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SOME PROPERTIES OF LINEAR OPERATORS DEFINED BY HYPERGEOMETRIC FUNCTIONS

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This paper is concerned with the connection of certain operators $\Phi_{b,c}$, defined by hypergeometric functions, with the α-logarithmic Bloch spaces $\mathcal{B}_{\log,\alpha}$ of analytic functions.

Key words : Fractional derivative; α-logarithmic Bloch spaces; nontangential limits; Pochhammer symbol; uniformly locally univalent.

1. Introduction

Let $\mathcal{H}(\mathbb{D})$ denote the class of analytic functions f on the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Also denote by $\mathcal{A}$ the subclass of $\mathcal{H}(\mathbb{D})$ consisting of functions normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

For $\alpha > 0$, the α-logarithmic Bloch space $\mathcal{B}_{\log,\alpha}$ is the Banach space of those functions $f \in \mathcal{H}(\mathbb{D})$ which satisfy

$$\| f \|_{\log,\alpha} := |f(0)| + \sup_{z\in\mathbb{D}}(1 - |z|^2) \left(\log \frac{2}{1 - |z|^2}\right)^{\alpha} |f'(z)| < \infty.$$

For simplicity, the space $\mathcal{B}_{\log,1}$ will be denoted by $\mathcal{B}_{\log}$.

The logarithmic Bloch spaces were studied by many authors (see for example [3] and [4]).

A typical example of an unbounded univalent function in the space $\mathcal{B}_{\log}$ is the function

$$f(z) = \log\log \frac{2}{1 - z}, \qquad (z \in \mathbb{D}).$$

Let us mention also that the logarithmic Bloch spaces play a basic role studying the boundedness of Hankel and Toeplitz operators in Bergman and Bloch-type spaces (see. e.g., [1] and [12]).

Various operators of fractional calculus (that is, fractional integral and fractional derivative) have been studied in the literature (cf. [5, 9-11]). We use the following definition used recently by Srivastava and Owa [10]. The fractional integral of order λ is defined, for a function f, by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}}\, d\zeta, \qquad (\lambda > 0),$$

where f is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta > 0$.

It is easy to check that for $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ we have

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(2+\lambda)} z^{\lambda+1} + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)}{\Gamma(\lambda+n+1)} a_n z^{n+\lambda}, \qquad (z \in \mathbb{D}). \tag{1.1}$$

Also the fractional derivative of order λ is defined, for a function f, by

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}}\, d\zeta, \qquad (0 \le \lambda < 1),$$

where f is a above, and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed, as in previous definition.

Also it is interesting to note that for $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ we have

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(2-\lambda)} z^{1-\lambda} + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+1-\lambda)} a_n z^{n-\lambda}, \qquad (z \in \mathbb{D}). \tag{1.2}$$

Jung *et al*. [5] investigated the linear operator $\Im_\beta^\alpha$ which is defined by

$$\Im_\beta^\alpha f(z) = \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \frac{\alpha}{z^\beta} \int_0^z \left(1 - \frac{t}{z}\right)^{\alpha-1} t^{\beta-1} f(t) dt, \qquad (f \in \mathcal{H}(\mathbb{D}), z \in \mathbb{D})$$
$$\alpha > 0; \beta > -1;$$

and they proved for $f \in \mathcal{A}$ with Re$f'(z) > 0$, $\Im_\beta^\alpha f$ is bounded at least for $\alpha > 1$ and $\beta > -1$. In view of ([5], Lemma 4), for function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ we have

$$\Im_\beta^\alpha f(z) = z + \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \sum_{n=2}^{\infty} \frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}\, a_n z^n, \qquad (\alpha > 0, \beta > -1). \tag{1.3}$$

Definition 1.1 — For $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with $f \in \mathcal{H}(\mathbb{D})$, we define

$$\Phi_{b,c} f(z) = \sum_{n=0}^{\infty} \frac{(b)_n}{(c)_n} a_n z^n, \qquad (z \in \mathbb{D}).$$

Here b, c are complex numbers such that $c \neq 0, -1, -2, ..., (b)_0 = 1$ for $b \neq 0$, and for each positive integer n, $(b)_n = b(b+1)(b+2)...(b+n-1)$ is the Pochhammer symbol.

The above series converges absolutely for all $z \in \mathbb{D}$, and hence represents an analytic function in the unit disc $\mathbb{D}$. It should be remarked that by specializing the parameters b, c and using (1.1), (1.2) and (1.3) one can obtain that

$$D_z^{-\lambda} f(z) = \frac{z^{\lambda}}{\Gamma(1+\lambda)} \Phi_{1,1+\lambda} f(z), \qquad (z \in \mathbb{D}, \lambda > 0), \tag{1.4}$$

$$D_z^{\lambda} f(z) = \frac{z^{-\lambda}}{\Gamma(1-\lambda)} \Phi_{1,1-\lambda} f(z), \qquad (z \in \mathbb{D}, 0 \leq \lambda < 1), \tag{1.5}$$

and

$$\Im_{\beta}^{\alpha} f(z) = \frac{\alpha+\beta}{\beta} \Phi_{\beta,\alpha+\beta} f(z), \qquad (z \in \mathbb{D}, \alpha > 0, \beta > 0), \tag{1.6}$$

where $f \in \mathcal{A}$.

It is natural to ask that for which functions f and parameters b, c, $\Phi_{b,c} f \in \mathcal{B}_{\log,\alpha} (\alpha > 0)$. The main purpose of this paper is finding conditions on the parameters b, c such that when f is chosen from suitable classes, $\Phi_{b,c} f \in \mathcal{B}_{\log,\alpha} (\alpha > 0)$.

Remark 1.1 : It is easy to see that the function $g(x) = (1-x)\left(\log \frac{2}{1-x}\right)^{\alpha}$ $(0 < x < 1)$ is bounded and decreasing for $0 \leq \alpha < \ln 2$.

Proving our results we shall use the following well known result

$$\int_0^1 t^{p-1}(1-t)^{q-1} dt = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

where Re $p > 0$ and Re $q > 0$.

2. Main Results

In the following Theorem we prove:

***Theorem* 2.1** — *Let b, c be complex numbers such that $-1 < Re\ c < Re\ b$. Also let $f \in \mathcal{H}(\mathbb{D})$ and $\Phi_{b,c} f \in \mathcal{B}_{\log,\alpha}$ where $0 \leq \alpha < \ln 2$. Then $f \in \mathcal{B}_{\log,\alpha}$. Moreover*

$$\|f\|_{\log,\alpha} \leq |f(0)| + \frac{|\Gamma(b)|\Gamma(Re(b-c))\Gamma(Re\ c+1)}{|\Gamma(c)||\Gamma(b-c)|\Gamma(Re\ b+1)} \|\Phi_{b,c} f\|_{\log,\alpha}.$$

Proof : Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}(\mathbb{D})$ and $\Phi_{b,c} f \in \mathcal{B}_{\log,\alpha}$. By the hypotheses of the theorem, for $z \in \mathbb{D}$ we have

$$\begin{aligned}
\int_0^1 (1-t)^{b-c-1} t^{c-1} \Phi_{b,c} f(tz) dt &= \frac{\Gamma(c)}{\Gamma(b)} \int_0^1 (1-t)^{b-c-1} t^{c-1} \sum_{n=0}^{\infty} \frac{\Gamma(b+n)}{\Gamma(c+n)} a_n t^n z^n dt \\
&= \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(b+n)}{\Gamma(c+n)} a_n z^n \int_0^1 (1-t)^{b-c-1} t^{n+c-1} dt \\
&= \frac{\Gamma(c)\Gamma(b-c)}{\Gamma(b)} \sum_{n=0}^{\infty} a_n z^n \\
&= \frac{\Gamma(c)\Gamma(b-c)}{\Gamma(b)} f(z). \qquad (2.1)
\end{aligned}$$

By Remark 1.1 and relation (2.1), we obtain

$$\begin{aligned}
&(1-|z|^2)\left(\log \frac{2}{1-|z|^2}\right)^{\alpha} |f'(z)| \\
&\leq \frac{|\Gamma(b)|}{|\Gamma(c)||\Gamma(b-c)|} \int_0^1 (1-t)^{Re(b-c)-1} t^{Re\ c} (1-|z|^2)\left(\log \frac{2}{1-|z|^2}\right)^{\alpha} |(\Phi_{b,c} f)'(tz)| dt \\
&\leq \frac{|\Gamma(b)|}{|\Gamma(c)||\Gamma(b-c)|} \int_0^1 (1-t)^{Re(b-c)-1} t^{Re\ c} (1-|tz|^2)\left(\log \frac{2}{1-|tz|^2}\right)^{\alpha} |(\Phi_{b,c} f)'(tz)| dt \\
&\leq \frac{|\Gamma(b)| \|\Phi_{b,c} f\|_{\log,\alpha}}{|\Gamma(c)||\Gamma(b-c)|} \int_0^1 (1-t)^{Re(b-c)-1} t^{Re\ c} dt \\
&< \infty, \qquad (z \in \mathbb{D}),
\end{aligned}$$

since $-1 < Re\ c < Re\ b$. Therefore $f \in \mathcal{B}_{\log,\alpha}$ and

$$\|f\|_{\log,\alpha} \leq |f(0)| + \frac{|\Gamma(b)|\Gamma(Re(b-c))\Gamma(Re\ c+1)}{|\Gamma(c)||\Gamma(b-c)|\Gamma(Re\ b+1)} \|\Phi_{b,c} f\|_{\log,\alpha}. \square$$

By putting $b = 1$ and $c = 1 - \lambda$ in the Theorem 2.1 and using (1.5) we obtain the following corollary:

Corollary 2.1 — Let $f \in \mathcal{A}$ and $z^{\lambda} D_z^{\lambda} f \in \mathcal{B}_{\log,\alpha}$ where $0 \leq \lambda < 1$ and $0 \leq \alpha < \ln 2$. Then $f \in \mathcal{B}_{\log,\alpha}$ and $\|f\|_{\log,\alpha} = (1-\lambda)\|z^{\lambda} D_z^{\lambda} f\|_{\log,\alpha}$.

A function $f \in \mathcal{H}(\mathbb{D})$ is called uniformly locally univalent if there exists a constant $\rho > 0$ such that f is univalent on the hyperbolic disc $|(z-a)/(1-\bar{a}z)| <$ tanh ρ of radius ρ for every $a \in \mathbb{D}$. It is known that a non-constant analytic function f is uniformly locally univalent if and only if the norm

$$\| f''/f' \| = \sup_{z \in \mathbb{D}} (1-|z|^2) \left| \frac{f''(z)}{f'(z)} \right|$$

of the pre-Schwarzian derivative f''/f' of f is finite. Let

$$B(\lambda) = \{f \in \mathcal{H}(\mathbb{D}); \| f''/f' \| \leq 2\lambda\}.$$

Kim and Sugawa [6, 7] investigated various properties of the functions which belong to the class $B(\lambda)$. In the next theorem we prove:

***Theorem* 2.2** — *Let $\lambda > 0$ and let b, c be complex numbers such that $-1 < Re\, b < Re\, c - \frac{\lambda}{2}$. Also suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is uniformly locally univalent with $\| f''/f' \| \leq \lambda$. Then* $\Phi_{b,c} f \in \mathcal{B}_{\log,\alpha}$ for $0 \leq \alpha < \ln 2$.

PROOF : Let $|z| = r$ $(0 < r < 1)$ and $\| f''/f' \| \leq \lambda$. Then we have

$$\begin{aligned} \log\left|\frac{f'(z)}{f'(0)}\right| &\leq \left|\log\frac{f'(z)}{f'(0)}\right| \\ &= \left|\int_0^z \frac{f''(w)}{f'(w)} dw\right| \\ &\leq r\int_0^1 \left|\frac{f''(tz)}{f'(tz)}\right| dt \\ &\leq r\int_0^1 \frac{\lambda}{1 - r^2t^2} dt \\ &= \lambda \log\sqrt{\frac{1+r}{1-r}}. \end{aligned}$$

This implies

$$|f'(z)| \leq |f'(0)| \left(\frac{1+r}{1-r}\right)^{\frac{\lambda}{2}}, \qquad (|z| = r < 1). \tag{2.2}$$

On the other hand, we have

$$\begin{aligned} \Phi_{b,c} f(z) &= \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(b+n)}{\Gamma(c+n)} . \frac{c+n}{c+n} a_n z^n \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b+1)} \sum_{n=0}^{\infty} \frac{\Gamma(b+n)\Gamma(c-b+1)}{\Gamma(c+n+1)} (c+n) a_n z^n \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b+1)} \int_0^1 t^{b-1}(1-t)^{c-b} \sum_{n=0}^{\infty} (c+n) a_n z^n t^n dt \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b+1)} \int_0^1 t^{b-1}(1-t)^{c-b} (cf(tz) + tzf'(tz)) dt. \end{aligned} \tag{2.3}$$

Set $h(z) = cf(z) + zf'(z)$. Since $\| f''/f' \| \leq \lambda$, we obtain $|f''(z)| \leq \frac{\lambda |f'(z)|}{1-|z|^2} (z \in \mathbb{D})$, and so

$$|h'(z)| \leq |f'(z)| \left(\frac{\lambda |z|}{1-|z|^2} + |c| + 1\right), \qquad (z \in \mathbb{D}). \tag{2.4}$$

Then in view of (2.3) and (2.4) we have

$$|(\Phi_{b,c}f)'(z)| \leq \frac{|\Gamma(c)|}{|\Gamma(b)||\Gamma(c-b+1)|} \int_0^1 t^{Re\ b}(1-t)^{Re(c-b)} |f'(tz)| \left(\frac{\lambda t|z|}{1-t^2|z|^2} + |c| + 1 \right) dt$$

and by relation (2.2), for $|z| = r$ $(0 < r < 1)$, we obtain

$$\begin{aligned}
&(1-|z|^2)\left(\log \frac{2}{1-|z|^2}\right)^{\alpha} |(\Phi_{b,c}f)'(z)| \\
&\quad \leq \frac{|\Gamma(c)|}{|\Gamma(b)||\Gamma(c-b+1)|} \int_0^1 t^{Re\ b}(1-t)^{Re(c-b)}(1-|z|^2)\left(\log \frac{2}{1-|z|^2}\right)^{\alpha} |f'(0)| \\
&\qquad \times \left(\frac{1+tr}{1-tr}\right)^{\frac{\lambda}{2}} \left(\frac{\lambda t|z|}{1-t^2|z|^2} + |c| + 1 \right) dt \\
&\quad \leq \frac{|\Gamma(c)||f'(0)|}{|\Gamma(b)||\Gamma(c-b+1)|} (\log 2)^{\alpha} \int_0^1 t^{Re\ b}(1-t)^{Re(c-b)} \left(\frac{2}{1-t}\right)^{\frac{\lambda}{2}} \left(\frac{\lambda t}{1-t} + |c| + 1 \right) dt \\
&\quad \leq \frac{|\Gamma(c)||f'(0)|2^{\frac{\lambda}{2}}}{|\Gamma(b)||\Gamma(c-b+1)|} \lambda (\log 2)^{\alpha} \int_0^1 t^{Re\ b+1}(1-t)^{Re(c-b)-\frac{\lambda}{2}-1} \\
&\qquad + \frac{|\Gamma(c)||f'(0)|2^{\frac{\lambda}{2}}}{|\Gamma(b)||\Gamma(c-b+1)|} (|c|+1)(\log 2)^{\alpha} \int_0^1 t^{Re\ b}(1-t)^{Re(c-b)-\frac{\lambda}{2}} \\
&\quad < \infty
\end{aligned}$$

since $-1 < \text{Re}\ b < \text{Re}\ c - \frac{\lambda}{2}$ and so $\Phi_{b,c}f \in \mathcal{B}_{\log,\alpha}$.

By putting $b = 1$ and $c = 1 + \lambda$ in the Theorem 2.2 and using (1.4) we obtain the following corollary:

Corollary 2.2 — Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in B(p)$ and $0 < p < \lambda$. Then $z^{-\lambda} D_z^{-\lambda} f \in \mathcal{B}_{\log,\alpha}$ for $0 \leq \alpha < \ln 2$.

Also by putting $b = \beta$ and $c = \gamma + \beta$ in the Theorem 2.2 and using (1.6) we obtain:

Corollary 2.3 — Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in B(p)$ and $0 < p < \gamma$. Then $\Im_{\beta}^{\gamma} f \in \mathcal{B}_{\log,\alpha}$ for $0 \leq \alpha < \ln 2$ and $\beta > 0$.

***Theorem* 2.3** — *Let* $0 \leq \alpha < \ln 2$ *and let* $0 < b < c - 1$. *Also suppose* $f(z) = \sum_{n=1}^{\infty} a_n z^n$ *and* $g(z) = \sum_{n=1}^{\infty} \frac{a_n}{n^{c-b}} z^n$ *be analytic in* $\mathbb{D}$ *where* $\{a_n\}$ *is a bounded sequence in* $\mathbb{C}$. *Then* $g \in \mathcal{B}_{\log,\alpha}$ *if and only if* $\Phi_{b,c}f \in \mathcal{B}_{\log,\alpha}$.

PROOF : From Stirlings formula, we have the following asymptotic expansion for the gamma function $(|\arg z| \leq \pi - \epsilon, \epsilon > 0)$ ([8], p. 88):

$$\Gamma(z) \approx e^{-z} z^z \sqrt{\frac{2\pi}{z}} \left(1 + \frac{1}{12z} + \frac{1}{288z^2} + ...\right) \qquad (z \to \infty).$$

Hence we obtain (as $n \to \infty$)

$$\frac{\Gamma(b+n)}{\Gamma(c+n)} \approx e^{c-b}\left(\frac{b+n}{c+n}\right)^{b+n-\frac{1}{2}}(c+n)^{b-c}\left(1+\frac{c-b}{12(b+n)(c+n)}+...\right). \tag{2.5}$$

Note that (as $n \to \infty$)

$$\left(\frac{b+n}{c+n}\right)^{b+n-\frac{1}{2}} \approx e^{-(c-b)}\left(1+\frac{c-b}{2(b+n)}+...\right) \tag{2.6}$$

and

$$(c+n)^{b-c} = n^{b-c}\left(1-\frac{-c}{n}\right)^{-(c-b)} = n^{b-c}\left(1+\frac{-c(c-b)}{n}+...\right). \tag{2.7}$$

Therefore by relations (2.5) to (2.7), we have

$$\frac{\Gamma(b+n)}{\Gamma(c+n)} \approx n^{b-c}\sum_{i=0}^{\infty} A_i n^{-i} \qquad (n \to \infty) \tag{2.8}$$

where the A_i are constants depending on b, c with $A_0 = 1$. From (2.8) we obtain

$$\begin{aligned}\Phi_{b,c}f(z) &= \frac{\Gamma(c)}{\Gamma(b)}\sum_{n=1}^{\infty}\frac{\Gamma(b+n)}{\Gamma(c+n)}a_n z^n \\ &\approx \frac{\Gamma(c)}{\Gamma(b)}\sum_{n=1}^{\infty} n^{b-c}\left\{\sum_{i=0}^{\infty} A_i n^{-i}\right\} a_n z^n,\end{aligned}$$

and so

$$\frac{\Gamma(b)}{\Gamma(c)}\Phi_{b,c}f(z) = g(z) + \sum_{n=1}^{\infty} n^{b-c} O\left(\frac{1}{n}\right) a_n z^n \tag{2.9}$$

where $g(z) = \sum_{n=1}^{\infty} \frac{a_n}{n^{c-b}} z^n$. Since $a_n = O(1)$, we have

$$\begin{aligned}\left|\sum_{n=1}^{\infty} n^{b-c+1} O\left(\frac{1}{n}\right) a_n z^{n-1}\right| &\le \left(\max_{n\in\mathbb{N}} |a_n|\right)\sum_{n=1}^{\infty}\left|O\left(\frac{1}{n^{c-b}}\right)\right| \\ &< \infty\end{aligned}$$

since $0 < b < c-1$ and so there exists $N > 0$ such that

$$\left|\sum_{n=1}^{\infty} n^{b-c+1} O\left(\frac{1}{n}\right) a_n z^{n-1}\right| < N, \qquad (z \in \mathbb{D}). \tag{2.10}$$

Now, suppose $g \in \mathcal{B}_{\log,\alpha}$ where $0 \le \alpha < \ln 2$. From (2.9) we have

$$(\Phi_{b,c}f)'(z) = \frac{\Gamma(c)}{\Gamma(b)}\left(g'(z) + \sum_{n=1}^{\infty} n^{b-c+1} O\left(\frac{1}{n}\right) a_n z^{n-1}\right), \qquad (z \in \mathbb{D}),$$

and so by (2.10)

$$(1-|z|^2)\left(\log\frac{2}{1-|z|^2}\right)^{\alpha}|(\Phi_{b,c}f)'(z)| \le \frac{|\Gamma(c)|}{|\Gamma(b)|}\left(\|g\|_{\log,\alpha}+N(1-|z|^2)\left(\log\frac{2}{1-|z|^2}\right)^{\alpha}\right)$$
$$\le \frac{|\Gamma(c)|}{|\Gamma(b)|}\left(\|g\|_{\log,\alpha}+N(\log 2)^{\alpha}\right)$$
$$<\infty, \qquad (z\in\mathbb{D}).$$

Therefore $\Phi_{b,c}f\in\mathcal{B}_{\log,\alpha}$.

Conversely, suppose $\Phi_{b,c}f\in\mathcal{B}_{\log,\alpha}$. From (2.9),

$$g'(z)=\frac{\Gamma(b)}{\Gamma(c)}(\Phi_{b,c}f)'(z)-\sum_{n=1}^{\infty}n^{b-c+1}O\left(\frac{1}{n}\right)a_nz^{n-1}, \qquad (z\in\mathbb{D}),$$

and therefore we obtain

$$\|g\|_{\log,\alpha}\le\frac{|\Gamma(b)|}{|\Gamma(c)|}\|\Phi_{b,c}f\|_{\log,\alpha}+\sup_{z\in\mathbb{D}}(1-|z|^2)\left(\log\frac{2}{1-|z|^2}\right)^{\alpha}N$$
$$\le\frac{|\Gamma(b)|}{|\Gamma(c)|}\|\Phi_{b,c}f\|_{\log,\alpha}+N(\log 2)^{\alpha}.$$

So $g\in\mathcal{B}_{\log,\alpha}$ and the proof is completed. □

By putting $b=1$ and $c=1+\lambda$ in the Theorem 2.3 and using (1.4) we obtain the following corollary:

Corollary 2.4 — Let $0\le\alpha<\ln 2$ and $\lambda>1$. Also let $f(z)=\sum_{n=1}^{\infty}a_nz^n\in\mathcal{A}$ where $\{a_n\}$ is a bounded sequence in $\mathbb{C}$. Then $\sum_{n=1}^{\infty}n^{-\lambda}a_nz^n\in\mathcal{B}_{\log,\alpha}$ if and only if $z^{-\lambda}D_z^{-\lambda}f\in\mathcal{B}_{\log,\alpha}$.

Also by putting $b=\beta$ and $c=\gamma+\beta$ in the Theorem 2.3 and using (1.6) we obtain:

Corollary 2.5 — Let $0\le\alpha<\ln 2$ and $\gamma>1$. Also let $f(z)=\sum_{n=1}^{\infty}a_nz^n\in\mathcal{A}$ where $\{a_n\}$ is a bounded sequence in $\mathbb{C}$. Then $\sum_{n=1}^{\infty}n^{-\gamma}a_nz^n\in\mathcal{B}_{\log,\alpha}$ if and only if $\Im_{\beta}^{\gamma}f\in\mathcal{B}_{\log,\alpha}$.

Corollary 2.6 — Suppose $0<b<c-1$ and $f(z)=\sum_{n=1}^{\infty}a_nz^n\in\mathcal{H}(\mathbb{D})$ where $\{a_n\}$ is a bounded sequence in $\mathbb{C}$. Then $\Phi_{b,c}f$ has nontangential limits in almost every direction.

PROOF : Set $g(z)=\sum_{n=1}^{\infty}\frac{a_n}{n^{c-b}}z^n$, $(z\in\mathbb{D})$. By relation (2.9) from Theorem 2.3, we have

$$\frac{\Gamma(b)}{\Gamma(c)}\Phi_{b,c}f(z)=g(z)+\sum_{n=1}^{\infty}n^{b-c}O\left(\frac{1}{n}\right)a_nz^n, \qquad (z\in\mathbb{D}).$$

Since $a_n=O(1)$ and $c-b>1$, we obtain

$$|g(z)|\le\left(\max_{n\in\mathbb{N}}|a_n|\right)\sum_{n=1}^{\infty}\frac{1}{n^{c-b}}<\infty, \qquad (z\in\mathbb{D})$$

and

$$\left|\sum_{n=1}^{\infty} n^{b-c} O\left(\frac{1}{n}\right) a_n z^n\right| \leq \left(\max_{n\in\mathbb{N}} |a_n|\right) \sum_{n=1}^{\infty} \left|O\left(\frac{1}{n^{c-b+1}}\right)\right|$$
$$< \infty, \qquad (z \in \mathbb{D}).$$

Therefore $\Phi_{b,c}f$ is bounded and so has nontangential limits in almost every direction. □

The next theorem deals with the case $0 < b < c+1$. For its proof we shall need the following result due to Clunie and Macgrogor [2].

***Theorem* 2.4** — *Let f be analytic and univalent in $\mathbb{D}$ and let $\gamma > 1/2$. Then there is a set E of measure 2π such that for all $\theta \in E$, if $\angle_\alpha$ is a Stolz angle with vertex $\alpha = e^{i\theta}$,*

$$\lim_{z\to\alpha} \frac{\log|f'(z)|}{\left(\log \frac{1}{1-|z|}\right)^\gamma} = 0, \qquad (z \in \angle_\alpha).$$

***Theorem* 2.5** — *Suppose f is analytic and univalent in $\mathbb{D}$ and $0 < b < c+1$. Then $\Phi_{b,c}f$ has nontangential limits in almost every direction.*

PROOF : Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic and univalent in $\mathbb{D}$ and fix γ in the interval $(1/2, 1)$. By Theorem 2.4, there exists some constant A and a set E of measure 2π such that for all $\theta \in E$

$$\log|f'(z)| \leq A\left(\log \frac{1}{1-|z|}\right)^\gamma$$

whenever $\alpha = e^{i\theta}$ and $z \in \angle_\alpha$. For any $\delta > 0$ we have

$$(1-|z|)^\delta |f'(z)| \leq (1-|z|)^\delta \exp\left\{A\left(\log\frac{1}{1-|z|}\right)^\gamma\right\}$$
$$= \exp\left\{A\left(\log\frac{1}{1-|z|}\right)^\gamma - \delta \log\frac{1}{1-|z|}\right\}.$$

Since $\gamma < 1$, it is clear that this last expression approaches zero as $|z| \to 1$. Hence

$$|f'(z)| \leq \frac{B}{(1-|z|)^\delta} \tag{2.11}$$

for some constant B whenever z lies in the Stolz angle $\angle_\alpha$. If $g(z) = \sum_{n=0}^{\infty}(c+n)a_n z^n = cf(z) + zf'(z)$, then from (2.11) we obtain

$$|g(z)| \leq |cf(0)| + |c|rB\int_0^1 \frac{1}{(1-tr)^\delta}dt + r\frac{B}{(1-r)^\delta}$$
$$\leq |cf(0)| + \frac{B|c|}{1-\delta} + \frac{rB}{(1-r)^\delta} \tag{2.12}$$

for all $z \in \angle_\alpha(|z| = r)$. Using the same argument as in the proof of Theorem 2.2 we know that

$$\Phi_{d,c}f(z) = \frac{\Gamma(c)}{\Gamma(d)\Gamma(c-d+1)} \int_0^1 t^{d-1}(1-t)^{c-d} g(tz)dt. \tag{2.13}$$

Let d be such that $b < d < c+1$, and choose $\delta > 0$ so that $\delta < c - d + 1$. Then in view of (2.12) and (2.13), for all $z \in \angle_\alpha(|z| = r)$, we have

$$\begin{aligned}
|\Phi_{d,c}f(z)| &\le \frac{|\Gamma(c)|}{\Gamma(d)\Gamma(c-d+1)} \int_0^1 t^{d-1}(1-t)^{c-d}|g(tz)|dt \\
&\le \frac{|\Gamma(c)|}{\Gamma(d)\Gamma(c-d+1)} \int_0^1 t^{d-1}(1-t)^{c-d}\left(|cf(0)| + \frac{B|c|}{1-\delta} + \frac{Btr}{(1-tr)^\delta}\right) dt \\
&\le \frac{|\Gamma(c)|}{\Gamma(d)\Gamma(c-d+1)} \left(\left(|cf(0)| + \frac{B|c|}{1-\delta}\right) \int_0^1 t^{d-1}(1-t)^{c-d}dt + B\int_0^1 t^d(1-t)^{c-d-\delta}dt\right) \\
&= K < \infty.
\end{aligned}$$

Hence $\Phi_{d,c}f$ is bounded inside $\angle_\alpha$. Also

$$\begin{aligned}
\Phi_{b,c}f(z) &= \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(b+n)}{\Gamma(d+n)} . \frac{\Gamma(d+n)}{\Gamma(c+n)} a_n z^n \\
&= \frac{\Gamma(c)}{\Gamma(b)\Gamma(d-b)} \int_0^1 t^{b-1}(1-t)^{d-b-1}\Psi(tz)dt
\end{aligned} \tag{2.14}$$

where

$$\Psi(z) = \sum_{n=0}^{\infty} \frac{\Gamma(d+n)}{\Gamma(c+n)} a_n z^n, \qquad (z \in \mathbb{D}).$$

Therefore

$$|\Psi(z)| = \frac{\Gamma(d)}{|\Gamma(c)|}|\Phi_{d,c}f(z)| < \frac{\Gamma(d)}{|\Gamma(c)|}K. \tag{2.15}$$

Let

$$h(t) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(d-b)} t^{b-1}(1-t)^{d-b-1}.$$

Then $h > 0$, $\int_0^1 h(t)dt = \frac{\Gamma(c)}{\Gamma(d)}$, and by (2.14),

$$\Phi_{b,c}f(z) = \int_0^1 h(t)\Psi(tz)dt. \tag{2.16}$$

These properties imply that $\Phi_{b,c}f$ is uniformly continuous on the set $\angle_\alpha$. To see this, let $z_1, z_2 \in \angle_\alpha$ and let $\epsilon > 0$ be given. Since $\int_0^1 h(t)dt$ exists, there is an $x \in (0,1)$ such that

$$\int_{1-x}^1 h(t)dt < \frac{\epsilon}{4K} \frac{|\Gamma(c)|}{\Gamma(d)}. \tag{2.17}$$

By the uniform continuity of Ψ on compact subset of $\mathbb{D}$, there exists $\lambda > 0$ such that

$$|\Psi(tz_2) - \Psi(tz_1)| < \frac{\epsilon\Gamma(d)}{2|\Gamma(c)|} \tag{2.18}$$

whenever $|z_2 - z_1| < \lambda$ with $z_1, z_2 \in \angle_\alpha$ and for all $t \in [0, 1-x]$. Therefore, by relations (2.15) to (2.18), for $|z_2 - z_1| < \lambda$ we have

$$\begin{aligned} |\Phi_{b,c}f(z_2) - \Phi_{b,c}f(z_1)| &\le \int_0^{1-x} h(t)|\Psi(tz_2) - \Psi(tz_1)|dt + \int_{1-x}^1 h(t)|\Psi(tz_2) - \Psi(tz_1)|dt \\ &< \frac{\epsilon\Gamma(d)}{2|\Gamma(c)|}\int_0^{1-x} h(t)dt + 2K\frac{\Gamma(d)}{|\Gamma(c)|}\int_{1-x}^1 h(t)dt \\ &< \frac{\epsilon\Gamma(d)}{2|\Gamma(c)|}\int_0^1 h(t)dt + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Hence $\Phi_{b,c}f$ is uniformly continuous inside $\angle_\alpha$ and so can be extended continuously to the boundary. This implies that

$$\lim_{z\to\alpha} \Phi_{b,c}f(z) \qquad (z \in \angle_\alpha),$$

exists for every $\theta \in E$, where $\alpha = e^{i\theta}$. Since E has measure 2π, this proves the theorem. □

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