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ASYMPTOTIC ANALYSIS OF MULTIPLE SOLUTIONS FOR PERTURBED CHOQUARD EQUATIONS [1]

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In this paper, we study the following Choquard equations with small perturbation f

$$-\Delta u+V(x)u=(I_\alpha*|u|^p)|u|^{p-2}u+f(x),\ x\in\mathbb{R}^N.$$

where $N\geq 3$ and I_α denotes the Riesz potential. As is known that the above equation has a ground state u_α and a bound state v_α by fibering maps (see [22] or [23]), our aim is to show that for fixed $p\in(1,\frac{N}{N-2})$, u_α and v_α converge to a ground state and a bound state of the limiting local problem respectively, as $\alpha\to 0$.

Key words : Choquard equation; convergence; Hartree type nonlocal term; perturbation; variational methods.

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1. INTRODUCTION

In this paper, we are concerned with the following nonlocal problem

$$-\Delta u+V(x)u=(I_\alpha*|u|^p)|u|^{p-2}u+f(x),\ x\in\mathbb{R}^N, \tag{1.1}$$

where $N\geq 3, p\in(1,\frac{N}{N-2})$, $\alpha\in(0,\min\{(p-1)N,N\})$ is a parameter, I_α is Riesz potential given by

$$I_\alpha(x)=\frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{N/2}2^\alpha|x|^{N-\alpha}} \tag{1.2}$$

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and Γ denotes the Gamma function. We assume $V(x)$ satisfies the following conditions.

(V) $V \in C(\mathbb{R}^N)$, $V_0 := \inf_{\mathbb{R}^N} V > 0$ and there exists a constant $r > 0$ such that, for any $M > 0$,

$$\text{meas}\{x \in \mathbb{R}^N : |x - y| \le r, V(x) \le M\} \to 0, \text{ as } |y| \to \infty,$$

where $meas$ stands for Lebesgue measure. One can refer to [1, 2] for more details.

When $N = 3$, $\alpha = 2$, $p = 2$ and $f = 0$, (1.1) arises in the study of nonlinear Choquard equations describing an electron trapped in its own hole, in a certain approximation to Hartree-Fock theory of one component plasma [10]. Recently, the existence and qualitative properties of Choquard type equations (1.1) have been widely and intensively studied in literatures. The existence of ground states, nodal solutions and multiple solutions to (1.1) is quite well known, see [4-8, 11, 13, 15, 16, 19, 20] and references therein. For the results about qualitative properties such as regularity, symmetry, uniqueness and decay, one can refer to for instance [12, 13, 15, 17, 21].

As stated in [18], the following local equation

$$-\Delta u + V(x)u = |u|^{2p-2}u + f(x), \tag{1.3}$$

can be viewed as a limit equation of (1.1) as $\alpha \to 0$. Moreover, the existence of ground state and bound state of (1.1) and (1.3) via fibering maps has been proved. One can refer to [3, 22, 23]. However, a natural interesting question arises whether both of the ground state and bound state of (1.1) converge to those of limit equation (1.3) as $\alpha \to 0$, respectively. This paper gives a complete answer.

We consider the Sobolev space $H := \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 dx < \infty\}$ with the norm $\|u\|^2 = \int_{\mathbb{R}^N}(|\nabla u|^2 + V(x)u^2)dx$. Under the assumption (V), the embedding $H \hookrightarrow H^1(\mathbb{R}^N)$ is continuous and H is a Hilbert space. Furthermore, the embedding from H into $L^s(\mathbb{R}^N)$ is compact for $s \in [2, \frac{2N}{N-2})$ (see [1]). Let H^* be the dual space of H and the norm on H^* is denoted by $\|\cdot\|_{H^*}$. Our main result is as follows.

***Theorem* 1.1** — *Assume $N \ge 3, p \in (1, \frac{N}{N-2})$ and (V) holds. Then there exists $\delta > 0$ small enough such that for any $f \in H^*\backslash\{0\}$ with $\|f\|_{H^*} < \delta$, equation (1.1) has a ground state u_α and a bound state v_α that converge to a ground state and a bound state of (1.3) as $\alpha \to 0$, respectively.*

Remark 1.1 : For fixed $p \in (1, \frac{N}{N-2})$, the energy functional E_α associated with (1.1) (see (2.1)) is well defined for every $\alpha \in (0, \min\{(p-1)N, N\})$.

The remainder of this paper is organized as follows. In Section 2, some notations and preliminary results are presented. In Section 3, we are devoted to the proof of Theorem 1.1.

2. Preliminaries

In this paper, we use the following notations.

• For $1 \leq s < \infty$, $L^s(\mathbb{R}^N)$ denotes the Lebesgue space with the norm $|u|_{L^s} = \left(\int_{\mathbb{R}^N} |u|^s dx\right)^{\frac{1}{s}}$.

• Let $\langle \cdot, \cdot \rangle$ be duality pairing between H and H^*.

• C denotes different positive constants and $C(\alpha)$ denotes different positive constants dependent on α.

Throughout the paper, we assume (V) holds and $f \in H^* \backslash \{0\}$. As usual, the corresponding energy functional $E_\alpha : H \to \mathbb{R}$ associated with (1.1) is

$$E_\alpha(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2p}\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p dx - \langle f, u \rangle. \tag{2.1}$$

In view of Remark 1.1, we can see that $E_\alpha \in C^1(H, \mathbb{R})$ whose Gateaux derivative is given by

$$\langle E'_\alpha(u), v \rangle = \int_{\mathbb{R}^N} \nabla u \nabla v + V(x)uv - \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^{p-2}uv dx - \langle f, v \rangle$$

for any $v \in H$. Recall that the critical points of E_α are solutions of (1.1) in the weak sense. For simplicity of notations, we denote $\mathbb{D}(u) = \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p dx$. Similarly, for problem (1.3), the energy functional is

$$E_0(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2p}\int_{\mathbb{R}^N} |u|^{2p} dx - \langle f, u \rangle$$

which is well defined in H and of C^1.

We consider the Nehari manifold $\mathcal{N}_\alpha = \{u \in H : \langle E'_\alpha(u), u \rangle = 0\}$. Let $J_\alpha(u) = \langle E'_\alpha(u), u \rangle$ and then $\langle J'_\alpha(u), u \rangle = 2\|u\|^2 - 2p\mathbb{D}(u) - \langle f, u \rangle$. As in [22] (or [23]), $\mathcal{N}_\alpha$ is split into three parts:

$$\begin{aligned} \mathcal{N}_\alpha^0 &= \{u \in \mathcal{N}_\alpha : \langle J'_\alpha(u), u \rangle = 0\}, \\ \mathcal{N}_\alpha^+ &= \{u \in \mathcal{N}_\alpha : \langle J'_\alpha(u), u \rangle > 0\}, \\ \mathcal{N}_\alpha^- &= \{u \in \mathcal{N}_\alpha : \langle J'_\alpha(u), u \rangle < 0\}. \end{aligned} \tag{2.2}$$

Set $\theta_\alpha^+ = \inf\limits_{\mathcal{N}_\alpha^+} E_\alpha(u)$ and $\theta_\alpha^- = \inf\limits_{\mathcal{N}_\alpha^-} E_\alpha(u)$. Similarly, we define $J_0(u), \mathcal{N}_0, \mathcal{N}_0^0, \mathcal{N}_0^+, \mathcal{N}_0^-, \theta_0^+, \theta_0^-$ by replacing $\mathbb{D}(u)$ by $\int_{\mathbb{R}^N} |u|^{2p}$ as above.

In the following, we give some preliminary results which are necessary in proving our main result.

Lemma 2.1 — [9, Theorem 4.3]. Let $s, t > 1$ and $0 < \alpha < N$ with $\frac{1}{s} + \frac{1}{t} = 1 + \frac{\alpha}{N}$, $f \in L^s(\mathbb{R}^N)$ and $h \in L^t(\mathbb{R}^N)$. There exists a sharp constant $C(N, \alpha, s)$ independent of f, h, such that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)h(y)}{|x - y|^{N-\alpha}} dx dy \leq C(N, \alpha, s)|f|_{L^s}|h|_{L^t}.$$

Here $C(N, \alpha, s)$ is a positive constant which depend only on N, α, s. When $s = t$, one has

$$\limsup_{\alpha \to 0} \alpha C(N, \alpha, s) \leq \frac{2}{s(s-1)} |S^{N-1}|, \tag{2.3}$$

where $|S^{N-1}|$ denotes the surface area of the $N-1$ dimensional unit sphere S^{N-1}.

Lemma 2.2 — [18, Proposition 2.1]. Let $\{\alpha_j\} > 0$ be a sequence converging to 0 and $\{u_j\} \subset H^1(\mathbb{R}^N)$ be a sequence converging to some $u^* \in H^1(\mathbb{R}^N)$ in $L^s(\mathbb{R}^N)$ for every $s \in (2, \frac{2N}{N-2})$ as $j \to \infty$. Then

$$\int_{\mathbb{R}^N} (I_{\alpha_j} * |u_j|^p)|u_j|^p dx \to \int_{\mathbb{R}^N} |u^*|^{2p} dx, \text{ as } j \to \infty.$$

In addition, for any $\phi \in H^1(\mathbb{R}^N)$, one has

$$\int_{\mathbb{R}^N} (I_{\alpha_j} * |u_j|^p)|u_j|^{p-2} u_j \phi dx \to \int_{\mathbb{R}^N} |u^*|^{2p-2} u^* \phi dx, \text{ as } j \to \infty.$$

3. Proof of Theorem 1.1

In this section, we are devoted to the proof of Theorem 1.1. First we list the following results that show the existence of a ground state and a bound state of (1.1) and (1.3).

Proposition 3.1 — Assume $N \geq 3, p \in (1, \frac{N}{N-2})$ and (V) holds. Then there exists $\delta > 0$ independent of α such that for any $f \in H^* \backslash \{0\}$ with $\|f\|_{H^*} < \delta$ small enough, there hold

(i) $\mathcal{N}_\alpha^0 = \{0\}$ and $\mathcal{N}_0^0 = \{0\}$.

(ii) for any $u \in H \backslash \{0\}$, there exists a unique $t_- > 0$ such that $t_- u \in \mathcal{N}_\alpha^-$; for any $u \in H$ with $\langle f, u \rangle > 0$, there exists a unique $t_+ > 0$ such that $t_+ u \in \mathcal{N}_\alpha^+$.

(iii) (1.1) has a ground state $u_\alpha \in \mathcal{N}_\alpha^+$ and a bound state $v_\alpha \in \mathcal{N}_\alpha^-$ such that $E_\alpha(u_\alpha) = \theta_\alpha^+ < 0$ and $E_\alpha(v_\alpha) = \theta_\alpha^- > 0$. Furthermore, if f is positive, u_α and v_α are positive.

(iv) (1.3) has a ground state $u_0 \in \mathcal{N}_0^+$ and a bound state $v_0 \in \mathcal{N}_0^-$ such that $E_0(u_0) = \theta_0^+ < 0$ and $E_0(v_0) = \theta_0^- > 0$. Moreover, if f is positive, u_0 and v_0 are positive.

Proof : The proofs of (i)-(iv) can be found in [22] (or [23]) with slight modifications. Furthermore, in view of Lemma 2.1, we can deduce that δ is independent of α. □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1 : First, by Proposition 3.1(iii), we obtain the existence of ground state u_α and bound state v_α of (.1) when $\|f\|_{H^*} < \delta$ for the α-uniformity. Now we prove the convergence of u_α and v_α as $\alpha \to 0$, respectively.

Step 1 : u_α tends to a ground state of (1.3) as $\alpha \to 0$.

Recall that $u_\alpha \in \mathcal{N}_\alpha^+$ and $E_\alpha(u_\alpha) = \theta_\alpha^+ < 0$. Then

$$\begin{aligned} E_\alpha(u_\alpha) &= E_\alpha(u_\alpha) - \tfrac{1}{2p}\langle E_\alpha'(u_\alpha), u_\alpha\rangle \\ &= (\tfrac{1}{2} - \tfrac{1}{2p})\|u_\alpha\|^2 - (1 - \tfrac{1}{2p})\langle f, u_\alpha\rangle < 0, \end{aligned} \tag{3.1}$$

which implies that $\|u_\alpha\| \le C\|f\|_{H^*} \le C\delta$ and so there exists a sequence $\{\alpha_j\} > 0$ with $\alpha_j \to 0$ as $j \to \infty$, such that $\{u_{\alpha_j}\}$ is bounded in H.

Up to a subsequence, $u_{\alpha_j} \rightharpoonup \bar{u}$ in H and $u_{\alpha_j} \to \bar{u}$ in $L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$ as $j \to \infty$. For any $v \in H$, we infer from Lemma 2.2 that $\langle E_{\alpha_j}'(u_{\alpha_j}), v\rangle \to \langle E_0'(\bar{u}), v\rangle$. Thus $E_0'(\bar{u}) = 0$ and $\bar{u} \neq 0$ due to the fact that $f \in H^*\backslash\{0\}$. So $\bar{u}$ is a nontrivial solution of (1.3).

In addition, by Lemma 2.2 again,

$$\begin{aligned} 0 &= \langle E_{\alpha_j}'(u_{\alpha_j}), u_{\alpha_j}\rangle - \langle E_0'(\bar{u}), \bar{u}\rangle \\ &= \|u_{\alpha_j}\|^2 - \int_{\mathbb{R}^N}(I_{\alpha_j} * |u_{\alpha_j}|^p|)|u_{\alpha_j}|^p dx - \langle f, u_{\alpha_j}\rangle \\ &\quad -(\|\bar{u}\|^2 - \int_{\mathbb{R}^N}|\bar{u}|^{2p}dx - \langle f, \bar{u}\rangle) \\ &= \|u_{\alpha_j}\|^2 - \|\bar{u}\|^2 + o(1). \end{aligned} \tag{3.2}$$

Here and in the following part, $o(1) \to 0$ as $j \to \infty$. Then we obtain $\|u_{\alpha_j}\| \to \|\bar{u}\|$. This combined with the fact $u_{\alpha_j} \rightharpoonup \bar{u}$, implies that $u_{\alpha_j} \to \bar{u}$ in H. Since $u_{\alpha_j} \in \mathcal{N}_{\alpha_j}^+$, we have $\langle J_{\alpha_j}'(u_{\alpha_j}), u_{\alpha_j}\rangle > 0$. Then $\langle J_0'(\bar{u}), \bar{u}\rangle \ge 0$. Note that $\bar{u} \neq 0$ and $\mathcal{N}_0^0 = \{0\}$ in Proposition 3.1(i). Thus we conclude that $\bar{u} \in \mathcal{N}_0^+$.

Now, we are going to prove that $\bar{u}$ is a ground state of (1.3). By Proposition 3.1(iv), we see $E_0(u_0) = \inf_{u \in \mathcal{N}_0^+} E_0(u) = \theta_0^+ < 0$. Then $\langle f, u_0\rangle > 0$. According to Proposition 3.1(ii), for each α_j, there exists $t_{\alpha_j} > 0$ such that $t_{\alpha_j}u_0 \in \mathcal{N}_{\alpha_j}^+$. In order to investigate the property of t_{α_j}, we define $\tilde{J} : (0, \infty) \times (-\alpha_0, \alpha_0) \to \mathbb{R}$ by

$$\tilde{J}(t, \alpha) = \begin{cases} t^2\|u_0\|^2 - t^{2p}\int_{\mathbb{R}^N}|u_0|^{2p}dx - t\langle f, u_0\rangle, \text{ if } \alpha = 0, \\ t^2\|u_0\|^2 - t^{2p}\int_{\mathbb{R}^N}(I_{|\alpha|} * |u_0|^p|)|u_0|^p dx - t\langle f, u_0\rangle, \text{ if } \alpha \neq 0. \end{cases} \tag{3.3}$$

Here we can choose some $\alpha_0 \in (0, \min\{(p-1)N, N\})$. By Lebesgue dominated convergence theorem, Lemmas 2.1 and 2.2, one can see that $\int_{\mathbb{R}^N}(I_{|\alpha|} * |u_0|^p|)|u_0|^p dx$ is continuous with respect to α, and so $\tilde{J}$ and $\frac{\partial \tilde{J}}{\partial t}$ are both continuous in $(0, \infty) \times (-\alpha_0, \alpha_0)$. Note that $\tilde{J}(1, 0) = 0$ and $\frac{\partial \tilde{J}}{\partial t}|_{(1,0)} > 0$ due to the fact $u_0 \in \mathcal{N}_0^+$. Then by applying the implicit function theorem, we have $t_{\alpha_j} \to 1$ as $j \to \infty$. Therefore, we deduce that

$$E_0(u_0) \le E_0(\bar{u}) = \lim_{j\to\infty} E_{\alpha_j}(u_{\alpha_j}) \le \lim_{j\to\infty} E_{\alpha_j}(t_{\alpha_j}u_0) = E_0(u_0).$$

Hence $E_0(\bar{u}) = E_0(u_0) = \inf\limits_{u\in\mathcal{N}_0^+} E_0(u) < 0$. This yields that $\bar{u}$ is a ground state of (1.3).

Step 2 : v_α tends to a bound state of (1.3) as $\alpha \to 0$.

Note that $v_\alpha \in \mathcal{N}_\alpha^-$ and $E_\alpha(v_\alpha) = \theta_\alpha^- > 0$. Choose a cut-off function $\eta \in C_c^\infty(\mathbb{R}^N)$. Since $\int_{\mathbb{R}^N}(I_\alpha * |\eta|^p|)|\eta|^p dx \to \int_{\mathbb{R}^N} |\eta|^{2p} dx$ as $\alpha \to 0$, it holds that

$$\frac{1}{2}\int_{\mathbb{R}^N} |\eta|^{2p} dx \le \int_{\mathbb{R}^N} (I_\alpha * |\eta|^p|)|\eta|^p dx \le \frac{3}{2}\int_{\mathbb{R}^N} |\eta|^{2p} dx$$

for α close to 0. On the other hand, for each α, we can find $t_\alpha > 0$ such that $t_\alpha\eta \in \mathcal{N}_\alpha^-$. Then

$$\begin{aligned} E_\alpha(v_\alpha) \le E_\alpha(t_\alpha\eta) &\le \tfrac{t_\alpha^2}{2}\|\eta\|^2 - \tfrac{t_\alpha^{2p}}{4p}\int_{\mathbb{R}^N} |\eta|^{2p} dx - t_\alpha\langle f,\eta\rangle \\ &\le \sup_{t\ge 0}\{\tfrac{t^2}{2}\|\eta\|^2 - \tfrac{t^{2p}}{4p}\int_{\mathbb{R}^N} |\eta|^{2p} dx - t\langle f,\eta\rangle\} \\ &= \sup_{t\ge 0}\{at^2 - bt^{2p} - ct\}, \end{aligned} \tag{3.4}$$

where $a = \frac{1}{2}\|\eta\|^2, b = \frac{1}{4p}\int_{\mathbb{R}^N} |\eta|^{2p} dx, c = \langle f,\eta\rangle$. Now consider $h(t) = at^2 - bt^{2p} - ct$. Since $h(t) \to -\infty$ as $t \to +\infty$ and $h(t) \to 0$ as $t \to 0$, there exists $D > 0$ independent of α such that $\sup\limits_{t\ge 0} h(t) \le D$. Thus $E_\alpha(v_\alpha) \le D$.

Observe that for any $u \in \mathcal{N}_\alpha$,

$$E_\alpha(u) = (\frac{1}{2} - \frac{1}{2p})\|u\|^2 - (1 - \frac{1}{2p})\langle f,u\rangle \ge (\frac{1}{2} - \frac{1}{2p})\|u\|^2 - C\|u\|. \tag{3.5}$$

Then E_α is coercive and bounded from below in $\mathcal{N}_\alpha$. Since $v_\alpha \in \mathcal{N}_\alpha^-$ and $E_\alpha(v_\alpha) \le D$, we can find a sequence $\{\alpha_j\}$ with $\alpha_j \to 0$ as $j \to \infty$, such that $\{v_{\alpha_j}\}$ is bounded in H.

Up to a subsequence, $v_{\alpha_j} \rightharpoonup \bar{v}$ in H and $v_{\alpha_j} \to \bar{v}$ in $L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$ as $j \to \infty$. We can derive from Lemma 2.2 that $\langle E'_{\alpha_j}(v_{\alpha_j}), w\rangle \to \langle E'_0(\bar{v}), w\rangle$ for any $w \in H$. Thus $E'_0(\bar{v}) = 0$ and $\bar{v} \ne 0$, that is, $\bar{v}$ is a nontrivial solution of (1.1).

Now we show that $\bar{v}$ is a bound state. Similar to (3.2), we get

$$0 = \langle E'_{\alpha_j}(v_{\alpha_j}), v_{\alpha_j}\rangle - \langle E'_0(\bar{v}), \bar{v}\rangle = \|v_{\alpha_j}\|^2 - \|\bar{v}\|^2 + o(1). \tag{3.6}$$

Then $\|v_{\alpha_j}\| \to \|\bar{v}\|$. So we infer from $v_{\alpha_j} \rightharpoonup \bar{v}$ that $v_{\alpha_j} \to \bar{v}$ in H. Note that $v_{\alpha_j} \in \mathcal{N}_{\alpha_j}^-$ and $\langle J'_{\alpha_j}(v_{\alpha_j}), v_{\alpha_j}\rangle < 0$. Then $\langle J'_0(\bar{v}), \bar{v}\rangle \le 0$. Since $\bar{v} \ne 0$ and $\mathcal{N}_0^0 = \{0\}$ in Proposition 3.1(i), we deduce that $\bar{v} \in \mathcal{N}_0^-$.

By Proposition 3.1(iv), we see $E_0(v_0) = \inf\limits_{v\in\mathcal{N}_0^-} E_0(v) = \theta_0^- > 0$. According to Proposition 3.1(ii), for each α_j, there exists $s_{\alpha_j} > 0$ such that $s_{\alpha_j} v_0 \in \mathcal{N}_{\alpha_j}^-$. We claim that $s_{\alpha_j} \to 1$ as $j \to \infty$.

Indeed, we define $\tilde{h} : (0,\infty) \times (-\alpha_0, \alpha_0) \to \mathbb{R}$ by

$$\tilde{h}(s,\alpha) = \begin{cases} s^2\|v_0\|^2 - s^{2p}\int_{\mathbb{R}^N} |v_0|^{2p}dx - s\langle f, v_0\rangle, \text{ if } \alpha = 0, \\ s^2\|v_0\|^2 - s^{2p}\int_{\mathbb{R}^N} (I_{|\alpha|} * |v_0|^p|)|v_0|^p dx - s\langle f, v_0\rangle, \text{ if } \alpha \neq 0. \end{cases} \tag{3.7}$$

In view of Lemmas 2.1 and 2.2, it follows from Lebesgue dominated convergence theorem that $\int_{\mathbb{R}^N} (I_{|\alpha|} * |v_0|^p|)|v_0|^p dx$ is continuous with respect to α. Furthermore, $\tilde{h}$ and $\frac{\partial \tilde{h}}{\partial s}$ are both continuous in $(0,\infty) \times (-\alpha_0, \alpha_0)$. Since $v_0 \in \mathcal{N}_0^-$, we have $\tilde{h}(1,0) = 0$ and $\frac{\partial \tilde{h}}{\partial s}|_{(1,0)} < 0$. So the claim follows from the implicit function theorem. Therefore,

$$E_0(v_0) \leq E_0(\bar{v}) = \lim_{j\to\infty} E_{\alpha_j}(v_{\alpha_j}) \leq \lim_{j\to\infty} E_{\alpha_j}(s_{\alpha_j} v_0) = E_0(v_0).$$

Hence $E_0(\bar{v}) = E_0(v_0) = \inf_{v\in\mathcal{N}_0^-} E_0(v) > 0$. This yields that $\bar{v}$ is a bound state of (1.3). The proof is completed.

Remark 3.1 : In view of Theorem 1.1 and Proposition 3.1, we see that u_0 and $\bar{u}$ are both ground states of (1.3) with $E_0(u_0) = E_0(\bar{u}) = \theta_0^+$ while v_0 and $\bar{v}$ are both bound states of (1.3) with $E_0(v_0) = E_0(\bar{v}) = \theta_0^-$. The solutions u_0, v_0 are obtained by fibering maps, and the solutions $\bar{u}, \bar{v}$ are the limits of solution sequences $\{u_\alpha\}$ and $\{v_\alpha\}$ of (1.1) as $\alpha \to 0$. But whether u_0 equals to $\bar{u}$ and so as v_0 and $\bar{v}$ is still worth studying.

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