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VISCOSITY EXTRAGRADIENT METHOD WITH ARMIJO LINESEARCH RULE FOR PSEUDOMONOTONE EQUILIBRIUM PROBLEM AND FIXED POINT PROBLEM IN HILBERT SPACES

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In this paper, we introduce a viscosity extragradient method with Armijo linesearch rule to find a common element of solution set of a pseudomonotone equilibrium problem and fixed point set of a nonexpansive nonself-mapping in Hilbert space. The strong convergence of the algorithm is proved. As the application, a common fixed point theorem for two nonexpansive nonself-mappings is proved. Finally, some numerical examples are given to illustrate the effectiveness of the algorithm. Our result improves the ones of others in the literature.

Key words : Nonexpansive mappings; equilibrium problems; Hilbert spaces.

1. INTRODUCTION

Let H be a real Hilbert space and C be a nonempty closed and convex subset of H. Let $f : C\times C \to \mathbb{R}$ be a bifunction with $f(x, x) = 0$ for all $x \in C$. The equilibrium problem due to Blum and Oettli [5] is to find $z \in C$ such that

$$f(z, y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $EP(f)$. Numerous problems in optimization, economics, information and communication technology reduce to find a solution of (1.1); see, for example, [9, 10, 14, 15] and the references quoted therein.

The equilibrium problem is pseudomonotone if

$$f(x,y) \geq 0 \Rightarrow f(y,x) \leq 0, \ \ \forall x, y \in C.$$

Recently, the algorithms of approximating the solutions of pseudomonotone equilibrium problems are investigated by many authors. In 2008, Tran *et al.* [18] introduced an extragradient method to solve a pseudomonotone equilibrium problem in $\mathbb{R}^n$ by the following manner: $x_0 \in C$ and

$$\begin{cases} y_n = \operatorname{argmin}_{y\in C}\{\lambda_n f(x_n, y) + \dfrac{1}{2}\|y - x_n\|^2\}, \\ x_{n+1} = \operatorname{argmin}_{y\in C}\{\lambda_n f(y_n, y) + \dfrac{1}{2}\|y - x_n\|^2\}, \ \ \forall n \in \mathbb{N}, \end{cases} \tag{1.2}$$

where $\{\lambda_n\} \subset (0,1]$ and f satisfies a Lipschitz-type property. The authors proved that the iterative scheme $\{x_n\}$ generated by (1.2) converges to some $x^* \in EP(f)$ under some certain conditions on f and $\{\lambda_n\}$.

In 2013, Vuong *et al.* [19] constructed a hybrid projection algorithm for finding a common element of fixed point set of a pseudo-contraction S and solution set of a pseudomonotone equilibrium problem by the following manner: $x_0 \in C$ and

$$\begin{cases} y_n = \operatorname{argmin}_{y\in C}\{\lambda_n f(x_n, y) + \dfrac{1}{2}\|y - x_n\|^2\}, \\ z_n = \operatorname{argmin}_{y\in C}\{\lambda_n f(y_n, y) + \dfrac{1}{2}\|y - x_n\|^2\}, \\ t_n = \alpha_n x_n + (1-\alpha_n)[\beta_n z_n + (1-\beta_n) S z_n], \\ C_n = \{z \in C : \|t_n - z\| \leq \|x_n - z\|\}, \\ D_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap D_n} x_0, \ \ \forall n \in \mathbb{N}, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\lambda_n\} \subset (0,1)$, and f satisfies a Lipschitz-type property. The authors proved the strong convergence of $\{x_n\}$ generated by (1.3).

In the literature, most authors use the hybrid methods like (1.3) to find the solution of a pseudomonotone equilibrium problem. Recently, Wang *et al.* [20] considered a viscosity extragradient method to find a common element of solution set of a pseudomonotone equilibrium problem and fixed point set of a nonexpansive mapping as follows: $x_0 \in C$ and

$$\begin{cases} y_n = \text{argmin}\{\frac{1}{2}\|y - x_n\|^2 + \lambda_n f(x_n, y) : y \in C\}, \\ t_n = \text{argmin}\{\frac{1}{2}\|t - x_n\|^2 + \lambda_n f(y_n, t) : t \in C\}, \\ x_{n+1} = \alpha_n p(x_n) + (1 - \alpha_n)(\beta_n x_n + (1 - \beta_n)Tt_n, \ \ \forall n \in \mathbb{N}, \end{cases} \tag{1.4}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\lambda_n\} \subset (0, 1)$, f is a bifunction on $C \times C$ satisfying a Lipschitz-type property, T is a nonexpansive mapping on C and p is a ρ-contraction on C. The authors proved the strong convergence of $\{x_n\}$ generated by (1.4).

On the other algorithms for finding the solution of pseudomontone equilibrium problem in which the bifunction f has Lipschitz-type property, the readers may refer to [1, 2, 11, 12].

However, it is sometimes difficult to check the Lipschitz-type property on the bifunction f. Hence some authors investigate the algorithms approximating the solution of pseudomonotone equilibrium problem without the restriction of Lipschitz-type property. In [4], Anh and Thi gave the following projection method with Armijo linesearch rule to find the solution of pseudomonotone equilibrium problem in which the bifunction f is not required to be Lipschitz-type continuous: $x_0 \in C$ and

$$\begin{cases} y_k = \text{argmin}_{y \in C}\{f(x_n, y) + \frac{\beta}{2}\|y - x_n\|^2\}, \\ \quad \text{find the smallest nonnegative integer } m_n \text{ such that} \\ \qquad f((1 - \gamma^{m_n})x_n + \gamma^{m_n} y_n, y_n) \leq -\sigma\|x_n - y_n\|^2, \\ C_n = \{z \in C : f((1 - \gamma^{m_n})x_n + \gamma^{m_n} y_n, x) \leq 0\}, \\ H_n = \{z \in C : \langle x - x_n, x_0 - x_n\rangle \leq 0\rangle\}, \\ x_{n+1} = P_{C_k \cap H_n} x_0, \quad \forall n \in \mathbb{N}, \end{cases} \tag{1.5}$$

where $\beta > 0, \gamma \in (0, 1)$ and $\sigma \in (0, \beta/2)$. The authors proved that $\{x_n\}$ generated by (1.5) converges to some $x^* \in EP(f)$.

On the more algorithms with Armijo linesearch rule for the pseudomonotone equilibrium problem without Lipschitz-type property, the readers may refer to [7, 8, 17].

In [4, 7, 8, 17], at each step x_n is obtained by $P_{C_n \cap H_n} x_0$, which is actually an optimization problem. It may be difficult to solve such an optimization problem when the subset C has the complex structure. In [3], the authors gave a weak convergent algorithm with Armijo linesearch rule in which the subsets C_n and H_n are not constructed to approximating the solution of the pseudomonotone equilibrium problem without Lipschitz-type property. In this paper, motivated by the results of Wang *et al*. [20] and Anh and Hien [3], we introduce a viscosity extragradient method to find a common element of solution set of a pseudomonotone equilibrium problem without the restriction of Lipschitz-type property and fixed point set of a nonexpansive nonself-mapping in Hilbert space. The strong convergence of the algorithm is proved. As the application, a common fixed point theorem for two nonexpansive nonself-mappings is proved. Finally, the first numerical example is given to illustrate the effectiveness of the algorithm and the second numerical example from the literature is used to compare the result with the one of other authors. Our result improves the ones of [20], Anh [1] and Anh and Thi [4] and others in the literature.

2. Preliminaries

Let H be a Hilbert space and C be a nonempty closed and convex subset of H. A mapping $T : C \to H$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \ \ \forall x, y \in C.$$

Denote the set of fixed points of T by $Fix(T)$, i.e., $Fix(T) = \{x \in C : x = Tx\}$. It is known that if $Fix(T) \neq \emptyset$, then $Fix(P_C T) = Fix(T)$; see [22].

Let $E \subset H$ be a nonempty closed and convex subset and set $d(x, E) = \inf\{\|x - y\| : y \in E\}$. If $E = \{y \in H : \langle w, y - z\rangle \leq 0\}$ for some $w, z \in H$ with $w \neq \theta$, where θ denotes the zero element in H, then

$$d(x, E) = \begin{cases} \dfrac{|\langle w, x - z\rangle|}{\|w\|}, & \text{if } x \notin E, \\ 0, & \text{if } x \in E. \end{cases}$$

Lemma 2.1 — For each point $x \in H$, there exists a unique nearest point of C, denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. Such a P_C is called the metric projection from H onto C. Then

(1) for $x \in H$ and $z \in C$, $z = P_C x$ if and only if

$$\langle x - z, z - y\rangle \geq 0, \ \ \forall y \in C;$$

(2) for all $x, y \in H$ and $z \in C$, it holds

$$\|P_C x - z\|^2 \leq \|x - z\|^2 - \|P_C x - x\|^2.$$

Lemma 2.2 — Let H be a real Hilbert space. For all $x, y \in H$, the following hold:

(1) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$;

(2) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2,$ for all $t \in [0,1]$.

Lemma 2.3 — [16]. Let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions that $f(x, \cdot)$ is convex on C for each $x \in C$ and f is jointly weakly continuous on $C \times C$. Let $\{x_n\}$ and $\{y_n\}$ in C converge weakly to $\bar{x}$ and $\bar{y}$, respectively. Then for any $\epsilon > 0$, there exist $\eta > 0$ and $n_\epsilon \in \mathbb{N}$ such that

$$\partial_2 f(x_n, y_n) \subset \partial_2 f(\bar{x}, \bar{y}) + \frac{\epsilon}{\eta} B$$

for each $n \in \mathbb{N}$ with $\geq n_\epsilon$, where B denotes the closed unit ball in H.

Lemma 2.4 — [7]. Let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions that $f(x, \cdot)$ is convex on C for each $x \in C$ and f is jointly weakly continuous on $C \times C$. Let $\{x_n\} \subset C$ be a bounded sequence, $\beta > 0$, and y_n be a sequence such that

$$y_n = \text{argmin}\big\{f(x_n, y) + \frac{\beta}{2}\|y - x_n\|^2 : y \in C\big\}.$$

Then $\{y_n\}$ is bounded.

Lemma 2.5 — Let $T : C \to C$ be a nonexpansive mapping. Assume that $\{x_n\} \subset C$ weakly converges to $x \in C$ and $\|x_n - Tx_n\| \to 0$, then $x \in Fix(T)$.

Lemma 2.6 — [21]. Let $\{a_n\}$ be a sequence of nonnegative real numbers. Suppose that

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n \delta_n, \quad \forall n \in \mathbb{N},$$

where $\{\gamma_n\} \subset (0,1)$ and $\{\delta_n\} \subset \mathbb{R}$ satisfy the conditions:

$$\lim_{n\to\infty} \gamma_n = 0, \ \sum_{n=1}^{\infty} \gamma_n = \infty, \ \limsup_{n\to\infty} \delta_n \leq 0.$$

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.7 — [13]. Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ as $k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:

$$a_{m_k} \leq a_{m_k+1}, \ a_k \leq a_{m_k+1}.$$

In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.

3. Main Result

In this section, let H be a Hilbert space and C be a nonempty closed and convex subset of H. Let $T : C \to H$ be a nonexpansive mapping, $g : C \to C$ be a ρ-contraction with $\rho \in (0,1)$ and $f : C \times C \to \mathbb{R}$ be a bifunction satisfying the following conditions:

(A1) $f(x,x) = 0$ for all $x \in C$ and f is pseudomonotone;

(A2) f is jointly weakly continuous on $C \times C$;

(A3) for each $x \in C$, $y \mapsto f(x,\cdot)$ is convex and subdifferentiable.

Assume that $EP(f) \cap Fix(T) \neq \emptyset$. We are in position to give the iterative algorithm as follows.

Algorithm : Initialization. Choose $x_1 \in C$, $\gamma \in (0,1)$, $\beta > 0$, $\sigma \in (0, \beta/2)$ and two sequences $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$. Set $n = 1$.

Step 1 : Solve the strongly convex problems:

$$\begin{cases} y_n = \operatorname{argmin}\{\frac{\beta}{2}\|y - x_n\|^2 + f(x_n, y) : y \in C\}, \\ r(x_n) = x_n - y_n. \end{cases}$$

If $r(x_n) = 0$ and $x_n = Tx_n$ for some $n \in \mathbb{N}$, stop. Otherwise, go to Step 2.

Step 2 : Find the smallest $m_n \in \mathbb{N}$ such that

$$f(x_n - \gamma^{m_n} r(x_n), y_n) \leq -\sigma \|r(x_n)\|^2.$$

Set $z_n = x_n - \gamma^{m_n} r(x_n)$.

Step 3. Compute

$$x_{n+1} = \beta_n g(x_n) + (1 - \beta_n)\big(\alpha_n x_n + (1 - \alpha_n) P_C T u_n\big),$$

where $u_n = P_{C_n} x_n$ and $C_n = \{x \in C : f(z_n, x) \leq 0\}$. Set $n = n + 1$.

Remark 3.1 : (1) From Lemma 3.1 of [11] it follows that if $r(x_n) = 0$ for some $n \in \mathbb{N}$, then $x_n \in EP(f)$. Therefore, if $r(x_n) = 0$ and $x_n = Tx_n$ for some $n \in \mathbb{N}$, then $x_n \in EP(f) \cap Fix(T)$;

(2) From Lemma 1 of [4] it follows that $EP(f) \subset C_n$ and $x_n \notin C_n$ for all $n \in \mathbb{N}$ and Step 2 is well defined, i.e., there exists the smallest $m_n \in \mathbb{N}$ such that

$$f(x_n - \gamma^{m_n} r(x_n), y_n) \leq -\sigma \|r(x_n)\|^2.$$

In the rest of the paper, for showing the convergence of algorithm, assume that $r(x_n) \neq 0$ and $x_n \neq Tx_n$ for all $n \in \mathbb{N}$.

Lemma 3.1 — The sequence $\{x_n\}$ is bounded.

PROOF : Since $Fix(T) \cap EP(f) \subset EP(f) \subset C_n$, from Lemma 2.1 it follows that

$$\|u_n - z\|^2 \leq \|x_n - z\|^2 - \|x_n - u_n\|^2 \leq \|x_n - z\|^2, \quad \forall z \in Fix(T) \cap EP(f). \tag{3.1}$$

Since $\{\alpha_n\}$ is strictly decreasing, by (3.1) and Lemma 2.2 we have

$$\begin{aligned}
\|x_{n+1} - z\| &\leq \beta_n\|g(x_n) - z\| + (1-\beta_n)\|\alpha_n x_n + (1-\alpha_n)P_C T u_n\| \\
&\leq \beta_n\|g(x_n) - g(z)\| + \beta_n\|g(z) - z\| + (1-\beta_n)\big[\alpha_n\|x_n - z\| \\
&\quad + (1-\alpha_n)\|P_C T u_n - z\|\big] \\
&\leq \beta_n\|g(x_n) - g(z)\| + \beta_n\|g(z) - z\| + (1-\beta_n)\big[\alpha_n\|x_n - z\| \\
&\quad + (1-\alpha_n)\|x_n - z\|\big] \\
&= \beta_n\|g(x_n) - g(z)\| + \beta_n\|g(z) - z\| + (1-\beta_n)\|x_n - z\| \\
&\leq \beta_n\rho\|x_n - z\| + \beta_n\|g(z) - z\| + (1-\beta_n)\|x_n - z\| \\
&= (1 - \beta_n(1-\rho))\|x_n - z\| + \beta_n\|g(z) - z\| \\
&\leq \max\{\|x_n - z\|, \|g(z) - z\|/(1-\rho)\} \\
&\leq \cdots \leq \max\{\|x_1 - z\|, \|g(z) - z\|/(1-\rho)\},\ \forall z \in Fix(T) \cap EP(f), \forall n \in \mathbb{N}.
\end{aligned}$$

So, $\{x_n\}$ is bounded. This completes the proof. □

Lemma 3.2 — If $\lim_{n\to\infty} \|P_C T u_n - u_n\| = 0$ and $\lim_{n\to\infty} \|x_n - u_n\| = 0$, then

$$\liminf_{n\to\infty} \langle x^* - g(x^*), x_n - x^*\rangle \geq 0,$$

where $x^* = P_{Fix(T)\cap EP(f)}\, g(x^*)$.

PROOF : To show this inequality, we choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\liminf_{n\to\infty} \langle x^* - g(x^*), x_n - x^*\rangle = \lim_{k\to\infty} \langle x^* - g(x^*), x_{n_k} - x^*\rangle. \tag{3.2}$$

Since $\{x_{n_k}\}$ is also bounded, there exists a subsequence $\{x_{n_{k_i}}\}$ of $\{x_{n_k}\}$ which converges weakly to $w \in C$. Without loss of generality, we can assume that $x_{n_k} \rightharpoonup w$ as $k \to \infty$, where $\rightharpoonup$ denotes the weak convergence.

From the boundedness of $\{x_n\}$ and Lemma 2.4 it follows that $\{y_n\}$ is bounded. Hence $\{z_n\}$ is also bounded. For each $w_n \in \partial_2 f(z_n, z_n)$, set

$$H_n = \{y \in C : \langle w_n, y - z_n\rangle \le 0\}.$$

Since $\{w_n\}$ is bounded by Lemma 2.3, then there exists $L > 0$ such that $\|w_n\| \le L$ for all $n \in \mathbb{N}$. From $w_n \in \partial_2 f(z_n, z_n)$ and $f(z_n, z_n) = 0$, we have

$$\langle w_n, y - z_n\rangle \le f(z_n, y),\ \forall y \in C. \tag{3.3}$$

In particular, $\langle w_n, y_n - z_n\rangle \le f(z_n, y_n)$. Since $f(z_n, y_n) \le -\sigma\|r(x_n)\|^2$, it follows that

$$\langle w_n, z_n - y_n\rangle \ge \sigma\|r(x_n)\|^2 > 0. \tag{3.4}$$

By (3.4) and $x_n - z_n = \frac{\gamma_n^{m_n}}{1-\gamma^{m_n}}(z_n - y_n)$, we have

$$\langle w_n, x_n - z_n\rangle = \frac{\gamma_n^{m_n}}{1-\gamma^{m_n}}\langle w_n, z_n - y_n\rangle \ge \frac{\gamma_n^{m_n}\sigma}{1-\gamma^{m_n}}\|r(x_n)\|^2 > 0.$$

Hence $x_n \notin H_n$ for all $n \in \mathbb{N}$. Note that (3.3) implies that $C_n \subset H_n$ for all $n \in \mathbb{N}$. By $u_n = P_{C_n}x_n \in C_n \subset H_n$, we have

$$\begin{aligned}\|u_n - x_n\| = d(x_n, C_n) \ge d(x_n, H_n) &= \frac{|\langle w_n, x_n - z_n\rangle|}{\|w_n\|}\\ &\ge \frac{\gamma^{m_n}\sigma\|x_n - y_n\|^2}{L},\quad \forall n \in \mathbb{N}.\end{aligned}$$

Since $\gamma \in (0, 1)$, it follows that

$$\lim_{n\to\infty}\gamma^{m_n}\|x_n - y_n\|^2 = 0.$$

In particular,

$$\lim_{k\to\infty}\gamma^{m_{n_k}}\|x_{n_k} - y_{n_k}\|^2 = 0. \tag{3.5}$$

Now we prove that $w \in EP(f)$ by the following cases:

Case 1 : $\limsup_{k\to\infty}\gamma^{m_{n_k}} > 0$. Then there exists $\bar{\gamma} > 0$ and a subsequence $\{\gamma^{m_{n_{k_i}}}\} \subset \{\gamma^{m_{n_k}}\}$ such that $\gamma^{m_{n_{k_i}}} > \bar{\gamma}$ for all $i \in \mathbb{N}$. By (3.5) we have

$$\lim_{i\to\infty}\|x_{n_{k_i}} - y_{n_{k_i}}\| = 0. \tag{3.6}$$

Hence $y_{n_{k_i}} \rightharpoonup w$ as $i \to \infty$. By the definition of $y_{n_{k_i}}$, we have

$$0 \in \partial_2 f(x_{n_{k_i}}, y_{n_{k_i}}) + \beta(y_{n_{k_i}} - x_{n_{k_i}}) + N_C(y_{n_{k_i}}).$$

Hence there exists $v_{n_{k_i}} \in \partial_2 f(x_{n_{k_i}}, y_{n_{k_i}})$ such that

$$\langle v_{n_{k_i}}, y - y_{n_{k_i}} \rangle + \beta\langle y_{n_{k_i}} - x_{n_{k_i}}, y - y_{n_{k_i}} \rangle \geq 0, \ \ \forall y \in C.$$

Combining with

$$f(x_{n_{k_i}}, y) - f(x_{n_{k_i}}, y_{n_{k_i}}) \geq \langle v_{n_{k_i}}, y - y_{n_{k_i}} \rangle, \ \forall y \in C,$$

yields

$$f(x_{n_{k_i}}, y) - f(x_{n_{k_i}}, y_{n_{k_i}}) + \beta\langle y_{n_{k_i}} - x_{n_{k_i}}, y - y_{n_{k_i}} \rangle \geq 0, \ \ \forall y \in C. \tag{3.7}$$

Hence

$$f(x_{n_{k_i}}, y) - f(x_{n_{k_i}}, y_{n_{k_i}}) + \beta\|y_{n_{k_i}} - x_{n_{k_i}}\|\|y - y_{n_{k_i}}\| \geq 0, \ \ \forall y \in C.$$

Letting $i \to \infty$, by the jointly weak continuity of f and (3.6), we obtain

$$f(w, y) \geq 0, \ \ \forall y \in C.$$

It follows that $w \in EP(f)$.

Case 2 : $\lim_{k\to\infty} \gamma^{m_{n_k}} = 0$.

Since $\{y_{n_k}\}$ is bounded. It deduces that there exists $\{y_{n_{k_i}}\} \subset \{y_{n_k}\}$ such that $y_{n_{k_i}} \rightharpoonup \overline{y}$ as $i \to \infty$. Replacing y by $x_{n_{k_i}}$ in (3.7), we get

$$f(x_{n_{k_i}}, y_{n_{k_i}}) + \beta\|y_{n_{k_i}} - x_{n_{k_i}}\|^2 \leq 0. \tag{3.8}$$

In the other hand, by the Armijo linesearch rule, for $m_{n_{k_i}} - 1$, it follows that

$$f(x_{n_{k_i}} - \gamma^{m_{n_{k_i}}-1} r(x_{n_{k_i}}), y_{n_{k_i}}) > -\sigma\|y_{n_{k_i}} - x_{n_{k_i}}\|^2. \tag{3.9}$$

Letting $i \to \infty$ in (3.8) and (3.9), by (i), we get

$$\frac{1}{\beta} f(w, \bar{y}) \leq -\lim_{i\to\infty} \|y_{n_{k_i}} - x_{n_{k_i}}\|^2 \leq \frac{1}{\sigma} f(w, \bar{y}).$$

It follows that $f(w, \bar{y}) = 0$ and hence $\lim_{i\to\infty} \|y_{n_{k_i}} - x_{n_{k_i}}\|^2 = 0$. By the Case 1, it is immediate that $w \in EP(f)$.

Since $\|u_n - x_n\| \to 0$ and $\|P_C T u_n - u_n\| \to 0$ as $n \to \infty$, by Lemma 2.5, we have $w \in Fix(P_C T)$, which implies that $w \in Fix(T)$ by [22]. Therefore, $w \in EP(f) \cap Fix(T)$.

Finally, by (3.2) and Lemma 2.1, we obtain

$$\liminf_{n\to\infty} \langle x^* - g(x^*), x_n - x^* \rangle = \langle x^* - g(x^*), w - x^* \rangle \geq 0.$$

This completes the proof. □

***Theorem* 3.1** — *Assume that the sequence $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following conditions:*

$$\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0, \quad \lim_{n\to\infty} \beta_n = 0 \quad \text{and} \quad \sum_{n=0}^{\infty} \beta_n = \infty.$$

Then $\{x_n\}$ converges strongly to the element $x^* = P_{EP(f)\cap Fix(T)} g(x^*)$, which is also the solution of the following variational inequality:

$$\langle x^* - g(x^*), y - x^* \rangle \geq 0, \ \ \forall y \in EP(f) \cap Fix(T).$$

PROOF : We show the proof process by the following two parts:

Case 1 : Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\|x_n - x^*\|\}_{n=n_0}^{\infty}$ is nonincreasing. In this situation, $\{\|x_n - x^*\|\}$ is convergent.

By Lemma 2.2, we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq \beta_n \|g(x_n) - x^*\|^2 + (1-\beta_n)\big[\alpha_n \|x_n - x^*\|^2 + (1-\alpha_n)\|P_C T u_n - x^*\|^2 \\ &\quad - \alpha_n(1-\alpha_n)\|x_n - P_C T u_n\|^2\big] \\ &\leq \beta_n \|g(x_n) - x^*\|^2 + (1-\beta_n)\big[\alpha_n \|x_n - x^*\|^2 + (1-\alpha_n)\|x_n - x^*\|^2 \\ &\quad - \alpha_n(1-\alpha_n)\|x_n - P_C T u_n\|^2\big] \\ &= \beta_n \|g(x_n) - x^*\|^2 + (1-\beta_n)\big[\|x_n - x^*\|^2 - \alpha_n(1-\alpha_n)\|x_n - P_C T u_n\|^2\big].\end{aligned}$$

Hence

$$\begin{aligned}\alpha_n(1-\alpha_n)\|x_n - P_C T u_n\|^2 &\leq \beta_n\big(\|g(x_n) - x^*\|^2 + \|x_n - P_C T u_n\|^2\big) \\ &\quad + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\ &\leq \beta_n M + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2, \ \forall n \in \mathbb{N},\end{aligned} \tag{3.10}$$

where $M = \sup_{n\in\mathbb{N}}\{\|g(x_n) - x^*\|^2 + \|x_n - P_C T u_n\|^2\}$. Since the limit of $\{\|x_n - x^*\|\}$ exists, by the hypothesis on $\{\alpha_n\}$ and $\{\beta_n\}$, we have

$$\lim_{n\to\infty} \|x_n - P_C T u_n\|^2 = 0. \tag{3.11}$$

On the other hand, by (3.1) we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le \beta_n\|g(x_n)-x^*\|^2+(1-\beta_n)\big[\alpha_n\|x_n-x^*\|^2+(1-\alpha_n)\|P_CTu_n-x^*\|^2\big]\\ &\le \beta_n\|g(x_n)-x^*\|^2+(1-\beta_n)\big[\alpha_n\|x_n-x^*\|^2+(1-\alpha_n)\|u_n-x^*\|^2\big]\\ &\le \beta_n\|g(x_n)-x^*\|^2+(1-\beta_n)\big[\alpha_n\|x_n-x^*\|^2+(1-\alpha_n)\|x_n-x^*\|^2\\ &\quad -(1-\alpha_n)\|x_n-u_n\|^2\big]\\ &= \beta_n\|g(x_n)-x^*\|^2+(1-\beta_n)\big[\|x_n-x^*\|^2-(1-\alpha_n)\|x_n-u_n\|^2\big].\end{aligned}$$

It follows that

$$(1-\alpha_n)\|x_n-u_n\|^2 \le \beta_n M_0+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2,\ \forall n\in\mathbb{N}, \tag{3.12}$$

where $M_0=\sup_{n\in\mathbb{N}}\{\|g(x_n)-x^*\|^2+\|x_n-u_n\|^2\}$. Since the limit of $\{\|x_n-x^*\|\}$ exists, by the hypothesis on $\{\alpha_n\}$ and $\{\beta_n\}$, we have

$$\lim_{n\to\infty}\|x_n-u_n\|^2=0. \tag{3.13}$$

Combining (3.12) with (3.13) we get

$$\lim_{n\to\infty}\|u_n-P_CTu_n\|=0. \tag{3.14}$$

From (3.13), (3.14) and Lemma 3.2 it follows that

$$\liminf_{n\to\infty}\langle x^*-g(x^*),x_n-x^*\rangle\ge 0. \tag{3.15}$$

By (3.1) again and Lemma 2.1 we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le (1-\beta_n)\big\|\alpha_n(x_n-x^*)+(1-\alpha_n)(P_CTu_n-x^*)\big\|^2\\ &\quad +2\beta_n\langle g(x_n)-x^*,x_{n+1}-x^*\rangle\\ &\le (1-\beta_n)\big[\alpha_n\|x_n-x^*\|^2+(1-\alpha_n)\|P_CTu_n-x^*\|^2\big]\\ &\quad +2\beta_n\langle g(x_n)-x^*,x_{n+1}-x^*\rangle\\ &\le (1-\beta_n)\big[\alpha_n\|x_n-x^*\|^2+(1-\alpha_n)\|x_n-x^*\|^2\big]\\ &\quad +2\beta_n\langle g(x_n)-x^*,x_{n+1}-x^*\rangle\\ &= (1-\beta_n)\|x_n-x^*\|^2+2\beta_n\langle g(x_n)-x^*,x_{n+1}-x^*\rangle.\end{aligned} \tag{3.16}$$

Since $\lim_{n\to\infty}\beta_n=0$ and $\sum_{n=1}^{\infty}\beta_n=\infty$, the desired conclusion is from (3.15), (3.16) and Lemma 2.6.

Case 2 : Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\|x_{n_i} - x^*\| < \|x_{n_i+1} - x^*\|, \ \ \forall i \in \mathbb{N}.$$

Then, by Lemma 2.7 there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$,

$$\|x_{m_k} - x^*\| \le \|x_{m_k+1} - x^*\| \quad \text{and} \ \|x_k - x^*\| \le \|x_{m_k+1} - x^*\|, \ \ \forall k \in \mathbb{N}.$$

This with (3.10) gives

$$\begin{aligned} \alpha_{m_k}(1-\alpha_{m_k})\|x_{m_k} - P_C T u_{m_k}\|^2 &\le \|x_{m_k} - x^*\|^2 - \|x_{m_k+1} - x^*\|^2 + \beta_{m_k} M \\ &\le \beta_{m_k} M \to 0, \ \ \text{as } k \to \infty. \end{aligned}$$

By the hypothesis on $\{\alpha_n\}$ and $\{\beta_n\}$, we have

$$\lim_{k\to\infty} \|x_{m_k} - P_C T u_{m_k}\| = 0. \tag{3.17}$$

By (3.12) we have

$$\begin{aligned} (1-\alpha_{m_k})\|x_{m_k} - u_{m_k}\|^2 &\le \|x_{m_k} - x^*\|^2 - \|x_{m_k+1} - x^*\|^2 + \beta_{m_k} M_0 \\ &\le \beta_{m_k} M_0 \\ &\to 0, \ \ \text{as } k \to \infty. \end{aligned}$$

By the hypothesis on $\{\alpha_n\}$ and $\{\beta_n\}$, we get

$$\lim_{k\to\infty} \|x_{m_k} - u_{m_k}\| = 0. \tag{3.18}$$

By (3.17), (3.18) and Lemma 2.2 we get

$$\limsup_{k\to\infty} \langle g(x^*) - x^*, x_{m_k+1} - x^* \rangle \le 0. \tag{3.19}$$

Note that (3.16) implies

$$\|x_{m_k+1} - x^*\|^2 \le (1-\beta_{m_k})\|x_{m_k} - x^*\|^2 + 2\beta_{m_k}\langle g(x^*) - x^*, x_{m_k+1} - x^* \rangle. \tag{3.20}$$

Since $\|x_{m_k} - x^*\| \le \|x_{m_k+1} - x^*\|$, we have

$$\begin{aligned} \beta_{m_k}\|x_{m_k} - x^*\|^2 &\le \|x_{m_k} - x^*\|^2 - \|x_{m_k+1} - x^*\|^2 + 2\beta_{m_k}\langle g(x^*) - x^*, x_{m_k+1} - x^* \rangle \\ &\le 2\beta_{m_k}\langle g(x^*) - x^*, x_{m_k+1} - x^* \rangle. \end{aligned}$$

Since $\beta_{m_k} > 0$, we get

$$\|x_{m_k} - x^*\|^2 \leq 2\langle g(x^*) - x^*, x_{m_k+1} - x^*\rangle.$$

It follows from (3.19) that $\|x_{m_k} - x^*\| \to 0$ as $k \to \infty$. This with (3.19) and (3.20) gives

$$\|x_{m_k+1} - x^*\| \to 0, \text{ as } k \to \infty.$$

But $\|x_k - x^*\| \leq \|x_{m_k+1} - x^*\|$ for all $k \in \mathbb{N}$, we conclude that $x_k \to x^*$ as $k \to \infty$. The proof is complete. □

Corollary 3.1 — Let H be a Hilbert space and C be a nonempty closed and convex subset of H. Let g be a ρ-contraction on C and $f : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions (A1)-(A3) with $EP(f) \neq \emptyset$. Define the sequence $\{x_n\}$ by the manner: $x_1 \in C$ and

$$\begin{cases} y_n = \text{argmin}\{\frac{\beta}{2}\|y - x_n\|^2 + f(x_n, y) : y \in C\}, \\ \text{put } r(x_n) = x_n - y_n \text{ and find the smallest } m_n \in \mathbb{N} \text{ such that} \\ \qquad f(x_n - \gamma^{m_n} r(x_n), y_n) \leq -\sigma\|r(x_n)\|^2, \\ \text{set } z_n = x_n - \gamma^{m_n} r(x_n), \\ C_n = \{x \in C : f(z_n, x) \leq 0\}, \\ x_{n+1} = \beta_n g(x_n) + (1 - \beta_n)\big(\alpha_n x_n + (1 - \alpha_n) P_{C_n} x_n\big), \ \forall\, n \in \mathbb{N}, \end{cases} \tag{3.21}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$, $\gamma \in (0, 1)$, $\beta > 0$, $\sigma \in (0, \beta/2)$. If the following conditions hold:

$$\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0, \quad \lim_{n\to\infty} \beta_n = 0, \quad \sum_{n=0}^{\infty} \beta_n = \infty.$$

Then $\{x_n\}$ generated by (3.21) converges strongly to the element $x^* = P_{EP(f)} g(x^*)$, which is also the solution of the following variational inequality:

$$\langle x^* - g(x^*), y - x^*\rangle \geq 0, \ \ \forall y \in EP(f).$$

Remark 3.2 : Our result improve the ones of Wang *et al*. [20] by removing the Lipschitz-type property on bifunction, Anh [1] and Anh and Thi [4] from nonexpansive self-mapping to nonexpansive nonself-mapping.

4. APPLICATION

Let $S, T : C \to H$ be two nonexpansive mappings. In this section we find a common fixed point of the mappings S and T by the result in Section 3.

Define the function $f : C \times C \to \mathbb{R}$ by

$$f(x, y) = \langle x - P_C Sx, y - x \rangle, \ \forall x, y \in C.$$

Obviously, $EP(f) = Fix(P_C S)$. Since $Fix(P_C S) = Fix(S)$ [22], it follows that $EP(f) = Fix(S)$.

We show that f is monotone. In fact, for all $x, y \in C$, we have

$$\begin{aligned} f(x, y) + f(y, x) &= \langle x - P_C Sx, y - x \rangle + \langle y - P_C Sy, x - y \rangle \\ &= \langle x - y - P_C Sx + P_C Sy, y - x \rangle \\ &= -\|x - y\|^2 + \langle P_C Sy - P_C Sx, y - x \rangle \\ &\leq -\|x - y\|^2 + \|y - x\|^2 \\ &= 0. \end{aligned}$$

So, f is monotone, which implies that f is pseudomonotone. It is easy to see that f satisfies the conditions (A1)-(A2).

Note that for each $x \in C$, we have

$$\begin{aligned} &\operatorname{argmin}\{\frac{\beta}{2}\|y - x\|^2 + f(x, y) : y \in C\} \\ &= \operatorname{argmin}\{\frac{\beta}{2}\|y - x\|^2 + \langle x - P_C Sx, y - x \rangle : y \in C\} \\ &= \operatorname{argmin}\left\{\frac{\beta}{2}\|y - x + \frac{1}{\beta}(x - P_C Sx)\|^2 - \frac{\|x - P_C Sx\|^2}{2\beta} : y \in C\right\} \\ &= \operatorname{argmin}\{\|y - x + \frac{1}{\beta}(x - P_C Sx)\|^2 : y \in C\} \\ &= P_C\big((1 - \frac{1}{\beta})x + \frac{1}{\beta} P_C Sx\big). \end{aligned}$$

Now, by the result in Section 3, we give the follows common fixed point theorem:

***Theorem* 4.1** — *Let H be a Hilbert space and C be a nonempty closed and convex subset of H. Let $S, T : C \to H$ be two nonexpansive mappings, $g : C \to C$ be a ρ-contraction. Set $\gamma \in (0, 1)$,*

$\beta > 0$, $\sigma \in (0, \beta/2)$. *Take* $x_1 \in C$ *and define the sequence* $\{x_n\}$ *by the following manner:*

$$\begin{cases} y_n = P_C\big((1-\frac{1}{\beta})x_n + \frac{1}{\beta}P_C S x_n\big), \\ put\ r(x_n) = x_n - y_n, \\ \quad find\ the\ smallest\ m_n \in \mathbb{N}\ such\ that \\ \qquad \langle v_n - P_C S v_n, y_n - v_n\rangle \le -\sigma\|r(x_n)\|^2, \\ \qquad\quad where\ v_n = x_n - (1-\gamma^{m_n})r(x_n), \\ set\ z_n = x_n - \gamma^{m_n} r(x_n)\ and\ C_n = \{x \in C : \langle z_n - P_C S z_n, x - z_n\rangle \le 0\}, \\ x_{n+1} = \beta_n g(x_n) + (1-\beta_n)\big(\alpha_n x_n + (1-\alpha_n)P_C T P_{C_n} x_n\big),\ \forall n \in \mathbb{N}, \end{cases} \tag{4.1}$$

where $\{\beta_n\}, \{\alpha_n\} \subset (0,1)$. *If* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *satisfy the following conditions:*

$$\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0, \quad \lim_{n\to\infty} \beta_n = 0, \quad \sum_{n=0}^{\infty} \beta_n = \infty,$$

then $\{x_n\}$ *generated by* (4.1) *converges strongly to the element* $x^* = P_{Fix(S)\cap Fix(T)} g(x^*)$*, which is also the solution of the following variational inequality:*

$$\langle x^* - g(x^*), y - x^*\rangle \ge 0, \ \ \forall y \in Fix(S) \cap Fix(T).$$

5. Numerical Examples

The programs for the following examples are performed by Matlab R2008a running on a PC Desktop with Core(TM) i3CPU M550 3.20GHz and 4GB Ram. The first example is used to show the effectiveness of the algorithm in Section 3.

Example 5.1 : Let $H = \mathbb{R}^m$ and $C = \{(x_1, x_2, \cdots, x_m) : x_1 \in [0,1], x_i \ge 0, i = 2, \cdots, m\}$. Let $g(x) = (x_1/2, x_2/3, \cdots, x_m/(m+1))$ and $Tx = (-x_1, -x_2/2, \cdots, -x_m/m)$ for all $x = (x_1, x_2, \cdots, x_m) \in C$. Define the bifunction f on $C \times C$ by

$$f(x,y) = \left(x_1 - \frac{x_1^2 \sin x_1}{3}\right)(y_1 - x_1) + \sum_{i=2}^{m}(y_i - x_i)$$

for all $x = (x_1, \cdots, x_m), y = (y_1, \cdots, y_m) \in C$. It is easy to check that f satisfies the conditions (A1)-(A3). Obviously, $Fix(T) \cap EP(f) = \{(0, \cdots, 0)\}$.

The program will stop if $\max\{\|x_n - y_n\|, \|x_n - Tx_n\|\} < 10^{-5}$.

Table 1 gives the results for $\{x_n\}$ with the initial point $x_1 = (2,2,2,2,2)$ under the control sequences $\{\alpha_n\}$ and $\{\beta_n\}$ and parameters γ, σ, β. From Table 1 we see that the program stops after 12 iterations.

Table 1 Results on $\{x_n\}$ and cpu times with $m = 5, \alpha_n = \frac{1+n}{4n}, \beta_n = \frac{1}{4n}$, $\gamma = 0.5$, $\beta = 2$ and $\sigma = 0.7$

Iterations	x_n^1	x_n^2	x_n^3	x_n^4	x_n^5
1	2.000000	2.000000	2.000000	2.000000	2.000000
2	1.000000	0.916666	0.875000	0.850000	0.833333
3	0.390625	0.338975	0.314453	0.300156	0.290798
4	0.135633	0.112991	0.102634	0.096717	0.092894
5	0.043974	0.035457	0.031672	0.029544	0.028182
6	0.013632	0.010696	0.009422	0.008715	0.008266
7	0.004094	0.003138	0.002731	0.002508	0.002368
8	0.001201	0.000901	0.000777	0.000709	0.000666
9	0.000346	0.000255	0.000217	0.000197	0.000185
10	0.000098	0.000071	0.000060	0.000054	0.000050
11	0.000027	0.000019	0.000016	0.000014	0.000013
12	0.000007	0.000005	0.000004	0.000004	0.000003
CPU times (s)	1.037073				

Table 2 gives the cpu times and iteration steps of performing the program with the different initial point and the different m under the same $\{\alpha_n\}$, $\{\beta_n\}$, γ, σ and β.

Table 2 Results with $\alpha_n = \frac{1+n}{4n}, \beta_n = \frac{1}{4n}, \gamma = 0.5$, $\beta = 2$ and $\sigma = 0.7$

	$x_1 = (1, \cdots, 1)$		$x_1 = (7, \cdots, 7)$	
Dimensions	CPU times (s)	No. iterations	CPU times (s)	No. iterations
$m = 10$	1.937971	12	1.923657	13
$m = 20$	5.978498	12	6.127072	12
$m = 40$	23.232948	12	19.364022	14
$m = 60$	47.219360	12	45.813779	14

Example 5.1 shows the effectiveness of algorithm in Section 3. Next, we use the the following example which is performed by the iterative scheme (3.21) to compare the computed results with the ones of others in the literature.

Example 5.2 : We consider the following bifunction used by Anh and Thi [4] and Bnouhachem [6]:

$$f(x,y) = \langle d, \arctan(x-y)\rangle + \langle q, x-y\rangle,$$

where $H = \mathbb{R}^5$, $d = (1,3,5,7,9)^T$, $q = (2,4,6,8,1)^T$, $\arctan(x) = (\arctan(x_1), \cdots, \arctan(x_5))^T$ and

$$C = \begin{cases} x \in \mathbb{R}^5_+, \\ x_1 + x_2 \geq 1.5, \\ x_1 + x_2 + x_3 + 2x_4 + x_5 \geq 5, \\ 3x_1 + 2x_2 + x_3 + 3x_4 + 4x_5 \leq 12. \end{cases}$$

For performing the iterative scheme (3.21), we take the ρ-contraction $g(x) = P_C(\rho x)$ with $\rho \in (0,1)$ for all $x \in C$. The program will stop if $\|x_n - y_n\| \leq \epsilon$.

For comparing the results with the ones in [4], we take the same initial point $x_1 = (1,1,1,2,0)^T$ and parameters for $\epsilon, \beta, \sigma, \gamma$ with the ones in [4].

Table 3 contains the results of performing the iteration (3.21) with $\beta_n = \frac{1}{40n}$, $\alpha_n = \frac{1+n}{4n}$ for all $n \in \mathbb{N}$.

Table 3 Results of the performing iteration (3.21) with $\epsilon = 10^{-4}, \rho = 0.99$

Case	β	σ	γ	No. iterations	CPU times (s)
1	2.5	1	0.7	23	1.752649
2	4	1.5	0.7	24	2.318744
3	3.5	1.5	0.8	23	1.839678
4	6	2	0.5	46	3.440802
5	5	1	0.7	29	2.593558
6	5	2	0.7	29	2.551811
7	5	2	0.9	27	2.496499

From Table 3 we see that the computed results are near with the ones of [4].

Table 4 gives the cpu times and iterations for the different ρ with $\epsilon = 10^{-4}, \beta = 4, \sigma = 1.5, \gamma = 0.7$, $\beta_n = \frac{1}{40n}$, $\alpha_n = \frac{1+n}{4n}$ for all $n \in \mathbb{N}$.

Table 4 Results of the performing iteration (3.21) with different ρ

Case	ρ	No. iterations	CPU times (s)
1	0.95	31	2.276954
2	0.85	85	5.440439
3	0.75	140	7.972300
4	0.65	195	10.611033
5	0.55	250	13.693944
6	0.45	305	16.199490
7	0.35	351	17.630689
8	0.25	384	19.763450
9	0.15	417	20.286794
10	0.1	433	20.483081
11	0.05	450	19.545507
12	0.01	463	20.699576
13	0.005	465	22.107884
14	0.001	466	21.313802
15	0.0008	466	20.507393
16	0.0006	466	22.082778

From Table 4, we see that the cpu times and iteration steps have the closed relation with the value of ρ and have been stable when $\rho < 0.001$.

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