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THE BEST POSSIBLE CONSTANTS OF THE INEQUALITIES WITH POWER EXPONENTIAL FUNCTIONS

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The author in [7] conjectured the following inequality; If a and b are nonnegative real numbers with $a + b = 1/2$, then the inequality $1/2 \leq a^{(2b)^k} + b^{(2a)^k} \leq 1$ holds for $0 \leq k \leq 1$. In this paper, we shall prove the conjecture affirmatively and give the upper and lower estimation of the power exponential functions $a^b + b^a$ for the nonnegative real numbers a and b with $a + b = 2$. Moreover, we pose some inequalities with power exponential functions.

Key words : Inequalities; power exponential functions; monotonically increasing functions; monotonically decreasing functions.

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1. INTRODUCTION

The inequality with double power exponential functions

$$a^{(2b)^k} + b^{(2a)^k} \leq 1 \tag{1}$$

holds for nonnegative real numbers a and b with $a + b = 1$ and $k \geq 1$, which is posed by Cîrtoaje [3] as Conjecture 5.1 and proved by Miyagi *et al.* in [5]. Also, the author [7] proved that the following inequality with the power exponential functions holds: If a and b are nonnegative real numbers with $a + b = c$, then the inequality

$$a^{2b} + b^{2a} \leq 1 \tag{2}$$

holds for $1/2 \leq c \leq 1$.

The above symmetric inequalities (1) and (2) with the power exponential functions look like very simple forms, but these proofs are not immediate. Moreover, the author [7] conjectured the following inequality with double power exponential functions.

Conjecture 2.10 — If a and b are nonnegative real numbers with $a+b=1/2$, then the inequality

$$\frac{1}{2} \leq a^{(2b)^k} + b^{(2a)^k} \leq 1$$

holds for $0 \leq k \leq 1$.

We shall prove the conjecture 2.10 affirmatively.

***Theorem* 1.1** — *The conjecture 2.10 is holds true.*

It is known that, for the case of $a+b=2$, the inequalities

$$a^{2b} + b^{2a} + \left(\frac{a-b}{2}\right)^2 \leq 2 \tag{3}$$

and

$$a^{3b} + b^{3a} + \left(\frac{a-b}{2}\right)^4 \leq 2 \tag{4}$$

hold.

The inequality (3) is posed by Cîrtoaje [2] as Proposition 4.5 and the inequality (4) is posed by Cîrtoaje [2] as Conjecture 4.7 and proved by Miyagi *et al.* [6]. It is known that, other result of the case of $a+b=2$, Matejíčka [4] proved; the inequality

$$a^{rb} + b^{ra} \leq 2$$

holds for nonnegative real numbers a and b with $a+b=2$, if and only if $0 < r \leq 3$.

The above claim is posed by Cîrtoaje [2] as Conjecture 4.6. We show the upper and lower estimations of the power exponential functions for the case of $a+b=2$ in Theorem 1.2. It is no known that the lower inequality with the power exponential functions for $a+b=2$, hence, the inequality of Theorem 1.2 is a new result.

***Theorem* 1.2** — *If a and b are nonnegative real numbers with $a+b=2$, then the inequality*

$$-\left(\frac{|a-b|}{2}\right)^{\alpha} + 2 \leq a^b + b^a \leq -\left(\frac{|a-b|}{2}\right)^{\beta} + 2$$

holds, where the constants $\alpha = \ln 2 \cong 0.693147$ and $\beta = 2$ are the best possible.

2. Proof of Theorem 1.1

If $a + b = c$ and $0 < c \le 1/2$, without loss of generality, we may assume that $0 < a \le c/2 \le b < c$. Here, we have

$$a^{(2b)^k} + b^{(2a)^k} = a^{(2c-2a)^k} + (c-a)^{(2a)^k}$$

and we set

$$F(k,a) = a^{(2c-2a)^k} + (c-a)^{(2a)^k}$$

for $k > 0$ and $0 < a \le c/2$.

The derivative of $F(k,a)$ is

$$\begin{aligned} F_k(k,a) =& \frac{\partial}{\partial k} F(k,a) \\ =& (2a)^k (\ln a + \ln 2)(c-a)^{(2a)^k} \ln (c-a) \\ &+ 2^k (\ln a) a^{(2c-2a)^k} (c-a)^k (\ln (c-a) + \ln 2)\,. \end{aligned}$$

Here, we have $\ln a + \ln 2 < 0$, $\ln (c-a) < 0$, $\ln a < 0$ and $\ln (c-a) + \ln 2 < 0$ for $0 < c \le 1/2$. Hence, we can get $F_k(k,a) > 0$ for $k > 0$ and $0 < c \le 1/2$. Therefore, $F(k,a)$ is strictly increasing for $k > 0$.

Proof of Theorem 1.1 : If $c = 1/2$ then we have $F(0,a) \le F(k,a) \le F(1,a)$ and $F(1,a) \le 1$ by the inequality (2). Thus, we obtain $1/2 \le F(k,a) \le 1$ and the proof of Theorem 1.1 is complete. □

3. Proof of Theorem 1.2

Proof of Theorem 1.2 : Consider the equation

$$\left(\frac{|a-b|}{2}\right)^n = 2 - a^b - b^a\,.$$

Using the substitution

$$a = 1-t, \quad b = 1+t, \quad 0 \le t \le 1,$$

the equation becomes

$$t^n = 2 - (1-t)^{1+t} - (1+t)^{1-t}\,.$$

Since $0 \le t \le 1$, we need to show that $\ln 2 \le n \le 2$, which is true if and only if

$$\ln 2 < G(t) < 2 \tag{5}$$

for $0 < t < 1$, and

$$\lim_{t\to 0+0} G(t) = 2\,, \quad \lim_{t\to 1-0} G(t) = \ln 2, \tag{6}$$

where

$$G(t) = \frac{\ln\left(2-(1-t)^{1+t}-(1+t)^{1-t}\right)}{\ln t}\,.$$

The relations (6) can be proved by l'Hopital's rule [8]. The right inequality (5) follows immediately from Bernoulli's inequality, as follows:

$$\begin{aligned} G(t) <& \frac{\ln\left(2-(1-t)(1-t^2)-1-t(1-t)\right)}{\ln t} \\ =& 2+\frac{\ln(2-t)}{\ln t} < 2. \end{aligned}$$

To prove the left inequality (5), we consider two cases: $0 < t \le 1/3$ and $1/3 \le t < 1$.

Case 1 : $0 < t \le 1/3$. By Bernoulli's inequality, we get

$$\begin{aligned} G(t) >& \frac{\ln\left(2-1+t(1+t)-(1+t)(1-t^2)\right)}{\ln t} \\ =& 2+\frac{\ln(2+t)}{\ln t} = \ln 2 + \frac{f(t)}{\ln t}\,, \end{aligned}$$

where

$$f(t) = \ln(2+t) + (2-\ln 2)\ln t\,.$$

Thus, it suffices to prove that $f(t) < 0$ for $0 < t \le 1/3$. Since $f(t)$ is strictly increasing, we have

$$f(t) \le f\left(\frac{1}{3}\right) = \ln 7 - (3-2\ln 2)\ln 3 \cong -0.588427 < 0\,.$$

Case 2 : $1/3 \le t < 1$. From the next Lemma 3.3 and Lemma 3.4, we have

$$G(t) > G_3(t) > \lim_{t\to 1-0} G_3(t)\,.$$

By l'Hopital's rule,

$$\lim_{t\to 1-0} G_3(t) = \lim_{t\to 1-0} \frac{t(-6t+4t\ln 2+6)}{-3t^2+2(t^2-1)\ln 2+6t+1} = \ln 2\,.$$

Thus, the proof is completed. □

Lemma 3.1 — We have

$$(1+t)^{-t} > \frac{1}{2} + (1-t)\left(\frac{1}{4}+\frac{\ln 2}{2}\right)$$

for $1/3 \leq t < 1$.

PROOF : We set

$$f(t) = (t+1)^{-t} - \frac{1}{2} - (1-t)\left(\frac{1}{4} + \frac{\ln 2}{2}\right)$$

then the derivatives of $f(t)$ are

$$f'(t) = -(t+1)^{-t}\left(-\frac{t}{t+1} - \ln(t+1)\right) + \frac{1}{4} + \frac{\ln 2}{2}$$

and

$$f''(t) = (t+1)^{-t-1} g(t)\,,$$

where

$$g(t) = t + (t+1)(\ln(t+1))^2 + 2t\ln(t+1) - 2\,.$$

The derivative of $g(t)$ is

$$g'(t) = \frac{3t + (t+1)(\ln(t+1))^2 + 4(t+1)\ln(t+1) + 1}{t+1} > 0.$$

Hence, $g(t)$ is strictly increasing for $1/3 < t < 1$ and

$$g\left(\frac{1}{3}\right) = -\frac{5}{3} + \frac{4}{3}\left(\ln\frac{4}{3}\right)^2 + \frac{2}{3}\ln\frac{4}{3} \cong -1.36453$$

and

$$g(1) = -1 + 2(\ln 2)^2 + 2\ln 2 = 1.3472\,.$$

Thus, there exists a unique real number t_0 such that $g(t) < 0$ for $1/3 \leq t < t_0$ and $g(t) > 0$ for $t_0 < t \leq 1$. $f'(t)$ is strictly decreasing for $1/3 < t < t_0$ and strictly increasing for $t_0 < t < 1$. From

$$f'\left(\frac{1}{3}\right) = \frac{1}{4} + \frac{\ln 2}{2} + \left(\frac{3}{4}\right)^{\frac{1}{3}}\left(-\frac{1}{4} - \ln\frac{4}{3}\right) \cong 0.108057$$

and $f'(1) = 0$, there exists a unique real number t_1 such that $f'(t) > 0$ for $1/3 \leq t < t_1$ and $f'(t) < 0$ for $t_1 < t < 1$. Hence, $f(t)$ is strictly increasing for $1/3 < t < t_1$ and strictly decreasing for $t_1 < t < 1$. By

$$f\left(\frac{1}{3}\right) = -\frac{1}{2} + \left(\frac{3}{4}\right)^{\frac{1}{3}} + \frac{2}{3}\left(-\frac{1}{4} - \frac{\ln 2}{2}\right) \cong 0.0108446$$

and $f(1)=0$, we obtain $f(t)>0$ for $1/3\le t<1$. □

The following Lemma 3.2 is given by Anderson *et al.* [1].

Lemma 3.2 — Let $f,\ g:[a,b]\to\mathbb{R}$ be two continuous functions which are differentiable on (a,b). Further, let $g'(x)\neq 0$ on (a,b). If

$$\frac{f'(x)}{g'(x)}$$

is increasing (or decreasing) on (a,b), then the functions

$$\frac{f(x)-f(b)}{g(x)-g(b)}$$

and

$$\frac{f(x)-f(a)}{g(x)-g(a)}$$

are also increasing (or decreasing) on (a, b).

Lemma 3.3 — For $1/3\le t<1$, we have

$$G(t)>G_3(t)\,,$$

where

$$G_3(t)=\frac{\ln\left(-3t^2+2t^2\ln 2+6t+1-2\ln 2\right)-2\ln 2}{\ln t}\,.$$

PROOF : By $(1-t)^{1+t}>(1-t)^2$ and Lemma 3.1, we can get $G(t)>G_3(t)$ for $1/3\le t<1$, where

$$\begin{aligned}G_3(t)=&\frac{\ln\left(2-(t-1)^2-(t+1)\left(\frac{1}{2}+(1-t)\left(\frac{1}{4}+\frac{\ln 2}{2}\right)\right)\right)}{\ln t}\\=&\frac{\ln\left(-3t^2+2t^2\ln 2+6t+1-2\ln 2\right)-2\ln 2}{\ln t}\,.\end{aligned}$$ □

Lemma 3.4 — $G_3(t)$ is strictly decreasing for $1/3\le t<1$.

PROOF : We set

$$f(x)=\ln\left(-3t^2+2t^2\ln 2+6t+1-2\ln 2\right)-2\ln 2$$

and

$$g(x)=\ln t\,.$$

We consider the function $h(t)=f'(t)/g'(t)$. The function $h(t)$ is

$$h(t)=\frac{t(-6t+4t\ln 2+6)}{-3t^2+2(t^2-1)\ln 2+6t+1}\,.$$

The derivative of $h(t)$ is

$$\begin{aligned} h'(t) =& \frac{2\left(-9t^2 + (6t^2 + 16t - 6)\ln 2 - 6t - 8t(\ln 2)^2 + 3\right)}{\left(-3t^2 + 2(t^2 - 1)\ln 2 + 6t + 1\right)^2} \\ =& \frac{2k(t)}{\left(-3t^2 + 2(t^2 - 1)\ln 2 + 6t + 1\right)^2} . \end{aligned}$$

The derivative of $k(t)$ is

$$\begin{aligned} k'(t) =& 2(2\ln 2 - 3)(3t + 1 - 2\ln 2) \\ \cong& 2 \times (-1.61371)(3t + 1 - 2\ln 2) \\ <& 2 \times (-1.61371)\left(3 \times \frac{1}{3} + 1 - 2\ln 2\right) \\ \cong& -1.98068 . \end{aligned}$$

Therefore, $k(t)$ is strictly decreasing for $1/3 \le t < 1$. From $k(1/3) = -8(\ln 2)^2/3 \cong -1.28121$ and $k(t) < 0$ for $1/3 \le t < 1$, $h(t)$ is strictly decreasing for $1/3 \le t < 1$. By Lemma 3.2, $(f(t) - f(1))/(g(t) - g(1)) = G_3(t)$ is strictly decreasing for $1/3 \le t < 1$. □

4. Conjectures

We shall pose some conjectures relate to the inequalities with power-exponential functions.

Conjecture 4.1 — If a and b are nonnegative real numbers with $a + b = 1$, then the inequality

$$a^{(2b)^k} + b^{(2a)^k} \ge 1$$

holds for $0 \le k \le \frac{2-3\ln 2}{(\ln 2)^2 - \ln 2} \cong 0.373501$, where the constant $\frac{2-3\ln 2}{(\ln 2)^2 - \ln 2}$ is the best possible.

Conjecture 4.2 — If a and b are nonnegative real numbers with $a + b = 2$, then the inequality

$$a^{(2b)^k} + b^{(2a)^k} \ge 2$$

holds for $0 \le k \le s \cong 2.65986$, where s is the solution of $2^s = 1 + 2s$.

Conjecture 4.3 — If a and b are nonnegative real numbers with $a + b = 2$, then the inequality

$$a^{(3b)^k} + b^{(3a)^k} \le 2$$

holds for $0 \le k \le 1$, where the constant 1 is the best possible.

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