Indian J. Pure Appl. Math., **51**(3): 1203-1223, September 2020

DOI: 10.1007/s13226-020-0459-8

A STUDY OF ERROR ESTIMATION FOR SECOND ORDER FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS

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(*Received* 23 *August* 2016; *after final revision* 12 *May* 2019; *accepted* 26 *June* 2019)

In this work, we study efficient asymptotically correct a posteriori error estimates for the numerical approximation of second order Fredholm integro-differential equations. We use the defect correction principle to find the deviation of the error estimation and show that collocation method by using m degree piecewise polynomial provides order $\mathcal{O}(h^{m+2})$ for the deviation of the error. Also, the theoretical behavior is tested on examples and it is shown that the numerical results confirm theoretical analysis.

Key words : Deviation of the error; collocation; finite difference; exact finite difference; integro-differential.

2010 Mathematics Subject Classification : 41A25; 45J05; 65N35.

1. Introduction

The second order Fredholm integro-differential (SFID) equations is defined in the following form

$$y''(t) = F\big(t, y(t), y'(t), z[y](t)\big), \quad t \in I := [a, b], \tag{1.1}$$

$$y(a) = r_1, \qquad y(b) = r_2, \tag{1.2}$$

with

$$z[y](t) := \int_a^b K\big(t, s, y(s)\big) ds,$$

where $a, b, r_1, r_2 \in (-\infty, \infty)$ and $b > a$. We define W and S as follows

$$W := \{(t, y, y', z)\,;\, t \in I \& y, y', z \in (-\infty, \infty)\},$$
$$S := \{(t, s, u)\,;\, t, s \in I \& u \in (-\infty, \infty)\}.$$

In this paper we shall assume that F and K are uniformly continuous in W and S, respectively. We say that $z[y](t)$ is linear if we can write $z[y](t)$ as

$$z[y](t) = \int_a^b \Lambda(t, s) y(s) ds,$$

where $\Lambda(t, s)$ is sufficiently smooth in $J := \{(t, s)\,; t, s \in I\}$. Also, we say that F is semilinear if we can write $F\big(t, y(t), y'(t), z[y](t)\big)$ as

$$F\big(t, y(t), y'(t), z[y](t)\big) = a_1(t)y'(t) + a_2(t)y(t) + a_3(t) + z[y](t).$$

In the nonlinear case, we assume that $F(t, y, y', z)$, $F_t(t, y, y', z)$, $F_y(t, y, y', z)$, $F_{y'}(t, y, y', z)$ and $F_z(t, y, y', z)$ are Lipschitz-continuous. Also when $z[y](t)$ is nonlinear we assume that $K(t, s, u)$ and $K_u(t, s, u)$ are Lipschitz-continuous. We say SFID equation with boundary condition (1.2) is linear if we can write (1.1) as follows

$$y''(t) = a_1(t)y'(t) + a_2(t)y(t) + a_3(t) + z[y](t), \quad t \in [a, b], \tag{1.3}$$

where $z[y](t)$ is linear. Also, in the linear case we assume that $a_1(t), a_2(t)$ and $a_3(t)$ are sufficiently smooth in I. The piecewise polynomial collocation method for integro-differential equations problem is studied in [2, 4, 9]. Also other methods for the integro-differential equations are studied in [7, 8, 10]. The defect correction principle is introduced in [3, 6]. The deviation of the error estimation for linear and nonlinear second order boundary value problem can be found in [1, 11]. The error estimation based on locally weighted defect that we will use in this manuscript, has been introduce in [1, 11].

The paper is organized as follows. In Section 2, the method is described and we introduce some details about piecewise polynomial collocation method, finite differences and exact difference schemes. In Section 3, the analysis of the deviation of the error for linear and nonlinear cases is given. In Section 4, we present the numerical experiments that demonstrate our theoretical results. A summary is given at the end of the paper in Section 5.

2. Description of the Method

In this section, we introduce piecewise polynomial collocation method, finite differences and exact difference schemes. Also, we describe some details about the deviation of the error estimation.

2.1 *Collocation method*

Let

$$a = \tau_0 < \tau_1 < \ldots < \tau_n = b, \ (n \geq 1),$$
$$0 = \rho_0 < \rho_1 < \ldots < \rho_m < \rho_{m+1} = 1.$$

We define $X_i,\ Z_n$ and $S^{(1)}_{m+1}(Z_n)$ as follows

$$X_i := \{t_{i,j} := \tau_i + \rho_j h_i; j = 1, \ldots, m\},$$

$$Z_n := \{t_{i,0} := \tau_i;\ i = 0, \ldots, n\},$$

$$S^{(1)}_{m+1}(Z_n) := \{p \in \mathcal{C}^1(I);\ p \restriction [\tau_i, \tau_{i+1}] \in \Pi_{m+1}([\tau_i, \tau_{i+1}])(i = 0, \ldots, n-1)\},$$

where $h_i := \tau_{i+1} - \tau_i$ and $\Pi_{m+1}([\tau_i, \tau_{i+1}])$ is a space of real polynomial functions on $[\tau_i, \tau_{i+1}]$ of degree$\leqslant m+1$. We define h (the diameter of gird Z_n) and h' as

$$h := \max\{h_i;\ i = 0, \ldots, n-1\},\ h' := \min\{h_i;\ i = 0, \ldots, n-1\}.$$

Also the set $X(n) := \bigcup_{i=0}^{n-1} X_i$ is called the set of collocation points. In the piecewise polynomial collocation method, we are looking to find a $p \in S^{(1)}_{m+1}(Z_n)$ so that (1.1)-(1.2) holds for all $t_{i,j} \in X(n)$. In the collocation method, we use the following quadrature method to determine $z[p](t)$

$$z[p](t_{i,j}) \approx \sum_{k=0}^{n-1}\sum_{z=0}^{m+1} \alpha_{k,z} K\big(t_{i,j}, t_{k,z}, p(t_{k,z})\big) =: \widetilde{z}\,[p](t_{i,j}), \tag{2.1}$$

where the quadrature weights are given by

$$\alpha_{k,z} := \int_{\tau_k}^{\tau_{k+1}} L_z^{[\tau_k, \tau_{k+1}]}(s)ds,$$

with (Lagrange polynomial)

$$L_j(\rho) := \prod_{\substack{i=0\\ i\neq j}}^{m+1} \frac{\rho - \rho_i}{\rho_j - \rho_i}, \quad L_j^{[a',b']}(\rho) := L_j(\frac{\rho - a'}{b' - a'}), \quad a \leq a' < b' \leq b.$$

Lemma 2.1 — For sufficiently smooth f, the following estimate holds

$$|z[f](t_{i,j}) - \widetilde{z}[f](t_{i,j})| = \mathcal{O}(h^{m+2}), \tag{2.2}$$

where $\widetilde{z}[\cdot](t_{i,j})$ is defined in (2.1).

PROOF : For nonlinear $z[\cdot](t)$ by using the Interpolation error theorem (see [5]), we can find

$$z[f](t_{i,j}) - \widetilde{z}[f](t_{i,j}) = \int_a^b K\big(t_{i,j}, s, f(s)\big)ds - \widetilde{z}[f](t_{i,j})$$
$$= \sum_{k=0}^{n-1} \int_{\tau_k}^{\tau_{k+1}} \underbrace{\Big(K\big(t_{i,j}, s, f(s)\big) - \sum_{z=0}^{m+1} L_z^{[\tau_k,\tau_{k+1}]}(s) K\big(t_{i,j}, t_{k,z}, f(t_{k,z})\big) \Big)}_{\mathcal{O}(h^{m+2})} ds$$
$$\leq n\, h\mathcal{O}(h^{m+2}) \leq \frac{h}{h'}(b-a)\mathcal{O}(h^{m+2}) = \mathcal{O}(h^{m+2}).$$

Similarly, we can obtain (2.2) for linear case. □

For the piecewise polynomial collocation method, we can find the following theorem [2, 9].

Theorem 2.2 — *Assume that the SFID problem (1.1)-(1.2) has a unique and sufficiently smooth solution $y(t)$. Also, assume that $p(t)$ is a piecewise polynomial collocation solution of degree $\leq m+1$. Then for sufficiently small h, the collocation solution $p(t)$ is well-defined and the following uniform estimates at least hold:*

$$\|y^{(j)}(t) - p^{(j)}(t)\|_\infty = \mathcal{O}(h^m),\ j = 0, 1, 2,$$
$$\|y^{(j)}(t) - p^{(j)}(t)\|_\infty = \mathcal{O}(h^{m+2-j}),\ j = 3, \ldots, m+1.$$

Remark 2.3 : In the piecewise polynomial collocation method when m is odd and the nodes ρ_i $(i = 1, \ldots, m)$ are symmetrically distributed, we have

$$\|y^{(j)}(t) - p^{(j)}(t)\|_\infty = \mathcal{O}(h^{m+1}),\ j = 0, 1.$$

Lemma 2.4 — For $z[\cdot](t)$, we have

$$|\widetilde{z}[p](t_{i,j}) - \widetilde{z}[y](t_{i,j})| = \mathcal{O}(h^m).$$

PROOF : For linear case by using Lemma 2.1, Theorem 2.2 and the Integral mean value theorem, we can write

$$\widetilde{z}[p](t_{i,j}) - \widetilde{z}[y](t_{i,j}) = z[e](t_{i,j}) + \mathcal{O}(h^{m+2}) =$$
$$= (b-a)\,\Lambda(t_{i,j}, \zeta_{i,j})\, \underbrace{e(\zeta_{i,j})}_{\mathcal{O}(h^m)} + \mathcal{O}(h^{m+2}) = \mathcal{O}(h^m),$$

where $\zeta_{i,j} \in [a,b]$. Also, when $z[\cdot](t)$ is nonlinear by using Lemma 2.1 and the Lipschitz condition for K, we get

$$
\begin{aligned}
&|\widetilde{z}[y](t_{i,j}) - \widetilde{z}[p](t_{i,j})| = |\widetilde{z}[y](t_{i,j}) - z[y](t_{i,j}) - \widetilde{z}[p](t_{i,j}) + z[p](t_{i,j}) \\
&+ z[y](t_{i,j}) - z[p](t_{i,j})| = |\int_a^b \Big(K\big(t_{i,j}, s, y(s)\big) - K\big(t_{i,j}, s, p(s)\big)\Big) ds| \\
&+ \mathcal{O}(h^{m+2}) \le C \int_a^b |y(s) - p(s)| ds + \mathcal{O}(h^{m+2}) = \mathcal{O}(h^m),
\end{aligned}
$$

which completes the proof. □

2.2 *Finite difference scheme*

We define $\mathcal{A}$ and $\mathcal{B}$ as follows

$$
\begin{aligned}
&\mathcal{A} := \{(i,j);\, t_{i,j} \in X(n) \cup Z_n\}, \qquad \mathcal{B} := \mathcal{A} - \{(0,0),(n,0)\}, \\
&\mathcal{T} := \mathcal{A} - \{(n,0)\}.
\end{aligned}
$$

Also, we define

$$
\begin{aligned}
&\delta_{i,j} := t_{i,j+1} - t_{i,j}, \qquad \widehat{\delta}_{i,j} := \frac{\delta_{i,j-1} + \delta_{i,j}}{2}, \\
&\widehat{\alpha}_{i,j} := \frac{\delta_{i,j-1}}{\widehat{\delta}_{i,j}}, \qquad \widehat{\beta}_{i,j} := \frac{\delta_{i,j}}{\widehat{\delta}_{i,j}}.
\end{aligned}
$$

Now, we write a general one-step finite difference scheme as

$$
\begin{aligned}
&(L_{\mathcal{A}}^{(2)}\eta)_{i,j} = F(t_{i,j}, \eta_{i,j}, (L_{\mathcal{A}}^{(1)}\eta)_{i,j}, \chi[\eta]_{i,j}), \quad (i,j) \in \mathcal{B}, \\
&\eta_{0,0} = r_1, \quad \eta_{n,0} = r_2,
\end{aligned}
\tag{2.3}
$$

where

$$
\begin{aligned}
&(L_{\mathcal{A}}^{(2)}\eta)_{i,j} := \frac{\widehat{\alpha}_{i,j}\eta_{i,j+1} - 2\eta_{i,j} + \widehat{\beta}_{i,j}\eta_{i,j-1}}{\widehat{\alpha}_{i,j}\widehat{\beta}_{i,j}\widehat{\delta}_{i,j}^2}, \\
&(L_{\mathcal{A}}^{(1)}\eta)_{i,j} := \frac{\eta_{i,j+1} - \eta_{i,j-1}}{2\widehat{\delta}_{i,j}},
\end{aligned}
$$

and

$$
\chi[\eta]_{i,j} := \sum_{(l,v)\in\mathcal{T}} \frac{\delta_{l,v}}{2}\big(K(t_{i,j}, t_{l,v}, \eta_{l,v}) + K(t_{i,j}, t_{l,v+1}, \eta_{l,v+1})\big).
$$

In fact, we use the trapezoidal rule to determine $z[\cdot](t)$. For the trapezoidal rule, we can find the following theorem.

***Theorem* 2.5** — *Let f be a sufficiently smooth function on the interval $[a,b]$. Then we have*

$$\big|\chi[f]-z[f]\big| \leq \frac{(b-a)h}{12h'}h^2 \max_{s\in[a,b]} \Big|\frac{\partial K\big(t_{i,j},s,f(s)\big)}{\partial s^2}\Big|.$$

Remark 2.6 : In this work, we find the finite difference scheme by using the trapezoidal rule. However, we can use the 2-point Gaussian quadrature method as follows

$$(L^{(2)}_{\mathcal{A}}\eta)_{i,j} = F\big(t_{i,j},\eta_{i,j},(L^{(1)}_{\mathcal{A}}\eta)_{i,j},\omega[\eta]_{i,j}\big), \quad (i,j)\in\mathcal{B},$$
$$\eta_{0,0}=r_1, \qquad \eta_{n,0}=r_2,$$

$$\omega[\eta]_{i,j} := \sum_{k=0}^{1}\sum_{(l,v)\in\mathcal{T}} w_k\frac{\delta_{l,v}}{2}K\big(t_{i,j},\lambda^{l,v}_k,\Upsilon_{l,v}(\lambda^{l,v}_k)\big),$$

where

$$\Upsilon_{l,v}(s) := \frac{s-t_{l,v+1}}{-\delta_{l,v}}\eta_{l,v}+\frac{s-t_{l,v}}{\delta_{l,v}}\eta_{l,v+1},$$

$$\lambda^{l,v}_k := \frac{\delta_{l,v}}{2}\varpi_k+\frac{\delta^+_{l,v}}{2},$$

with $w_0=w_1=1$, $-\varpi_0=\varpi_1=\frac{\sqrt{3}}{3}$ and $\delta^+_{l,v}:=t_{l,v+1}+t_{l,v}$. This change will have no effect in the order of the finite difference scheme and the deviation of the error. And a similar argument can be given for this finite difference scheme. In Section 4, we study this case by numerical experiments.

Definition 2.7 — For any function u, we define

$$\mathcal{R}(u) := \{u(t_{i,j})\,;\,(i,j)\in\mathcal{A}\},$$

also we define

$$\eta := \{\eta_{i,j}\,;\,(i,j)\in\mathcal{A}\}, \qquad L^{(l)}_{\mathcal{A}}\eta := \{(L^{(l)}_{\mathcal{A}}\eta)_{i,j}\,;(i,j)\in\mathcal{A}\}, \;\; l=1,2.$$

For the above finite difference scheme we have the following estimate

$$\|\eta-\mathcal{R}(y)\|_\infty = \mathcal{O}(h^2),$$
$$\|L^{(l)}_{\mathcal{A}}\eta-\mathcal{R}(y^{(l)})\|_\infty = \mathcal{O}(h^2), \quad l=1,2,$$

where η and $L_{\mathcal{A}}^{(l)}\eta$ is defined in the Definition 2.7.

2.3 *Exact finite difference scheme and Deviation of the error estimation*

Now, we study the deviation of the error estimation for (1.1)-(1.2). In the first step, we consider the Dirichlet problem

$$y''(t) = f(t),\ a \le t \le b, \quad y(a) = y(b) = 0, \tag{2.4}$$

where $f(t)$ is permitted to have jump discontinuities in the points belonging to Z_n. For the discretization form of (2.4), i.e.,

$$(L_{\mathcal{A}}^{(2)}\eta)_{i,j} = f(t_{i,j}), \quad (i,j) \in \mathcal{B}, \tag{2.5}$$

according to [1, 11], we have the following lemmas.

Lemma 2.8 — The unique solution η of (2.5) is given by

$$\eta_{i,j} = \sum_{(l,v)\in\mathcal{B}} G(t_{i,j}, t_{l,v})\delta_{l,v} f(t_{l,v}),$$

where $G(t,\tau)$ is Green's function

$$G(t,\tau) = \begin{cases} \frac{(b-t)(a-\tau)}{b-a}, & a \le \tau \le t \le b, \\ \frac{(b-\tau)(a-t)}{b-a}, & a \le t \le \tau \le b. \end{cases}$$

Lemma 2.9 — For $v \in \widehat{\mathcal{C}}_2[t_{i,j-1}, t_{i,j}, t_{i,j+1}]$, where

$$\widehat{\mathcal{C}}_2[t_{i,j-1}, t_{i,j}, t_{i,j+1}] := \{v \in C^1[t_{i,j-1}, t_{i,j+1}] : v''\text{continuous on } [t_{i,j-1}, t_{i,j}) \cup (t_{i,j}, t_{i,j+1}], \lim_{t\uparrow t_{i,j}} v'' \in \mathbb{R}, \lim_{t\downarrow t_{i,j}} v'' \in \mathbb{R} \text{ exist}\},$$

we have

$$(L_{\mathcal{A}}^{(2)}v)_{i,j} = \int_{-\widehat{\alpha}_{i,j}}^{\widehat{\beta}_{i,j}} R_{i,j}(\xi) v''(t_{i,j} + \widehat{\delta}_{i,j}\xi) d\xi,$$

with kernel

$$R_{i,j}(\xi) = \begin{cases} 1 + \frac{\xi}{\widehat{\alpha}_{i,j}}, & \xi \in [-\alpha_{i,j}, 0], \\ 1 - \frac{\xi}{\widehat{\beta}_{i,j}}, & \xi \in [0, \beta_{i,j}]. \end{cases}$$

Therefore as [1, 11], we can find "the exact finite difference scheme" for (1.1) as follows

$$(L_{\mathcal{A}}^{(2)}p)_{i,j} = \mathcal{I}_{\mathcal{A}}\big(F(\cdot, p, p', z[p]), t_{i,j}\big),$$

where

$$\mathcal{I}_{\mathcal{A}}\big(w,t_{i,j}\big):=\int_{-\widehat{\alpha}_{i,j}}^{\widehat{\beta}_{i,j}} R_{i,j}(\xi)w(t_{i,j}+\widehat{\delta}_{i,j}\xi)d\xi.$$

Then we can say that a solution of problem (1.1)-(1.2) satisfies in the exact finite difference scheme. Also according to the collocation method, we have the following relation.

$$p''(t_{i,j})-F\big(t_{i,j},p(t_{i,j}),p'(t_{i,j}),z[p](t_{i,j})\big)\equiv 0,\ (i,j)\in X(n).$$

We define defect at $t_{i,j}$ as follows

$$D_{i,j}:=(L_{\mathcal{A}}^{(2)}p)_{i,j}-\mathcal{I}_{\mathcal{A}}\big(F(\cdot,p,p',z[p]),t_{i,j}\big),\ (i,j)\in\mathcal{B}. \tag{2.6}$$

In order to compute integral in (2.6), we use quadrature formula. When $t_{i,j}\in X(n)$, we have [1, 11]

$$\begin{aligned}\mathcal{I}_{\mathcal{A}}\big(F(\cdot,p,p',z[p]),t_{i,j}\big)&\approx Q_{\mathcal{A}}\big(F(\cdot,p,\widetilde{z}[p]),t_{i,j}\big)\\&:=\sum_{k=0}^{m+1}\gamma_{i,j}^{k}F\big(t_{i,k},p(t_{i,k}),p'(t_{i,k}),\widetilde{z}[p](t_{i,k})\big),\end{aligned}$$

where

$$\gamma_{i,j}^{k}=\int_{-\widehat{\alpha}_{i,j}}^{\widehat{\beta}_{i,j}}R_{i,j}(\xi)L_k(\rho_j+\xi\frac{\widehat{\delta}_{i,j}}{h_i})d\xi.$$

Also for $t_{i,0}=\tau_i$, we have [1, 11]

$$\begin{aligned}\mathcal{I}_{\mathcal{A}}\big(F(\cdot,p,p',z[p]),t_{i,0}\big)&\approx Q_{\mathcal{A}}\big(F(\cdot,p,p',\widetilde{z}[p]),t_{i,0}\big)\\&:=\sum_{k=0}^{m+1}\gamma_{i,0}^{k+}F\big(t_{i,k},p(t_{i,k}),p'(t_{i,k}),\widetilde{z}[p](t_{i,k})\big)\\&+\sum_{k=0}^{m+1}\gamma_{i,0}^{k-}F\big(t_{i-1,k},p(t_{i-1,k}),p'(t_{i-1,k}),\widetilde{z}[p](t_{i-1,k})\big),\end{aligned}$$

where

$$\begin{aligned}\gamma_{i,0}^{k+}&=\int_{0}^{\widehat{\beta}_{i,0}}R_{i,0}(\xi)L_k(\xi\frac{\widehat{\delta}_{i,0}}{h_i})d\xi,\\\gamma_{i,0}^{k-}&=\int_{-\widehat{\alpha}_{i,0}}^{0}R_{i,0}(\xi)L_k(1+\xi\frac{\widehat{\delta}_{i,0}}{h_i})d\xi.\end{aligned}$$

For the above quadrature formula, we can find the following lemma.

Lemma 2.10 — For sufficiently smooth f the following error holds

$$\mathcal{I}_{\mathcal{A}}\big(f, t_{i,j}\big) - Q_{\mathcal{A}}\big(f, t_{i,j}\big) = \mathcal{O}(h^{m+2}).$$

Also, when m is odd and the nodes ρ_i are symmetrically, we can see the following relation.

$$\mathcal{I}_{\mathcal{A}}\big(f, t_{i,j}\big) - Q_{\mathcal{A}}\big(f, t_{i,j}\big) = \mathcal{O}(h^{m+3}).$$

In this step we define $\pi = \{\pi_{i,j}\,;\,(i,j) \in \mathcal{A}\}$ as the solution of the following finite difference

$$\begin{aligned}(L_{\mathcal{A}}^{(2)}\pi)_{i,j} &= F\big(t_{i,j}, \pi_{i,j}, (L_{\mathcal{A}}^{(1)}\pi)_{i,j}, \chi[\pi]_{i,j}\big) + D_{i,j},\ (i,j) \in \mathcal{B}, \\ \pi_{0,0} &= r_1, \quad \pi_{n,0} = r_2.\end{aligned} \tag{2.7}$$

We define $\mathbf{D} := \{D_{i,j}\,;\,(i,j) \in \mathcal{B}\}$. For small value $\mathbf{D}$, we have

$$\pi - \mathcal{R}(p) \approx \eta - \mathcal{R}(y).$$

We define ε and e as

$$\varepsilon := \pi - \eta \approx \mathcal{R}(p) - \mathcal{R}(y) =: e.$$

An estimate for the error e can be found in Theorem 2.2. The deviation of the error can be defined as

$$\theta := e - \varepsilon.$$

By using (2.41) and Lemma 2.10 we can easily find the following lemmas.

Lemma 2.11 — The defined defect in (2.41) has order $\mathcal{O}(h^m)$.

Lemma 2.12 — The $\pi - \eta$ has order $\mathcal{O}(h^m)$.

In the next section, we will study the order of the deviation of the error estimate for SFID equation.

3. Analysis of the Deviation of the Error

3.1 *Linear case*

Lemma 3.1 — For the linear and nonlinear $z[\cdot](t)$, we have

$$\big|\chi[e]_{i,j} - \widetilde{z}\,[e](t_{i,j})\big| = \mathcal{O}(h^{m+2}). \tag{3.1}$$

PROOF : By using the triangle inequality and Lemma 2.1, we can write

$$\big|\chi[e]_{i,j} - \widetilde{z}[e](t_{i,j})\big| \leq \big|\chi[e]_{i,j} - z[e](t_{i,j})\big| + \underbrace{\big|z[e](t_{i,j}) - \widetilde{z}[e](t_{i,j})\big|}_{\mathcal{O}(h^{m+2})}, \tag{3.2}$$

also by using Theorem 2.5, we can write

$$\big|\chi[e]_{i,j} - z[e](t_{i,j})\big| \leq \frac{(b-a)h}{12h'} h^2 \max_{s\in[a,b]} \big| \underbrace{\frac{\partial^2\Big(K\big(t_{i,j}, s, e(s)\big)\Big)}{\partial s^2}}_{\mathcal{O}(h^m)} \big| = \mathcal{O}(h^{m+2}), \tag{3.3}$$

therefore from (3.2) and (3.3), we can find (3.1). □

Theorem 3.2 — *Consider the SFID equation (1.3) with boundary conditions (1.2). Assume that the SFID problem has a unique and sufficiently smooth solution. Then the following estimate holds*

$$||\theta||_\infty = ||e - \varepsilon||_\infty = \mathcal{O}(h^{m+2}),$$

where e is error, ε is the error estimate and θ is the deviation of the error estimate.

PROOF : Since F is semilinear then by using (2.3) and (2.7), we get

$$\begin{aligned}
(L_{\mathcal{A}}^{(2)}\varepsilon)_{i,j} = &\quad a_1(t_{i,j})(L_{\mathcal{A}}^{(1)}\varepsilon)_{i,j} + a_2(t_{i,j})\varepsilon_{i,j} + \chi[\varepsilon]_{i,j} + D_{i,j},\\
(L_{\mathcal{A}}^{(2)}e)_{i,j} = &\quad Q_{\mathcal{A}}\big(a_1p' + a_2p + a_3 + \widetilde{z}[p], t_{i,j}\big) - \mathcal{I}_{\mathcal{A}}\big(a_1y' + a_2y + a_3 + z[y], t_{i,j}\big)\\
&\quad + D_{i,j} + \mathcal{O}(h^{m+2}).
\end{aligned}$$

Therefore we can write

$$\begin{aligned}
(L_{\mathcal{A}}^{(2)}\theta)_{i,j} &= a_1(t_{i,j})(L_{\mathcal{A}}^{(1)}\theta)_{i,j} + a_2(t_{i,j})\theta_{i,j} + \chi[\theta]_{i,j} +\\
&\quad \underbrace{\mathcal{I}_{\mathcal{A}}(a_1e' + a_2e, t_{i,j}) - (a_1(t_{i,j})(L_{\mathcal{A}}^{(1)}e)_{i,j} + a_2(t_{i,j})e_{i,j})}_{I_1}\\
&\quad + \underbrace{(\mathcal{Q}_{\mathcal{A}} - \mathcal{I}_{\mathcal{A}})(a_1p' + a_2p + a_3, t_{i,j})}_{I_2}\\
&\quad + \underbrace{\mathcal{I}_A(z[e], t_{i,j}) - \chi[e]_{i,j}}_{I_3} + \underbrace{Q_A\big(\widetilde{z}[p](t_{i,j})\big) - \mathcal{I}_A(z[p], t_{i,j})}_{I_4} + \mathcal{O}(h^{m+2}),
\end{aligned} \tag{3.4}$$

by using Lemma 6.1 in [11] and [1], we can say

$$I_1 = \frac{1}{\widehat{\delta}_{i,j}}(\delta_{i,j-1}g_{i,j-\frac{1}{2}} - \delta_{i,j}g_{i,j+\frac{1}{2}}) + \frac{1}{2\widehat{\delta}_{i,j}}(\delta_{i,j}\phi_{i,j+\frac{1}{2}} - \delta_{i,j-1}\phi_{i,j-\frac{1}{2}}) + \mathcal{O}(h^{m+2}), \tag{3.5}$$

where $g_{i,j\pm\frac{1}{2}} = \mathcal{O}(h^{m+1})$ and $\phi_{i,j\pm\frac{1}{2}} = \mathcal{O}(h^{m+1})$. Also according to Theorem 6.1 in [11] and [1], we can obtain $I_2 = \mathcal{O}(h^{m+2})$. Also since $p \in \Pi_{m+1}$ and $\Lambda(t,s)$ is sufficiently smooth then by using Lemma 2.10, we can say that $I_4 = \mathcal{O}(h^{m+2})$. Now we study I_3. By using Lemma 3.1, we obtain

$$\begin{aligned}
I_3 &= \mathcal{I}_A(z[e], t_{i,j}) - \chi[e]_{i,j} = \mathcal{I}_A(\widetilde{z}[e], t_{i,j}) - \widetilde{z}[e](t_{i,j}) + \mathcal{O}(h^{m+2}) \\
&= \sum_{(l,v)\in\mathcal{T}} \alpha_{l,v} e(t_{l,v}) \big(\underbrace{\int_{-\widehat{\alpha}_{i,j}}^{\widehat{\beta}_{i,j}} R_{i,j}(\xi)\Lambda(t_{i,j} + \xi\delta_{i,j}, t_{l,v}) d\xi - \Lambda(t_{i,j}, t_{l,v})}_{I_5} \big) \\
&\quad + \mathcal{O}(h^{m+2}).
\end{aligned}$$

For I_5, we obtain

$$I_5 = \frac{1}{2\widehat{\delta}_{i,j}} \big(\delta_{i,j}^2 \Psi_{i,j+\frac{1}{2}}(t_{l,v}) - \delta_{i,j-1}^2 \Psi_{i,j-\frac{1}{2}}(t_{l,v}) \big) + S_1(t_{l,v}),$$

where

$$\begin{aligned}
\Psi_{i,j+\frac{1}{2}}(t_{l,v}) &:= \int_{-\frac{1}{2}}^{\frac{1}{2}} (u^2 + \frac{1}{4}) \Lambda_t(t_{i,j+\frac{1}{2}} + \delta_{i,j} u, t_{l,v}) du, \\
\Psi_{i,j-\frac{1}{2}}(t_{l,v}) &:= \int_{-\frac{1}{2}}^{\frac{1}{2}} (u^2 + \frac{1}{4}) \Lambda_t(t_{i,j-\frac{1}{2}} + \delta_{i,j-1} u, t_{l,v}) du, \\
S_1(t_{l,v}) &:= \frac{-1}{2\widehat{\delta}_{i,j}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \Big(\delta_{i,j}^2 u \Lambda_t(t_{i,j+\frac{1}{2}} + \delta_{i,j} u, t_{l,v}) + \delta_{i,j-1}^2 u \Lambda_t(t_{i,j-\frac{1}{2}} + \delta_{i,j} u, t_{l,v}) \Big) du.
\end{aligned}$$

We can see that

$$\begin{aligned}
\Psi_{i,j+\frac{1}{2}}(t_{l,v}) &= \mathcal{O}(1), \\
\Psi_{i,j-\frac{1}{2}}(t_{l,v}) &= \mathcal{O}(1), \\
S_1(t_{l,v}) &= \mathcal{O}(h^2).
\end{aligned}$$

Therefore we can rewrite (3.4) as follows

$$\begin{aligned}
(L_{\mathcal{A}}^{(2)}\theta)_{i,j} &= a_1(t_{i,j})(L_{\mathcal{A}}^{(1)}\theta)_{i,j} + a_2(t_{i,j})\theta_{i,j} + \chi[\theta]_{i,j} \\
&\quad + \frac{1}{\widehat{\delta}_{i,j}} (\delta_{i,j-1} g_{i,j-\frac{1}{2}} - \delta_{i,j} g_{i,j+\frac{1}{2}}) + \frac{1}{2\widehat{\delta}_{i,j}} (\delta_{i,j}\phi_{i,j+\frac{1}{2}} - \delta_{i,j-1}\phi_{i,j-\frac{1}{2}}) \\
&\quad + \sum_{(l,v)\in\mathcal{T}} \alpha_{l,v} e(t_{l,v}) \Big(\frac{1}{2\widehat{\delta}_{i,j}} \big(\delta_{i,j}^2 \Psi_{i,j+\frac{1}{2}}(t_{l,v}) - \delta_{i,j-1}^2 \Psi_{i,j-\frac{1}{2}}(t_{l,v}) \big) \Big) \\
&\quad + \sum_{(l,v)\in\mathcal{T}} \alpha_{l,v} \underbrace{e(t_{l,v})}_{\mathcal{O}(h^m)} \underbrace{S_1(t_{l,v})}_{\mathcal{O}(h^2)} + \mathcal{O}(h^{m+2})
\end{aligned}$$

$$
\begin{aligned}
&= a_1(t_{i,j})(L_{\mathcal{A}}^{(1)}\theta)_{i,j} + a_2(t_{i,j})\theta_{i,j} + \chi[\theta]_{i,j} \\
&+ \frac{1}{\widehat{\delta}_{i,j}}(\delta_{i,j-1}g_{i,j-\frac{1}{2}} - \delta_{i,j}g_{i,j+\frac{1}{2}}) + \frac{1}{2\widehat{\delta}_{i,j}}(\delta_{i,j}\phi_{i,j+\frac{1}{2}} - \delta_{i,j-1}\phi_{i,j-\frac{1}{2}}) \\
&+ \sum_{(l,v)\in\mathcal{T}} \alpha_{l,v} e(t_{l,v})\Big(\frac{1}{2\widehat{\delta}_{i,j}}\big(\delta_{i,j}^2\Psi_{i,j+\frac{1}{2}}(t_{l,v}) - \delta_{i,j-1}^2\Psi_{i,j-\frac{1}{2}}(t_{l,v})\big)\Big) \\
&+ \mathcal{O}(h^{m+2}).
\end{aligned} \tag{3.6}
$$

The proof is continued by considering the following scheme:

$$
\begin{aligned}
(L_{\mathcal{A}}^{(2)}\widehat{\theta})_{i,j} &= \frac{1}{\widehat{\delta}_{i,j}}(\delta_{i,j-1}g_{i,j-\frac{1}{2}} - \delta_{i,j}g_{i,j+\frac{1}{2}}) + \frac{1}{2\widehat{\delta}_{i,j}}(\delta_{i,j}\phi_{i,j+\frac{1}{2}} - \delta_{i,j-1}\phi_{i,j-\frac{1}{2}}) \\
&\quad + \sum_{(l,v)\in\mathcal{T}} \alpha_{l,v} e(t_{l,v})\Big(\frac{1}{2\widehat{\delta}_{i,j}}\big(\delta_{i,j}^2\Psi_{i,j+\frac{1}{2}}(t_{l,v}) - \delta_{i,j-1}^2\Psi_{i,j-\frac{1}{2}}(t_{l,v})\big)\Big).
\end{aligned} \tag{3.7}
$$

In this step, we define

$$
\begin{aligned}
&H := \{\delta_{i,j-1}g_{i,j-\frac{1}{2}} - \delta_{i,j}g_{i,j+\frac{1}{2}};\ (i,j)\in\mathcal{B}\}, \\
&\Phi := \{\delta_{i,j}\phi_{i,j+\frac{1}{2}} - \delta_{i,j-1}\phi_{i,j-\frac{1}{2}};\ (i,j)\in\mathcal{B}\}, \\
&\Theta^{l,v} := \{\delta_{i,j}^2\Psi_{i,j+\frac{1}{2}}(t_{l,v}) - \delta_{i,j-1}^2\Psi_{i,j-\frac{1}{2}}(t_{l,v});\ (i,j)\in\mathcal{B}\}.
\end{aligned}
$$

Then by using Lemma 2.8, we find

$$
\widehat{\theta}_{i,j} = \left(\frac{1}{\widehat{\delta}_{i,j}}(L_{\mathcal{A}}^2)^{-1}H\right)_{i,j} + \left(\frac{1}{2\widehat{\delta}_{i,j}}(L_{\mathcal{A}}^2)^{-1}\Phi\right)_{i,j} + \sum_{(l,v)\in\mathcal{T}} \alpha_{l,v} e(t_{l,v}) \left(\frac{1}{2\widehat{\delta}_{i,j}}(L_{\mathcal{A}}^2)^{-1}\Theta^{l,v}\right)_{i,j}.
$$

Let

$$
\begin{aligned}
&g^* = \max_{i,j}|g_{i,j\pm\frac{1}{2}}| = \mathcal{O}(h^{m+1}), \\
&\phi^* = \max_{i,j}|\phi_{i,j\pm\frac{1}{2}}| = \mathcal{O}(h^{m+1}), \\
&\Psi_{l,v}^* = |\Psi_{i,j\pm\frac{1}{2}}(t_{l,v})| = \mathcal{O}(1),
\end{aligned}
$$

therefore we get

$$
\begin{aligned}
\|\widehat{\theta}\|_\infty &\le 2h(\max_{w,x}\sum_{(i,j)\in\mathcal{B}} G(t_{w,x},t_{i,j})g^*) + h(\max_{w,x}\sum_{(i,j)\in\mathcal{B}} G(t_{w,x},t_{i,j})\phi^*) \\
&\quad + h^2\sum_{(l,v)\in\mathcal{T}} \alpha_{l,v} e(t_{l,v})\big(\max_{w,x}\sum_{(i,j)\in\mathcal{B}} G(t_{w,x},t_{i,j})\Psi_{l,v}^*\big) = \mathcal{O}(h^{m+2}).
\end{aligned} \tag{3.8}
$$

Now, we have:

$$\|\theta\|_\infty \leq \|\theta - \widehat{\theta}\|_\infty + \|\widehat{\theta}\|_\infty = \|\theta - \widehat{\theta}\|_\infty + \mathcal{O}(h^{m+2}).$$

To end the proof, we prove that $\|\theta - \widehat{\theta}\,\|_\infty = \mathcal{O}(h^{m+2})$. By using (3.6)-(3.7), we get:

$$\begin{aligned}\big(L_{\mathcal{A}}^{(2)}(\theta - \widehat{\theta}\,)\big)_{i,j} &= a_1(t_{i,j})\big(L_{\mathcal{A}}^{(1)}(\theta - \widehat{\theta}\,)\big)_{i,j} + a_2(t_{i,j})(\theta_{i,j} - \widehat{\theta}_{i,j}) + \chi[\theta - \widehat{\theta}\,]_{i,j} \\ &\quad + \underbrace{a_1(t_{i,j})(L_{\mathcal{A}}^{(1)}\widehat{\theta}\,)_{i,j}}_{Y_1} + \underbrace{a_2(t_{i,j})\widehat{\theta}_{i,j}}_{Y_2} + \underbrace{\chi[\widehat{\theta}\,]_{i,j}}_{Y_3} + \mathcal{O}(h^{m+2}).\end{aligned}$$

By using (3.8) and the definition of $\chi[\cdot]$, we can say that $Y_2 = \mathcal{O}(h^{m+2})$ and $Y_3 = \mathcal{O}(h^{m+2})$. Also for the term Y_1, in a similar way to (6.10) in [11], we can find $Y_1 = \mathcal{O}(h^{m+2})$. Then we have

$$\begin{aligned}\big(L_{\mathcal{A}}^{(2)}(\theta - \widehat{\theta}\,)\big)_{i,j} &= a_1(t_{i,j})\big(L_{\mathcal{A}}^{(1)}(\theta - \widehat{\theta}\,)\big)_{i,j} + a_2(t_{i,j})(\theta_{i,j} - \widehat{\theta}_{i,j}) \\ &\quad + \chi[\theta - \widehat{\theta}\,]_{i,j} + \mathcal{O}(h^{m+2}).\end{aligned} \tag{3.9}$$

Now by using stability of the finite difference scheme, we can say $|\theta - \widehat{\theta}\,| = \mathcal{O}(h^{m+2})$. Therefore we get $\|\theta\|_\infty = \mathcal{O}(h^{m+2})$. □

Nonlinear case

Definition 3.3 — For nonlinear and linear $z[\cdot]$, we define

$$\overline{\chi}[\varepsilon]_{i,j} := \sum_{(l,v)\in\mathcal{T}} \frac{\delta_{l,v}}{2}\big(\Gamma_\varepsilon(t_{i,j}, t_{l,v})\varepsilon_{l,v} + \Gamma_\varepsilon(t_{i,j}, t_{l,v+1})\varepsilon_{l,v+1}\big),$$

where

$$\Gamma_\varepsilon(t_{i,j}, t_{l,v}) := \begin{cases} \Lambda(t_{i,j}, t_{l,v}), & \text{when } z[\cdot] \text{ is linear,} \\ \int_0^1 K_u(t_{i,j}, t_{l,v}, \eta_{l,v} + \tau\varepsilon_{l,v})d\tau, & \text{when } z[\cdot] \text{ is nonlinear,} \end{cases}$$

and

$$\overline{z}[e] := \int_a^b \Gamma_e(t_{i,j}, s)e(s)ds,$$

where

$$\Gamma_e(t_{i,j}, s) := \begin{cases} \Lambda(t_{i,j}, s), & \text{when } z[\cdot] \text{ is linear,} \\ \int_0^1 K_u\big(t, s, y(s) + \tau e(s)\big)d\tau, & \text{when } z[\cdot] \text{ is nonlinear.} \end{cases}$$

Now we can easily see

$$\chi[\pi]_{i,j} - \chi[\eta]_{i,j} = \overline{\chi}[\varepsilon]_{i,j}, \tag{3.10}$$

$$z[p](t_{i,j}) - z[y](t_{i,j}) = \overline{z}[e](t_{i,j}). \tag{3.11}$$

Lemma 3.4 — For linear and nonlinear $z[\cdot](t)$, we have

$$|\overline{\chi}[\varepsilon]_{i,j} - \overline{z}[e](t_{i,j})| = \mathcal{O}(h^m).$$

PROOF : In the linear case by using Lemma 2.12, Theorem 2.2 and the Integral mean value theorem, we get

$$\begin{aligned}
\overline{\chi}[\varepsilon]_{i,j} - \overline{z}[e](t_{i,j}) &= \chi[\varepsilon]_{i,j} - z[e](t_{i,j}) \\
&= \sum_{(l,v)\in\mathcal{T}} \frac{\delta_{l,v}}{2}\big(\Lambda(t_{i,j}, t_{l,v}) \underbrace{\varepsilon_{l,v}}_{\mathcal{O}(h^m)} + \Lambda(t_{i,j}, t_{l,v+1}) \underbrace{\varepsilon_{l,v+1}}_{\mathcal{O}(h^m)}\big) \\
&- \int_a^b \Lambda(t_{i,j}, s) \underbrace{e(s)}_{\mathcal{O}(h^m)} \, ds \le \frac{(b-a)(m+1)h}{h'}\mathcal{O}(h^m) \max_{(l,v)\in\mathcal{A}} \Lambda(t_{i,j}, t_{l,v}) \\
&+ \mathcal{O}(h^m)(b-a)\Lambda(t_{i,j}, \zeta_{i,j}) = \mathcal{O}(h^m),
\end{aligned}$$

where $\zeta_{i,j} \in (a,b)$. In this step, we study nonlinear case. According to (3.10) and (3.11), we obtain

$$\begin{aligned}
|\overline{\chi}[\varepsilon]_{i,j}| = |\chi[\pi]_{i,j} - \chi[\eta]_{i,j}| &= |\sum_{(l,v)\in\mathcal{T}} \frac{\delta_{l,v}}{2}\big(K(t_{i,j}, t_{l,v}, \pi_{l,v}) - K(t_{i,j}, t_{l,v}, \eta_{l,v})\big) \\
&+ \sum_{(l,v)\in\mathcal{T}} \frac{\delta_{l,v}}{2}\big(K(t_{i,j}, t_{l,v+1}, \pi_{l,v+1}) - K(t_{i,j}, t_{l,v+1}, \eta_{l,v+1})\big)| \\
&\le C \sum_{(l,v)\in\mathcal{T}} \frac{\delta_{l,v}}{2} |\pi_{l,v} - \eta_{l,v}| + C \sum_{(l,v)\in\mathcal{B}} \frac{\delta_{l,v}}{2} |\pi_{l,v+1} - \eta_{l,v+1}| \\
&\le C\mathcal{O}(h^m)hn(m+1) \le C\frac{(b-a)(m+1)h}{h'}\mathcal{O}(h^m) = \mathcal{O}(h^m),
\end{aligned}$$

also

$$\begin{aligned}
|\overline{z}[e](t_{i,j})| = |z[p](t_{i,j}) - z[y](t_{i,j})| &\le \int_a^b |K\big(t_{i,j}, s, p(s)\big) - K\big(t_{i,j}, s, y(s)\big)| ds \\
&\le C(b-a)|p(s) - y(s)| = \mathcal{O}(h^m),
\end{aligned}$$

therefore by using the triangle inequality, we have

$$|\overline{z}[e](t_{i,j}) - \overline{\chi}[\varepsilon]_{i,j}| \le |\overline{z}[e](t_{i,j})| + |\overline{\chi}[\varepsilon]_{i,j}| = \mathcal{O}(h^m).$$

Definition 3.5 — By using (2.1), $\widetilde{\widetilde{z}}[e](t_{i,j})$ is defined as

$$\widetilde{\widetilde{z}}[e](t_{i,j}) := \sum_{(l,v)\in\mathcal{T}} \alpha_{l,v}\Gamma_e(t_{i,j}, t_{l,v})e(t_{l,v}).$$

As Lemma 3.1, we can write the following lemma.

Definition 3.5 — For linear and nonlinear $z[\cdot](t)$, we have

$$|\overline{\chi}[e]_{i,j} - \widetilde{\widetilde{z}}[e](t_{i,j})| = \mathcal{O}(h^{m+2}).$$

When F is nonlinear, we have the following theorem.

***Theorem* 3.7** — *Consider the SFID equation (1.1) with boundary conditions (1.1), where $F(t, y, y', z), F_t(t, y, y', z), F_y(t, y, y', z), F_{y'}(t, y, y', z)$ and $F_z(t, y, y', z)$ are Lipschitz-continuous. Also for nonlinear $z[\cdot](t)$, we let $K(t, s, u)$ and $K_u(t, s, u)$ are Lipschitz-continuous. Assume that the SFID problem has a unique and sufficiently smooth solution. Then the following estimate holds*

$$||\theta||_\infty = ||e - \varepsilon||_\infty = \mathcal{O}(h^{m+2}),$$

where e is error, ε is the error estimate and θ is the deviation of the error estimate.

PROOF : We have

$$\begin{aligned}(L^{(2)}_{\mathcal{A}}\theta)_{i,j} &= (L^{(2)}_{\mathcal{A}}e)_{i,j} - (L^{(2)}_{\mathcal{A}}\varepsilon)_{i,j} \\ &= \mathcal{I}_{\mathcal{A}}\big(\underbrace{F(\cdot, p, p', z[p]) - F(\cdot, y, y', z[y]), t_{i,j}}_{I_6}\big) \\ &- \big(\underbrace{F(t_{i,j}, \pi_{i,j}, (L^{(1)}_{\mathcal{A}}\pi)_{i,j}, \chi[\pi]_{i,j}) - F(t_{i,j}, \eta_{i,j}, (L^{(1)}_{\mathcal{A}}\eta)_{i,j}, \chi[\eta]_{i,j})}_{I_7}\big) \\ &+ Q_{\mathcal{A}}\big(F(, p, p', \widetilde{z}[p]), t_{i,j}\big) - \mathcal{I}_{\mathcal{A}}\big(F(\cdot, p, p', z[p]), t_{i,j}\big) + \mathcal{O}(h^{m+2}). \end{aligned} \tag{3.12}$$

We can get

$$\begin{aligned} I_6 &= c_1(t_{i,j})e(t_{i,j}) + c_2(t_{i,j})e'(t_{i,j}) + c_3(t_{i,j})\overline{z}[e](t_{i,j}), \\ I_7 &= b_1(t_{i,j})\varepsilon_{i,j} + b_2(t_{i,j})(L^{(1)}_{\mathcal{A}}\varepsilon)_{i,j} + b_3(t_{i,j})\overline{\chi}[\varepsilon]_{i,j}, \end{aligned}$$

where

$$
\begin{aligned}
b_1(t_{i,j}) &:= \int_0^1 F_y\big(t_{i,j}, \eta_{i,j} + \tau\varepsilon_{i,j}, (L_{\mathcal{A}}^{(1)}\pi)_{i,j}, \chi[\pi]_{i,j}\big)d\tau,\\
b_2(t_{i,j}) &:= \int_0^1 F_{y'}\big(t_{i,j}, \eta_{i,j}, (L_{\mathcal{A}}^{(1)}\eta)_{i,j} + \tau(L_{\mathcal{A}}^{(1)}\varepsilon)_{i,j}, \chi[\pi]_{i,j}\big)d\tau,\\
b_3(t_{i,j}) &:= \int_0^1 F_z\big(t_{i,j}, \eta_{i,j}, (L_{\mathcal{A}}^{(1)}\eta)_{i,j}, \chi[\eta]_{i,j} + \tau\overline{\chi}[\varepsilon]_{i,j}\big)d\tau,\\
c_1(t_{i,j}) &:= \int_0^1 F_y\big(t_{i,j}, y(t_{i,j}) + \tau e(t_{i,j}), p'(t_{i,j}), z[p](t_{i,j})\big)d\tau,\\
c_2(t_{i,j}) &:= \int_0^1 F_{y'}\big(t_{i,j}, y(t_{i,j}), y'(t_{i,j}) + \tau e'(t_{i,j}), z[p](t_{i,j})\big)d\tau,\\
c_3(t_{i,j}) &:= \int_0^1 F_z\big(t_{i,j}, y(t_{i,j}), y'(t_{i,j}), z[y](t_{i,j}) + \tau\overline{z}[e](t_{i,j})\big)d\tau.
\end{aligned}
$$

Also by using the Lipschitz condition for F_y, $F_{y'}$ and F_z, we get

$$
\begin{aligned}
&\Big|F_y\big(t_{i,j}, \eta_{i,j} + \tau\varepsilon_{i,j}, (L_{\mathcal{A}}^{(1)}\pi)_{i,j}, \chi[\pi]_{i,j}\big) - F_y\big(t_{i,j}, y(t_{i,j}) + \tau e(t_{i,j}), p'(t_{i,j}), z[p](t_{i,j})\big)\Big|\\
&\le C\Big(\big|\eta_{i,j} - y(t_{i,j})\big| + \tau\big|\varepsilon_{i,j} - e(t_{i,j})\big|\Big) + C\Big|(L_{\mathcal{A}}^{(1)}\pi)_{i,j} - p'(t_{i,j})\Big| + C\Big|\chi[\pi]_{i,j} - z[p](t_{i,j})\Big|\\
&= \mathcal{O}(h^2), \qquad (3.13)
\end{aligned}
$$

$$
\begin{aligned}
&\big|F_{y'}\big(t_{i,j}, \eta_{i,j}, (L_{\mathcal{A}}^{(1)}\eta)_{i,j} + \tau(L_{\mathcal{A}}^{(1)}\varepsilon)_{i,j}, \chi[\pi]_{i,j}\big) - F_{y'}\big(t_{i,j}, y(t_{i,j}), y'(t_{i,j}) + \tau e'(t_{i,j}), z[p](t_{i,j})\big)\big|\\
&\le C\big|\eta_{i,j} - y(t_{i,j})\big| + C\Big(\big|(L_{\mathcal{A}}^{(1)}\eta)_{i,j} - y'(t_{i,j})\big| + \tau\big|(L_{\mathcal{A}}^{(1)}\varepsilon)_{i,j} - e'(t_{i,j})\big|\Big) + C\Big|\chi[\pi]_{i,j} - z[p](t_{i,j})\Big|\\
&= \mathcal{O}(h^2), \qquad (3.14)
\end{aligned}
$$

$$
\begin{aligned}
&\big|F_z\big(t_{i,j}, \eta_{i,j}, (L_{\mathcal{A}}^{(1)}\eta)_{i,j}, \chi[\eta]_{i,j} + \tau\chi[\varepsilon]_{i,j}\big) - F_z\big(t_{i,j}, y(t_{i,j}), y'(t_{i,j}), z[y](t_{i,j}) + \tau\overline{z}[e](t_{i,j})\big)\big|\\
&\le C\big|\eta_{i,j} - y(t_{i,j})\big| + C\big|(L_{\mathcal{A}}^{(1)}\eta)_{i,j} - y'(t_{i,j})\big| + C\Big(\big|\chi[\eta]_{i,j} - z[y](t_{i,j})\big| + \tau\big|\overline{\chi}[\varepsilon]_{i,j} - \overline{z}[e](t_{i,j})\big|\Big)\\
&= \mathcal{O}(h^2). \qquad (3.15)
\end{aligned}
$$

Therefore by using (3.13), (3.14) and (3.15), we obtain

$$
c_k(t_{i,j}) - b_k(t_{i,j}) = \mathcal{O}(h^2), \qquad k = 1, 2, 3.
$$

So we can rewrite (3.12) as follows

$$
\begin{aligned}
(L_{\mathcal{A}}^{(2)}\theta)_{i,j} &= \mathcal{I}_{\mathcal{A}}\big(c_1(t_{i,j})e(t_{i,j}) + c_2(t_{i,j})e'(t_{i,j}) + c_3(t_{i,j})\overline{z}[e](t_{i,j})\big)\\
&\quad - \big(b_1(t_{i,j})\varepsilon_{i,j} + b_2(t_{i,j})(L_{\mathcal{A}}^{(1)}\varepsilon)_{i,j} + b_3(t_{i,j})\overline{\chi}[\varepsilon]_{i,j}\big)\\
&\quad + Q_{\mathcal{A}}\big(F(, p, p', \widetilde{z}[p]), t_{i,j}\big) - \mathcal{I}_{\mathcal{A}}\big(F(\cdot, p, p', z[p]), t_{i,j}\big) + \mathcal{O}(h^{m+2})
\end{aligned}
$$

$$
\begin{aligned}
&-\big(b_1(t_{i,j})e_{i,j}+b_2(t_{i,j})(L_{\mathcal{A}}^{(1)}e)_{i,j}+b_3(t_{i,j})\overline{\chi}[e]_{i,j}\big)\\
&+\big(b_1(t_{i,j})e_{i,j}+b_2(t_{i,j})(L_{\mathcal{A}}^{(1)}e)_{i,j}+b_3(t_{i,j})\overline{\chi}[e]_{i,j}\big)\\
&-\big(c_1(t_{i,j})e(t_{i,j})+c_2(t_{i,j})e'(t_{i,j})+c_3(t_{i,j})\overline{\chi}[e](t_{i,j})\big)\\
&+\big(c_1(t_{i,j})e(t_{i,j})+c_2(t_{i,j})e'(t_{i,j})+c_3(t_{i,j})\overline{\chi}[e](t_{i,j})\big)\\
&=b_1(t_{i,j})\theta_{i,j}+b_2(t_{i,j})(L_{\mathcal{A}}^{(1)}\theta)_{i,j}+b_3(t_{i,j})\overline{\chi}[\theta]_{i,j}\\
&+\mathcal{I}_{\mathcal{A}}\big(c_1(t_{i,j})e(t_{i,j})+c_2(t_{i,j})e'(t_{i,j})+c_3(t_{i,j})\overline{z}[e](t_{i,j})\big)\\
&-\big(c_1(t_{i,j})e(t_{i,j})+c_2(t_{i,j})e'(t_{i,j})+c_3(t_{i,j})\overline{\chi}[e](t_{i,j})\big)\\
&+\underbrace{(c_1-b_1)(t_{i,j})}_{\mathcal{O}(h^2)}\underbrace{e(t_{i,j})}_{\mathcal{O}(h^m)}+\underbrace{(c_2-b_2)(t_{i,j})}_{\mathcal{O}(h^2)}\underbrace{e'(t_{i,j})}_{\mathcal{O}(h^m)}+\underbrace{(c_3-b_3)(t_{i,j})}_{\mathcal{O}(h^2)}\underbrace{\overline{\chi}[e](t_{i,j})}_{\mathcal{O}(h^m)}\\
&+\underbrace{Q_{\mathcal{A}}\big(F(,p,p',\widetilde{z}[p]),t_{i,j}\big)-\mathcal{I}_{\mathcal{A}}\big(F(\cdot,p,p',z[p]),t_{i,j}\big)}_{\mathcal{O}(h^{m+2})}+\mathcal{O}(h^{m+2})\\
&=b_1(t_{i,j})\theta_{i,j}+b_2(t_{i,j})(L_{\mathcal{A}}^{(1)}\theta)_{i,j}+b_3(t_{i,j})\overline{\chi}[\theta]_{i,j}\\
&+\underbrace{\mathcal{I}_{\mathcal{A}}\big(c_1(t_{i,j})e(t_{i,j})+c_2(t_{i,j})e'(t_{i,j})\big)-\big(c_1(t_{i,j})e(t_{i,j})+c_2(t_{i,j})e'(t_{i,j})\big)}_{I_1}\\
&+c_3(t_{i,j})\big(\underbrace{\mathcal{I}_{\mathcal{A}}\big(\overline{z}[e](t_{i,j})\big)-\overline{\chi}[e](t_{i,j})}_{I_2}\big)+\mathcal{O}(h^{m+2}).
\end{aligned}
\tag{3.16}
$$

In the above relation, I_1 is obtained as (3.5). For I_2, by using Lemma 3.6, we get

$$
\begin{aligned}
I_2&=\mathcal{I}_{\mathcal{A}}\big(\overline{z}[e](t_{i,j})\big)-\overline{\chi}[e](t_{i,j})=\mathcal{I}_{\mathcal{A}}\big(\widetilde{\overline{z}}[e](t_{i,j})\big)-\widetilde{\overline{z}}[e](t_{i,j})+\mathcal{O}(h^{m+2})\\
&=\sum_{(l,v)\in\mathcal{T}}\alpha_{l,v}e(t_{l,v})\Big(\frac{1}{2\widehat{\delta}_{i,j}}\big(\delta_{i,j}^2\Xi_{i,j+\frac12}(t_{l,v})-\delta_{i,j-1}^2\Xi_{i,j-\frac12}(t_{l,v})\big)\Big)+\mathcal{O}(h^{m+2}).
\end{aligned}
$$

where

$$
\begin{aligned}
\Xi_{i,j+\frac12}(t_{l,v})&:=\int_{-\frac12}^{\frac12}(u^2+\frac14)\Gamma_{e\,t}(t_{i,j+\frac12}+\delta_{i,j}u,t_{l,v})du,\\
\Xi_{i,j-\frac12}(t_{l,v})&:=\int_{-\frac12}^{\frac12}(u^2+\frac14)\Gamma_{e\,t}(t_{i,j-\frac12}+\delta_{i,j-1}u,t_{l,v})du.
\end{aligned}
$$

We can see that $\Xi_{i,j+\frac12}(t_{l,v})=\mathcal{O}(1),\Xi_{i,j-\frac12}(t_{l,v})=\mathcal{O}(1)$. Then we rewrite (3.16) as follows

$$
\begin{aligned}
(L_{\mathcal{A}}^{(2)}\theta)_{i,j}&=b_1(t_{i,j})\theta_{i,j}+b_2(t_{i,j})(L_{\mathcal{A}}^{(1)}\theta)_{i,j}+b_3(t_{i,j})\overline{\chi}[\theta]_{i,j}\\
&\quad+(L_{\mathcal{A}}^{(2)}\widehat{\theta}\,)_{i,j}+\mathcal{O}(h^{m+2}),
\end{aligned}
$$

where

$$(L_{\mathcal{A}}^{(2)}\widehat{\theta})_{i,j} = \frac{1}{\widehat{\delta}_{i,j}}(\delta_{i,j-1}g_{i,j-\frac{1}{2}} - \delta_{i,j}g_{i,j+\frac{1}{2}}) + \frac{1}{2\widehat{\delta}_{i,j}}(\delta_{i,j}\phi_{i,j+\frac{1}{2}} - \delta_{i,j-1}\phi_{i,j-\frac{1}{2}})$$
$$+ \sum_{(l,v)\in\mathcal{T}} \alpha_{l,v}e(t_{l,v})\Big(\frac{1}{2\widehat{\delta}_{i,j}}\big(\delta_{i,j}^2\Xi_{i,j+\frac{1}{2}}(t_{l,v}) - \delta_{i,j-1}^2\Xi_{i,j-\frac{1}{2}}(t_{l,v})\big)\Big)$$
$$+ \mathcal{O}(h^{m+2}).$$

As (3.8) and (3.9), we can prove that $\|\widehat{\theta}\|_\infty = \mathcal{O}(h^{m+2})$ and $\|\theta - \widehat{\theta}\|_\infty = \mathcal{O}(h^{m+2})$. So

$$\|\theta\|_\infty \leq \|\theta - \widehat{\theta}\|_\infty + \|\widehat{\theta}\|_\infty = \mathcal{O}(h^{m+2}).\square$$

Remark 3.8 : In special case, when m is odd and the nodes ρ_i $(i = 1, \dots, m)$ are symmetrically distributed by using Remark 2.3 and the above discussion, we can say that

$$||\theta||_\infty = ||e - \varepsilon||_\infty = \mathcal{O}(h^{m+3}).$$

In next section by numerical experiments we study this case (see example 2).

4. Numerical Examples

We now obtain the numerical results. In this section we have computed the numerical results by Mathematica-9 programming. A numerical order is calculated according to

$$Order := \frac{\ln(\|e_{n-1}\|_\infty/\|e_n\|_\infty)}{\ln 2}.$$

Also in the examples, the boundary conditions are taken from the exact solution.

Example 1 : In this example we consider, the linear case as follows

$$y''(t) = ty'(t) + y(t) + a_3(t) + \int_0^1 tsy(s)ds.$$

In this case $a_3(t)$ is chosen such that exact solution is $y(t) = t\exp(2t)$. For this example, we choose n collocation subintervals of length $1/n$. In the Table 1 and 2 we choose $m = 2$ and assume that $\{\rho_0 = 0, \rho_1 = 0.2, \rho_2 = 0.65, \rho_3 = 1\}$. We use the finite difference scheme in the Remark 2.6 to find numerical results for Table 2. Also in the Table 3, we assume that $\rho_i(i = 0, \dots, 5)$ are equidistant point.

Table 1: Numerical results for Example 1.

n	$\|\|e\|\|_\infty$	$Order$	$\|\|\theta\|\|_\infty$	$Order$
4	1.37091e-2	$----$	3.76668e-5	$----$
8	3.96024e-3	1.79148	1.54185e-6	4.61056
16	1.03416e-3	1.93713	1.06939e-7	3.84981

Table 2: Numerical results for Example 1.

n	$\|\|e\|\|_\infty$	$Order$	$\|\|\theta\|\|_\infty$	$Order$
4	1.37091e-2	$----$	3.77005e-5	$----$
8	3.96024e-3	1.79148	1.53945e-6	4.61410
16	1.03416e-3	1.93713	1.06864e-7	3.84857

Table 3: Numerical results for Example 1 with $m = 4$.

n	$\|\|e\|\|_\infty$	$Order$	$\|\|\theta\|\|_\infty$	$Order$
4	6.349980e-5	$----$	3.13330e-8	$----$
8	4.136140e-6	3.94040	3.95715e-10	6.30708
16	2.576416e-7	4.00485	5.66214e-12	6.12697

Example 2 : For nonlinear case we consider the problem

$$y''(t) = (ty'(t))^2 + y^3(t) + a_3(t) + \int_0^1 tsy(s)ds,$$

$a_3(t)$ is chosen such that exact solution is $y(t) = \exp(2t)$. For this example, we choose n collocation subintervals of length $1/n$. In the Table 4 we choose $m = 4$ and assume that $\rho_i(i = 0, \ldots, 5)$ are equidistant points. Also in the Table 4, we we choose $m = 3$ and $\{\rho_0 = 0, \rho_1 = 0.2, \rho_2 = 0.65, \rho_3 = 0.8, \rho_4 = 1\}$. Numerical results are tabulated in Table 6 by using the finite difference scheme in the Remark 2.6.

Table 4: Numerical results for Example 2.

n	$\|\|e\|\|_\infty$	$Order$	$\|\|\theta\|\|_\infty$	$Order$
4	5.61277e-5	$----$	6.96467e-7	$----$
8	3.20050e-6	4.13235	9.89168e-9	6.13770
16	1.94647e-7	4.03937	1.34356e-10	6.20208

Table 5: Numerical results for Example 2 with $m = 3$ and equidistant points.

n	$\|\|e\|\|_\infty$	$Order$	$\|\|\theta\|\|_\infty$	$Order$
4	5.13598e-4	$----$	4.19134e-5	$----$
8	6.29898e-5	3.02745	8.28122e-7	5.66142
16	7.89056e-6	2.99692	1.45131e-8	5.83442

Table 6: Numerical results for Example 2 with $m = 5$ and equidistant points.

n	$\|\|e\|\|_\infty$	$Order$	$\|\|\theta\|\|_\infty$	$Order$
4	1.87219e-6	$----$	6.96467e-7	$----$
8	2.79428e-8	6.06611	1.69114e-10	7.55132
16	4.32541e-10	6.01349	7.30527e-13	7.85484

Example 3 : As a last study, we consider here the following nonlinear problem

$$y''(t) = (y'(t))^2 + y(t) + a_3(t) + \int_0^1 y^2(s)ds,$$

$a_3(t)$ is chosen such that exact solution is $y(t) = \exp(3t)$. For this example, we choose n collocation subintervals of length $1/n$. In the Table 7, we choose $m = 2$ and assume that $\rho_i(i = 0, ..., 3)$ are equidistant points.

Table 7: Numerical results for Example 3 with $m = 2$ and equidistant points.

n	$\|\|e\|\|_\infty$	$Order$	$\|\|\theta\|\|_\infty$	$Order$
16	1.17596e-3	$----$	9.64867e-5	$----$
32	2.67957e-4	2.13377	9.71034e-6	3.31273
64	6.41159e-5	2.06325	6.93182e-7	3.80822
128	1.57799e-5	2.02259	4.50846e-8	3.94253

5. Conclusion

In this work, we study efficient asymptotically correct a posteriori error estimates for the numerical approximation of second order Fredholm integro-differential equations. In addition, it is shown that when we use m degree piecewise polynomial collocation method, the order of the deviation of the error estimation is $\mathcal{O}(h^{m+2})$. Also, numerical results confirm our theoretical analysis.

ACKNOWLEDGEMENT

We would like to thank the anonymous referees for their valuable comments and suggestions, which helped improve the original manuscript of this paper.

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