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THEORETICAL DEVELOPMENTS IN THE STUDY OF PARTIAL DIFFERENTIAL EQUATIONS

K. T. Joseph and K. Sandeep

TIFR Centre for Applicable Mathematics, Bengaluru, India
e-mails: ktj@math.tifrbng.res.in; sandeep@math.tifrbng.res.in

In this article we review some of the main contributions of Indian mathematicians in the theoretical analysis of partial differential equations in the last decade.

Key words : PDE; elliptic; hyperbolic.

1. Introduction

Most of the important partial differential equations (PDEs) has its origin in physics and geometry. Applications of PDEs go in to many different areas of mathematical physics. Its connections with other branches of mathematics are many. One example is the work of Petrowsky in 1945 on support of fundamental solutions of hyperbolic operators with constant coefficients. Petrowsky gave a necessary and sufficient conditions for stable lacunas in terms of the homology of algebraic hyper surface given by the symbol of the operator. This work was corrected, clarified and generalized after more than twenty five years, by Atiyah, Bott and Garding [31, 32] in their two papers. The theory of integrable systems have applications in algebraic geometry. Application of heat equation in the proof of index theorems is well known. The spectral theory of Laplace-Beltrami operators and scattering theory for wave equations are used in the study of automorphic forms in number theory.

All the modern developments in the theory of PDEs were intimately linked to the progress in geometric measure theory, complex analysis, harmonic analysis, functional analysis, topology, and algebraic geometry. The theory of PDE is vast and diverse. Even in the case of linear PDEs, in the class of elliptic, hyperbolic, parabolic and dispersive equations some questions can be answered in a general framework but the qualitative properties and analysis of solutions differ significantly. When it comes to the nonlinear case each PDE is a world in itself. New ideas and techniques are required for progress in each case and so research in the theory of PDEs remains always as a challenging field.

During the past ten years there were tremendous progress in the field of linear and nonlinear partial differential equations. The aim of this article is to review some of the important contributions made by mathematicians in India. These include works in elliptic equations, hyperbolic equations and the dispersive equations. The main works are on well posedness, existence and regularity results and analysis of control, homogenization and inverse problems.

The numerical and computational aspects of the theory of PDEs is vey important for both theoretical and applications point of view and substantial works were done in India during the last decade, however in this article we are focussing only the theoretical aspects of PDE.

2. Elliptic PDE

The second half of 20th century witnessed a lot of developments in the study of nonlinear elliptic partial differential equations. A lot of progress have already been made in the study of linear elliptic equations. Together with various tools from nonlinear analysis like degree theory, various index theories, variational calculus, Morse theory, etc., many spectacular results were proved on semilinear elliptic equations originating from geometry, physics, biology etc. The work on Yamabe problem from differential geometry by Trudinger, Aubin and Schoen and the works of Nirenberg and Brezis on its Euclidean counterpart brought out the role of sharp inequalities in the analysis of these problems. Alexandroff's moving plane method and its adaptation to the PDE setting by Serrin lead to the discovery of many interesting qualitative properties of solutions.

Starting from the mid 80's there have been many important works by Indian mathematicians in this area which were well noticed by the international community.These contributions have been on many aspects of this theory namely the study of existence of solutions, uniqueness/multiplicity of solutions, establishing various inequalities/embeddings, qualitative properties of solutions etc. Below we will explain some of the main contributions made in the last decade.

2.1 *Sharp inequalities*

It is well known that sharp inequalities play an important role in the study of partial differential equations. Many sharp embeddings were developed by the Indian PDE community, below we will explain some of these inequalities.

It was known that the classical Hardy inequality which states for $u \in C_c^\infty(\mathbb{R}^n), n \geq 3$, $[\int_{\mathbb{R}^n}[|\nabla u|^2 - \frac{(n-2)^2}{4}\frac{u^2}{|x|^2}]\ dx\ \geq 0$ is optimal. However Brezis and Vazques [44] showed that the inequality can be improved if we restrict $u \in C_c^\infty(\Omega)$ for a bounded open set $\Omega \subset \mathbb{R}^n, n \geq 3$ and subsequently there was a conjecture of Brezis on the improvement of this inequality. In a significant work Adimurthi, Chaudhuri and Ramaswamy [1] proved this conjecture showing that the above inequality can be improved by adding infinitely many weighted L^2 norms. This work stimulated a lot of study on Hardy type inequalities. In [8] Adimurthi and Sekar established that the fundamental solution plays a crucial role in establishing Hardy type inequalities. Using this idea in [8] a Hardy type inequality where the $\int_{\mathbb{R}^n} |\nabla u|^2$ is replaced by $\int_{\mathbb{R}^n} a_{i,j}(x) u_{x_i} u_{x_j}$ for an elliptic matrix $a_{i,j}(x)$ is obtained and as a consequence sharp Hardy type inequalities were proved in general manifolds and in the Heisenberg group. Another interesting result obtained in this direction is a Hardy inequality developed by Adimurthi and Tintarev for the Dirac operator in [12].

Hardy inequality has its second order counterpart namely the Hardy Rellich inequality which states that for $u \in C_c^\infty(\mathbb{R}^n), n \geq 5$, $\int_{\mathbb{R}^n} [|\Delta u|^2 - \frac{n^2(n-4)^2}{16} \frac{u^2}{|x|^4}]\ dx \ \ \geq 0$. Motivated by the developments on the Hardy inequality one wonders whether improvements are possible when one restricts this inequality to functions which are supported in a bounded domain. The answers were already known in the work of Tertikas and his collaborators. In dimension 4 the form of this Hardy Rellich inequality was not known, in a nontrivial work [7], Adimurthi et al. solved this problem by proving this inequality in dimension 4.

Another type of embeddings where the Indian PDE community contributed significantly is the limiting case of the Sobolev embedding of the nth order Sobolev space in $\mathbb{R}^n$ namely the Moser-Trudinger and Adams inequality. The classical Moser-Trudinger inequality states that

$$\sup \left\{ \int\limits_{\Omega} e^{4\pi u^2} : u \in W_0^{1,2}(\Omega), ||\nabla u||_2 \leq 1 \right\} < \infty$$

for any bounded domain $\Omega \subset \mathbb{R}^2$. These type of inequalities are an active area of research due to its applications in PDE problems with exponential nonlinearity coming from geometry and physics. In an important piece of work [3] Adimurthi, etc., established an improved version of this inequality making precise a result of Lions. The work is known for the beautiful blow-up analysis. In another work [9] Adimurthi and Sandeep established the following singular version of this embedding $\sup\{\int\limits_{\Omega} \frac{e^{4\pi(1-\beta)u^2}}{|x|^{2\beta}} : u \in W_0^{1,2}(\Omega), ||\nabla u||_2 \leq 1\} < \infty$. The main contribution of this work was the discovery of a conformal map which maps this inequality to the classical one when one restricts to radial functions. These transformations are playing a major role in the study of cocompactness properties of problems in $\mathbb{R}^2$ with exponential nonlinearities. These embeddings have been extensively studied for functions defined on compact manifolds. However sharp embeddings were missing in noncompact manifolds. In the case of Hyperbolic space Mancini and Sandeep established the sharp Moser-Trudinger inequality [103] (see [13] for a different proof) and in fact classified the hyperbolic metric on the unit disc as an optimal case of the conformal metrics on the Euclidean unit disc for which Moser-Trudinger inequality holds. The higher order version of this embedding namely the Adams inequality in the case of Hyperbolic space was established by Karmakar and Sandeep in [90]. This result is obtained by introducing a conformally invariant norm using the geometric GJMS operator and it yielded a much sharper result comparing with some of the results proved at the same time.

One of the important question one studies with these inequalities is about the existence of extremal functions for this inequalities. This question for the singular Moser-Trudinger embedding was open for quite some time. The corresponding result for the standard Moser-Trudinger is already known and the proof uses the isoperimetric inequality directly or indirectly. The main difficulty in the singular case is that a weighted version of this isoperimetric inequality which is required was missing. In a very interesting work Csató [56] and Csató and Roy [57, 58] solved this problem in the two dimensional case.

2.2 *Qualitative properties of solutions*

The eigenvalue problems for the Laplace operator and other degenerate elliptic operators are important and very challenging. We know from classical results that among smooth bounded domains with the same measure the first eigenvalue is minimal iff the domain is a ball. In a very nice work [91] Kesavan studied similar issues for domains of the form $B_1 \setminus B_2$ where B_1 is a fixed ball and B_2 is a ball of fixed radius such that $B_2 \subset B_1$. In this work he showed that the first Dirichlet eigenvalue of Laplacian is maximum iff the balls B_1 and B_2 are concentric. The results were extended to space forms by Anisa and Aithal in [26] and to rank one symmetric spaces of non-compact type by Anisa and Vemuri in [46]. The same problem for the p-Laplace equation was studied by Anisa and Rajesh [47] and obtained some partial results. The main issue in this case is due to the lack of strong comparison principle and the problem is solved in a recent work of Anoop and his collaborators in [28]. In another important work [27] Anoop and his collaborators have shown that the second eigenfunction for the p-Laplacian with Dirichlet boundary condition can not be radial.

In [98], Lucia and Prashanth studied the simplicity and uniqueness of the positive principal eigenvalue of the weighted p-Laplacian eigenvalue problem under various assumptions on the weight. The main issue with the problem is the nonavailability of Harnack's inequality. However using a capacity argument the authors established the simplicity of the first eigenvalue.

Another major symmetry result obtained was on the positive solutions of the Hardy-Sobolev-Mazya equation. In Mancini and Sandeep [102] and Castorina *et al*. [45], showed that these equation has a hidden symmetry namely the hyperbolic symmetry which eventually resulted in the classification of extremals of Hardy-Sobolev-Mazya inequalities.

In [95] Lin and Prajapat studied the C^2 solutions of the equation $\Delta u - \mu u + k(x)u^{\frac{n+2}{n-2}}$ in the punctured unit disc under suitable assumptions on k and the solutions were shown to be asymptotically symmetric.

2.3 *Concentration phenomenon and lack of compactness*

One of the main difficulty in the analysis of elliptic problems coming from geometry and physics is their lack of compactness. One studies how the compactness fails and then uses this information to study the existence of solutions.

One of the main tool to study the lack of compactness is the blow-up analysis. However for problems in $\mathbb{R}^2$ of the form $-\Delta u = f(u)$ where f is like an exponential function, there was no effective blow-up analysis. Adimurthi and Druet developed a technique for this blow-up analysis through a linearization argument and which was further developed in [3]. Blow-up analysis was developed for the fourth order problem in [6]. In [4] Adimurthi and Grossi proved a conjecture of Ni regarding the asymptotic behaviour of positive solutions when $f(u) = u^p$ as $p \to \infty$. Most of the difficulties in these problems comes from the lack of compactness for the Moser-Trudinger embedding which is too complicated to analyse. Adimurthi and Tintarev analysed this phenomenon in [19] and showed that the weak continuity of the Moser functional on the unit ball of the Sobolev space $H_0^1(B)$, where B is the unit ball in $\mathbb{R}^2$ fails only on the translations of concentrating Moser functions, up to a remainder vanishing in the Sobolev norm. This type of analysis also known as profile decomposition was established for the space of bounded variation functions in [20] by Adimurthi and Tintarev.

In these problems with lack of compactness, the noncompact sequences generally arises as a scaled form of a fixed profile which happens to be the solution of a limiting partial differential equation. Thus one of the crucial problems in this analysis is to classify the solutions of these type of limiting PDEs. One such limiting problem coming from a problem in an astrophysics model is the Hardy-Sobolev Mazya equation. A complete classification of positive solution of this equation was established by Sandeep and his collaborators in [45, 101] and consequently the lack of compactness was analyzed in [38] by Bhakta and Sandeep.

One of the effective tool in establishing the existence of solution for this type of problems is through an approach known as finite dimensional reduction in which one constructs solutions which are small perturbations of a scaled form of the limiting profile mentioned above. This approach depends heavily on classifying the positive solutions of the limiting problem and then establishing whether these solutions are nondegenerate or not. In this direction, in [70] Ganguly and Sandeep established that the limiting solution for a semilinear elliptic problem in the hyperbolic space is degenerate and showed that the degeneracy occurs only though a finite dimensional subspace. The finite dimensional reduction approach was successfully used by Prashanth and his collaborators to establish various existence and exact multiplicity results results for the perturbed Gaussian curvature problem on S^2 (see [78]), the perturbed Q-curvature problem on S^4 (see [117]) and for the perturbed scalar curvature problem (see [116]).

In recent years there have been lot of interest in understanding the concentrating pattern of singularly perturbed equations. Lot of work has been done in the past on solutions concentrating at a single point or at a finite number of points. However very little was known on solutions concentrating on higher dimensional manifolds. Srikanth and his collaborators developed various tools to study these problems. In [65] Esposito *et al.* the information on Morse index was used to identify the concentration pattern of singularly perturbed elliptic problems in an annulus. In [126], positive solutions concentrating on a one dimensional orbit was constructed for the singularly perturbed problem $-\epsilon\Delta u+\lambda u = u^p$ as the parameter $\epsilon \to 0$ in an annulus in $\mathbb{R}^4$ with Dirichlet boundary condition. This result was extended to annulus in R^{2n} and solutions concentrating on an $n-1$ dimensional manifold in [113]. These results were proved by reducing the problem to a lower dimension and using the known tools for this reduced problem. This reduction procedure was shown to be related to the Hopf fibration and using the Hopf fibration of the complex projective space Ruf and Srikanth proved the existence of solutions concentrating on one dimensional orbits in [127]. These techniques were used further to construct solutions concentrating on orthogonal spheres in [106] by Manna and Srikanth.

2.4 *Uniqueness and multiplicity*

One of the fundamental question about any PDE is the existence and uniqueness of solution. If it is not unique one also investigates the structure of the solution set. Below we will describe some of the Indian contributions in this direction in the last decade.

Finite energy solutions of the semilinear elliptic problem $\Delta_{\mathbb{H}}u + \lambda u + u^p = 0$ in the hyperbolic space were completely analysed in [102] by Mancini and Sandeep and conditions under which the solution exists is obtained. It was also shown that the solution is unique. One of the remarkable result obtained in this work was a nonexistence phenomenon in the three dimensional hyperbolic space.

The uniqueness of positive solutions of the Dirichlet problem in the unit ball B for $-div(|\nabla u|^{p-2}\nabla u) = f(u)$ where f is an Emden-Fowler type nonlinearity is a long standing open problem. In a recent work [21] Adimurthi *et al.* proved that the solutions are unique in the class of solutions $\{u : u(0) > c\}$ for a suitable constant c.

In [96], Lin and Prajapat studied the self-dual vortex equations on a torus which arises in the relativistic abelian Chern-Simons model involving two Higgs particles and two Gauge fields. They proved the existence of maximal solutions and the invertibility of the linearized operator at the maximal solutions. They also proved the existence of a local minimizer for the associated energy functional and the existence of a second mountain pass critical point.

An analytic global unbounded branch of solution was established by Bougherara *et al.* for a singular bifurcation problem in [43]. Another important contribution in the existence of solution was the work of Bhakta and Marcus [39] for a semilinear elliptic problem with singularity where they establish the existence and uniqueness for the problem.

The bifurcation problem for the Dirichlet problem $-\Delta_n u = \lambda f(u)$ in the unit ball in $\mathbb{R}^n$ was studied in [73] by Giacomoni *et al.* and it was shown that there exists a $\lambda_0 > 0$ such that for $\lambda < \lambda_0$ the problem has two solutions, uniqueness for $\lambda = \lambda_0$ and nonexistence for $\lambda > \lambda_0$. Various multiplicity results have been proved by Sreenadh and his collaborators using the Nehari manifold approach for various equations which are active topics of research at present like nonlocal equations, semilinear problems involving $p(x)$-Laplacian, Kirchhoff equation etc., see [107, 130, 131] and the references therein. In a nice work [77] Goyal and Sreenadh studied the Fucik spectrum of nonlocal elliptic operators and showed that the lines $\mathbb{R} \times \{\lambda_1\}$ and $\{\lambda_1\} \times \mathbb{R}$ are isolated in the Fucik spectrum.

Uniqueness and asymptotic profile of least energy solutions for a critical exponent problem with Hardy potential in the unit ball was obtained in [122] by Ramaswamy and Santra.

In [99], the structure of stationary isothermic surfaces for the solution of a heat equation with initial data as the characteristic function of a domain was established.

2.5 *Systems*

There has been some interesting work on systems. In [92] Kesavan established a generalized version of the Poincare Lemma which states that any irrotational vector field in the negative order Sobolev space $H^{-1}(\Omega)$ where Ω is simply connected is the gradient of an $L^2(\Omega)$ function.

In [109], Musina and Sreenadh established the existence of non-trivial and radially symmetric solutions to the Henon-Lane-Emden system with weights.

There has been some interesting work on differential forms in open subsets of $\mathbb{R}^n$ by Saugata and his collaborators. Some of the issues studied are differential inclusion where one studies the solvability of the equation $dw \in E$ where E is a given subset consisting of $k + 1$ forms and d is the exterior derivative, the pull back equation which studies whether two differential forms are related via a pull back of one of them under a diffeomorphism and minimization of convex integrals. Saugata Bandyopadhyay has made important contributions towards these problems in [33-35] and [36].

2.6 *Inverse problems*

Inverse problems is an active area of research at present and there have been some nice works on this topic from India. Venkateswaran and his collaborators studied uniqueness and stability questions related to lower order perturbations of biharmonic and polyharmonic operators in [49, 132]. They have also derived exact inversion formulas for integral transforms arising from problems in radar, seimic and medical imaging, and tensor tomography [69]. In several instances, deriving exact inversion formulas for integral transforms may be difficult, in which case, approximate inversion methods such as microlocal inversion methods may be useful. Instead of reconstructing the function itself, one hopes to reconstruct the singularities of the function. In this context, they have made important contributions in [68, 133].

3. Hyperbolic Partial Differential Equations

First order systems of partial differential equations of the form

$$A_0(u,x,t)\partial_t u + \sum_{j=1}^{k} A_j(u,x,t)\partial_{x_j} u + B(u,x,t) = 0, \tag{1}$$

appear in many physical applications when higher order terms which account for viscosity and heat conduction effects are ignored. Here u is a function of the space variable $x \in R^k$, and the time variable $t \in R$ and taking values in open connected subset Ω of R^n and $A_j(u,x,t)$ and $B(u,x,t)$ are $n \times n$ matrices depending on (u,x,t). The compressible Euler equations in 3-space dimensions, a 5×5 system describing conservation of mass, momentum and energy is such a system with a long history which dates back to Euler [66].

One of the basic question is global in time, well-posedness of the initial value problem. The notion of hyperbolicity comes in this context. The mathematical theory of linear hyperbolic equations were started by Hadamard and developed into a beautiful theory by subsequent works of Schauder, Petrowsky, Friedrichs, John, Leray, Garding, Lax, Hormander and others. For nonlinear systems the theory is not yet well developed. Due to nonlinearity and absence of regularising effects, solutions which are initially smooth becomes discontinuous in finite time. There is no well developed theory for general multidimensional case, except for scalar equations and systems of strictly hyperbolic equations in one space variable in conservation form. Systems of conservation laws in one space variable with flux $f : \Omega \to R^n$, takes the form

$$\partial_t u + \partial_x (f(u)) = 0. \tag{2}$$

This system is called strictly hyperbolic if the Jacobian matrix $Df(u)$ has real distinct eigenvalues, indexed in increasing order, $\lambda_1(u) < \lambda_2(u) < ... < \lambda_n(u)$. They are called the characteristic speeds of the system. Rigorous mathematical theory of conservation laws started with the works of Hopf [81] on Burgers equation. It was found that the right function space to work with is BV space and solution to (2) with the initial condition $u(x,0) = u_0(x), x \in R^1$ should be formulated in the weak sense. Weak formulation leads to a severe restriction of the discontinuities through the Rankine Hugoniot conditions, but weak solutions are not unique. Physical solution is selected by imposing admissibility criteria on the solution. Here the small scale effects which was ignored in the equation (2) plays an important role.

An early example of a shock admissibility criterion in gas dynamics is that only compressive shocks are admissible. Riemann [125], in 1860, observed that this is equivalent to the requirement that shock be supersonic relative to the state at front and subsonic relative to the state on the back. Lax [94] fomulated this as a general shock condition, for genuinely nonlinear case. Liu [97] extended this to a comprehensive shock admissibility criterion which work for more general characteristic fields. Indeed there are many selection principles, all of them are not equivalent but they are related. Viscosity admissibility criteria and the Lax/Liu-condition are sufficiently powerful to give uniqueness for strictly hyperbolic systems when shocks are of moderate strength.

The work of Lax [94] on the Riemann problem and Glimm's work [76] on general initial value problem were important milestones in the development of the theory of systems of conservation laws. The contributions by Liu, Dafermos, Bressan, Serre, LeFloch and their collaborators lead to a well-posedness theory. The book of Dafermos [59] is a thorough treatise of the theory of hyperbolic conservation laws, covering different aspects with a detailed list of references. Now we highlight some of the important contributions by Indian mathematicians.

3.1 *Scalar conservation laws with discontinuous flux*

Hyperbolic equations with discontinuous flux appear in many physical situations and classical theory described before does not apply here. There are many theories and [25] gives a unified treatment of research in this topics which includes contributions from India. In a series of papers, Adimurthi and Gowda [2], Adimurthi, Misra and Gowda [5, 10, 11] and Adimurthi, Dutta, Ghoshal and Gowda [14] developed a new well-posedness theory of scalar conservation laws with discontinuous flux.The class of flux functions they consider are some what restricted because of convexity assumptions but results are sharp.

In the paper [14], total variation bound of (A, B) entropy solutions to the Cauchy problem for a single conservation law of the form

$$u_t + (H(x)f(u) + (1 - H(x))g(u))_x = 0, \;\; u(x, 0) = u_0(x),$$

where $f(u), g(u)$ are strictly convex functions and $H(x)$ the Heaviside function is studied. The main result of this paper is that the (A, B) entropy solutions are of bounded variation if A and B are not critical points of g and f respectively. If either A or B is a critical point of g or f then an example is constructed where the (A, B) entropy solution has unbounded total variation.

New structure theorem for entropy weak solutions [15] and stability results in L^p norm for $1 \leq p < \infty$ [18] are other important works done in the case of scalar conservation laws with some convexity assumptions on the flux. They also studied optimal control problem for scalar conservation laws with strictly convex flux in [17]. In another paper [16] exact controllability of entropy solution of scalar conservation laws with strictly convex flux was studied using initial or boundary data control. In these analysis, explicit formula for convex conservation laws, derived by Lax [94] and by Joseph and Gowda [83] and generalized characteristics introduced by Dafermos were crucial.

3.2 *Stochastic partial differential equations*

Conservation laws with stochastic forcing term is an active field of research at present. The method of deterministic entropy inequalities fail to capture the noise-noise interactions and the Krushkov approach cannot be directly adapted for a well posedness theory. Feng and Nualart [67] resolved this difficulty by introducing a strong in time entropy formulation. Biswas and Majee [41] gave a more appropriate weak in time and weak in space entropy formulation and developed a well posedness theory. In another paper [42] Biswas and his collaborators derived an explicit continuous dependence estimate for multidimensional stochastic balance laws driven by Levy processes and then they get error estimate for solutions for the stochastic vanishing viscosity method. In addition, they established fractional BV estimate for vanishing viscosity approximations in case the noise coefficient depends on both the solution and spatial variable.

In [71], Gawarecki, Mandrekar and Rajeev proved the existence and uniqueness of strong solutions for linear stochastic differential equations in the space dual to a multi-Hilbertian space driven by a finite-dimensional Brownian motion under relaxed assumptions on the coefficients. In another interesting paper [72], they proved a monotonicity inequality for linear stochastic partial differential equations. More recently, Rajeev and Suresh Kumar [121] proposed a new method for proving the existence and pathwise uniqueness of strong solutions of stochastic differential equations with irregular diffusion and drift coefficients without the assumption of nondegeneracy.

In [104] Manna and Mohan proved the existence and uniqueness of the strong solution of a stochastic infinite dimensional shell model of turbulence. In another work [105] Manna *et al.* studied the stochastic Navier-Stokes equations in an admissible unbounded multi-channel domain consisting of several outlets smoothly connected to a bounded domain and proved the existence and uniqueness of a strong path-wise solution.

3.3 *Nonlinear Dispersive Equations*

Nonlinear Dispersive Equations is an important class of PDEs and in the last couple of decades there have been many important works on the analysis of well-posedness for nonlinear dispersive PDEs by world's leading experts. Local and global existence of Schrodinger equation corresponding to the twisted Laplacian and in modulation spaces were studied by Ratnakumar and Sohani in [123] and Bhimani and Ratnakumar [40] respectively. Existing numerical analysis lags far behind in handling various issues related to dispersive equations, in this context one task is to fill the gap between the state of the art in the numerical analysis and the continuous PDE theory of these problems. Koley and his collaborators proved convergence of a fully discrete finite difference scheme for the Korteweg-de Vries (KdV) equation in [80], while convergence of a higher order Galerkin scheme for KdV is proved in [61]. They also analyzed operator splitting schemes for Benjamin-Ono (BO) equation. In particular they proved convergence of both Godunov and Strang splittings in [62]. Furthermore, in [63], they proved convergence of finite difference schemes for the BO equation.

3.4 *Initial boundary value problem for hyperbolic systems*

Initial boundary value problem for (2) in $x > 0, t > 0$ with initial condition $u(x, 0) = u_0(x)$, for $x > 0$ with a Dirichlet type boundary condition $u(0, t) = u_B(t), t > 0$ generally has no solution. Since the characteristic speed depends on the solution and we need to prescribe data only on the entering characteristic directions, the boundary condition has to be prescribed in a weak form $u(0, t) \in \mathcal{A}(u_B(t))$ where $\mathcal{A}(u_B(t))$ is admissible set depending on $u_B(t)$ and certain other physical features coming from the small scale physical features in the system. For scalar case, Bardos, Leroux and Nedelec [37], derived such an admissible set based on boundary entropy inequalities. For scalar conservation laws there are enough entropy-entropy flux pairs to characterize the admissible set. For systems, admissible set given by this method is in general too large to give a well-posed problem. Alternately, Gisclon and Serre [74], Gisclon [75] and Joseph and LeFloch [84, 85] formulated boundary conditions in terms of Boundary layers, for strictly hyperbolic systems with p negative and $n - p$ positive characteristic speeds. For vanishing viscosity approximation

$$u_t + f(u)_x = \epsilon(B(u)_x)_x, \;\; x > 0, t > 0$$

with positive matrix $B(u)$, the boundary layer is given by

$$B(v)v' = f(v) - f(\bar{v}), \ \ v(0) = u_B(t), \ \ v(\infty) = \bar{v}.$$

Admissible boundary data based on boundary layer is the set of all $\bar{v}$ for which this ODE problem has a solution, denoted by $\mathcal{A}(u_B(t))$. They analysed the boundary layers in vanishing viscosity limit and showed that the admissible set $\mathcal{A}(u_B(t))$ contains the point $u_B(t)$ and a manifold with dimension p and its tangent space at the point $u_B(t)$ is spanned by $r_j(u_B(t)) : j = 1, 2, ...p$ - the eigenvectors corresponding to the negative characteristic speeds. Structure of the admissible set for difference approximations, relaxation approximations and Dafermos self-similar diffusive approximations with general diffusive matrix are also studied in Joseph and LeFloch [85-87] where as the diffusive-dispersive effects in the boundary layers is analysed in Choudhury, Joseph and LeFloch [50]. The analysis shows that different regularizations give different admissible set and different solutions. These works clarify the importance of physical regularizations for the formulation of boundary conditions for hyperbolic systems. In another paper Joseph and LeFloch [88] analyzed physical viscosity and capillarity effects for self-similar solutions to the Riemann problem for a system which is not hyperbolic, that is arising in liquid-vapour phase dynamics.

3.5 *Riemann problems and interactions*

Sharma and his collaborators studied several nonlinear hyperbolic equations coming in practical problems with arbitrary data, but satisfying some natural physical conditions. They include study of the evolutionary behaviour of an unsteady three dimensional motion of a shock wave of arbitrary strength in a non-ideal gas [114], using singular surface theory and Riemann problem for isentropic magnetogasdynamics system and interaction of elementary waves [119, 120]. In the paper [118], Radha and Sharma solved the Riemann problem for the one-dimensional Euler equation governing the flow of ideal polytropic gases. Then evolution of the amplitudes of C^1 discontinuities and interaction of C^1 wave with shock waves were studied. In the papers [128] by Sharma and Radha and [114] by Pandey and Sharma, Lie group methods and similarity analysis are used to study many systems such as ideal gas equations, magnetogasdynamics equations, viscous compressible fluid, and reflection of a shock waves in a plane flow.

3.6 δ - *waves*

The multi-dimensional zero-pressure gas dynamics system is an analytical model proposed to describe the large-scale structure of the universe, see Gurbatov *et al.* [79] and the references there in. The new feature of this system is that the velocity component remains in the BV space, where as the density component is a measure, see Joseph [82]. There is no mathematical theory for this system but some progress is made recently by Albeverio and Shelkovich [22]. In the paper [48], Choudhury, Joseph and Sahoo constructed radial solutions with different behaviours at the origin. Also Joseph and Sahoo [89] analyzed the development of $\delta, \delta', \delta''$ waves for special systems using vanishing viscosity method and gave a formulation of solution and derivation of Rankine-Hugoniot conditions.

3.7 *Kinematical Conservation Laws and Applications*

The kinematical conservation laws (KCL) are equations of evolution of a moving surface Ω_t in d-dimensional (d-D) space R^d. The KCL are derived in a specially defined ray coordinate system $(\xi_1, \xi_2, \ldots, \xi_{d-1}, t)$, where $\xi_1, \xi_2, \ldots, \xi_{d-1}$ are surface coordinates on Ω_t and $t > 0$ is time. The analysis of KCL system was completed by Arun and Prasad [29] and Arun *et al*. [30]. Since the KCL constitutes a system of conservation laws, its physically valid weak solutions can contain discontinuities like shocks. The successive positions of the surface Ω_t can be obtained by mapping a solution of the KCL into the physical space, via solving the ray equations corresponding to the motion. The image of a discontinuous solution of the KCL, containing shocks, gives rise to singularities on Ω_t, known as kinks, which are points on Ω_t when Ω_t is a curve in R^2 and curves on Ω_t when it is a surface in R^3. Across a kink, geometrical quantities, such as the normal *n* to Ω_t, the metrics associated with $\xi_1, \xi_2, \ldots, \xi_{d-1}$ etc. and dynamical variables, such as an amplitude w on Ω_t may be discontinuous. On the other hand, the differential form of KCL can be shown to be equivalent to the ray equations for Ω_t as long as the solution remains smooth. The KCL is a purely geometric result and its derivation does not take into account any dynamics driving the surface. Hence, the KCL leads to an undetermined system of equations and additional closure relations are necessary to get a completely determined set of equations. One of the most important application is in the so-called weakly nonlinear ray theory, which is a powerful perturbation method to study the propagation of a small amplitude nonlinear wave front in a polytropic gas. Here, an energy transport equation involving an amplitude w, which is related to the normal velocity m of the wave front, serves the role of a closure relation.

4. Homogenization and Control Problems

Homogenization is one of the very active areas of research in the field of partial differential equations. It has applications in several applied problems including material science, porous media, thin structures, oscillatory domains etc.

Initially, Kesavan has studied the homogenization of elliptic eigenvalue problem and later he has studied several problem on the homogenization of optimal control problem and also thin plates with his collaborators and students Rajesh Mahadevan, N. Sabu and T. Muthukumar. For references and further recent developments see [93, 100, 110, 111].

Vanninathan has contributed very significantly to the theory beginning with the work on the eigenvalue problem in perforated domains. In fact, he with his collaborators developed the method of Bloch Wave analysis to study homogenization problems. For example, he has studied the first and second order correctors for elliptic PDEs with rapidly oscillating coefficients by using Bloch waves. This is the continuation of the authors earlier works. He has also studied the homogenization of a Schrodinger equation in a periodic medium with a time dependent potential. With Sivaji Ganesh, the homogenization of a periodic linear elasticity problem in three dimensions is carried out using the Bloch wave method. The authors highlight various phenomena in the Fourier space and to illustrate how Fourier techniques can be used to study homogenization problems with the help of Bloch approximation. In another significant work, Vanninathan with his collaborator has revisited the well-known homogenization results concerning elliptic second-order operators with oscillating coefficients by applying the Bloch wave approximation. In another paper, he has considered the sequence of operators in a bounded domain. The aim of the paper was to find a necessary and sufficient condition on the coefficients for the sequence of solution to be bounded in certain Sobolev spaces. He also presented a new approach to study the problem of homogenization of periodic structures. By using the Bloch wave decomposition, some classical results on homogenization are re-established. We refer [23, 54, 129] and the references there for details.

Now we describe Vanninathan's work on comparison between the two-scale asymptotic expansion method for periodic homogenization and the so-called Bloch wave method. It is well-known that the homogenized tensor coincides with the Hessian matrix of the first Bloch eigenvalue when the Bloch parameter vanishes. In the context of the two-scale asymptotic expansion method, there is the notion of high order homogenized equation where the homogenized equation can be improved by adding small additional higher order differential terms. The next non-zero high order term is a fourth-order term, accounting for dispersion effects. This homogenized fourth-order tensor is not equal to the fourth-order tensor arising in the Taylor expansion of the first Bloch eigenvalue, which is often called Burnett tensor. In [24], Allaire, Briane and Vanninathan establish an exact relation between the homogenized fourth-order tensor and the Burnett fourth-order tensor. For the special case of a simple laminate they prove that the homogenized fourth-order tensor may change sign. In the elliptic case they explain the difference between the homogenized and Burnett fourth-order tensors by a difference in the source term which features an additional corrector term. Finally, for the wave equation, the two fourth-order tensors coincide again, so dispersion is unambiguously defined, and only the source terms differ as in the elliptic case. In an earlier paper Conca, Orive and Vanninathan [55] studied the case where the Burnett fourth-order tensor has a sign.

Nandakumaran has studied homogenization of Stokes problems and eigenvalue problems in perforated domains. His main tool was two-scale convergence. Later with Rajesh [112], he has studied few nonlinear parabolic problems with nonlinearity in the coefficient of time derivative as well. They have also studied homogenization in the set-up of viscosity solutions. In [110] with T. Muthukumar, he has studied certain low cost control problems, later in a significant paper [100] Mahadevan and Muthkumar answered the question which could not be carried out earlier. Now for the last several years, he with his students are studying problems in domains with rapidly oscillating and there are many articles in this direction. This type of domains appears in several applications. They have some significant contribution to the optimal control problems in oscillating domains. They use the method of unfolding to study such problems [111]. In a novel approach, recently they have characterized optimal controls via the unfolding operators which is quite new.

Now we look at some important contributions in control problem. We start with the work of Chowdhury, Ramaswamy and Raymond [53], who studied the one-dimensional compressible Navier-Stokes system linearized about a constant steady state. The null controllability for regular initial data by an interior control acting everywhere in the velocity equation was established. This result was proved to be sharp by showing that the null controllability cannot be achieved by a localized interior control or by a boundary control acting only in the velocity equation. The system was proved to be approximately controllable, but not stabilizable with a decay rate greater than the value of an accumulation point of the real eigenvalues of the linearized operator. In [51], Chowdhury and Ramaswamy study optimal control problems for the two-dimensional unsteady linearized compressible Navier-Stokes equations in a rectangle. The control acts through a Dirichlet boundary condition. The authors first study the existence and uniqueness of the solution for the two-dimensional linearized compressible Navier-Stokes equations in a rectangle with nonhomogeneous Dirichlet boundary data, not necessarily smooth, by the transposition method. They next prove the existence and uniqueness of optimal controls. Finally, they derive first-order necessary and sufficient optimality conditions. The paper of Chowdhury, Maity, Ramaswamy and Raymond [52] provides a first result on the problem of finding a feedback control law that stabilizes compressible fluid flows, and proposes a localized distributed control, acting only in the velocity equation of the compressible Navier-Stokes system. In the paper of Mitra, Ramaswamy and Raymond [108], the compressible Navier-Stokes equations are studied in a bounded interval with an internal square-integrable control is considered acting on the velocity equation.

In the paper [111], Nandakumaran, Prakash and Sardar were concerned with an optimal control problem for the Laplace equation posed in a two-dimensional domain having an oscillating boundary. Periodic controls in thin periodic slabs of period $\epsilon > 0$ are considered. It was proved that for each $\epsilon > 0$, the optimal control problems considered are well-posed. Secondly, optimal controls were characterized via the unfolding operator method. Finally, a convergence analysis, as the parameter ϵ goes to 0 was carried out. In [60], Nandakumaran and his collaborators studied exact internal controllability for the wave equation in a domain with oscillating boundary with Neumann boundary condition.

In [64], Ervedoza and Vanninathan studied controllability of a simplified model of fluid-structure interaction. In a significant paper [124], Raymond and Vanninathan, studied null controllability in a fluid-solid structure model, coupling the Stokes equations in a two-dimensional domain with a system of ordinary differential equations corresponding to a finite-dimensional approximation of equations modelling deformations of an elastic body or vibrations of a rigid body. A null controllability result in time $T > 0$ was obtained by means of a distributed control acting only in the fluid equation and located in relatively compact subset of the domain. The proof is based on a Carleman estimate for the related adjoint system.

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