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RECTIFYING CURVES ON A SMOOTH SURFACE IMMERSED IN THE EUCLIDEAN SPACE

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The main objective of the present paper is to investigate a sufficient condition for which a rectifying curve on a smooth surface remains invariant under isometry of surfaces, and also it is shown that under such an isometry the component of the position vector of a rectifying curve on a smooth surface along the normal to the surface is invariant.

Key words : Rectifying curve; Frenet-Serret equation; isometry of surfaces; first fundamental form.

1. INTRODUCTION

In 2003, Chen [3] introduced the notion of the rectifying curve in the Euclidean space $\mathbb{R}^3$ as a curve whose position vector lies in the rectifying plane and such a curve classified by an unit speed curve in an unit sphere S^2 and also obtained some of its characterization. For further properties of rectifying curves we refer the reader to see [4] and [5]. By motivating the above studies, the main goal of this paper is to investigate the nature of rectifying curves on a smooth surface S under an isometry to another smooth surface $\bar{S}$. Then we obtain a sufficient condition for which a rectifying curve on S remains invariant under isometry $F : S \to \bar{S}$. We also note that under isometry of $\mathbb{R}^3$, a rectifying curve on $\mathbb{R}^3$ is not necessarily transformed to a rectifying curve on $\mathbb{R}^3$. It is also shown that the component of the position vector of a rectifying curve on a smooth surface along the normal to the surface is invariant under the rectifying curve preserving isometry of surfaces.

The structure of the paper is as follows. Section 2 deals with the discussion of some rudimentary facts of Frenet-Serret equations and rectifying curves. Section 3 is devoted to the study of rectifying curves on a smooth surface and deduced the components of position vectors of such a curve along the normal to the surface. The last section is concerned with the main results (see Theorem 4.1, Theorem 4.2).

2. Preliminaries

In this section, we recall some rudimentary facts of rectifying curves, isometry of surfaces and first fundamental form (for details see, [1, 2]) which will be used throughout the paper.

Let $\gamma(s) : I \to \mathbb{R}^3$, where $I = (\alpha, \beta) \subset \mathbb{R}$, be an unit speed parametrized curve having at least fourth order continuous derivatives. Let the tangent vector of the curve $\gamma(s)$ be denoted by $\vec{t}$. We consider $\vec{t'}(s) \neq 0$, so that there is an unit normal vector $\vec{n}$ along $\vec{t'}(s)$ and also a positive function $\kappa(s)$ such that $\vec{t'}(s) = \kappa(s)\vec{n}(s)$, where $\vec{t'}$ denote the derivative with respect to the arc length parameter s. The binormal vector field is defined by $\vec{b} = \vec{t} \times \vec{n}$. There is another curvature function $\tau(s)$, called torsion, and is given by the equation $\vec{b'}(s) = \tau(s)\vec{n}(s)$. At each point on $\gamma(s)$, $\{\vec{t}, \vec{n}, \vec{b}\}$ forms an orthonormal frame. At every point of the curve $\gamma(s)$, the planes generating by $\{\vec{t}, \vec{n}\}$,$\{\vec{n}, \vec{b}\}$ and $\{\vec{b}, \vec{t}\}$ are called osculating plane, normal plane and rectifying plane respectively. The quantity $\|\vec{b'}(s)\|$ measures the rate of change of the neighbouring osculating plane with the osculating plane at s. The Frenet-Serret equations are given by

$$\begin{aligned} \vec{t'} &= \kappa\vec{n}, \\ \vec{n'} &= -\kappa\vec{t} + \tau\vec{b}, \\ \vec{b'} &= -\tau\vec{n}. \end{aligned}$$

A curve in $\mathbb{R}^3$ is called rectifying [3] if its position vector always lies in the rectifying plane of that curve. The position vector $\gamma(s)$ satisfies the equation

$$\gamma(s) = \lambda(s)\vec{t}(s) + \mu(s)\vec{b}(s),$$

for some functions $\lambda(s)$ and $\mu(s)$.

Let $\gamma(t) = \phi(u(t), v(t))$, where $t \in (a, b) \subset \mathbb{R}$, be a curve in a surface patch ϕ. Then $\{\phi_u, \phi_v\}$ are linearly independent, and hence generates the tangent space $T_p\phi$ at a point $p \in \phi$. Thus we have

$$\begin{aligned} \|\dot{\gamma(t)}\|^2 &= (\phi_u\dot{u} + \phi_v\dot{v}) \cdot (\phi_u\dot{u} + \phi_v\dot{v}), \\ &= (\phi_u \cdot \phi_u)\dot{u}^2 + 2(\phi_u \cdot \phi_v)\dot{u}\dot{v} + (\phi_v \cdot \phi_v)\dot{v}^2, \\ &= \mathbf{E}\dot{u}^2 + 2\mathbf{F}\dot{u}\dot{v} + \mathbf{G}\dot{v}^2, \end{aligned}$$

where $\dot{\gamma}(t)$ denotes the derivative with respect to the parameter t.

A surface S is said to be regular if, for each $p \in S$ there exists a neighbourhood $V \subset \mathbb{R}^3$ and a map $\psi : U \to V \cap S$ of an open set $U \subset \mathbb{R}^2$ onto $V \cap S \subset \mathbb{R}^3$ such that ψ is differentiable, homeomorphism and the differential $d\psi_q$ is one to one for all $q \in U$.

Definition 2.1 — The first fundamental form of a regular surface S at a point p is a quadratic form $I_p : T_pS \to \mathbb{R}$ given by

$$I_p(\dot{\gamma}(t)) =< \dot{\gamma}(t), \dot{\gamma}(t) >= \|\dot{\gamma}(t)\|^2.$$

Definition 2.2 — A diffeomorphism $F : S \to \bar{S}$, where S and $\bar{S}$ are smooth surfaces in $\mathbb{R}^3$, is an isometry if F takes a curve from S to a curve of same length on $\bar{S}$.

Isometry of $\mathbb{R}^3$ is uniquely described as an orthogonal transformation followed by a translation. If we rotate the rectifying curve $\gamma(s)$ by fixing a point $\gamma(s_0)$ then at $\gamma(s_0)$, the Frenet-Serret frame transforms into another frame. Hence at $\gamma(s_0)$ the corresponding rectifying plane transforms into another rectifying plane. But the position vector of the curve $\gamma(s)$ does not change before and after the rotation. Therefore, generally, rectifying curves are not invariant under the isometry of $\mathbb{R}^3$.

3. Rectifying Curves on Smooth Surfaces

Let $\phi : U \to S$ be the coordinate chart for a smooth surface S and the unit speed parametrized curve $\gamma(s) : (\alpha, \beta) \to S$, where$(\alpha, \beta) \subset \mathbb{R}$, contained in the image of a surface patch ϕ in the atlas of S. Then $\gamma(s)$ is given by

$$\begin{aligned} (\alpha, \beta) \to U, \quad s \rightarrowtail (u(s), v(s)), \\ \gamma(s) = \phi(u(s), v(s)). \end{aligned} \tag{1}$$

Differentiating (1) with respect to s, we get

$$\begin{aligned} \gamma'(s) &= \phi_u u' + \phi_v v', \qquad (2) \\ \text{i.e., } \vec{t}(s) &= \gamma'(s) = \phi_u u' + \phi_v v', \\ \text{hence, } \vec{t'}(s) &= u''\phi_u + v''\phi_v + u'^2\phi_{uu} + 2u'v'\phi_{uv} + v'^2\phi_{vv}. \end{aligned}$$

If $k(s)$ is the curvature of $\gamma(s)$ and $\vec{N}$ is normal to S then the normal $\vec{n}(s)$ is given by

$$\begin{aligned} \vec{n}(s) &= \frac{1}{\kappa(s)}(u''\phi_u + v''\phi_v + u'^2\phi_{uu} + 2u'v'\phi_{uv} + v'^2\phi_{vv}). \\ \vec{b}(s) &= \vec{t}(s) \times \vec{n}(s) = \vec{t}(s) \times \frac{\vec{t'}(s)}{\kappa(s)}, \\ &= \frac{1}{k(s)}\Big[(\phi_u u' + \phi_v v') \times (u''\phi_u + v''\phi_v + u'^2\phi_{uu} + 2u'v'\phi_{uv} + v'^2\phi_{vv})\Big], \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{k(s)}\Big[u'v''\vec{N} + u'^3\phi_u \times \phi_{uu} + 2u'^2v'\phi_u \times \phi_{uv} + u'v'^2\phi_u \times \phi_{vv} \\
&\quad -u''v'\vec{N} + u'^2v'\phi_v \times \phi_{uu} + 2u'v'^2\phi_v \times \phi_{uv} + v'^3\phi_v \times \phi_{vv}\Big], \\
&= \frac{1}{k(s)}\Big[\{u'v'' - u''v'\}\vec{N} + u'^3\phi_u \times \phi_{uu} + 2u'^2v'\phi_u \times \phi_{uv} \\
&\quad +u'v'^2\phi_u \times \phi_{vv} + u'^2v'\phi_v \times \phi_{uu} + 2u'v'^2\phi_v \times \phi_{uv} + v'^3\phi_v \times \phi_{vv}\Big].
\end{aligned}
$$

So, $\gamma(s)$ in S will be rectifying curve if $\gamma(s) = \lambda(s)t(s) + \mu(s)b(s)$, for some functions $\lambda(s)$ and $\mu(s)$. i.e.,

$$
\begin{aligned}
\gamma(s) &= \lambda(s)(\phi_u u' + \phi_v v') + \frac{\mu(s)}{k(s)}\Big[\{u'v'' - u''v'\}\vec{N} + u'^3\phi_u \times \phi_{uu} + 2u'^2v'\phi_u \times \phi_{uv} \\
&\quad +u'v'^2\phi_u \times \phi_{vv} + u'^2v'\phi_v \times \phi_{uu} + 2u'v'^2\phi_v \times \phi_{uv} + v'^3\phi_v \times \phi_{vv}\Big]
\end{aligned}
$$

for some functions $\lambda(s)$ and $\mu(s)$.

Now we find component of the position vector of the curve $\gamma(s)$ along the normal $\vec{N}$ to the surface S at a point $\gamma(s)$ and obtain

$$
\begin{aligned}
\gamma(s)\cdot\vec{N} &= \lambda(s)(\phi_u u' + \phi_v v') + \frac{\mu(s)}{k(s)}\Big[\{u'v'' - u''v'\}\vec{N} + u'^3\phi_u \times \phi_{uu} + 2u'^2v'\phi_u \times \phi_{uv} \\
&\quad +u'v'^2\phi_u \times \phi_{vv} + u'^2v'\phi_v \times \phi_{uu} + 2u'v'^2\phi_v \times \phi_{uv} + v'^3\phi_v \times \phi_{vv}\Big]\cdot\vec{N}, \\
&= \frac{\mu(s)}{k(s)}\Big[(u'v'' - u''v)(\mathbf{EG} - \mathbf{F^2}) + u'^3(\phi_u \times \phi_{uu})\cdot\vec{N} + 2u'^2v'(\phi_u \times \phi_{uv})\cdot\vec{N} \\
&\quad +u'v'^2(\phi_u \times \phi_{vv})\cdot\vec{N} + u'^2v'(\phi_v \times \phi_{uu})\cdot\vec{N} + 2u'v'^2(\phi_v \times \phi_{uv})\cdot\vec{N} \\
&\quad +v'^3(\phi_v \times \phi_{vv})\cdot\vec{N}\Big], \\
&= \frac{\mu(s)}{k(s)}\Big[(u'v'' - u''v)(\mathbf{EG} - \mathbf{F^2}) + u'^3\{\mathbf{E}(\phi_{uu}\cdot\phi_v) - \mathbf{F}(\phi_{uu}\cdot\phi_u)\} \qquad (3) \\
&\quad +2u'^2v'\{\mathbf{E}(\phi_{uv}\cdot\phi_v) - \mathbf{F}(\phi_{uv}\cdot\phi_u)\} + u'v'^2\{\mathbf{E}(\phi_{vv}\cdot\phi_v) - \mathbf{F}(\phi_{vv}\cdot\phi_u)\} \\
&\quad +u'^2v'\{\mathbf{F}(\phi_{uu}\cdot\phi_v) - \mathbf{G}(\phi_{uu}\cdot\phi_u)\} + 2u'v'^2\{\mathbf{F}(\phi_{uv}\cdot\phi_v) - \mathbf{G}(\phi_{uv}\cdot\phi_u)\} \\
&\quad +v'^3\{\mathbf{F}(\phi_{vv}\cdot\phi_v) - \mathbf{G}(\phi_{vv}\cdot\phi_u)\}\Big].
\end{aligned}
$$

4. MAIN RESULTS

In the following theorem we consider the expression $F_*(\gamma(s))$ as a product of a 3×3 matrix F_* and a 3×1 matrix $\gamma(s)$.

***Theorem* 4.1** — *Let $F : S \to \bar{S}$ be an isometry, where S and $\bar{S}$ are smooth surfaces and $\gamma(s)$ be a rectifying curve on S. Then $\bar{\gamma}(s)$ is a rectifying curve on $\bar{S}$ if*

$$\begin{aligned}\bar{\gamma}(s) \quad - \quad & F_*(\gamma(s)) = \frac{\mu(s)}{k(s)}\Big[u'^3\Big(F_*\phi_u \times \frac{\partial F_*}{\partial u}\phi_u\Big) + 2u'^2v'\Big(F_*\phi_u \times \frac{\partial F_*}{\partial u}\phi_v\Big) \\ & +u'v'^2\Big(F_*\phi_u \times \frac{\partial F_*}{\partial v}\phi_v\Big) + u'^2v'\Big(F_*\phi_v \times \frac{\partial F_*}{\partial u}\phi_u\Big) + 2u'v'^2\Big(F_*\phi_v \times \frac{\partial F_*}{\partial u}\phi_v\Big) \\ & +v'^3\Big(F_*\phi_v \times \frac{\partial F_*}{\partial v}\phi_v\Big)\Big]. \qquad (4)\end{aligned}$$

PROOF : Let ϕ and $\bar{\phi}$ be the coordinate charts for S and $\bar{S}$ respectively, where

$$\bar{\phi} = F \circ \phi.$$

The tangent plane at a point p on S is generated by two vectors ϕ_u and ϕ_v. Since F is an isometry between S and $\bar{S}$, the differential map F_* of F is a 3×3 orthogonal matrix. Therefore F_* takes linearly independent vectors ϕ_u and ϕ_v of T_pS to $\bar{\phi}_u$ and $\bar{\phi}_v$ of $T_{F(p)}S$. Also $\vec{N}$ and $\vec{\bar{N}}$ are normals to S and $\bar{S}$ respectively.

$$\bar{\phi}_u(u,v) = F_*\phi_u = F_*(\phi(u,v))\phi_u, \qquad (5)$$

$$\bar{\phi}_v(u,v) = F_*\phi_v = F_*(\phi(u,v))\phi_v. \qquad (6)$$

Again differentiating (5) and (6) partially with respect to both u and v respectively, we get

$$\begin{cases} \bar{\phi}_{uu} & = \frac{\partial F_*}{\partial u}\phi_u + F_*\phi_{uu}, \\ \bar{\phi}_{vv} & = \frac{\partial F_*}{\partial v}\phi_v + F_*\phi_{vv}, \\ \bar{\phi}_{uv} & = \frac{\partial F_*}{\partial u}\phi_v + F_*\phi_{uv}, \\ & = \frac{\partial F_*}{\partial v}\phi_u + F_*\phi_{uv}. \end{cases} \qquad (*)$$

Now

$$F_*\phi_u \times \frac{\partial F_*}{\partial u}\phi_u = F_*\phi_u \times \Big(\frac{\partial F_*}{\partial u}\phi_u + F * \phi_{uu}\Big) - F_*(\phi_u \times \phi_{uu}) = \bar{\phi}_u \times \bar{\phi}_{uu} - F_*(\phi_u \times \phi_{uu}). \quad (7)$$

Similarly

$$\begin{cases} F_*\phi_u \times \frac{\partial F_*}{\partial u}\phi_v & = \bar{\phi}_u \times \bar{\phi}_{uv} - F_*(\phi_u \times \phi_{uv}), \\ F_*\phi_u \times \frac{\partial F_*}{\partial v}\phi_v & = \bar{\phi}_u \times \bar{\phi}_{vv} - F_*(\phi_u \times \phi_{vv}), \\ F_*\phi_v \times \frac{\partial F_*}{\partial u}\phi_u & = \bar{\phi}_v \times \bar{\phi}_{uu} - F_*(\phi_v \times \phi_{uu}), \\ F_*\phi_v \times \frac{\partial F_*}{\partial u}\phi_v & = \bar{\phi}_v \times \bar{\phi}_{uv} - F_*(\phi_v \times \phi_{uv}), \\ F_*\phi_v \times \frac{\partial F_*}{\partial v}\phi_v & = \bar{\phi}_v \times \bar{\phi}_{vv} - F_*(\phi_v \times \phi_{vv}). \end{cases} \qquad (**)$$

In view of (4), (7) and (**) we get

$$\begin{aligned}\bar{\gamma}(s) &= \lambda(s)(u'F_*\phi_u + v'F_*\phi_v) + \frac{\mu(s)}{k(s)}\Big[\{u'v'' - u''v'\}F_*\vec{N} + u'^3F_*(\phi_u \times \phi_{uu}) \\ &+2u'^2v'F_*(\phi_u \times \phi_{uv}) + u'v'^2F_*(\phi_u \times \phi_{vv}) + u'^2v'F_*(\phi_v \times \phi_{uu}) + 2u'v'^2F_*(\phi_v \times \phi_{uv}) \\ &+v'^3F_*(\phi_v \times \phi_{vv}) + u'^3\Big(F_*\phi_u \times \frac{\partial F_*}{\partial u}\phi_u\Big) + 2u'^2v'\Big(F_*\phi_u \times \frac{\partial F_*}{\partial u}\phi_v\Big) \\ &+u'v'^2\Big(F_*\phi_u \times \frac{\partial F_*}{\partial v}\phi_v\Big) + u'^2v'\Big(F_*\phi_v \times \frac{\partial F_*}{\partial u}\phi_u\Big) + 2u'v'^2\Big(F_*\phi_v \times \frac{\partial F_*}{\partial u}\phi_v\Big) \\ &+v'^3\Big(F_*\phi_v \times \frac{\partial F_*}{\partial v}\phi_v\Big)\Big],\end{aligned}$$

which can be written as

$$\begin{aligned}\bar{\gamma}(s) = \lambda(s)\Big(u'\bar{\phi}_u + v'\bar{\phi}_v\Big) + \frac{\mu(s)}{k(s)}\Big[&\{u'v'' - u''v'\}\vec{\bar{N}} + u'^3\bar{\phi}_u \times \bar{\phi}_{uu} + 2u'^2v'\bar{\phi}_u \times \bar{\phi}_{uv} \\ &+u'v'^2\bar{\phi}_u \times \bar{\phi}_{vv} + u'^2\dot{v}\bar{\phi}_v \times \bar{\phi}_{uu} + 2u'v'^2\bar{\phi}_v \times \bar{\phi}_{uv} + v'^3\bar{\phi}_v \times \bar{\phi}_{vv}\Big],\end{aligned}$$

and hence

$$\bar{\gamma}(s) = \bar{\lambda}(s)\vec{\bar{t}}(s) + \frac{\bar{\mu}(s)}{\bar{k}(s)}\vec{\bar{b}}(s),$$

for some functions $\bar{\lambda}(s)$ and $\bar{\mu}(s)$. Therefore $\bar{\gamma}(s)$ is a rectifying curve on $\bar{S}$. $\square$

Note : In the above theorem we see that the functions $\lambda(s)$ and $\bar{\lambda}(s)$ for the rectifying curves $\gamma(s)$ and $\gamma\bar{(s)}$ on S and $\bar{S}$ respectively does not change while taking an isometry on S to $\bar{S}$. Also $\frac{\bar{\mu}(s)}{\bar{k}(s)} = \frac{\mu(s)}{k(s)}$, i.e., $\mu(s)$ and $\bar{\mu}(s)$ for the rectifying curves $\gamma(s)$ and $\gamma\bar{(s)}$ respectively are related by the curvature functions $k(s)$ and $\bar{k}(s)$.

***Theorem* 4.2** — *Let F be an isometry of two smooth surfaces S and $\bar{S}$. For the rectifying curves $\gamma(s)$ and $\bar{\gamma}(s)$ on S and $\bar{S}$ respectively the component of the position vector of the rectifying curve along normal to the surface is invariant under the isometry F, i.e., $\gamma(s) \cdot \vec{N} = \bar{\gamma}(s) \cdot \vec{\bar{N}}$.*

PROOF : Since $F : S \to \bar{S}$ is an isometry and $\gamma(s)$, $\bar{\gamma}(s)$ are rectifying curves on S and $\bar{S}$ respectively, the relations (5), (6) and $(*)$ hold. Since S and $\bar{S}$ are isometric, we have

$$\mathbf{E} = \bar{\mathbf{E}}, \quad \mathbf{F} = \bar{\mathbf{F}}, \quad \mathbf{G} = \bar{\mathbf{G}}, \tag{8}$$

and hence

$$\begin{aligned}\mathbf{E} = \bar{\mathbf{E}} = \bar{\phi_u} \cdot \bar{\phi_u} &= (F_*\phi_u) \cdot (F_*\phi_u), \\ \text{i.e., } (F_*\phi_u) \cdot (F_*\phi_u) &= \phi_u \cdot \phi_u. \end{aligned} \tag{9}$$

Differentiating (9) partially with respect to u we get

$$2\Big(\frac{\partial F_*}{\partial u}\phi_u + F_*\phi_{uu}\Big) \cdot (F_*\phi_u) = 2\phi_{uu} \cdot \phi_u,$$

$$\text{i.e., } \bar{\phi_{uu}} \cdot \bar{\phi_u} = \phi_{uu} \cdot \phi_u. \tag{10}$$

Again differentiating (9) partially with respect to v we get

$$2\Big(\frac{\partial F_*}{\partial v}\phi_u + F_*\phi_{uv}\Big) \cdot (F_*\phi_u) = 2\phi_{uv} \cdot \phi_u,$$

$$\text{i.e., } \bar{\phi_{uv}} \cdot \bar{\phi_u} = \phi_{uv} \cdot \phi_u. \tag{11}$$

Again

$$\mathbf{G} = \bar{\mathbf{G}} = \bar{\phi_v} \cdot \bar{\phi_v} = (F_*\phi_v) \cdot (F_*\phi_v),$$

$$\text{i.e., } (F_*\phi_v) \cdot (F_*\phi_v) = \phi_v \cdot \phi_v. \tag{12}$$

Similarly differentiating (12) partially with respect to u and v we get

$$\bar{\phi_{uv}} \cdot \bar{\phi_v} = \phi_{uv} \cdot \phi_v, \tag{13}$$

and

$$\bar{\phi_{vv}} \cdot \bar{\phi_v} = \phi_{vv} \cdot \phi_v. \tag{14}$$

Again also

$$\mathbf{F} = \bar{\mathbf{F}} = \bar{\phi_u} \cdot \bar{\phi_v} = (F_*\phi_u) \cdot (F_*\phi_v),$$

$$\text{i.e., } (F_*\phi_u) \cdot (F_*\phi_v) = \phi_u \cdot \phi_v. \tag{15}$$

Differentiating (15) partially with respect to u we get

$$\Big(\frac{\partial F_*}{\partial u}\phi_u + F_*\phi_{uu}\Big) \cdot (F_*\phi_v) + (F_*\phi_u) \cdot \Big(\frac{\partial F_*}{\partial u}\phi_v + F_*\phi_{uv}\Big) = \phi_{uu} \cdot \phi_u + \phi_u \cdot \phi_{uv},$$

$$\text{i.e., } \bar{\phi_{uu}} \cdot \bar{\phi_v} + \bar{\phi_u} \cdot \bar{\phi_{uv}} = \phi_{uu} \cdot \phi_v + \phi_u \cdot \phi_{uv}. \tag{16}$$

Using equation (11) we can write equation (16) as

$$\bar{\phi_{uu}} \cdot \bar{\phi_v} = \phi_{uu} \cdot \phi_v. \tag{17}$$

Differentiating (17) partially with respect to v we get

$$\Big(\frac{\partial F_*}{\partial v}\phi_u + F_*\phi_{uv}\Big) \cdot (F_*\phi_v) + (F_*\phi_u) \cdot \Big(\frac{\partial F_*}{\partial v}\phi_v + F_*\phi_{vv}\Big) = \phi_{uv} \cdot \phi_v + \phi_u \cdot \phi_{vv},$$

$$\text{i.e., } \bar{\phi_{uv}} \cdot \bar{\phi_v} + \bar{\phi_u} \cdot \bar{\phi_{vv}} = \phi_{uv} \cdot \phi_v + \phi_u \cdot \phi_{vv}. \tag{18}$$

Using equation (13) we can write equation (18) as

$$\bar{\phi_{vv}} \cdot \bar{\phi_u} = \phi_{vv} \cdot \phi_u. \tag{19}$$

Equation (3) for the rectifying curve $\bar{\gamma}(s)$ can be written as

$$\begin{aligned}\bar{\gamma}(s)\cdot\vec{\bar{N}} &= \frac{\bar{\mu}(s)}{\bar{k}(s)}\Big[(u'v''-u''v)(\bar{\mathbf{E}}\bar{\mathbf{G}}-\bar{\mathbf{F}}^2)+u'^3\{\bar{\mathbf{E}}(\bar{\phi}_{uu}\cdot\bar{\phi}_v)-\bar{\mathbf{F}}(\bar{\phi}_{uu}\cdot\bar{\phi}_u)\}\\ &\quad+2u'^2v'\{\bar{\mathbf{E}}(\bar{\phi}_{uv}\cdot\bar{\phi}_v)-\bar{\mathbf{F}}(\bar{\phi}_{uv}\cdot\bar{\phi}_u)\}+u'v'^2\{\bar{\mathbf{E}}(\bar{\phi}_{vv}\cdot\bar{\phi}_v)-\bar{\mathbf{F}}(\bar{\phi}_{vv}\cdot\bar{\phi}_u)\}\\ &\quad+u'^2v'\{\bar{\mathbf{F}}(\bar{\phi}_{uu}\cdot\bar{\phi}_v)-\bar{\mathbf{G}}(\bar{\phi}_{uu}\cdot\bar{\phi}_u)\}+2u'v'^2\{\bar{\mathbf{F}}(\bar{\phi}_{uv}\cdot\bar{\phi}_v)-\bar{\mathbf{G}}(\bar{\phi}_{uv}\cdot\bar{\phi}_u)\}\\ &\quad+v'^3\{\bar{\mathbf{F}}(\bar{\phi}_{vv}\cdot\bar{\phi}_v)-\bar{\mathbf{G}}(\bar{\phi}_{vv}\cdot\bar{\phi}_u)\}\Big].\end{aligned}$$

By virtue of (8), (10), (11), (13), (14), (17) and (19), the last relation yields

$$\bar{\gamma}(s)\cdot\vec{\bar{N}}=\gamma(s)\cdot\vec{N}.$$

Therefore the component of a rectifying curve $\gamma(s)$ along normal to the surface S is invariant under the rectifying curve preserving isomerty of surfaces. □

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REFERENCES

1. A. Pressley, *Elementary differential geometry*, Springer-Verlag, (2001).
2. M. P. do Carmo, *Differential geometry of curves and surfaces*, Prentice-Hall, Inc, New Jersey, (1976).
3. B.-Y. Chen, What does the position vector of a space curve always lie in its rectifying plane?, *Amer. Math. Monthly*, **110** (2003), 147-152.
4. B.-Y. Chen and F. Dillen, Rectifying curve as centrode and extremal curve, *Bull. Inst. Math. Acad. Sinica*, **33**(2) (2005), 77-90.
5. S. Deshmukh, B.-Y. Chen, and S. H. Alshammari, On a rectifying curves in Euclidean 3-space, *Turk. J. Math.*, **42** (2018), 609-620.