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DYNAMICS OF THE NONLINEAR RATIONAL DIFFERENCE EQUATION $x_{n+1} = \frac{Ax_{n-\alpha}x_{n-\beta} + Bx_{n-\gamma}}{Cx_{n-\alpha}x_{n-\beta} + Dx_{n-\gamma}}$

Abdualrazaq Sanbo[*,***], E. M. Elsayed[*,**] and Faris Alzahrani[*]

[*]*Mathematics Department, Faculty of Science, King Abdulaziz University, P. O. Box* 80203, *Jeddah* 21589, *Saudi Arabia*

[**]*Department of Mathematics, Faculty of Science, Mansoura University, Mansoura* 35516, *Egypt*

[***]*General Studies Department, Jeddah College of Telecom and Electronics, TVTC, B.P.* 2816, *Jeddah* 21461, *Saudi Arabia*

e-mails: sanbo@cte.edu.sa; emmelsayed@yahoo.com; faris.kau@hotmail.com

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In this article, we study the global stability and the asymptotic properties of the non-negative solutions of the non-linear difference equation:

$$x_{n+1} = \frac{Ax_{n-\alpha}x_{n-\beta} + Bx_{n-\gamma}}{Cx_{n-\alpha}x_{n-\beta} + Dx_{n-\gamma}}, \quad n = 0, 1, ...,$$

where α, β, γ are positive integers, A, B, C, D are positive real numbers and the initial conditions $x_{-p}, x_{-p+1}, ..., x_{-1}, x_0$ for $p = \max\{\alpha, \beta, \gamma\}$ are arbitrary positive real numbers.

Key words : Difference equations; recursive sequences; local stability; global stability; boundedness; prime period two solution.

1. Introduction

The qualitative study of difference equations is a fertile research area and increasingly attracts many mathematicians. This topic draws its importance from the fact that many real life phenomena are modeled using difference equations. Examples from economy, biology, etc. can be found in [6, 9, 10, 24, 29].

It is known that nonlinear difference equations are capable of producing a complicated behavior regardless its order. There has been a great interest in studying the global attractivity, the boundedness character and the periodicity nature of nonlinear difference equations. For example, in the articles [6, 9, 10, 25, 28] closely related global convergence results were obtained which can be applied to nonlinear difference equations in proving that every solution of these equations conver to a period two solution. The study of these equations is challenging and rewarding and is still in its infancy. We believe that the nonlinear rational difference equations are of paramount importance in their own right. Furthermore the results about such equations offer prototypes for the development of the basic theory of the global behavior of nonlinear difference equations.

Elabbasy *et al*. [7, 8] investigated the global stability and the periodicity of the solutions of the recursive sequence

$$x_{n+1}=\frac{\alpha x_n+\beta x_{n-1}+\gamma x_{n-2}}{Ax_n+Bx_{n-1}+Cx_{n-2}} \quad \text{and} \quad x_{n+1}=\frac{\alpha x_{n-l}+\beta x_{n-k}}{Ax_{n-l}+Bx_{n-k}}$$

Zayed [38] studied the global stability and the asymptotic properties of the nonnegative solutions of the nonlinear difference equation

$$x_{n+1}=Ax_n+Bx_{n-k}+\frac{px_n+x_{n-k}}{q+x_{n-k}}$$

Our aim in this paper is to investigate the behavior of the solution of the following nonlinear difference equation

$$x_{n+1}=\frac{Ax_{n-\alpha}x_{n-\beta}+Bx_{n-\gamma}}{Cx_{n-\alpha}x_{n-\beta}+Dx_{n-\gamma}}, \quad n=0,1,..., \tag{1}$$

where A, B, C, D are positive real numbers, α, β, γ are positive integers and the initial conditions $x_{-p}, x_{-p+1}, ..., x_{-1}, x_0$ for $p=\max\{\alpha,\beta,\gamma\}$ are arbitrary positive real numbers.

Here, we recall some notations and results which will be useful in our investigation. Let I be some interval of real numbers and let

$$f: I^{k+1}\rightarrow I,$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, x_{-k+2}, ...,$ $x_0\in I$, the difference equation

$$x_{n+1}=f(x_n, x_{n-1}, ..., x_{n-k}), \quad n=0,1,..., \tag{2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 1 — The difference equation (2) is said to be persistence if there exist numbers m and M with $0<m\leq M<\infty$ such that for any initial conditions $x_{-k}, x_{-k+1}, ..., x_0\in(0,\infty)$ there exists a positive integer N, which depends on the initial conditions, such that

$$m \leq x_n \leq M \quad \text{for all} \quad n \geq N.$$

Definition 2 — (Equilibrium Point). A point $\bar{x} \in I$ is called an equilibrium point of Eq. (2) if $\bar{x} = f(\bar{x}, \bar{x}, ..., \bar{x})$. That is, $x_n = \bar{x}$ for $n \geq 0$, is a solution of Eq. (2), or equivalently, $\bar{x}$ is a fixed point of f.

Definition 3 — (Stability).

- The equilibrium point $\bar{x}$ of Eq. (2) is locally stable if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, x_{-k+2}, ..., x_0 \in I$, with

$$|x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + |x_{-k+2} - \bar{x}| + ... + |x_0 - \bar{x}| < \delta,$$

  we have $|x_n - \bar{x}| < \varepsilon$, for all $n \geq -k$.

- The equilibrium point $\bar{x}$ of Eq. (2) is locally asymptotically stable if $\bar{x}$ is locally stable solution of Eq. (2) and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, x_{-k+2}, ..., x_0 \in I$, with

$$|x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + |x_{-k+2} - \bar{x}| + ... + |x_0 - \bar{x}| < \delta,$$

  we have $\lim_{n\to\infty} x_n = \bar{x}$.

- The equilibrium point $\bar{x}$ of Eq. (2) is global attractor if for all $x_{-k}, x_{-k+1}, ..., x_0 \in I$ we have

$$\lim_{n\to\infty} x_n = \bar{x}.$$

- The equilibrium point $\bar{x}$ of Eq. (2) is globally asymptotically stable if $\bar{x}$ is locally stable, and $\bar{x}$ is also a global attractor of Eq. (2).

- The equilibrium point $\bar{x}$ of Eq. (2) is unstable if $\bar{x}$ is not locally stable.

The linearized equation of Eq. (2) about the equilibrium $\bar{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f(\bar{x}, \bar{x}, ..., \bar{x})}{\partial x_{n-i}} y_{n-i}$$

***Theorem* 1** — *Assume that $p, q \in \mathbb{R}$ and $k \in \{0, 1, 2, ...\}$. Then $|p| + |q| < 1$ is a sufficient condition for the asymptotic stability of the difference equation*

$$x_{n+1} + px_n + qx_{n-k} = 0, n = 0, 1, ...$$

Remark 1 : The theorem can be easily extended to a general linear equations of the form

$$x_{n+k} + p_1 x_{n+k-1} + ... + p_k x_n = 0, \quad n = 0, 1, ..., \tag{3}$$

where $p_1, p_2, ..., p_k \in \mathbb{R}$ and $k \in \{0, 1, 2, ...\}$. Then Eq. (3) is asymptotically stable provided that $\sum_{i=0}^{k} |p_i| < 1$.

***Theorem* 2** — *Let $f \in C[I^{k+1}, I]$ for some interval I of the real numbers and for some non-negative integer k, and consider the difference equation (2). Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of Eq.(2), and suppose that there exist constants $h \in I$ and $H \in I$ such that*

$$h \le x_n \le H \quad \text{for all} \quad n \ge -k.$$

Let l_0 be a limit point of the sequence $\{x_n\}_{n=-k}^{\infty}$. Then the following statements are true.

(i) *There exists a solution $\{L_n\}_{n=-\infty}^{\infty}$ of Eq. (2), called a full limiting sequence of $\{x_n\}_{n=-k}^{\infty}$, such that $L_0 = l_0$, and such that for every $N \in \{..., -1, 0, 1, ...\}$, L_N is a limit point of $\{x_n\}_{n=-k}^{\infty}$.*

(ii) For every $i_0 \le -k$, there exists a subsequence $\{x_{r_i}\}_{i=0}^{\infty}$ of $\{x_n\}_{n=-k}^{\infty}$ such that

$$L_N = \lim_{n\to\infty} x_{r_i+N} \quad \text{for every} \quad N \ge -i_0.$$

2. Local Stability of Eq. (1)

In this section we investigate the local stability character of the solutions of Eq. (1). If Eq. (1) admits an equilibrium point $\bar{x}$ then $\bar{x} > 0$ and

$$\bar{x} = \frac{A\bar{x}^2 + B\bar{x}}{C\bar{x}^2 + D\bar{x}} = \frac{A\bar{x} + B}{C\bar{x} + D},$$

or

$$\bar{x}(C\bar{x} + D) = A\bar{x} + B,$$

or also

$$C\bar{x}^2 + (D - A)\bar{x} - B = 0,$$

Let $\Delta = (D - A)^2 + 4BC$. Then Eq. (1) admits a unique positive equilibrium point given by

$$\bar{x} = \frac{A - D + \sqrt{(D - A)^2 + 4BC}}{2C}. \tag{4}$$

Define the following function

$$\begin{aligned} &f : (0,\infty)^3 \to (0,\infty) \\ &f(u,v,w) = \frac{Auv + Bw}{Cuv + Dw}. \end{aligned} \tag{5}$$

Therefore it follows that

$$f_u(u,v,w) = \frac{(AD - BC)vw}{(Cuv + Dw)^2}, \quad f_v(u,v,w) = \frac{(AD - BC)uw}{(Cuv + Dw)^2}, \quad f_w(u,v,w) = -\frac{(AD - BC)uv}{(Cuv + Dw)^2}.$$

Then

$$f_u(\bar{x},\bar{x},\bar{x}) = \frac{(AD - BC)}{(C\bar{x} + D)^2}, \quad f_v(\bar{x},\bar{x},\bar{x}) = \frac{(AD - BC)}{(C\bar{x} + D)^2}, \quad f_w(\bar{x},\bar{x},\bar{x}) = -\frac{(AD - BC)}{(C\bar{x} + D)^2}.$$

Let $a = \dfrac{(AD - BC)}{(C\bar{x} + D)^2}$ then the linearized equation of Eq. (1) about $\bar{x}$ is

$$y_{n+1} - ay_{n-\alpha} - ay_{n-\beta} + ay_{n-\gamma} = 0. \tag{6}$$

whose characteristic equation is given by

$$\lambda^{\gamma+1} - a\lambda^{\gamma-\alpha} - a\lambda^{\gamma-\beta} + a = 0.$$

***Theorem* 3** — *Assume that*

$$|AD - BC| < \frac{(C\bar{x} + D)^2}{3}.$$

Then the equilibrium point of Eq. (1) is locally asymptotically stable.

PROOF : It follows from Theorem 1 that Eq. (6) is asymptotically stable if

$$|a| + |a| + |a| = \left|\frac{(AD - BC)}{(C\bar{x} + D)^2}\right| + \left|\frac{(AD - BC)}{(C\bar{x} + D)^2}\right| + \left|\frac{(AD - BC)}{(C\bar{x} + D)^2}\right| = 3\frac{|AD - BC|}{(C\bar{x} + D)^2} < 1,$$

and so,

$$|AD - BC| < \frac{(C\bar{x} + D)^2}{3}.$$

The proof is complete.

3. GLOBAL ATTRACTOR OF THE EQUILIBRIUM POINT OF EQ. (1)

In this section we investigate the global attractivity character of solutions of Eq. (1).

***Theorem* 4** — *The equilibrium point $\bar{x}$ of Eq. (1) is global attractor if $D \geq A$.*

PROOF : Let $\{x_n\}_{n=-p}^{\infty}$ be a solution of Eq. (1) and again let f be the function defined by Eq. (5), then we can easily see that the function $f(u,v,w)$ is increasing in u, v and decreasing in w. Thus from Eq. (1), we see that

$$x_{n+1} = \frac{Ax_{n-\alpha}x_{n-\beta} + Bx_{n-\gamma}}{Cx_{n-\alpha}x_{n-\beta} + Dx_{n-\gamma}} \leq \frac{Ax_{n-\alpha}x_{n-\beta} + B \times 0}{Cx_{n-\alpha}x_{n-\beta} + D \times 0} = \frac{A}{C}.$$

Then

$$x_n \leq \frac{A}{C} = H \quad \text{for all} \quad n \geq 1.$$

$$x_{n+1} = \frac{Ax_{n-\alpha}x_{n-\beta} + Bx_{n-\gamma}}{Cx_{n-\alpha}x_{n-\beta} + Dx_{n-\gamma}} \geq \frac{A \times 0 + Bx_{n-\gamma}}{C \times 0 + Dx_{n-\gamma}} = \frac{B}{D}.$$

Then

$$x_n \geq \frac{B}{D} = h \quad \text{for all} \quad n \geq 1.$$

We see therefore that

$$h = \frac{B}{D} \leq x_n \leq \frac{A}{C} = H \quad \text{for all} \quad n \geq 1.$$

It follows by the method of Full Limiting sequences that there exist solutions $\{I_n\}_{n=-\infty}^{\infty}$ and $\{S_n\}_{n=-\infty}^{\infty}$ of Eq.(1) with

$$I = I_0 = \lim_{n\to\infty} \inf x_n \leq \lim_{n\to\infty} \sup x_n = S_0 = S,$$

where

$$I_n, S_n \in [I, S], \quad n = 0, -1, ...$$

It suffices to show that $I = S$.

Now it follows from Eq.(1) that

$$I = \frac{AI_{n-\alpha-1}I_{n-\beta-1} + BI_{n-\gamma-1}}{CI_{n-\alpha-1}I_{n-\beta-1} + DI_{n-\gamma-1}} \leq \frac{AS^2 + BI}{CS^2 + DI},$$

and so

$$AS^2 + BI - DI^2 \geq CS^2 I \quad \to \quad AS^2 I + BI^2 - DI^3 \geq CS^2 I^2.$$

Similarly, it follows from Eq.(1) that

$$S = \frac{AS_{n-\alpha-1}S_{n-\beta-1} + BS_{n-\gamma-1}}{CS_{n-\alpha-1}S_{n-\beta-1} + DS_{n-\gamma-1}} \geq \frac{AI^2 + BS}{CI^2 + DS},$$

and so

$$AI^2 + BS - DS^2 \leq CI^2 S \quad \to \quad AI^2 S + BS^2 - DS^3 \leq CI^2 S^2.$$

It follows that

$$AI^2S + BS^2 - DS^3 \leq CI^2S^2 \leq AS^2I + BI^2 - DI^3.$$

or also

$$AS^2I+BI^2-DI^3-AI^2S-BS^2+DS^3 = (S-I)\left[(A-D)SI - (BS + BI + DS^2 + DI^2)\right] \geq 0.$$

Hence

$$I \geq S \quad \text{if} \quad (A-D)SI - (BS + BI + DS^2 + DI^2) < 0.$$

Now, we know that $D \geq A$ and so it follows that $I \geq S$. Therefore $I = S$. This completes the proof.

For confirming the local and global stability results, we consider two numerical examples. For $A = 1, B = 1, C = 1, D = 4, \alpha = 3, \beta = 1, \gamma = 2, x_{-3} = 1, x_{-2} = 2, x_{-1} = 3, x_0 = 1$ which satisfy the stability conditions then Eq.(1) admits an equilibrium point $\bar{x} = 0.30278$ which is a global attractor of Eq.(1) (See Figure 1, left). For $A = 2, B = 5, C = 5, D = 3, \alpha = 1, \beta = 2, \gamma = 3, x_{-3} = 1, x_{-2} = 2, x_{-1} = 3, x_0 = 1$ which satisfy also the stability conditions then Eq.(1) admits an equilibrium point $\bar{x} = 0.9050$ which is a global attractor of Eq.(1) (See Figure 1, right).

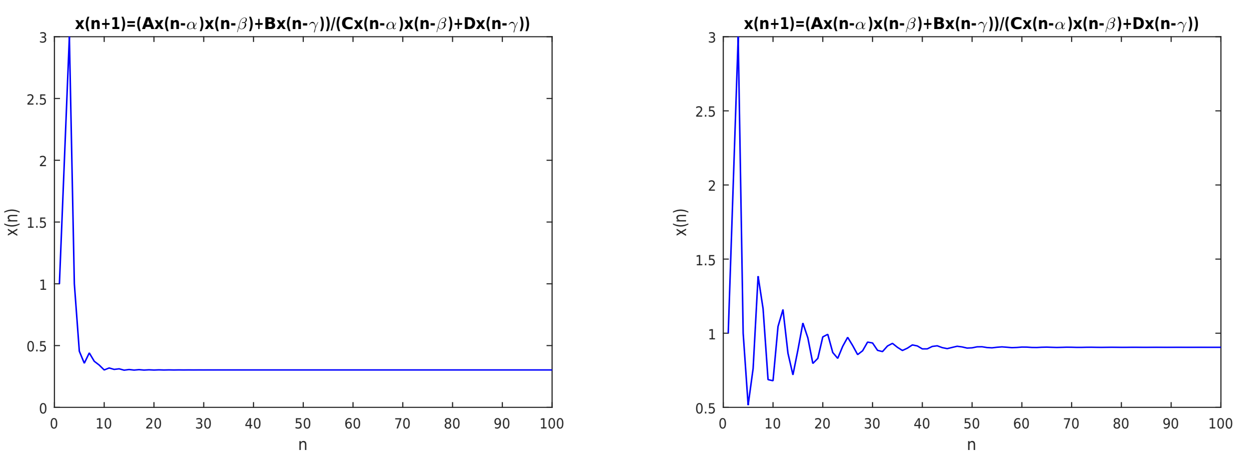

Figure 1: Right $A = 1, B = 1, C = 1, D = 4, \alpha = 3, \beta = 1, \gamma = 2, x_{-3} = 1, x_{-2} = 2, x_{-1} = 3, x_0 = 1$ and left $A = 2, B = 5, C = 5, D = 3, \alpha = 1, \beta = 2, \gamma = 3, x_{-3} = 1, x_{-2} = 2, x_{-1} = 3, x_0 = 1$.

4. Boundedness of Solutions of Eq. (1)

In this section we study the boundedness of solutions of Eq. (1).

***Theorem* 5** — *Every solution of Eq. (1) is bounded and persists.*

PROOF : Let $\{x_n\}_{n=-p}^{\infty}$ be a solution of Eq. (1). It follows from Eq. (1) that

$$\begin{aligned} x_{n+1} &= \frac{Ax_{n-\alpha}x_{n-\beta} + Bx_{n-\gamma}}{Cx_{n-\alpha}x_{n-\beta} + Dx_{n-\gamma}} \\ &= \frac{Ax_{n-\alpha}x_{n-\beta}}{Cx_{n-\alpha}x_{n-\beta} + Dx_{n-\gamma}} + \frac{Bx_{n-\gamma}}{Cx_{n-\alpha}x_{n-\beta} + Dx_{n-\gamma}} \\ &\leq \frac{Ax_{n-\alpha}x_{n-\beta}}{Cx_{n-\alpha}x_{n-\beta}} + \frac{Bx_{n-\gamma}}{Dx_{n-\gamma}} = \left(\frac{A}{C} + \frac{B}{D}\right) = M \end{aligned}$$

Hence

$$x_n \leq \left(\frac{A}{C} + \frac{B}{D}\right) = M \quad \text{for all} \quad n \geq 1. \tag{7}$$

Now we wish to show that there exists $m > 0$ such that

$$x_n \geq m \quad \text{for all} \quad n \geq 1.$$

The change of variables, $y_n = \dfrac{1}{x_n}$, gives Eq. (1) in the form

$$\frac{1}{y_{n+1}} = \frac{\dfrac{A}{y_{n-\alpha}y_{n-\beta}} + \dfrac{B}{y_{n-\gamma}}}{\dfrac{C}{y_{n-\alpha}y_{n-\beta}} + \dfrac{D}{y_{n-\gamma}}} = \frac{Ay_{n-\gamma} + By_{n-\alpha}y_{n-\beta}}{Cy_{n-\gamma} + Dy_{n-\alpha}y_{n-\beta}},$$

or in the equivalent form

$$\begin{aligned} y_{n+1} &= \frac{Cy_{n-\gamma} + Dy_{n-\alpha}y_{n-\beta}}{Ay_{n-\gamma} + By_{n-\alpha}y_{n-\beta}} \\ &= \frac{Cy_{n-\gamma}}{Ay_{n-\gamma} + By_{n-\alpha}y_{n-\beta}} + \frac{Dy_{n-\alpha}y_{n-\beta}}{Ay_{n-\gamma} + By_{n-\alpha}y_{n-\beta}} \\ &\leq \frac{Cy_{n-\gamma}}{Ay_{n-\gamma}} + \frac{Dy_{n-\alpha}y_{n-\beta}}{By_{n-\alpha}y_{n-\beta}} = \left(\frac{C}{A} + \frac{D}{B}\right) \end{aligned}$$

Thus we obtain

$$x_n = \frac{1}{y_n} \geq \frac{1}{\left(\dfrac{C}{A} + \dfrac{D}{B}\right)} = \frac{AB}{BC + AD} = m \quad \text{for all} \quad n \geq 1. \tag{8}$$

One deduces from (7) and (8) that

$$m \leq x_n \leq M \quad \text{for all} \quad n \geq 1.$$

Therefore every solution of Eq.(1) is bounded and persists.

For confirming the boundedness results, we consider the following two numerical examples. For $A = 2, B = 1, C = 1, D = 4, \alpha = 2, \beta = 2, \gamma = 3, x_{-3} = 10, x_{-2} = 2, x_{-1} = 3, x_0 = 4$ and $A = 2, B = 5, C = 5, D = 3, \alpha = 1, \beta = 2, \gamma = 3, x_{-3} = 1, x_{-2} = 2, x_{-1} = 3, x_0 = 1$. The solution is bounded and persists (See Figure 2).

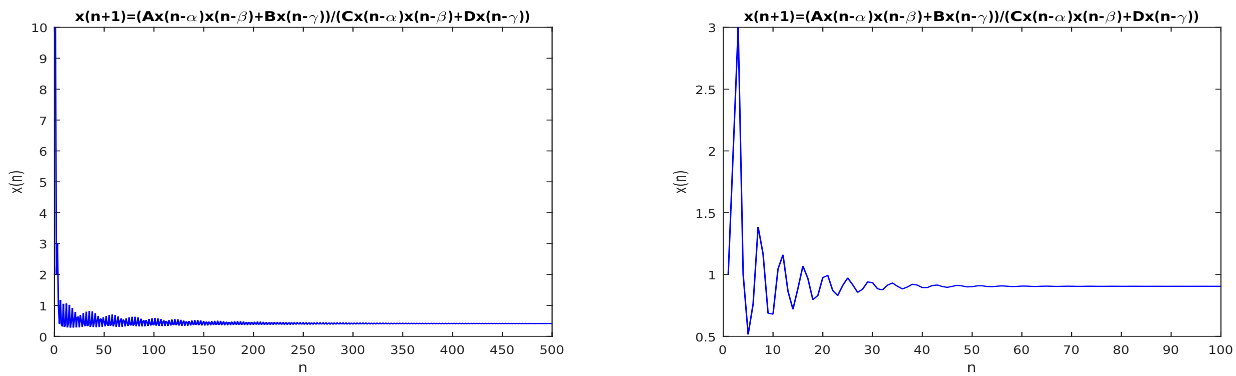

Figure 2: Right $A = 2, B = 1, C = 1, D = 4, \alpha = 2, \beta = 2, \gamma = 3, x_{-3} = 10, x_{-2} = 2, x_{-1} = 3, x_0 = 4$ and left $A = 2, B = 5, C = 5, D = 3, \alpha = 1, \beta = 2, \gamma = 3, x_{-3} = 1, x_{-2} = 2, x_{-1} = 3, x_0 = 1$.

5. Periodicity of Solutions of Eq. (1)

In this section we study the existence of prime period two solutions of Eq. (1).

***Theorem* 6** — *For $A = D$, Eq. (1) has prime period two solutions if and only if $BC > 4A^2$, γ is odd, α and β are even.*

Proof : First suppose that there exists a prime period two solutions

$$..., p, q, p, q, ...$$

of Eq.(1). We see from Eq.(1) that

$$p = \frac{Aq^2 + Bp}{Cq^2 + Dp}$$

and

$$q = \frac{Ap^2 + Bq}{Cp^2 + Dq}.$$

Hence

$$Cq^2p + Dp^2 = Aq^2 + Bp \tag{9}$$

and

$$Cqp^2 + Dq^2 = Ap^2 + Bq. \tag{10}$$

Subtracting Eq.(10) from Eq.(9) gives

$$Cqp(q-p) + D(p^2 - q^2) = A(q^2 - p^2) + B(p-q).$$

Since $p \neq q$, it follows that

$$Cqp + D(p+q) = A(q+p) + B. \tag{11}$$

Again adding Eq.(9) and Eq.(10) yields

$$Cqp(q+p) + D(p^2 + q^2) = A(q^2 + p^2) + B(p+q).$$

As $p + q \neq 0$ then

$$Cqp + D\frac{(p^2+q^2)}{p+q} = A\frac{(p^2+q^2)}{p+q} + B. \tag{12}$$

Subtracting Eq.(12) from Eq.(11) gives

$$D\left(\frac{(p^2+q^2)}{p+q} - (p+q)\right) = A\left(\frac{(p^2+q^2)}{p+q} - (p+q)\right).$$

Since $\frac{(p^2+q^2)}{p+q} \neq (p+q)$ then $A = D$. From Eq.(11), we deduce that

$$qp = \frac{B}{C}.$$

From Eq.(9), we deduce that

$$Bq + Dp^2 = Aq^2 + Bp \tag{13}$$

and from Eq. (10), we deduce that

$$Bp + Dq^2 = Ap^2 + Bq. \tag{14}$$

Subtracting Eq.(14) from Eq.(13) we obtain

$$2A(p^2 - q^2) + 2B(q-p) = 0.$$

Since $p \neq q$, then

$$p + q = \frac{B}{A}.$$

Then p and q are positive solutions of equation

$$t^2 - \frac{B}{A}t + \frac{B}{C} = 0. \tag{15}$$

This means that

$$\Delta = \frac{B^2}{A^2} - 4\frac{B}{C} > 0$$

and then

$$\frac{B}{A^2} > \frac{4}{C} \quad \rightarrow \quad BC > 4A^2$$

Conversely suppose that condition (i) is true. We will show that Eq.(1) has a prime period two solution. Assume that

$$p = \frac{\frac{B}{A} - \sqrt{\frac{B^2}{A^2} - 4\frac{B}{C}}}{2} = \frac{B}{2A}\Big(1 - \sqrt{1 - \frac{4A^2}{BC}}\Big)$$

and

$$q = \frac{\frac{B}{A} + \sqrt{\frac{B^2}{A^2} - 4\frac{B}{C}}}{2} = \frac{B}{2A}\Big(1 + \sqrt{1 - \frac{4A^2}{BC}}\Big)$$

We see from condition (i) that $1 - \dfrac{4A^2}{BC} > 0$, therefore p and q are distinct positive real numbers. Set

$$x_{-\alpha} = q, x_{-\gamma} = p, x_{-\beta} = q, ..., \quad \text{and} \quad x_0 = p.$$

We wish to show that

$$x_1 = x_{-1} = q \quad \text{and} \quad x_2 = x_0 = p.$$

It follows from Eq.(1) that

$$x_1 = \frac{Ax_{-\alpha}x_{-\beta} + Bx_{-\gamma}}{Cx_{-\alpha}x_{-\beta} + Dx_{-\gamma}} = \frac{Ap^2 + Bq}{Cp^2 + Aq} = \frac{A\left(\frac{B}{2A}\Big(1 - \sqrt{1 - \frac{4A^2}{BC}}\Big)\right)^2 + B\frac{B}{2A}\Big(1 + \sqrt{1 - \frac{4A^2}{BC}}\Big)}{C\left(\frac{B}{2A}\Big(1 - \sqrt{1 - \frac{4A^2}{BC}}\Big)\right)^2 + A\frac{B}{2A}\Big(1 + \sqrt{1 - \frac{4A^2}{BC}}\Big)}$$

$$= \frac{B}{A}\frac{\left(1 - \sqrt{1 - \frac{4A^2}{BC}}\right)^2 + 2\Big(1 + \sqrt{1 - \frac{4A^2}{BC}}\Big)}{\frac{BC}{A^2}\left(1 - \sqrt{1 - \frac{4A^2}{BC}}\right)^2 + 2\Big(1 + \sqrt{1 - \frac{4A^2}{BC}}\Big)}$$

$$= \frac{B}{A}\frac{4-\dfrac{4A^2}{BC}}{\dfrac{BC}{A^2}\left(2-\dfrac{4A^2}{BC}-2\sqrt{1-\dfrac{4A^2}{BC}}\right)+2\left(1+\sqrt{1-\dfrac{4A^2}{BC}}\right)}$$

$$= \frac{4(\dfrac{B}{A}-\dfrac{A}{C})}{\dfrac{2BC}{A^2}-4-\dfrac{2BC}{A^2}\sqrt{1-\dfrac{4A^2}{BC}}+2+2\sqrt{1-\dfrac{4A^2}{BC}}}$$

$$= \frac{4(\dfrac{B}{A}-\dfrac{A}{C})}{\left(\dfrac{2BC}{A^2}-2\right)\left(1-\sqrt{1-\dfrac{4A^2}{BC}}\right)}$$

Multiplying the denominator and numerator by $\left(1+\sqrt{1-\dfrac{4A^2}{BC}}\right)=\dfrac{2Aq}{B}$ gives

$$x_1 = \frac{4(\dfrac{B}{A}-\dfrac{A}{C})\left(1+\sqrt{1-\dfrac{4A^2}{BC}}\right)}{\left(\dfrac{2BC}{A^2}-2\right)\dfrac{4A^2}{BC}} = \frac{4(\dfrac{B}{A}-\dfrac{A}{C})\dfrac{2Aq}{B}}{\left(\dfrac{2BC}{A^2}-2\right)\dfrac{4A^2}{BC}} = \frac{\left(1-\dfrac{A^2}{BC}\right)}{\left(1-\dfrac{A^2}{BC}\right)}q = q.$$

Similarly as before one can easly show that

$$x_2 = p.$$

Then it follows by induction that

$$x_{2n} = p \quad \text{and} \quad x_{2n+1} = q \quad \text{for all} \quad n \geq -1.$$

Thus Eq.(1) has the positive prime period two solution

$$..., p, q, p, q, ...$$

where p and q are the distinct roots of the quadratic equation (15) and the proof is complete.

For confirming the periodicity results, we consider the following two numerical examples. For $A = 6, B = 30, C = 5, D = 6, \alpha = 4, \beta = 2, \gamma = 1, x_{-4} = 2, x_{-3} = 3, x_{-2} = 2, x_{-1} = 3, x_0 = 2$ which satisfy the periodicity conditions then Eq.(1) has positive prime period two solutions $..., 2, 3, 2, 3, ...$ (See Figure 3, right).

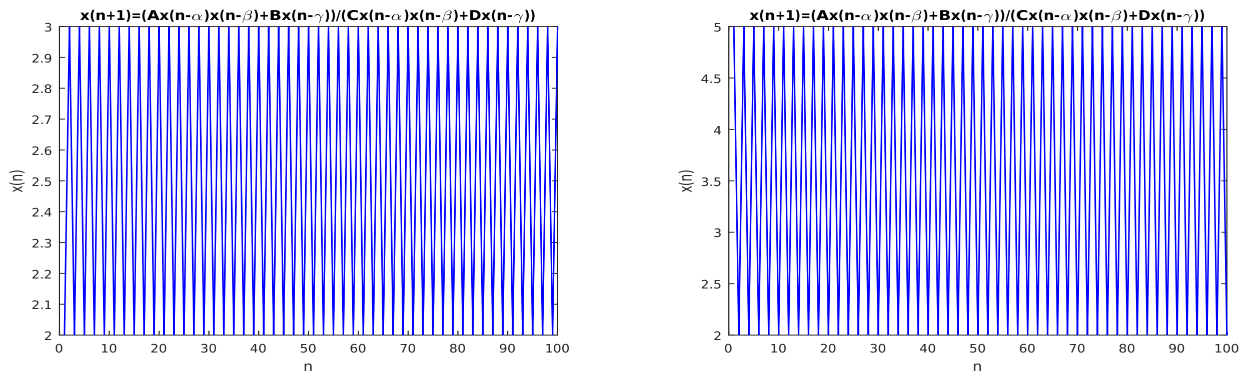

Figure 3: Right $A=6, B=30, C=5, D=6, \alpha=4, \beta=2, \gamma=1, x_{-4}=2, x_{-3}=3, x_{-2}=2, x_{-1}=3, x_0=2$ and left $A=10, B=70, C=7, D=10, \alpha=4, \beta=4, \gamma=5, x_{-5}=5, x_{-4}=2, x_{-3}=5, x_{-2}=2, x_{-1}=5, x_0=2$.

Since $A=10, B=70, C=7, D=10, \alpha=4, \beta=4, \gamma=5, x_{-5}=5, x_{-4}=2, x_{-3}=5, x_{-2}=2, x_{-2}=5, x_0=2$ which satisfy the periodicity conditions then Eq.(1) has positive prime period two solutions $\ldots, 5, 2, 5, 2, \ldots$ (See Figure 3, left).

Lemma 1 —

(i) If α, β and γ are odd, then Eq. (1) has no positive solutions of prime period two.

(ii) If α, β and γ are even, then Eq. (1) has no positive solutions of prime period two.

(iii) If α, γ are odd and β is even (respectively if β, γ are odd and α is even), then Eq. (1) has no positive solutions of prime period two.

(iv) If α is odd, β and γ are even (respectively if α, γ are even and β is odd), then Eq. (1) has no positive solutions of prime period two.

PROOF : Suppose that there exists a prime period two solutions

$$\ldots, p, q, p, q, \ldots$$

of Eq.(1).

We prove this for the first two cases where γ, α and β are odd and for the case where α, β and γ are even. The other cases are similar and will be omitted.

(i) Assume that α, β and γ are odd. We see from Eq.(1) that

$$p = \frac{Ap^2 + Bp}{Cp^2 + Dp} = \frac{Ap + B}{Cp + D},$$

and

$$q = \frac{Aq^2 + Bq}{Cq^2 + Dq} = \frac{Aq + B}{Cq + D}.$$

Hence

$$Cp^2 + (D - A)p - B = 0,$$

and

$$Cq^2 + (D - A)q - B = 0.$$

Then p and q are positive solutions of equation

$$Ct^2 + (D - A)t - B = 0.$$

Since $B > 0$, then one of the solutions is negative. This is a contradiction. Thus Eq.(1) has no prime period two solution.

(ii) Assume that α, β and γ are even. We see from Eq.(1) that

$$q = \frac{Ap^2 + Bp}{Cp^2 + Dp} = \frac{Ap + B}{Cp + D},$$

and

$$p = \frac{Aq^2 + Bq}{Cq^2 + Dq} = \frac{Aq + B}{Cq + D}.$$

Hence

$$Cpq + Dq = Ap + B, \tag{16}$$

and

$$Cpq + Dp = Aq + B. \tag{17}$$

Subtracting Eq.(17) from Eq.(16) we obtain

$$(D + A)(q - p) = 0.$$

Since $D + A \neq 0$, then

$$p = q$$

which is a contradiction and then Eq.(1) has no prime period two solution.

6. CONCLUSION

This paper discussed local and global stability, boundedness and periodicity of the solutions of Eq.(1). In Section 2 we proved that Eq.(1) admits a unique positive equilibrium point given by $\bar{x} = \dfrac{D - A + \sqrt{(D-A)^2 + 4BC}}{2C}$ and that if $|AD - BC| < \dfrac{(C\bar{x} + D)^2}{3}$ then this equilibrium point is locally asymptotically stable. In Section 3 we showed that the unique equilibrium of Eq.(1) is globally asymptotically stable if $D \geq A$. In Section 4 we proved that the solution of Eq.(1) is always bounded and persists. In Section 5 we gave some conditions on the periodicity of solutions of Eq.(1).

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