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ON SOFT INTERSECTION LEIBNIZ ALGEBRAS

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The main goal of this study is to introduce the description of soft Leibniz subalgebras (respectively soft Leibniz ideals) and to state some properties. Moreover, in this note, we investigate the concept of soft intersection Leibniz subalgebras (respectively soft intersection Leibniz ideals).

Key words : Soft Leibniz subalgebra; soft Leibniz ideal; soft intersection Leibniz ideal.

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1. INTRODUCTION

The concept of the soft set theory was introduced as a new mathematical tool dealing with uncertainties and vagueness by Molodtsov [11] in 1999. The soft set theory is an interesting area, which was studied in many papers (see [1, 2]). Loday [10] discovered Leibniz algebras which are certain non-(anti)commutatible analogs of Lie algebras. Then Akram [3] investigated the Fuzzy Lie algebras. Our main starting point is given by the paper [4] Akram *et al.* in 2013 which initiated the study of soft Lie subalgebras (resp. soft Lie ideals) and soft intersection Lie subalgebras (resp. soft intersection Lie ideals). Moreover, in the paper [4], the authors gave some of fundamental properties on soft Lie subalgebras (resp. soft Lie ideals) and soft intersection Lie subalgebras (resp. soft intersection Lie ideals). The aim of this paper is to define the concepts of soft Leibniz subalgebras (resp. soft Leibniz ideals) and soft intersection Leibniz subalgebras (resp. soft intersection Leibniz ideals). Moreover, we state some main properties on soft (intersection) Leibniz subalgebras and soft (intersection) Leibniz ideals.

2. Preliminaries

In this section we begin by setting up some definitions and notations which we need for our aims throughout this paper. We refer to [5-9] for more details. Let F be a field with characteristic zero and L be an algebra over F with the multiplication $[,]: L \times L \to L$. If L satisfies the Leibniz identity

$$[[a,b],c] = [a,[b,c]] - [b,[a,c]]$$

for all $a, b, c \in L$, then L is called a (left) Leibniz algebra. Leibniz algebras are non-anticommutative generalization of Lie algebras. As an immediate consequence, every Lie algebras are Leibniz algebras. For the subspaces U and W of L, $[U, W]$ is a subspace generated by the elements $[u, w]$ where $u \in U$ and $w \in W$. A subspace V is said to be a Leibniz subalgebra of L, if $[a, b] \in V$ for all $a, b \in V$. A subalgebra V is called an ideal of L, if $[a, b], [b, a] \in V$ for all $a \in V$ and $b \in L$. Let L_1 and L_2 be two Leibniz algebras. A linear map $\varphi : L_1 \to L_2$ is called a homomorphism if $\varphi([x, y]) = [\varphi(x), \varphi(y)]$ for all $x, y \in L_1$.

Let X be an initial universe and let E be a set of all possible parameters which are words or sentences related to the objects in X. We denote a soft universe and the power set of X respectively by the pair (X, E) and $P(X)$.

Definition 2.1 — A pair $S_A = (S, A)$ is called soft set over X, where $A \subseteq E$, S is a set-valued function $S : A \to P(X)$, that is, a soft set over X is a parameterized family of subsets of X. For any $a \in A$, $S(a)$ is a set of a-approximate elements of soft set (S, A). A soft set S_A over E is represented by the set of ordered pairs $S_A = \{(a, S(a))|a \in E, S(a) \in P(X)\}$.

Example 2.2 : Let $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $E = \{a_1, a_2, a_3, a_4\}$ be the set of parameters such that a_1 represents the parameter divisibility by 3, a_2 represents the parameter divisibility by 2, a_3 represents the parameter divisibility by 4, a_4 represents the parameter divisibility by 5.
Let $A = \{a_1, a_2, a_4\}$, then the soft set $S_A = (S, A) = \{S(a_1), S(a_2), S(a_4)\}$, where $S(a_1) = \{3, 6, 9\}, S(a_2) = \{2, 4, 6, 8, 10\}, S(a_4) = \{5, 10\}$.

Definition 2.3 — Let S_A and T_B be two soft sets over a common universe X. Then we define the intersection $S_A \cap T_B$ as $S_A \cap T_B(a) = S(a) \cap T(a)$ for all $a \in E$. Moreover, the product $S_A \wedge T_B$ is defined as $S_A \wedge T_B(a, b) = S(a) \cap T(b)$ for $(a, b) \in E \times E$.

3. The Main Results

In this section we give our main definitions and results.

Definition 3.1 — Let $S_A = (S, A)$ be a soft set over a Leibniz algebra L. If $S(a)$ is a Leibniz subalgebra (respectively Leibniz ideal) of L for every $a \in A$, then the soft set S_A is said to be a soft Leibniz subalgebra (respectively soft Leibniz ideal) over a Leibniz algebra L.

We construct the following example.

Example 3.2 : Let L be a Leibniz algebra over a field F with the basis $A = \{e_1, e_2, e_3, e_4, e_5\}$ by the following multiplication rule:

$$[e_2, e_1] = -e_2, [e_1, e_2] = e_2, [e_1, e_4] = e_4, [e_1, e_5] = e_5,$$
$$[e_2, e_3] = e_4, [e_3, e_2] = e_5, [e_4, e_1] = e_5, [e_5, e_1] = -e_5,$$

other products are zero and define

$$S(a) = \begin{cases} < e_2, e_4, e_5 > & \text{if } a = e_2, e_4, e_5, \\ < e_6 > & \text{if } a = e_1, \\ < e_3, e_4, e_5 > & \text{if } a = e_3, \\ L & \text{otherwise.} \end{cases}$$

For all $a \in A$, $S(a)$ is a Leibniz subalgebra (ideal). Thus, the soft set S_A becomes a soft Leibniz subalgebra (ideal) over L.

Proposition 3.3 — Let S_A and T_B be two soft Leibniz subalgebras (resp. soft Leibniz ideals) over a Leibniz algebra L. Then $S_A \cap T_B$ and $S_A \wedge T_B$ are also soft Leibniz subalgebras (resp. soft Leibniz ideals) over L.

PROOF : We need to show that for all $a \in E$ and every $x, y \in S_A \cap T_B(a) = S(a) \cap T(a)$, $[x, y] \in S_A \cap T_B(a) = S(a) \cap T(a)$. For every $x, y \in S(a) \cap T(a)$, we have $x, y \in S(a)$ and $x, y \in T(a)$. Since $S(a)$ and $T(a)$ are Leibniz subalgebras, we have $[x, y] \in S(a)$ and $[x, y] \in T(a)$. Therefore, $[x, y] \in S_A \cap T_B(a) = S(a) \cap T(a)$, that is, $S_A \cap T_B$ is a soft Leibniz subalgebra over a Leibniz algebra L. By a similar way, we prove that $S_A \wedge T_B$ is a soft Leibniz subalgebra.

Now we prove that for all $a \in E$ and every $x \in S_A \cap T_B(a) = S(a) \cap T(a), z \in L, [x, z], [z, x] \in S_A \cap T_B(a) = S(a) \cap T(a)$. Since $S(a)$ and $T(a)$ are Leibniz ideals, $[x, z], [z, x] \in S(a)$ and $[x, z], [z, x] \in T(a)$. Thus, $[x, z], [z, x] \in S_A \cap T_B(a) = S(a) \cap T(a)$, this means that $S_A \cap T_B$ is a soft Leibniz ideal over L. □

Remark 3.4 : If S_A and T_B are two soft Leibniz subalgebras over a Leibniz algebra L, then $[S_A, T_B]$ is a soft Leibniz subalgebra over L. But if S_A and T_B are two soft Leibniz ideals over a

Leibniz algebra L, then since the product space of two Leibniz ideals need not be ideal, in general, $[S_A, T_B]$ is not a soft Leibniz ideal over L.

Definition 3.5 — Let $S_A = (S, A)$ be a soft set a Leibniz algebra L. If for every $a \in A$, $S(a) = \{0\}$, then the soft set S_A is called trivial over L. The soft set S_A is called whole over L, if $S_A(a) = L$ for every $a \in A$.

Definition 3.6 — Let L_1 and L_2 be two Leibniz algebras and $\varphi : L_1 \to L_2$ is a homomorphism of Leibniz algebras. If S_A is a soft set over L_1, then $\varphi(S_A)$ is a soft set over L_2 where $\varphi(S) : E \to P(L_2)$ is defined by $\varphi(S)(a) = \varphi(S(a))$ for every $a \in E$.

Proposition 3.7 — Let L_1 and L_2 be two Leibniz algebras and $\varphi : L_1 \to L_2$ is a homomorphism of Leibniz algebras. If S_A is a soft Leibniz algebra over L_1, then $\varphi(S_A)$ is a soft Leibniz algebra over L_2.

PROOF Since S_A is a soft Leibniz algebra over L_1, for all $a \in A$, $S(a)$ is a Leibniz subalgebra. Since S_A is soft set of L_1, then $\varphi(S_A)$ is a soft set over L_2 where $\varphi(S) : E \to P(L_2)$ is defined by $\varphi(S)(a) = \varphi(S(a))$ for all $a \in E$. $\varphi(S(a))$ is a Leibniz subalgebra of L_2, so $\varphi(S_A)$ is soft Leibniz algebra over L_2.

***Theorem* 3.8** — *Let $\varphi : L_1 \to L_2$ be a homomorphism of Leibniz algebras and S_A is a soft Leibniz algebra over L_1.*

(i) If $S(a) = Ker\varphi$ for all $a \in A$, then $\varphi(S_A)$ is the trivial soft Leibniz algebra over L_2.

(ii) If φ is surjective and S_A is the whole soft Leibniz algebra over L_1, then $\varphi(S_A)$ is the whole soft Leibniz algebra over L_2.

PROOF : (i) Since $Ker\varphi$ is an ideal of L_1, for all $a \in A$, $S(a) = Ker\varphi$ is a Leibniz algebra. Hence S_A is a soft Leibniz algebra. By Proposition 3.7, $\varphi(S_A)$ is a soft Leibniz algebra over L_2 and since for every $a \in A$, $\varphi(S(a)) = \{0\}$, the soft Leibniz algebra $\varphi(S_A)$ is trivial.

(ii) Let φ be onto, so $\varphi(L_1) = L_2$. By Proposition 3.7, $\varphi(S_A)$ is a soft Leibniz algebra over L_2. Since $\varphi(S_A) = L_2$, namely, for each $a \in A$, $\varphi(S(a)) = L_2$, $\varphi(S_A)$ is whole over L_2. □

Definition 3.9 — Let $L = E$ be a Leibniz algebra and S_A be a soft set over U. Then S_A is satisfying the following conditions:

(i) $S(a) \cap S(b) \subseteq S(a + b)$

(ii) $S(a) \subseteq S(\alpha a)$

(iii) $S(a) \cap S(b) \subseteq S([a, b])$

for all $a, b \in A, \alpha \in F$, S_A is called a soft intersection Leibniz subalgebra over X.

Definition 3.10 — Let $L = E$ be a Leibniz algebra and S_A be a soft set over U. Then S_A is satisfying the following conditions:

(i) $S(a) \cap S(b) \subseteq S(a + b)$

(ii) $S(a) \subseteq S(\alpha a)$

(iii) $S(a) \subseteq S([a, b])$ and $S(b) \subseteq S([a, b])$ for all $a, b \in A, \alpha \in F$, S_A is called a soft intersection Leibniz ideal over X.

Example 3.11 : Let L be a Leibniz algebra over a field F with the basis $A = \{e_1, e_2, e_3, e_4\}$ by the following multiplication rule:

$$[e_1, e_1] = e_2, [e_1, e_2] = -e_2 - e_3, [e_2, e_1] = 0, [e_3, e_1] = 0,$$
$$[e_1, e_3] = e_2 + e_3, [e_4, e_1] = e_2 + e_3, [e_1, e_4] = 0,$$

other products are zero and define

$$S(a) = \begin{cases} < e_2, e_3 > & \text{if } a = e_2, e_3, \\ < e_2 > & \text{if } a = e_1, e_4, \\ L & \text{otherwise.} \end{cases}$$

For all $a \in A$, $S(a)$ is a Leibniz ideal. Thus, the soft set S_A becomes a soft Leibniz ideal over L. Furthermore, the soft set S_A satisfies all conditions at Definition 3.10, that is, this set becomes a soft intersection Leibniz ideal over L.

Proposition 3.12 — Let L be a Leibniz algebra and A be a Leibniz subalgebra (respectively Leibniz ideal) of L. If S_A is a soft intersection Leibniz subalgebra (resp. soft intersection Leibniz ideal) over X, then for every $a \in A$ $S(-a) = S(a)$ and $S(a) \subseteq S(0)$.

PROOF : $S(a) \cap S(0) \subseteq S(a + 0)$, so $S(0) \subseteq S(a)$. Applying Definition 3.10 part (ii) with $\alpha = -1$, we get $S(-a) \subseteq S((-1)(-a)$, namely $S(-a) \subseteq S(a)$ and $S(a) \subseteq S((-1)(a))$. Therefore $S(-a) = S(a)$. □

***Theorem* 3.13** — *Let L be a Leibniz algebra and A, B be two Leibniz subalgebras (respectively Leibniz ideals) of L. If S_A and T_B are two soft intersection Leibniz subalgebras (resp. soft intersection Leibniz ideals) over X, then $S_A \wedge T_B$ is a soft intersection Leibniz subalgebra (resp. soft intersection Leibniz ideal) over X.*

PROOF : Let $(a_1, b_1), (a_2, b_2) \in A \times B$ and $\alpha \in F$. Now we check all conditions for soft intersection Leibniz subalgebra and soft intersection Leibniz ideal.

$$\begin{aligned}
(S_A \wedge T_B)(a_1, b_1) \cap (S_A \wedge T_B)(a_2, b_2) &= (S(a_1) \cap T(b_1)) \cap (S(a_2) \cap T(b_2)) \\
&= (S(a_1) \cap S(a_2)) \cap (T(b_1) \cap T(b_2)) \\
&\subseteq S(a_1 + a_2) \cap T(b_1 + b_2) \\
&= (S_A \wedge T_B)((a_1 + a_2, b_1 + b_2)) \\
&= (S_A \wedge T_B)((a_1, b_1) + (a_2, b_2)). \\
(S_A \wedge T_B)(a_1, b_1) &= S(a_1) \cap T(b_1) \\
&\subseteq S(\alpha a_1) \cap T(\alpha b_1) \\
&= (S_A \wedge T_B)(\alpha a_1, \alpha b_1) \\
&= (S_A \wedge T_B)(\alpha(a_1, b_1)).
\end{aligned}$$

$$\begin{aligned}
(S_A \wedge T_B)[a_1, b_1] \cap (S_A \wedge T_B)[a_2, b_2] &= [S(a_1), T(b_1)] \cap [S(a_2), T(b_2)] \\
&= ([S(a_1), S(a_2)] \cap [T(b_1), T(b_2)]) \\
&\subseteq S([a_1, a_2]) \cap T([b_1, b_2]) \\
&= (S_A \wedge T_B)([a_1, a_2], [b_1, b_2]) \\
&= (S_A \wedge T_B)([(a_1, b_1), (a_2, b_2)]). \\
(S_A \wedge T_B)[a_1, b_1] &= S(a_1) \cap T(b_1) \\
&\subseteq S([a_1, a_2]) \cap T([b_1, b_2]) \\
&= (S_A \wedge T_B)([a_1, a_2], [b_1, b_2]) \\
&= (S_A \wedge T_B)([(a_1, b_1), (a_2, b_2)]). \\
(S_A \wedge T_B)[a_2, b_2] &= S(a_2) \cap T(b_2) \\
&\subseteq S([a_1, a_2]) \cap T([b_1, b_2]) \\
&= (S_A \wedge T_B)([a_1, a_2], [b_1, b_2]) \\
&= (S_A \wedge T_B)([(a_1, b_1), (a_2, b_2)]).
\end{aligned}$$

***Theorem* 3.14** — *Let L be a Leibniz algebra and A be a Leibniz subalgebra (respectively Leibniz ideal) of L. If S_A and T_A are two soft intersection Leibniz subalgebras (resp. soft intersection Leibniz ideals) over X, then $S_A \cap T_A$ is a soft intersection Leibniz subalgebra (resp. soft intersection Leibniz ideal) over X.*

PROOF : The proof of theorem is similar to the proof of Theorem 3.13.

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