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ORIGINAL RESEARCH

# A quantum Laguerre semigroup

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**Abstract** In this paper we construct a quantum extension of the Laguerre semigroup and study its properties. In particular we show that it has a unique pure invariant state and any initial state converges to this invariant state. For initial states satisfying a finite energy condition, convergence is exponentially fast.

**Keywords** Quantum dynamical semigroup · Non-commutative Markov process · Laugerre semigroup

**AMS Subject Classification:** 46L55 · 46N50 · 47D07

## 1 Introduction

Quantum Markov semigroups (QMS) are weakly *-continuous semigroups of completely positive, identity preserving and normal maps on a von Neumann algebra. They are a natural generalization of classical Markov semigroups on a function space, which is replaced in quantum theory by a non-commutative operator algebra. Moreover, they also arise from scaling limits of quantum systems interacting with external environments (see e.g. [2, 7]).

Often a QMS on a non-commutative von Neumann algebra leaves invariant an abelian subalgebra. In this way one identifies a classical Markov process by restriction of a quantum Markov process.

P.-A. Meyer pointed out that a classical finite Markov chain in continuous time could be found in this way and also viewed upon as an Evans-Hudson diffusion (see [11]). In a subsequent paper, K.R. Parthasarathy and K.B. Sinha [19] showed that whenever the structure maps of Evans-Hudson are defined on a commutative $^*$-algebra of operators, the whole Evans-Hudson diffusion is commutative or, equivalently, is a classical stochastic process even though the driving quantum noises are non-commutative. This fact enabled them to construct a whole class of continuous-time Markov chains as Evans-Hudson diffusions by using general group actions.

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Later the first author [12, 14] showed that also elliptic diffusions on $\mathbb{R}^d$ with smooth covariance and drift can be realised in the same way. Several other processes have also been considered in the literature, one can see, for example the recent paper [4] and the references therein.

These constructions are based on the identification of a generalized Gorini-Kossakowski-Sudharshan-Lindblad (GKSL) representation of the generator and the proof that the corresponding minimal QMS is Markov. The main difficulty stems from the proof that the minimal QMS is identity preserving.

Carrying on this program becomes much more difficult when one considers domains with boundaries or elliptic degenerate diffusion (see [5, 13]). This happens also in the classical case where the construction of a Markov semigroup and the characterisation of the domain of the generator are typically a non-trivial problem (see [3, 10]).

In this paper we construct a quantum extension of the Laguerre semigroup whose generator (2.1) is elliptic degenerate on a domain with boundary and we study its properties. In particular we show (Theorem 4.1) that it has a unique pure invariant state, the ground state of the Laguerre operator. Moreover, any initial state converges to this invariant state and, if it satisfies a finite energy condition, convergence is exponentially fast.

The paper is organised as follows. In Section 2 we find a generalized GKSL representation of the generator of the classical Laguerre semigroup. Then, in Section 3 we prove (Theorem 3.2) that the minimal semigroup is identity preserving for $\alpha \geq 1$ and, in this case, it is an extension of the classical Laguerre semigroup (Theorem 3.3). Finally, in Section 4, we show that, the quantum extension of the classical Laguerre semigroup that we constructed for $\alpha \geq 1$ has a unique pure invariant state and initial states satisfying a finite energy condition converge exponentially fast to this invariant state (Theorem 4.1).

## 2 GKSL representation of the generator

The Laguerre operator, as Markovian pre-generator of a diffusion process on $(0, +\infty)$, is typically defined on the space $C_c^\infty((0, +\infty); \mathbb{C})$ with compact support by

$$(A_\alpha f)(x) = \frac{1}{2}\left(xf''(x) + (\alpha + 1 - x)f'(x)\right) \tag{2.1}$$

where $\alpha \geq 0$. It is well-known (see e.g. [10] Chapter 8, Theorem 2.1) that $A_\alpha$ (actually $2A_\alpha$ but the constant 2, of course, plays no role) is closable and its closure generates a strongly continuous contraction semigroup $(T_t)_{t\geq 0}$ of positive maps on the Banach space of complex-valued continuous functions on $(0, +\infty)$ vanishing at 0 and $+\infty$. Moreover, adjoining the constant function 1, and defining $T_t 1 = 1$ for all $t \geq 0$, one finds the Markov semigroup of the classical Laguerre diffusion process (see [3] Sect. 2.7.3, page 113). This is a strongly continuous semigroup on the abelian $C^*$-algebra $\mathcal{A}$ of continuous function on $(0, +\infty)$ with the same limits at 0 and $+\infty$. It is worth noticing here that these boundary conditions on the generator play some role only for $\alpha < 1$(see [9] Th. 2.1).

We now define a quantum extension (see [14], Chapter 4). Throughout the paper $\mathcal{B}(\mathsf{h})$ will denote the von Neumann algebra of all bounded operators on a complex separable Hilbert space $\mathsf{h}$.

**Definition 2.1** *A quantum Markov semigroup $(\mathcal{T}_t)_{t\geq 0}$ on $\mathcal{B}(\mathsf{h})$ is a quantum extension of a classical Markov semigroup $(T_t)_{t\geq 0}$ on a commutative unital $C^*$ algebra $\mathcal{A}$ if $\mathcal{A}$ is a subalgebra of $\mathcal{B}(\mathsf{h})$ and $\mathcal{T}_t(a) = T_t(a)$ for all $t \geq 0$ and $a \in \mathcal{A}$.*

Since the operator $A_\alpha$ is unbounded, the candidate generator of a QMS extending the semigroup generated by $A_\alpha$ will also be so. QMSs with unbounded generators can be effectively constructed by the minimal semigroup method (see e.g. [14, 17] and the references therein) whenever one can find a generalized GKSL representation of the generator which can be formally written as $\mathcal{L}(X) = G^*X + \sum_\ell L_\ell^* X L_\ell + XG$. To be precise, we make the following definition (see [14, Chapter 4]).

**Definition 2.2** *The infinitesimal generator $A$ of a strongly continuous semigroup $(T_t)_{t\geq 0}$ defined on an abelian $C^*$ algebra $\mathcal{A}$ is said to be represented in a generalized GKSL form on $\mathcal{B}(\mathsf{h})$ if $\mathcal{A}$ is a subalgebra of $\mathcal{B}(\mathsf{h})$ and there exist operators $G$ and $\{L_\ell\}_{\ell\geq 1}$ on $\mathsf{h}$ such that*

(i) *the operator $G$ is the infinitesimal generator of a strongly continuous contraction semigroup $(P_t)_{t\geq 0}$ on $\mathsf{h}$,*
(ii) *for $\ell \geq 1$, the domain of the operators $L_\ell$ contains the domain of $G$,*

(iii) *for all* $u, v \in \mathrm{Dom}(G)$, *we have*

$$\langle v, Gu\rangle + \sum_{\ell \geq 1} \langle L_\ell v, L_\ell u\rangle + \langle Gv, u\rangle = 0,$$

*where the series is absolutely convergent*,

(iv) *for all* $u, v \in \mathrm{Dom}(G)$ *and all* $f \in \mathrm{Dom}(A)$, *we have*

$$\langle v, (Af)u\rangle \ = \ \langle v, fGu\rangle + \sum_{\ell \geq 1} \langle L_\ell v, fL_\ell u\rangle + \langle Gv, fu\rangle.$$

Assumptions (i), (ii) and (iii) are needed for constructing a quantum Markov semigroup by the minimal semigroup method.

In this section, we show how to choose the unbounded operators for a generalized GKSL representation of a candidate generator of a quantum extension of the Laguerre semigroup. In other words, we decide what choices can be made for $\mathsf{h}$, $G$ and $L_\ell$ in Definition 2.2 when $A$ is as given by (2.1). Note that in this case, (iv) of Definition 2.2 reduces to the following condition:

for all $u, v \in \mathrm{Dom}(G)$ and all $f \in C_c^\infty((0, +\infty), \mathbb{C})$, or $f$ constant, we have

$$\langle v, (A_\alpha f)u\rangle \ = \ \langle v, fGu\rangle + \sum_{\ell \geq 1} \langle L_\ell v, fL_\ell u\rangle + \langle Gv, fu\rangle.$$

For the moment, we only do algebraic calculations, analytical and technical aspects will be considered later.

As a first step, following the scheme introduced in (Sections 4.1 and 4.2, [14]) we first compute

$$(A_\alpha fg)(x) - f(x)(A_\alpha g)(x) - (A_\alpha f)(x)g(x) = xf'(x)g'(x).$$

Comparison with the known formula for the (bounded) generator $\mathcal{L}$ of a QMS

$$\mathcal{L}(X^*Y) - X^*\mathcal{L}(Y) - \mathcal{L}(X^*)Y = \sum_\ell [L_\ell, X]^*[L_\ell, Y]$$

for all $X$, $Y$ in $\mathcal{B}(\mathsf{h})$, leads us to consider a GKSL generator with only one operator $L$ with domain containing functions in $C_c^\infty((0, +\infty); \mathbb{C})$ acting as follows

$$(Lu)(x) = x^{1/2}(\partial_x u)(x) = x^{1/2}u'(x). \tag{2.2}$$

where $\partial_x$ denotes the derivative with respect to $x$. Indeed, denoting by $M(f)$ the multiplication operator by a smooth function $f$, the commutator $[L, M(f)]$ is the multiplication operator $M(x^{1/2}f')$ by the function $x \to x^{1/2}f'(x)$ and so

$$[L, M(f)]^*[L, M(g)] = M(x\bar{f}'g').$$

As Hilbert space $\mathsf{h}$ we consider $L^2((0, +\infty), x^\alpha \mathrm{e}^{-x}dx)$ where the choice of the weight function $x^\alpha \mathrm{e}^{-x}$ is motivated by the fact that it is the unique invariant measure of the classical Laguerre diffusion process. A simple computation shows that the adjoint of $L$ is an extension of the operator $L^\dagger$ on $C_c^\infty((0, +\infty); \mathbb{C})$

$$(L^\dagger v)(x) = -x^{1/2}v'(x) - \left(\alpha + \frac{1}{2}\right)\frac{v(x)}{x^{1/2}} + x^{1/2}v(x).$$

With the above choice of $L$ and $L^\dagger$ we can consider the formal GKSL generator

$$\mathcal{L}(X) = -\frac{1}{2}L^\dagger LX + L^\dagger XL - \frac{1}{2}XL^\dagger L$$

and compute

$$
\begin{aligned}
\mathcal{L}(M(f)) =& \frac{1}{2}\left(L^\dagger[M(f), L] + [L^\dagger, M(f)]L\right) \\
=& \frac{1}{2}\left(\partial_x M(x^{1/2})M(x^{1/2}f') - M(x^{1/2}f')\,M(x^{1/2})\,\partial_x\right) \\
& + \frac{1}{2}M(\alpha x^{-1/2} - x^{1/2})M(x^{1/2}f') \\
=& \frac{1}{2}[\partial_x, M(xf')] + \frac{1}{2}M((\alpha - x)f') \\
=& \frac{1}{2}M(xf'' + (\alpha + 1 - x)f') \\
=& M(A_\alpha f).
\end{aligned} \tag{2.3}
$$

Therefore it follows that the condition (iv) of Definition 2.2 is also satisfied as (2.3) and the condition (iv) of Definition 2.2 are equivalent due to the scheme introduced in (Sections 4.1 and 4.2, [14]).

It is worth noticing here that the operator $-L^\dagger L$ acts on $C_c^\infty((0, +\infty); \mathbb{C})$ as follows

$$
-(L^\dagger Lu)(x) = xu''(x) + (\alpha + 1 - x)u'(x) \tag{2.4}
$$

therefore it coincides with the Laguerre operator $2A_\alpha$ on $C_c^\infty((0, +\infty); \mathbb{C})$ and $(-1/2)L^\dagger L$ is a good candidate for a generator of a strongly continuous contraction semigroup the Hilbert space $\mathsf{h}$.

***Remark*** The above choice of the GKSL representation is not unique. One could check by similar computations that the following operators defined on smooth functions with compact support

$$
\begin{aligned}
(Lu)(x) =& x^{1/2}u'(x) + \eta(x)u(x) \\
(L^\dagger u)(x) =& -x^{1/2}u'(x) - \left(\alpha + \frac{1}{2}\right)\frac{u(x)}{x^{1/2}} + x^{1/2}u(x) + \overline{\eta}(x)u(x) \\
(Hu)(x) =& -\frac{\mathrm{i}}{2}\xi(x)u'(x) - \frac{\mathrm{i}}{2}(\overline{\xi}u(x))' + \frac{\mathrm{i}}{2}\left(1 - \frac{\alpha}{x}\right)\overline{\xi}(x)u(x)
\end{aligned}
$$

where $\eta$ and $\xi$ are two functions on $(0, +\infty)$ satisfying

$$
\xi(x) + \overline{\xi}(x) = x^{1/2}(\eta(x) + \overline{\eta}(x))
$$

would give another GKSL representation of $A_\alpha$, namely for all smooth function $f$,

$$
\mathcal{L}(M(f)) + \mathrm{i}[H, M(f)] = M(A_\alpha f).
$$

However, here we consider $\eta = \xi = 0$ for which $G$ coincides with a Laguerre operator as an operator on $\mathsf{h}$; other choices lead to difficult problems on semigroup generation. Indeed, one may work with another operator $G$ which is a perturbation of the Laguerre operator with a singular potential.

We end this section by addressing the problem of self-adjoint extensions of the operator $L^\dagger L$ defined on smooth functions with compact support.

**Proposition 2.3** *The closure of the operator $L^\dagger L$ defined on the space $C_c^\infty((0, +\infty), \mathbb{C})$ by (2.4) is self-adjoint for $\alpha \geq 1$.*

*Proof* The operator $L^\dagger L$ is clearly closable because it is symmetric on $C_c^\infty((0, +\infty), \mathbb{C})$ and so it has a densely defined adjoint

By Theorem 2.1 of [9] its closure generates a $C_0$ semigroup, and therefore is self-adjoint, if the function

$$
x \mapsto \int_x^1 y^{-(\alpha+1)}\mathrm{e}^y\,\mathrm{d}y \int_y^1 z^\alpha \mathrm{e}^{-z}\,\mathrm{d}z \tag{2.5}
$$

is not in $L^2((0, 1); x^\alpha \mathrm{e}^{-x}dx)$ and not in $L^2((1, \infty); x^\alpha \mathrm{e}^{-x}dx)$. Square integrability in a neighborhood of 0 depends in the behaviour of the integrand near 0. Now, for $x \sim 0$ and $\alpha > -1$, we have

$$y^{-(\alpha+1)}\mathrm{e}^{y}\int_{y}^{1} z^{\alpha}\mathrm{e}^{-z}\,\mathrm{d}z \sim c_{\alpha}y^{-(\alpha+1)}\mathrm{e}^{y}$$

where $c_\alpha = \int_0^1 z^\alpha \mathrm{e}^{-z}\,\mathrm{d}z$ so that the leading part of (2.5) in a neighbourhood of 0 is

$$x \to c_{\alpha}\int_{x}^{1} y^{-(\alpha+1)}\mathrm{d}y \sim \alpha^{-1}c_{\alpha}x^{-\alpha}$$

for $\alpha > 0$. As a consequence (2.5) belongs to $L^2((0,1);x^\alpha \mathrm{e}^{-x}dx)$ if and only if $0<\alpha<1$. In a similar way, for $\alpha = 0$, the leading term of (2.5) in a neighbourhhood of 0 is $-c_0\log(x)$ and (2.5) belongs to $L^2((0,1);x^\alpha \mathrm{e}^{-x}dx)$. Summing up, (2.5) is not in $L^2((0,1);x^\alpha \mathrm{e}^{-x}dx))$ if and only if $\alpha \geq 1$.

Now, for all $x \sim +\infty$, we have

$$\int_{x}^{1} y^{-(\alpha+1)}\mathrm{e}^{y}\,\mathrm{d}y\int_{y}^{1} z^{\alpha}\mathrm{e}^{-z}\,\mathrm{d}z = \int_{1}^{x} y^{-(\alpha+1)}\mathrm{e}^{y}\,\mathrm{d}y\int_{1}^{y} z^{\alpha}\mathrm{e}^{-z}\,\mathrm{d}z$$

The asymptotic behaviour of integrand, for large $y$, is $\sim\Gamma(\alpha+1)y^{-(\alpha+1)}\mathrm{e}^{y}$ and (2.5) diverges as $x$ goes to $+\infty$. Moreover, by L'Hôpital's rule

$$\begin{aligned}\lim_{x\to+\infty}\frac{\int_1^x y^{-(\alpha+1)}\mathrm{e}^{y}\,\mathrm{d}y\int_1^y z^{\alpha}\mathrm{e}^{-z}\,\mathrm{d}z}{x^{-(\alpha+1)}\mathrm{e}^{x}} &= \lim_{x\to+\infty}\frac{x^{-(\alpha+1)}\mathrm{e}^{x}\int_1^x z^{\alpha}\mathrm{e}^{-z}\,\mathrm{d}z}{x^{-(\alpha+1)}\mathrm{e}^{x}-(\alpha+1)x^{-(\alpha+2)}\mathrm{e}^{x}}\\ &= \int_1^{+\infty} z^{\alpha}\mathrm{e}^{-z}\,\mathrm{d}z\\ &<\infty.\end{aligned}$$

Therefore (2.5) is not in $L^2((1,\infty);x^\alpha \mathrm{e}^{-x}\mathrm{d}x)$ for all $\alpha \geq 0$ and the conclusion follows. □

***Remark*** By applying Theorem 2.2 of [9] one can see, by the same arguments, that the closure of $L^\dagger L$ defined on $C_c^\infty((0,+\infty),\mathbb{C})$ by (2.4) is *not* self-adjoint for $0 \leq \alpha < 1$.

## 3 Minimal semigroup and conservativity

The discussion in the previous section shows that a natural candidate as operator $G$ is an extension of $-\frac{1}{2}L^\dagger L$ that generates a strongly continuous contraction semigroup on the complex Hilbert space $\mathsf{h} = L^2((0,+\infty);x^\alpha \mathrm{e}^{-x}dx)$. Moreover, some extension of the operator $L$ defined by (2.2) is a natural candidate as $L_1$ and all the remaining $L_\ell$ are zero.

As we noted in the previous section, for all $u \in C_c^\infty((0,+\infty),\mathbb{C})$ we have

$$-\frac{1}{2}L^\dagger L u = A_\alpha u. \tag{3.1}$$

By Proposition 2.3, the closure in $\mathsf{h}$ of $L^\dagger L$ defined on $C_c^\infty((0,+\infty),\mathbb{C})$ is self-adjoint for $\alpha \geq 1$. In this case it coincides with $-2G$ where $G$ is the self-adjoint operator defined through its spectral decomposition

$$\begin{aligned}\mathrm{Dom}(G) &= \left\{ u\in\mathsf{h} \;\middle|\; \sum_{n\geq 0} n^2|\langle u, e_n\rangle|^2 < \infty \right\}\\ Gu &= -\sum_{n\geq 1}\frac{n}{2}\,|e_n\rangle\langle e_n|\,u\\ &= -\sum_{n\geq 1}\frac{n}{2}\,u_n e_n\end{aligned} \tag{3.2}$$

where $u_n := \langle e_n, u\rangle, n\geq 1$, and $(e_n)_{n\geq 0}$ is the orthonormal system obtained by normalization of generalized Laguerre polynomials $(p_n)_{n\geq 0}$ satisfying

$$xp_n''(x) + (\alpha+1-x)p_n'(x) = -np_n(x).$$

In particular

$$p_0(x) = 1, \quad p_1(x) = -x + \alpha + 1, \ \ldots\ , \ p_n(x) = \frac{x^{-\alpha} e^x}{n!} \frac{d^n}{dx^n} (x^{\alpha+n} e^{-x})$$

and

$$e_n(x) = \left( \frac{n!}{(n+\alpha)!} \right)^{1/2} p_n(x).$$

We refer to [3, 21, 22] for more details. The case where $\alpha < 1$, in which the domain $C_c^\infty((0,+\infty),\mathbb{C})$ is no more a core for $G$, will be studied in another paper.

We now complete the construction of the minimal semigroup. Let $G$ be the negative self-adjoint operator defined by (3.2), let $(P_t)_{t\geq 0}$ be the strongly continuous contraction semigroup on $\mathsf{h}$ generated by $G$ which is also analytic for $\alpha \geq 1$ because $G$ is negative self-adjoint.

Let $L$ be the operator defined as:

$$\mathrm{Dom}(L) = \mathrm{Dom}(G), \qquad (Lu)(x) = x^{1/2} u'(x) \tag{3.3}$$

Then obviously $G$ and $L$ satisfy conditions (i) (ii) and (iii) of Definition 2.2. We have already noted from (2.3) that the condition (iv) of Definition 2.2 holds. Therefore we found a generalized GKSL representation of the generator (2.1).

The *minimal* semigroup associated with operators $G$, $L$ (see, for instance, [6, 14]) is constructed, on elements $x$ of $\mathcal{B}(\mathsf{h})$, by means of the non decreasing sequence of positive maps $(\mathcal{T}_t^{(n)})_{n\geq 0}$ defined, by recurrence, as follows

$$\begin{aligned} \mathcal{T}_t^{(0)}(x) &= P_t^* x P_t, \\ \left\langle v, \mathcal{T}_t^{(n+1)}(x) u \right\rangle &= \langle P_t v, x P_t u \rangle + \int_0^t \left\langle L P_{t-s} v, \mathcal{T}_s^{(n)}(x) L P_{t-s} u \right\rangle \mathrm{d}s \end{aligned} \tag{3.4}$$

for all $x \in \mathcal{B}(\mathsf{h})$, $t \geq 0$, $v, u \in \mathrm{Dom}(G)$. Indeed, we have

$$\mathcal{T}_t(x) = \sup_{n\geq 0} \mathcal{T}_t^{(n)}(x)$$

for all positive $x \in \mathcal{B}(\mathsf{h})$ and all $t \geq 0$. The definition of positive maps $\mathcal{T}_t$ is then extended to all the elements of $\mathcal{B}(\mathsf{h})$ by linearity. The minimal semigroup associated with $G$, $L$ satisfies the integral equation

$$\begin{aligned} \langle v, \mathcal{T}_t(x) u \rangle =& \langle v, xu \rangle \\ &+ \int_0^t (\langle Gv, \mathcal{T}_s(x) u \rangle + \langle Lv, \mathcal{T}_s(x) Lu \rangle + \langle v, \mathcal{T}_s(x) Gu \rangle) \mathrm{d}s \end{aligned} \tag{3.5}$$

for all $x \in \mathcal{B}(\mathsf{h})$, $t \geq 0$, $v, u \in \mathrm{Dom}(G)$. Moreover, it is the unique solution to the above equation if and only if it is conservative (or Markov) i.e. $\mathcal{T}_t(1) = 1$ for all $t \geq 0$ (see e.g. reference [14]).

We will achieve conservativity by applying Corollary 3.41 of [14] that we now recall here (in the special case where there is only one non-zero operator $L$) for the reader's convenience.

**Proposition 3.1** *Let $G$ and $L$ as in (3.2) and (3.3) above. Suppose that there exist a self-adjoint operator $C$ with domain coinciding with the domain of $G$ and a core $D$ for $C$ with the following properties:*

(i) $L(D) \subseteq D(C^{1/2})$,

(ii) *there exists a self-adjoint operator $\Phi$ such that*

$$-2\Re\mathrm{e}\langle u, Gu \rangle = \langle u, \Phi u \rangle \leq \langle u, Cu \rangle$$

*for all $u \in D$,*

(iii) *there exists a non-negative constant $b$ such that the inequality*

$$2\Re\mathrm{e}\langle Cu, Gu \rangle + \left\langle C^{1/2} Lu, C^{1/2} Lu \right\rangle \leq b \langle u, Cu \rangle \tag{3.6}$$

*holds for every $u \in D$.*

*Then the minimal quantum dynamical semigroup is Markov.*

Applying the above result we can prove the following

**Theorem 3.2** *The minimal semigroup $(\mathcal{T}_t)_{t\geq 0}$ associated with the above G, L is conservative for $\alpha \geq 1$.*

*Proof* Let $C = -2G$ so that $C$ coincides with the closure of $L^{\dagger}L$ and let $D = C_c^{\infty}((0,+\infty);\mathbb{C})$. Clearly, $D$ is an invariant domain for $L$ and so $L(D) \subseteq D \subseteq \mathrm{Dom}(C) \subseteq \mathrm{Dom}(C^{1/2})$. It follows that assumption (i) of Proposition 3.1 holds.

Assumption (ii) also holds because $G$ is self-adjoint. We finally check assumption (iii). For every $u \in D$ we have

$$\begin{aligned} 2\Re\mathrm{e}\langle Cu, Gu\rangle + \left\langle C^{1/2}Lu, C^{1/2}Lu\right\rangle &= -\left\langle L^{\dagger}Lu, L^{\dagger}Lu\right\rangle + \left\langle Lu, L^{\dagger}L\,Lu\right\rangle \\ &= \left\langle Lu, \left[L^{\dagger}, L\right]Lu\right\rangle \end{aligned}$$

where $\left[L^{\dagger}, L\right]$ denotes the commutator of $L^{\dagger}$ and $L$. Since $Lu$ belongs to $D$ the following formal computation makes sense as an operator on $D$

$$\begin{aligned} [L^{\dagger}, L] &= \left[-x^{1/2}\partial_x - (\alpha + 1/2)x^{-1/2} + x^{1/2}, x^{1/2}\partial_x\right] \\ &= \left[-(\alpha + 1/2)x^{-1/2} + x^{1/2}, x^{1/2}\partial_x\right] \\ &= -\frac{1}{2}\left(1 + \left(\alpha + \frac{1}{2}\right)\frac{1}{x}\right). \end{aligned}$$

It follows that

$$\begin{aligned} 2\Re\mathrm{e}\langle Cu, Gu\rangle + \left\langle C^{1/2}Lu, C^{1/2}Lu\right\rangle &\leq -\frac{1}{2}\langle Lu, Lu\rangle \\ &= -\frac{1}{2}\langle u, Cu\rangle \leq 0 \end{aligned} \tag{3.7}$$

for all $u \in D$ and the proof is complete. □

***Remark*** By a known characterization of the domain of the generator of a conservative quantum Markov semigroup (Proposition 3.32 [14]), Theorem 3.2 implies that the domain of the generator of $(\mathcal{T}_t)_{t\geq 0}$ consists of all operators $X \in \mathcal{B}(\mathsf{h})$ such that the bilinear form

$$\mathrm{Dom}(G) \times \mathrm{Dom}(G) \ni (v, u) \to \langle Gv, Xu\rangle + \langle Lv, XLu\rangle + \langle v, XGu\rangle$$

is bounded. Moreover, the linear manifold generated by rank one operators $|u\rangle\langle v|$ is an essential domain for the generator of the predual semigroup.

We can now prove that the QMS $\mathcal{T}$ extends the classical Markov semigroup generated by $A_{\alpha}$.

**Theorem 3.3** *The abelian $C^*$-algebra $\mathcal{A}$ of complex-valued continuous functions on $(0,+\infty)$ with the same limits at 0 and $+\infty$ is invariant for the minimal semigroup and $\mathcal{T}_t(M(f)) = M(T_t f)$ for all $f \in \mathcal{A}$.*

*Proof* Clearly the identity (iv) of Definition 2.2 holds by our choice of operators $G$ and $L$ and, denoting by tr the usual trace, we have

$$\mathrm{tr}(|u\rangle\langle v|M(A_{\alpha}f)) = \mathrm{tr}(\mathcal{L}_*(|u\rangle\langle v|)M(f))$$

where $\mathcal{L}_*$ is the generator of the predual semigroup. The linear span of rank one operators $|u\rangle\langle v|$ for $u, v \in \mathrm{Dom}(G)$ is a core for $\mathcal{L}_*$ because the minimal QMS is conservative (see [14] Proposition 3.32), therefore we have

$$\mathrm{tr}(\eta M(A_{\alpha}f)) = \mathrm{tr}(\mathcal{L}_*(\eta)M(f))$$

for all $\eta \in \mathrm{Dom}(\mathcal{L}_*)$ and all $f \in \mathrm{Dom}(A_{\alpha})$. As a consequence, for all $t > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{tr}(\mathcal{T}_{*s}(\eta)M(T_{t-s}f)) = \mathrm{tr}(\mathcal{L}_*(\mathcal{T}_{*s}(\eta))M(T_{t-s}f)) - \mathrm{tr}(\mathcal{T}_{*s}(\eta)M(A_\alpha T_{t-s}f)) = 0.$$

It follows that

$$\mathrm{tr}(\eta\mathcal{T}_t(M(f))) = \mathrm{tr}(\mathcal{T}_{*t}(\eta)M(f)) = \mathrm{tr}(\eta M(T_t f))$$

for all $t \geq 0$ and all $\eta$ trace class operator, therefore $\mathcal{T}_t(M(f)) = M(T_t f)$ and the proof is complete. □

## 4 The quantum Laguerre semigroup

In this section we will study the structure of the quantum Laguerre semigroup on $\mathcal{B}(\mathsf{h})$ that we have constructed. Throughout this section $\alpha \geq 1$.

It is well-known (see e.g. [3]) that the classical Laguerre semigroup of the Laguerre diffusion process is irreducible and admits the function

$$(0, +\infty) \ni x \mapsto \frac{x^\alpha}{\Gamma(\alpha+1)}\mathrm{e}^{-x} \in \mathbb{R}$$

where $\Gamma$ denotes the Euler Gamma function, as the unique invariant measure. This defines a faithful normal state on the von Neumann algebra $L^\infty((0, +\infty); \mathbb{C})$.

Surprisingly, the quantum Laguerre semigroup has other qualitative features that we will now investigate.

Let $(\mathcal{T}_{*t})_{t\geq 0}$ denote the predual semigroup.

**Theorem 4.1** *The constant function on* $(0, +\infty)$

$$e_0(x) = \Gamma(\alpha+1)^{-1/2}$$

*defines a norm one vector in* $\mathsf{h}$. *The pure state* $|e_0\rangle\langle e_0|$ *is the unique, invariant state for the quantum Laguerre semigroup* $\mathcal{T}$ *and for all initial normal states* $\rho$,

$$\lim_{t\to\infty}\mathcal{T}_{*t}(\rho) = |e_0\rangle\langle e_0|$$

*in the trace norm topology. Moreover, for all initial state* $\rho$ *satisfying the finite energy condition* $\mathrm{tr}(\rho C) < \infty$, *where* $C = -2G$, *we have*

$$\|\mathcal{T}_{*t}(\rho) - |e_0\rangle\langle e_0|\|_1 \leq 2\,\mathrm{e}^{-t/4}\,\mathrm{tr}(\rho C)^{1/2}.$$

We begin by a useful Lemma which is essentially a byproduct of our proof of conservativity.

**Lemma 4.2** *Let* $C = -2G$ *as in the proof of Theorem 3.2 and, for all* $\varepsilon > 0$ *consider bounded approximations* $C_\varepsilon = C(\mathbb{1} + \varepsilon C)^{-1}$. *For all* $t \geq 0$ *and* $\varepsilon > 0$ *the following inequality holds*

$$\mathcal{T}_t(C_\varepsilon) \leq \mathrm{e}^{-t/2}C.$$

*Proof* Fix $\varepsilon > 0$. We note that $C_\varepsilon \leq C$. Recalling the construction of the minimal semigroup as limit of a non decreasing sequence of positive maps $(\mathcal{T}_t^{(n)})_{n\geq 0}$ defined, by recurrence as in (3.4), we check that $\mathcal{T}_t^{(n)}(C_\varepsilon) \leq \mathrm{e}^{-t/2}C$ for all $t \geq 0$ and $n \geq 0$.

For $n = 0$, for all $z \in \mathbb{C}$ and $u \in \mathsf{h}$ orthogonal to $e_0, u_k := \langle e_k, u\rangle, k \geq 1$, by explicit computation we find

$$\begin{aligned}\left\langle ze_0+u,\mathcal{T}_t^{(0)}(C_\varepsilon)(ze_0+u)\right\rangle &=\langle P_t(ze_0+u),C_\varepsilon P_t(ze_0+u)\rangle\\ &=\sum_{k\geq 1}\mathrm{e}^{-kt}|u_k|^2\frac{k}{1+\varepsilon k}\\ &\leq \mathrm{e}^{-t}\sum_{k\geq 1}|u_k|^2\frac{k}{1+\varepsilon k}\\ &=\mathrm{e}^{-t}\langle ze_0+u,C_\varepsilon(ze_0+u)\rangle\end{aligned}$$

and so $\mathcal{T}_t^{(0)}(C_\varepsilon)\leq \mathrm{e}^{-t}C_\varepsilon\leq \mathrm{e}^{-t/2}C$.

If $\mathcal{T}_t^{(n)}(C_\varepsilon)\leq \mathrm{e}^{-t/2}C$ for a certain $n$, then recalling that the semigroup $(P_t)_{t\geq 0}$ is analytic for all $u\in\mathrm{Dom}(G)$, $P_t$ for $t>0$ maps $\mathrm{Dom}(G)$ to $\mathrm{Dom}(G^m)$ $\forall m>0$ and operator compositions $CLP_t$, $C^{1/2}LP_t$ make sense $\forall t>0$,

$$\begin{aligned}\left\langle u,\mathcal{T}_t^{(n+1)}(C_\varepsilon)u\right\rangle &=\langle P_tu,C_\varepsilon P_tu\rangle+\int_0^t\left\langle LP_{t-s}u,\mathcal{T}_s^{(n)}(C_\varepsilon)LP_{t-s}u\right\rangle\mathrm{d}s\\ &\leq\langle P_tu,C_\varepsilon P_tu\rangle+\int_0^t\mathrm{e}^{-s/2}\langle LP_{t-s}u,CLP_{t-s}u\rangle\mathrm{d}s\\ &=\langle P_tu,C_\varepsilon P_tu\rangle+\int_0^t\mathrm{e}^{-s/2}\left\langle C^{1/2}LP_{t-s}u,C^{1/2}LP_{t-s}u\right\rangle\mathrm{d}s.\end{aligned}$$

By the inequality (3.7) in the proof of Theorem 3.2, we have

$$\begin{aligned}\left\langle u,\mathcal{T}_t^{(n+1)}(C_\varepsilon)u\right\rangle \leq &-\int_0^t\mathrm{e}^{-s/2}\,2\Re\mathfrak{e}\left\langle\mathfrak{C}^{/}\mathfrak{P}_{\mathfrak{t}-\mathfrak{s}}\mathfrak{u},\mathfrak{G}\mathfrak{C}^{/}\mathfrak{P}_{\mathfrak{t}-\mathfrak{s}}\mathfrak{u}\right\rangle\mathfrak{d}\mathfrak{s}\\ &-\frac{1}{2}\int_0^t\mathrm{e}^{-s/2}\langle P_{t-s}u,CP_{t-s}u\rangle\mathrm{d}s+\langle P_tu,C_\varepsilon P_tu\rangle\\ =&\ \langle P_tu,C_\varepsilon P_tu\rangle+\int_0^t\mathrm{e}^{-s/2}\frac{\mathrm{d}}{\mathrm{d}s}\left\|C^{1/2}P_{t-s}u\right\|^2\mathrm{d}s\\ &-\frac{1}{2}\int_0^t\mathrm{e}^{-s/2}\left\|C^{1/2}P_{t-s}u\right\|^2\mathrm{d}s\end{aligned}$$

Integrating by parts the first integral, terms with integrals on [0, $t$] cancel. Using $C_\varepsilon\leq C$, we get the inequality

$$\begin{aligned}\left\langle u,\mathcal{T}_t^{(n+1)}(C_\varepsilon)u\right\rangle &\leq\langle P_tu,CP_tu\rangle+\left[\mathrm{e}^{-s/2}\left\|C^{1/2}P_{t-s}u\right\|^2\right]_0^t\\ &\leq \mathrm{e}^{-t/2}\left\|C^{1/2}u\right\|^2=\mathrm{e}^{-t/2}\langle u,Cu\rangle.\end{aligned}$$

Therefore $\mathcal{T}_t^{(n)}(C_\varepsilon)\leq \mathrm{e}^{-t/2}C$ for all $n\geq 0$. □

We now recall the following inequality on the trace norm distance from a pure state (see Theorem 4.1 of [1])

**Theorem 4.3** *Let $\rho$ be a normal state and $e_0$ a unit vector. Then one has*

$$\|\,\rho-|e_0\rangle\langle e_0|\,\|_1\leq 2(1-\rho_{00})^{1/2}$$

where $\rho_{00}=\langle e_0,\rho\, e_0\rangle$.

We can now prove Theorem 4.1.

*Proof of Theorem 4.1* By Proposition 3.32 of [14] all rank one operators belong to the domain of the generator $\mathcal{L}_*$ of the predual semigroup. Therefore, since $Ge_0=Le_0=0$, we have

$$\mathcal{L}_*(|e_0\rangle\langle e_0|)=|Ge_0\rangle\langle e_0|+|Le_0\rangle\langle Le_0|+|e_0\rangle\langle Ge_0|=0$$

and $|e_0\rangle\langle e_0|$ is an invariant state.

Let $\rho$ be a state satisfying the finite energy condition $\mathrm{tr}(\rho\, C) < \infty$ and let $\rho_t = \mathcal{T}_{*t}(\rho)$. By Theorem 4.3 we have

$$\begin{aligned}\| \rho_t - |e_0\rangle\langle e_0| \|_1 &\leq 2(1 - \langle e_0, \rho_t\, e_0\rangle)^{1/2} \\ &= 2(1 - \mathrm{tr}(\rho\mathcal{T}_t(|e_0\rangle\langle e_0|)))^{1/2} \\ &= 2\Big(\mathrm{tr}(\rho\mathcal{T}_t(|e_0\rangle\langle e_0|^{\perp}))\Big)^{1/2},\end{aligned}$$

where $|e_0\rangle\langle e_0|^{\perp}$ denotes the projection orthogonal to the rank one projection $|e_0\rangle\langle e_0|$.

First note that $|e_0\rangle\langle e_0|^{\perp} \leq C$, and $\mathrm{tr}(\omega C) = \sup_{\varepsilon > 0} \mathrm{tr}(\omega C_\varepsilon)$ for all state $\omega$ such that the left-hand side is finite. By Lemma 4.2, implying, in particular, that the projection $|e_0\rangle\langle e_0|^{\perp}$, which is superharmonic in the sense of [15] converges to 0 exponentially fast in the norm topology, we immediately find

$$\begin{aligned}\| \rho_t - |e_0\rangle\langle e_0| \|_1 &\leq 2(\mathrm{tr}(\rho_t\, C))^{1/2} \\ &= 2\,(\sup_{\varepsilon > 0} \mathrm{tr}(\rho_t\, C_\varepsilon))^{1/2} \\ &= 2\,(\sup_{\varepsilon > 0} \mathrm{tr}(\rho\mathcal{T}_t(C_\varepsilon)))^{1/2} \\ &\leq 2\,\mathrm{e}^{-t/4}\,\mathrm{tr}(\rho\, C)^{1/2}.\end{aligned}$$

Convergence towards the invariant state $|e_0\rangle\langle e_0|$ for an arbitrary initial state follows from the trace norm density of states satisfying the finite energy condition (in particular finite rank states) in the set of all states.

This also proves the uniqueness of the pure invariant state $|e_0\rangle\langle e_0|$. □

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## References


1. L. Accardi, F. Fagnola and S. Hachicha, Generic q-Markov semigroups and speed of convergence of q-algorithms, Infin. Dimens. Anal. Quantum Probab. Relat. Top., **9** (2006), no. 4, 567-594.
2. L. Accardi, Y.G. Lu and I. Volovich, Quantum theory and its stochastic limit, Springer-Verlag, Berlin, (2002).
3. D. Bakry, I. Gentil and M. Ledoux, Analysis and geometry of Markov diffusion operators, Grundlehren der Mathematischen Wissenschaften, **348**. Springer, Cham, 2014.
4. K. Bessadok, F. Fagnola and S. Hachicha, Classical and Quantum Markov processes associated with $q$-Bessel operators, Open Sys. Inf. Dyn., (2020) To appear.
5. B. V. Rajarama Bhat, F. Fagnola and Kalyan B. Sinha, On quantum extensions of semigroups of Brownian motion on a half-line, Russian J. Math. Phys., **4** (1996), no. 1, 13-28.
6. A.M. Chebotarev and F. Fagnola, Sufficient conditions for conservativity of minimal quantum dynamical semigroups, J. Funct. Anal., **153** (1998), no. 2, 382-404.
7. E.B. Davies, Quantum Theory of Open Systems, Academic Press, London 1976.
8. A. Dhahri, F. Fagnola, R. Rebolledo, The decoherence-free subalgebra of a quantum Markov semigroup with unbounded generator, Infin. Dimens. Anal. Quantum Probab. Relat. Top., **13** (2010), no. 3, 413-433.
9. A. Eberle, Uniqueness and non-uniqueness of semigroups generated by singular diffusion operators, Lecture Notes in Mathematics, 1718. Springer-Verlag, Berlin, (1999).
10. S.N. Ethier and T.G. Kurtz, Markov processes: Characterization and convergence, Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley, New York (1986).
11. M. Evans and R. L. Hudson, Multidimensional quantum diffusions, In: Quantum probability and applications, III (Oberwolfach, 1987), 69-88, Lecture Notes in Math., **1303**, Springer, Berlin (1988).
12. F. Fagnola, Diffusion processes in Fock space, Quantum Probability and Related Topics **IX**, (1994), 189-214.
13. F. Fagnola and R. Monte, Quantum extensions of semigroups generated by Bessel processes, Math. Notes **60** (1996), no. 3-4, 389-401.
14. F. Fagnola, Quantum Markov Semigroups and Quantum Markov Flows, Proyecciones **18** (1999) no. 3, 1-144.
15. F. Fagnola and R. Rebolledo, Subharmonic projections for a quantum Markov semigroup, J. Math. Phys., **43** (2002), no. 2, 1074-1082.
16. L. Forzani, E. Sasso, and R. Scotto, Weak-type inequalities for higher order Riesz-Laguerre transforms, J. Funct. Anal., **256** (2009), no. 1, 258-274.

17. A. Mohari and K.B. Sinha, Stochastic dilation of minimal quantum dynamical semigroup, Proc. Indian Acad. Sci. Math. Sci., **102** (1992), 159-173.
18. P.-A. Meyer, Recent progress in quantum stochastic calculus Séminaire Bourbaki, Vol. 1992/93, Astérisque **216** (1993), Exp. No. 761, 3, 35-47.
19. K. R. Parthasarathy and K.B. Sinha, Markov chains as Evans-Hudson diffusions in Fock space, Séminaire de Probabilités, **XXIV** 1988/89, 362-369, Lecture Notes in Math., 1426, Springer, Berlin (1990).
20. M. Reed and B. Simon, Methods of Modern Mathematical Physics, Vol. II, Fourier Analysis, Selfadjointness, Academic Press, (1975).
21. E. Sasso, Maximal operators for the holomorphic Laguerre semigroup, Math. Scand., **97** (2005), no. 2, 235-265.
22. E. Sasso, Weak type estimates for Riesz-Laguerre transforms, Bull. Austral. Math. Soc., **75** (2007), no. 3, 397-408.