^+\setminus\{x_n\}}(w_i-1)+w_n$. By the same arguement as in Case-I and Corollary 2.3, $\operatorname{reg}(I(D))=\operatorname{reg}(I(G))+\sum_{x_i\in V^+}(w_i-1)$.

Case-III: Assume that $x_{n-1}\in V^+$.

Let $J=(x_nx_{n-1}^{w_{n-1}})$ and $K=I(D_1)$. Since J has linear resolution, $I(D)=J+K$ is a Betti splitting. Here $\operatorname{reg}(J)=w_{n-1}+1$ and by the same arguement as in Case-I, $\operatorname{reg}(K)=\operatorname{reg}(I(H_1))+\sum_{x_i\in V^+}(w_i-1)$ and $\operatorname{reg}(J\cap K)=\operatorname{reg}(I(G))+\sum_{x_i\in V^+\setminus\{x_{n-1}\}}(w_i-1)+w_{n-1}$. By the same arguement as in Case-I and Corollary 2.3, $\operatorname{reg}(I(D))=\operatorname{reg}(I(G))+\sum_{x_i\in V^+}(w_i-1)$.

Case-IV: Assume that $x_{n-2}\in V^+$ and $x_n\notin V^+$.

Let $J=(x_{n-1}x_n)$ and $K=I(D_1)$. Since J has linear resolution, $I(D)=J+K$ is a Betti splitting. Here $\operatorname{reg}(J)=2$ and by the same arguement as in Case-I, $\operatorname{reg}(K)=\operatorname{reg}(I(H_1))+\sum_{x_i\in V^+}(w_i-1)$ and $\operatorname{reg}(J\cap K)=\operatorname{reg}(I(G))+\sum_{x_i\in V^+\setminus\{x_{n-2}\}}(w_i-1)+w_{n-2}$. By the same arguement as in Case-I and Corollary 2.3, $\operatorname{reg}(I(D))=\operatorname{reg}(I(G))+\sum_{x_i\in V^+}(w_i-1)$.

Case-V: Assume that x_{n-2},x_{n-1} and $x_n\notin V^+$.

Let $J=(x_{n-1}x_n)$ and $K=I(D_1)$. Since J has linear resolution, $I(D)=J+K$ is a Betti splitting. Here $\operatorname{reg}(J)=2$ and by the same arguement as in Case-I, $\operatorname{reg}(K)=\operatorname{reg}(I(H_1))+\sum_{x_i\in V^+}(w_i-1)$ and $\operatorname{reg}(J\cap K)=\operatorname{reg}(I(G))+\sum_{x_i\in V^+}(w_i-1)+1$. By the same arguement as in Case-I and Corollary 2.3, $\operatorname{reg}(I(D))=\operatorname{reg}(I(G))+\sum_{x_i\in V^+}(w_i-1)$.

Hence for any weighted oriented path D of length n in T_2,

$$\operatorname{reg}(I(D))=\operatorname{reg}(I(G))+\sum_{x_i\in V^+}(w_i-1) \text{ where } w_i=w(x_i) \text{ for } x_i\in V^+.$$

□

Theorem 4.3 *Let D be a weighted oriented path of length n in T_1 with underlying graph $G=P_n=x_0x_1\cdots x_n$. Then $\operatorname{reg}(I(D))=\operatorname{reg}(I(G))+\sum_{x_i\in V^+\setminus\{x_j\}}(w_i-1)$ where $w_i=w(x_i)$ for $x_i\in V^+$ and x_j is one of the vertices of V^+ with minimum weight.*

Proof Here $V(D)=\{x_0,x_1,\ldots,x_n\}$. By the definition of T_1, $G=P_n=P_{3k}$ for some $k\in\mathbb{N}$. We use the method of induction on k (Fig. 5).

Base Case: If $k=1$, then $I(D)=(x_0^{w_0}x_1,x_1x_2,x_2x_3^{w_3})$. Let $J=(x_2x_3^{w_3})$ and $K=(x_0^{w_0}x_1,x_1x_2)$. Since J has linear resolution, $I(D)=J+K$ is a Betti splitting. Here $\operatorname{reg}(J)=w_3+1$ and $\operatorname{reg}(K)=w_0+1$. Let $J\cap$

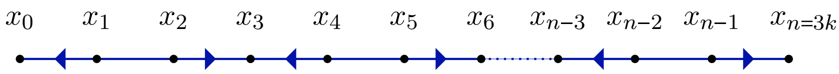

Figure 5 A weighted oriented path in T_1

$K = JL$ where $L = (x_0^{w_0}x_1, x_1)$ which implies $\operatorname{reg}(J \cap K) = w_3 + 2$. Thus by Corollary 2.3, we have $\operatorname{reg}(I(D)) = \max\{\operatorname{reg}(J), \operatorname{reg}(K), \operatorname{reg}(J \cap K) - 1\} = \max\{w_0 + 1, w_3 + 1\} = 2 + \max\{w_0 - 1, w_3 - 1\} = \operatorname{reg}(I(G)) + \max\{w_0 - 1, w_3 - 1\}$.

Now we consider the case $n = 3k$ for some $k > 1$. Let $D_1 = D \setminus \{x_n\}$, $D_2 = D \setminus \{x_{n-2}, x_{n-1}, x_n\}$, $H_1 = G \setminus \{x_n\}$ and $H_2 = G \setminus \{x_{n-2}, x_{n-1}, x_n\}$ i.e. H_1 and H_2 are the corresponding underlying graphs of D_1 and D_2 respectively. Here $I(D) = (x_0^{w_0}x_1, x_1x_2, x_2x_3^{w_3}, \ldots, x_{n-3}^{w_{n-3}}x_{n-2}, x_{n-2}x_{n-1}, x_{n-1}x_n^{w_n})$. Let $J = (x_{n-1}x_n^{w_n})$ and $K = I(D_1)$. Since J has linear resolution, $I(D) = J + K$ is a Betti splitting. Here $\operatorname{reg}(J) = w_n + 1$. Since $x_{n-1} \notin V^+$, D_1 is a weighted oriented path of length $n - 1$ in T_2. By Corollary 2.8, $\operatorname{reg}(I(G)) = \operatorname{reg}(I(H_1))$. Thus by Theorem 4.2, we have

$$\operatorname{reg}(K) = \operatorname{reg}(I(D_1)) = \operatorname{reg}(I(H_1)) + \sum_{x_i \in V^+ \setminus \{x_n\}} (w_i - 1) = \operatorname{reg}(I(G)) + \sum_{x_i \in V^+ \setminus \{x_n\}} (w_i - 1).$$

Let $J \cap K = JL$ where $L = (I(D_2), x_{n-2})$ and D_2 is a weighted oriented path of length $n - 3 = 3(k-1)$ in T_1. Thus by the induction hypothesis, we get

$$\operatorname{reg}(L) = \operatorname{reg}(I(D_2)) = \operatorname{reg}(I(H_2)) + \sum_{x_i \in V^+ \setminus \{x_n, x_m\}} (w_i - 1)$$

where x_m is one of vertices of $V^+(D_2)$ with minimum weight. By Corollary 2.8, $\operatorname{reg}(I(G)) = \operatorname{reg}(I(H_2)) + 1$. Thus by Lemma 2.5, we have

$$\begin{aligned}
\operatorname{reg}(J \cap K) &= \operatorname{reg}(J) + \operatorname{reg}(L) \\
&= \operatorname{reg}(J) + \operatorname{reg}(I(D_2)) \\
&= (w_n + 1) + \operatorname{reg}(I(H_2)) + \sum_{x_i \in V^+ \setminus \{x_n, x_m\}} (w_i - 1) \\
&= \operatorname{reg}(I(H_2)) + 1 + \sum_{x_i \in V^+ \setminus \{x_n, x_m\}} (w_i - 1) + w_n \\
&= \operatorname{reg}(I(G)) + \sum_{x_i \in V^+ \setminus \{x_n, x_m\}} (w_i - 1) + w_n
\end{aligned}$$

Therefore by Corollary 2.3, we get

$$\begin{aligned}
\operatorname{reg}(I(D)) &= \max\{\operatorname{reg}(J), \operatorname{reg}(K), \operatorname{reg}(J \cap K) - 1\} \\
&= \max\left\{ w_n + 1, \operatorname{reg}(I(G)) + \sum_{x_i \in V^+ \setminus \{x_n\}} (w_i - 1), \operatorname{reg}(I(G)) + \sum_{x_i \in V^+ \setminus \{x_m\}} (w_i - 1) \right\} \\
&= \operatorname{reg}(I(G)) + \sum_{x_i \in V^+ \setminus \{x_j\}} (w_i - 1)
\end{aligned}$$

where $x_j = \min\{x_n, x_m\}$ i.e. x_j is one of the vertices of V^+ with minimum weight.

Hence for any weighted oriented path D of length n in T_1,

$$\operatorname{reg}(I(D)) = \operatorname{reg}(I(G)) + \sum_{x_i \in V^+ \setminus \{x_j\}} (w_i - 1)$$

where $w_i = w(x_i)$ for $x_i \in V^+$ and x_j is one of the vertices of V^+ with minimum weight. □

Theorem 4.4 *Let D be a weighted oriented cycle of length n for $n \equiv 0, 1 (\bmod 3)$ with underlying graph $G = C_n = x_1 \ldots x_n x_1$ and vertices of V^+ are sinks. Then $\operatorname{reg}(I(D)) = \operatorname{reg}(I(G)) + \sum_{x_i \in V^+} (w_i - 1)$ where $w_i = w(x_i)$ for $x_i \in V^+$.*

Proof Here $V(D)=\{x_1,\ldots,x_n\}$. Without loss of generality, let $x_k \neq x_1, x_n$ be one of the vertices of V^+. Let $D_1 = D \setminus \{x_k\}$, $D_2 = D \setminus N_D[x_k]$, $H_1 = G \setminus \{x_k\}$ and $H_2 = G \setminus N_G[x_k]$ i.e. H_1 and H_2 are the corresponding underlying graphs of D_1 and D_2 respectively. By Corollary 2.9, $\text{reg}(I(G)) = \text{reg}(I(H_1)) = \text{reg}(I(H_2)) + 1$ except $n = 3, 4$ and $\text{reg}(I(G)) = \text{reg}(I(H_1)) = 2$ for $n = 3, 4$.

Now consider the exact sequence

$$0 \longrightarrow \frac{R}{(I(D):x_k^{w_k})}(-w_k) \xrightarrow{.x_k^{w_k}} \frac{R}{I(D)} \longrightarrow \frac{R}{(I(D),x_k^{w_k})} \longrightarrow 0 \tag{1}$$

Here $(I(D), x_k^{w_k}) = (I(D_1), x_k^{w_k})$ where D_1 is a weighted oriented path of length $n-2$. Since the end vertices of D_1 can not be in V^+, D_1 is in T_2. Thus by Lemma 2.4 and Theorem 4.2, we have

$$\begin{aligned}\text{reg}(I(D), x_k^{w_k}) &= \text{reg}(I(D_1)) + w_k - 1\\ &= \text{reg}(I(H_1)) + \sum_{x_i \in V^+ \setminus \{x_k\}} (w_i - 1) + w_k - 1\\ &= \text{reg}(I(H_1)) + \sum_{x_i \in V^+} (w_i - 1).\end{aligned}$$

Here $(I(D):x_k^{w_k}) = (I(D_2), x_{k-1}, x_{k+1})$ except $n = 3, 4$ and $(I(D):x_k^{w_k}) = (x_{k-1}, x_{k+1})$ for $n = 3, 4$.

For $n \neq 3, 4$, since D_2 is a weighted oriented path of length $n-4$ in T_1 or T_2, by Theorem 4.2 and Theorem 4.3, we have

$$\begin{aligned}\text{reg}((I(D):x_k^{w_k})(-(w_k))) &= \text{reg}(I(D_2)) + w_k\\ &\leq \text{reg}(I(H_2)) + \sum_{x_i \in V^+ \setminus \{x_k\}} (w_i - 1) + w_k\\ &= \text{reg}(I(H_2)) + \sum_{x_i \in V^+} (w_i - 1) + 1\\ &= \text{reg}(I(H_1)) + \sum_{x_i \in V^+} (w_i - 1).\end{aligned}$$

$$\begin{aligned}\text{For } n = 3, 4,\ \text{reg}((I(D):x_k^{w_k})(-w_k)) &= 1 + w_k\\ &= \text{reg}(I(H_1)) + w_k - 1\\ &\leq \text{reg}(I(H_1)) + \sum_{x_i \in V^+} (w_i - 1).\end{aligned}$$

By Lemma 2.6 and exact sequence (1), we get

$$\text{reg}(I(D)) \leq \max\{\text{reg}((I(D):x_k^{w_k})(-w_k)), \text{reg}(I(D), x_k^{w_k})\}.$$

Since $\text{reg}((I(D):x_k^{w_k})(-w_k)) - 1 \neq \text{reg}(I(D), x_k^{w_k})$, by Lemma 2.6 and exact sequence (1) we have

$$\begin{aligned}\text{reg}(I(D)) &= \text{reg}(I(D), x_k^{w_k})\\ &= \text{reg}(I(H_1)) + \sum_{x_i \in V^+} (w_i - 1)\\ &= \text{reg}(I(G)) + \sum_{x_i \in V^+} (w_i - 1).\end{aligned}$$

□

Theorem 4.5 *Let D be a weighted oriented cycle of length n for $n \equiv 2 (\text{ mod } 3)$ with underlying graph $G = C_n = x_1 \ldots x_n x_1$ and vertices of V^+ are sinks. Then $\text{reg}(I(D)) = \text{reg}(I(G)) + \sum_{x_i \in V^+} (w_i - 1)$ where $w_i = w(x_i)$ for $x_i \in V^+$.*

Proof Here $V(D) = \{x_1, \ldots, x_n\}$. We use the method of induction on the number of vertices of V^+.

Base Case: Assume that V^+ contains no vertex. Then the proof follows trivially.

Now we consider the case when V^+ contains m number of vertices for some $m \geq 1$. Without loss of generality, let $x_k \neq x_1, x_n$ be one of the m vertices of V^+. Let $D_1 = D \setminus \{x_k\}$ and $H_1 = G \setminus \{x_k\}$ i.e. H_1 is the corresponding underlying graph of D_1. By Corollary 2.9, we have $\text{reg}(I(G)) = \text{reg}(I(H_1)) + 1$. Consider the exact sequence

$$0 \longrightarrow \frac{R}{(I(D) : x_k^{w_k-1})}(-(w_k-1)) \xrightarrow{x_k^{w_k-1}} \frac{R}{I(D)} \longrightarrow \frac{R}{(I(D), x_k^{w_k-1})} \longrightarrow 0 \qquad (2)$$

Here $(I(D), x_k^{w_k-1}) = (I(D_1), x_k^{w_k-1})$ where D_1 is a weighted oriented path of length $n-2$. Since the end vertices of D_1 can not be in V^+, D_1 is in T_2. Then by Theorem 4.2, we have $\text{reg}(I(D_1)) = \text{reg}(I(H_1)) + \sum_{x_i \in V^+ \setminus \{x_k\}} (w_i - 1)$.

Thus by Lemma 2.4, we have

$$\begin{aligned}\text{reg}(I(D), x_k^{w_k-1}) &= \text{reg}(I(D_1)) + (w_k - 1) - 1\\ &= \text{reg}(I(H_1)) + \sum_{x_i \in V^+} (w_i - 1) - 1\\ &= \text{reg}(I(G)) + \sum_{x_i \in V^+} (w_i - 1) - 2.\end{aligned}$$

Here $(I(D) : x_k^{w_k-1}) = I(D_3)$ where D_3 is a weighted oriented cycle with $m-1$ vertices in $V^+(D_3)$. Thus by using the induction hypothesis,

$$\text{reg}(I(D_3)) = \text{reg}(I(G)) + \sum_{x_i \in V^+ \setminus \{x_k\}} (w_i - 1).$$

Then $\text{reg}((I(D) : x_k^{w_k-1})(-(w_k-1))) = \text{reg}(I(D_3)) + w_k - 1 = \text{reg}(I(G)) + \sum_{x_i \in V^+} (w_i - 1)$. By Lemma 2.6 and exact sequence (2), we have

$$\text{reg}(I(D)) \leq \max\{\text{reg}((I(D) : x_k^{w_k-1})(-(w_k-1))), \text{reg}(I(D), x_k^{w_k-1})\}.$$

Since $\text{reg}((I(D) : x_k^{w_k-1})(-(w_k-1))) - 1 \neq \text{reg}(I(D), x_k^{w_k-1})$, by Lemma 2.6 and exact sequence (2), we get

$$\text{reg}(I(D)) = \text{reg}((I(D) : x_k^{w_k-1})(-(w_k-1))) = \text{reg}(I(G)) + \sum_{x_i \in V^+} (w_i - 1).$$

Hence for any weighted oriented cycle D of length n where $n \equiv 2(\bmod 3)$ with vertices of V^+ are sinks, $\text{reg}(I(D)) = \text{reg}(I(G)) + \sum_{x_i \in V^+} (w_i - 1)$ where $w_i = w(x_i)$ for $x_i \in V^+$. □

By the computations in Macaulay 2, we have seen that it is even a hard job to relate the regularity of the edge ideal of a weighted oriented tree with the regularity of edge ideal of its underlying graph when the vertices of V^+ are sink.

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